The Euler-Maclaurin Formula

Daniel Rostovtsev
Date: June 19, 2017

The Euler-Maclaurin formula describes the difference between the sum and integral of a function over a given interval using nth derivatives and Bernoulli numbers. It was discovered at the same time, without collaboration, by Euler and Maclaurin in the 1730s. Euler used the formula to compute the values of infinite series, and Maclaurin used to compute the values of inevaluable integrals.

Take B_i and ω_i to be the ith Bernoulli numbers and periodic polynomials, respectively. For any m-differntiable function the difference between the sum of f and its integral is as follows:

$$\int_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = \sum_{n=1}^{m} (-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+1}}{m!} \int_a^b \omega_m(t) f^m(t)dt$$

First, a few quick facts about Bernoulli polynomials:

Definition: Periodic Bernoulli Polynomials

A "periodic" Bernoulli polynomial is defined as follows:

$$\omega_i(t) = \beta_i(t - \lfloor t \rfloor) = \beta_i(\{t\})$$

Where $\lfloor t \rfloor$ is the greatest integer less than t, and $\{t\}$ is the fractional part of t.

Proposition 1

For $i \neq 1$, $\omega_i(t)$ is continuous and $\omega_i(0) = \omega_i(k) = B_i$ for all $k \in \mathbb{Z}$

Proof. First, it must be shown that $\beta_i(0) = \beta_i(1) = B_i$, and the rest follows from the definition of ω_i.

$$\beta_n(0) = \sum_{i=0}^{n} B_i(0)^{n-i} = B_n$$

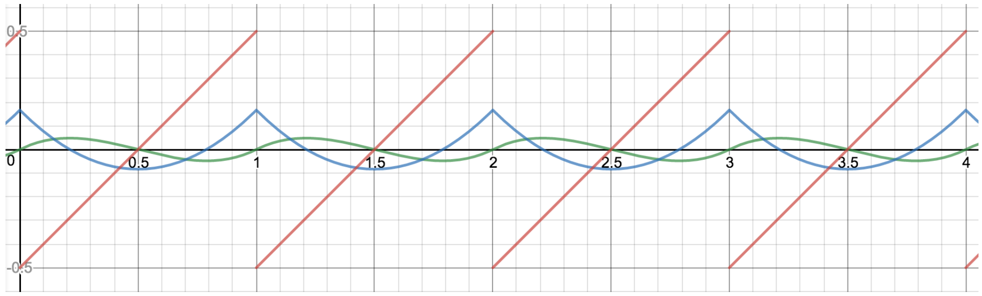
$$\beta_n(1) = \sum_{i=1} \binom{n}{i} B_i(1)^{n-1} = \sum_{i=0}^{n} \binom{n}{i} B_i = B_n + \sum_{i=0}^{n-1} \binom{n}{i} B_i$$

By definition, however $\displaystyle\sum_{i=0}^{n-1} \binom{n}{i} B_i =$ for all $n \neq 1$.

Since $\beta_i(0) = \beta_i(1) = 0$, and β_i is continuous (as it is simply a polynomial), it follows that ω_i is continuously "linked" and $\omega_i(k) = B_i$ for all $k \in \mathbb{Z}$ as long as $i \neq 1$.

□

As a sidenote, here is a graph of the first three periodic Bernoulli polynomials: (red is first, blue is second and green is third)



Proposition 2

$\frac{d}{dt}\beta_n(t) = n\beta_{n-1}(t)$, and excluding any discontinuities at $t = 1$, $\frac{d}{dt}\omega_n(t) = n\omega_{n-1}(t)$

Proof. Computing $\frac{d}{dt}\beta_n$:

$$\frac{d}{dt}B_n(t) = \frac{d}{dt}\sum_{i=0}^{n}\binom{n}{i}B_i(t)^{n-i} = \sum_{i=0}^{n}\binom{n}{i}B_i\frac{d}{dt}(t)^{n-i} = \sum_{i=0}^{n}\binom{n}{i}B_i \cdot (n-i)(t)^{n-i-1}$$

