Topological Phases of Matter*

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I. INTRODUCTION

The Nobel Prize in Physics 2016 was sahared among David J.Thouless (half of the prize), F. Duncan Haldane ($\frac{1}{4}$ of the prize), and J. Michael Korstelitz ($\frac{1}{4}$ of the prize), for ***"theoretical discoveries of topological phase transitions and topological phases of matter"***. The Nobel prize rewarded Thouless and Haldane for their contributions in understanding topological phases of matter [1], and rewarded Korstelitz and Thouless for their contributions to understanding topological phase transitions [2]. The idea could be simplified as: looking at an electron moving in a solid and exploring its wave function using topological concepts, one can explain a number of experiments. Kosterlit and Thouless introduced a new order parameter for phase transition based on principles of topology. They introduced a new definition of long range order based on golbal properties of a low-dimensional solid, as opposed to interactions between two points (spin-spin interactions for example) in the solid [2]. This long range, "topological order" exists in 2D solids and neutral superfluids at a finite temperature, and is due to the proliferation of the unbinding of vortex-antivortex pairs also referred to as *topological defects* in the 2D solid. The idea of having a phase transition based on topological defects is now studied in many low-dimensional systems such as superfluid ^4He, 2D Bose-Einstein condensates, superconductors.

Order parameters vs topological invariants: what

II. PHASE TRANSITIONS AND ORDER PARAMETERS

The microscopic realization of a crystal reveals a lattice of ordered atoms. The lattice has a large N number of atoms arranged in an ordered manner, with translational and rotational symmetry. To classify and explain the properties of materials, many theories were proposed. The *principle of emergence* motivated the development of phenomenological methods to explain the properties of matter, which are based on a local *order parameter* such as the magnetization M for ferromagnetic phase transition as opposed to microscopic features such as spins, atoms, and molecules. These *order parameters* characterize the organization of microscopic particles in the system, and are functions of coarse-grained variables such as temperatures and pressure. In his development of a theory of phase transition, Landau and Ginzburg [3–5] successfully associated different orders and hence different phases of materials, to different symmetries. He proposed that a discontinuity of the order parameter would cause a change of phase (or a phase transition), by spontaneously breaking the symmetry of the material.

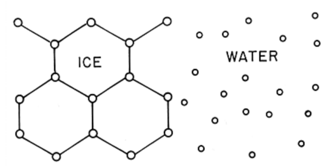

FIG. 1: Ice breaks translational and rotational symmetries of water.

Crystals can be classified according to what types of symmetries are broken by their lattices. Crystals break translational and rotational symmetries of free space (e.g water losing translational and rotational symmetry by transition into ice). Liquid crystals only break the rotational symmetries. Magnets break time-reversal symmetry and the rotational symmetry of spin space. Superfluids (e.g low temperature ^4He at low temperature) break the $U(1)$ symmetry associated with the conservation of particles.

III. BAND THEORY AND THE HALL EXPERIMENT

Band Theory

In the free electron model of a metal, non-interacting electrons form an electron gas and freely move around

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the solid and fill up energy levels while obeying the exclusion principle [6]. At $T = 0\ K$, the highest-filled energy level is the Fermi Energy, E_F. With the potential energy $V(r) = 0$, the electron wave function is a solution to the time-independent Schrodinger equation:

$$-\frac{\hbar^2}{2m_e}\nabla^2 |\phi(k)\rangle = E\,|\phi(k)\rangle \tag{1}$$

$$\text{with } |\phi(k)\rangle = Ae^{ikr} \tag{2}$$

$$\text{and energy eigenvalues } E = \frac{\hbar^2 k^2}{2m_e} \tag{3}$$

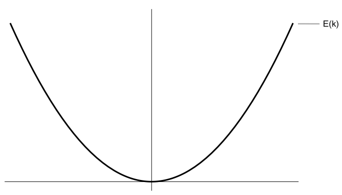

FIG. 2: Free electron dispersion relation.

The free electron model assumes a constant potential in the solid, as well as a impenetrable barrier at the edges of the solid. Modeling electrons in a solid as being in a periodic potential rather than completely free provides a better understanding of material properties (e.g insulator, metal, semimetal, semiconductor). Here, the potential drops at each lattice points where ion cores (sources of the potential) are located, with the periodicity of the inter-atomic separation a. Due to the interaction between electrons (fermions) and the exclusion principle, degeneracy is split for the large number (N) of electrons in the lattice's periodic potential, which forms energy bands. Given the symmetry of the periodic potential $(U(r + na) = U(r))$, solutions to the Schrodinger equation are limited to so-called Bloch wavefunctions. Using **Bloch's Theorem** to represent the wavefunctions, we have:

$$|\Psi_{\mathbf{k}}(\mathbf{r})\rangle = u_{\mathbf{k}}(\mathbf{r})e^{i\mathbf{k}.\mathbf{r}} \tag{4}$$

$$\text{with } u_{\mathbf{k}}(\mathbf{r} + n\mathbf{a}) = u_{\mathbf{k}}(\mathbf{r}) \tag{5}$$

The dispersion relation in the reduced first Brillouin Zone (BZ) shows the expected energy bands, with band gaps due to Bragg diffraction of the electron waves at the edges of the BZ. The energy gap (E_g) reveals a *forbidden zone* that cannot be occupied by the electrons' energy levels. Depending on where E_F lies for the system and the number of electrons per atom, bands can be half-filled (metal), or filled (insulators). This gap is already indicating different states of matter, and a phase transition...

