Bézier Curves

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Bézier Curve Formulation

Given a series of lines $\overline{P_0P_1}, ..., \overline{P_{n-1}P_n}$, the resulting Bézier curve, $B_{P_0...P_n}(t)$ can be described as follows:

$B_{P_0...P_n}(t) = \sum_{i=0}^{n} \binom{n}{i} P_i(1-t)^i t^{n-i}$

Where $0 \leq t \leq 1$.

Proof. Using the algorithm for the construction of Bézier curves and induction:

The trivial case, describing $B_{P_0P_1}$:

$B_{P_0P_1} = P_0 + (P_1 - P_0)t$ since the Bézier curve is an interpolation of P_0 and P_1 along $\overline{P_0P_1}$.
$\therefore B_{P_0P_1} = P_0(1-t) + P_1t = \sum_{i=0}^{1} \binom{1}{i} P_i(1-t)^i t^{1-i}$

So the hypothesis is true for the base case. Applying the Interpolation Lemma to the induction hypothesis, it can be shown that the hypothesis is true for any $(n+1)^{\text{th}}$ iteration if it is true for the n^{th}.

It follows from the algorithm for the construction of Bézier curves that $B_{P_0...P_{n+1}}(t) = B_{B_{P_0...P_n}(t)B_{P_1...P_{n+1}}(t)}(t)$
$\therefore B_{P_0...P_{n+1}}(t) = [B_{P_0...P_n}] \cdot (1-t) + [B_{P_1...P_{n+1}}] \cdot (t)$

Which, by the induction hypothesis, is equal to the following:

$[\sum_{i=0}^{n} \binom{n}{i} P_i(1-t)^i t^{n-i}] \cdot (1-t) + [\sum_{i=0}^{n} \binom{n}{i} P_{i+1}(1-t)^i t^{n-i}] \cdot (t)$

$= \sum_{i=0}^{n} \binom{n}{i} P_i(1-t)^{i+1} t^{n-i} + \sum_{i=0}^{n} \binom{n}{i} P_{i+1}(1-t)^i t^{(n+1)-i}$

$= \sum_{i=0}^{n+1} [\binom{n}{i} + \binom{n}{i-1}] P_i(1-t)^i t^{n-i}$

$= \sum_{i=0}^{n+1} [\binom{n+1}{i}] P_i(1-t)^i t^{(n+1)-i}$ by Pascal's Triangle Binomial Lemma

This proves that the following is true:
$B_{P_0...P_n}(t) = \sum_{i=0}^{n} \binom{n}{i} P_i(1-t)^i t^{n-i} \implies B_{P_0...P_{n+1}}(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} P_i(1-t)^i t^{(n+1)-i}$

So $B_{P_0...P_n}(t) = \sum_{i=0}^{n} \binom{n}{i} P_i(1-t)^i t^{n-i}$

And this concludes the proof.

□

(Pascal's Triangle Binomial Lemma is proved on the next page)

Pascal's Triangle Binomial Lemma

$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$

Proof. This is provable by direct computation:

Simply expand $\binom{n}{i}$:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Then expand $\binom{n}{i-1}$:

$$\binom{n}{i-1} = \frac{n!}{(i-1)!(n-i+1)!}$$

Then add the two:

$$\binom{n}{i} + \binom{n}{i-1} = \frac{n!}{i!(n-i)!} \cdot \frac{(n-i+1)}{(n-i+1)} + \frac{n!}{(i-1)!(n-i+1)!} \cdot \frac{(i)}{(i)} = \frac{n!(n-i+1) + n!(i)}{i!(n+1-i)!}$$

$$= \frac{n!(n+1)}{i!(n+1-i)!} = \frac{(n+1)!}{i!(n+1-i)!} = \binom{n+1}{i}$$

□

Notes

Credit for showing Bézier curves to me goes to Vsevolod Rostovtsev

The animations and explanations of Jason Davies and Mike Kamermans were accessible, mesmerizing and intuitive. This is a rigorous proof that the explicit parametric formula given to describe Bézier curves is in fact correct. Hopefully this provides some helpful insight.

Jason Davies' page: https://www.jasondavies.com/animated-bezier/
Mike Kamerman's page: https://pomax.github.io/bezierinfo/