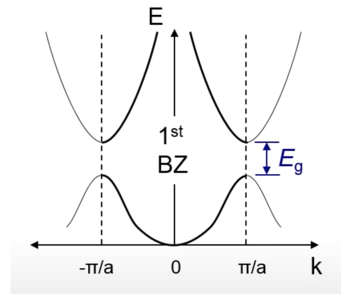

FIG. 3: Reduced Zone Disperson Relation of an Electron in a Periodic Potential

The Classical Hall experiment

In an experiment in 1879, Edwin Hall discovered that when a 2D conductor with current passing through it is placed in a strong magnetic field perpendicular to the plane of the electrons motion, the moving charges experience a Lorentz force ($F \propto j_x, B_z$) which pushes carriers on one side of the conductor, deflecting j_x). The concentration of charges of different signs on two edges of the conductor establishes an electric field which, in steady state, balances the Lorentz force. The from the electric field emerges a potential proportional to the current j_x and the magnetic B_z field, and perpendicular to both.

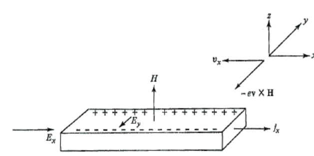

FIG. 4: Hall Experiment

From the Hall experiment it is possible to leverage the Drude model of metals and derive the following quantities that are relevant for the study of phases of matter based on topological invariants.

- Lorentz Force: $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ (6)
- Equilibrium condition: $j_y = 0$ (7)
- Magnetoresistivity: $\rho_{xx} = \dfrac{E_x}{j_x}$ (8)
- Hall (off-diagonal) resistivity: $\rho_{yx} = \dfrac{E_y}{j_x}$ (9)
- Hall coefficient: $R_H \equiv \dfrac{E_y}{j_x B_z} = \dfrac{1}{nq}$ (10)

Where n is the carrier density, and q is the carrier charge ($-e$ for electrons). Given that R_H shares the same sign as the carriers in the solid, the hall coefficient which can be measured experimentally is instrumental in deriving both the sign and the density of the carriers, from the relation in Eq (10).

IV. TOPOLOGICAL INVARIANTS

A. Topology in Physics

Topology is the study of invariants under arbitrary continous transformations of geometrical objects. A topological space can be thought of as a set from which has been swept away all structure irrelevant to the continuity of functions defined on it.

We now turn our attention to phases transitions in electronic matter that are not governed by an order parameter, but rather by a topological invariant.

B. Landau levels

Feels like a SHO
$|\Psi\rangle \otimes_R |\Phi\rangle$
$\partial_x A_y$

$$H = \frac{1}{2m}\left(P_x - \frac{q}{c}(-By)\right)^2 + \frac{1}{2m}P_y^2 + \frac{1}{2m}P_z^2$$

$\implies [H, P_x] = 0$
$\therefore \psi(x,y) = \psi(y)e^{ik_x x}$
$P_x = \hbar k_x$

$$H_{k_x} = \frac{P_y^2}{2m} + \frac{1}{2m}\left(\frac{qBy}{c} + \hbar k_x\right)^2$$

$$H_{k_x} = \frac{P_y^2}{2m} + \frac{1}{2}m\left(\frac{qB}{mc}\right)^2\left(y - \left(-\frac{\hbar k_x c}{qB}\right)\right)^2$$

$$H_{k_x} = \frac{P_y^2}{2m} + \frac{1}{2}m\left(\frac{qB}{mc}\right)^2(y - y_0)^2$$

$$y_0 \equiv -\frac{\hbar k_x c}{qB}$$

C. The Integer Quantum Hall Effect (IQHE)

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & -\nu\frac{e^2}{h} \\ \nu\frac{e^2}{h} & 0 \end{pmatrix}$$

D. Berry Phase

E. TKNN

$$\mathbf{A}(\mathbf{k}) = -i\langle u_i(\mathbf{k})|\nabla_k|u_i(\mathbf{k})\rangle \tag{11}$$

$$\mathbf{F}(\mathbf{k}) = \nabla_k \times \mathbf{A}(\mathbf{k}) \tag{12}$$

$$v = \frac{1}{2\pi}\oint_C \mathbf{A}(\mathbf{k})\cdot d\mathbf{k} = \frac{1}{2\pi}\int_{BZ}\mathbf{F}(\mathbf{k})d^2\mathbf{k} \tag{13}$$

V. EXAMPLES AND CURRENT WORK

Appendix A: Details 1

Appendix A: Details 2

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. Den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Physical Review Letters **49**, 405 (1982), publisher: American Physical Society (APS).

[2] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, Journal of Physics C: Solid State Physics **6**, 1181 (1973), publisher: IOP Publishing.

[3] P. Hohenberg and A. Krekhov, An introduction to the Ginzburg–Landau theory of phase transitions and nonequilibrium patterns, Physics Reports **572**, 1 (2015), publisher: Elsevier BV.

[4] 29 - ON THE THEORY OF PHASE TRANSITIONS, in *Collected Papers of L.D. Landau*, edited by D. TER HAAR (Pergamon, 1965) pp. 193–216.

[5] X.-G. Wen, Colloquium : Zoo of quantum-topological phases of matter, Reviews of Modern Physics **89**, 10.1103/revmodphys.89.041004 (2017), publisher: American Physical Society (APS).

[6] C. Kittel, *Introduction to Solid State Physics* (Wiley, 2004).