However, $\binom{n}{i}(n-i)$ simplifies to $n\binom{n-1}{i}$ since:

$$\binom{n}{i}(n-i) = \frac{n!}{i!(n-i)!}(n-i) = \frac{n\cdot(n-1)!}{i!(n-i-1)!} = n\binom{n-1}{i}$$

It follows then that:

$$\frac{d}{dt}\beta_n(t) = n\sum_{i=0}^{n-1}\binom{n-1}{i}B_i(t)^{n-i-1} = n\,\beta_{n-1}(t)$$

Also, by the definiton of $\omega_n(t)$, if t is not an integer, then $\frac{d}{dt}\omega_n(t) = n\,\omega_{n-1}(t)$

□

Proposition 3

$\int\beta_n(t)dt = \frac{1}{n+1}\beta_{n+1}(t)$ and $\int\omega_n(t)dt = \frac{1}{n+1}\omega_{n+1}(t)$

Proof. This follows directly from the second proposition.

□

Now to derive the Euler-Maclaurin Formula. Consider the diffence between $f(k)$ and $\int_k^{k+1} f(t)dt$ for some $k \in \mathbb{Z}$. Using integration by parts, the integral of $f(t)$ over the unit interval can be expressed in terms of periodic Bernoulli polynomials. Then the difference can be computed as the sums of a sequence of Bernoulli numbers and nth derivatives of f.

Lemma 1

For a differentaible function f defined on $[k, k+1]$:

$$\int_k^{k+1} f(t)dt = \lim_{n\to(k+1)^-} \omega_1(t)f(t) - \lim_{n\to k^+} \omega_1(t)f(t) - \int_k^{k+1} \omega_1(t)f'(t)dt$$

Equivalently: $\displaystyle\int_k^{k+1} f(t)dt = \frac{1}{2}(f(k+1)+f(k)) - \int_k^{k+1} \omega_1(t)f'(t)dt$

Proof. This follows from integration by parts and the fact that $\omega_0(t) = 1$.

$$\int_k^{k+1} f(t)dt = \int_k^{k+1} (1)\cdot f(t)dt = \int_k^{k+1} \omega_0(t)f(t)dt$$

Using integration by parts, let $dv = \omega_0(t)dt$ and $u = f(t)$. Then $v = \frac{1}{n+1}\omega_1(t)$ (from proposition 3) and $du = f'(t)dt$. Continuing with integration by parts, conventionally:

$$\int_k^{k+1} f(t)dt = f(t)\omega_1(t)|_k^{k+1} - \int_k^{k+1} \omega_1(t)f'(t)dt$$

Yet $\omega_1(t)$ is discontinuous at $t = 1$. Using a visual argument, it is clear from the graph earlier that the following difference of limits should be computed instead, as follows:

$$\int_k^{k+1} f(t)dt = \lim_{n\to(k+1)^-} \omega_1(t)f(t) - \lim_{n\to k^+} \omega_1(t)f(t) - \int_k^{k+1} \omega_1(t)f'(t)dt$$

Since $\lim_{n\to k^+} \omega_1(t) = -\frac{1}{2}$ and $\lim_{n\to(k+1)^-} \omega_1(t) = \frac{1}{2}$:

$$\int_k^{k+1} f(t)dt = \frac{1}{2}(f(k+1)+f(k)) - \int_k^{k+1} \omega_1(t)f'(t)dt$$

Later, to prove the Euler-Maclaurin formula, integration by parts will be used again to compute $\int_k^{k+1} \omega_n(t)f^n(t)dt$ for higher n. In those cases, however, there will be no discontinuities of ω_n, so the integral will be easier to evaluate.

□

Lemma 2

$\forall\ n \geq 1$ and $\forall\ a, b \in \mathbb{Z}$: $\displaystyle\int_a^b \omega_n(t) f^n(t)dt = \frac{B_{n+1}}{n+1}[f^n(b) - f^n(a)] - \frac{1}{n+1}\int_a^b \omega_{n+1}(t) f^{n+1} dt$

Proof. Using integration by parts, where $u = f^n(t)$ and $dv = \omega_n(t)dt$:

$$du = f^{n+1}(t) \text{ and } v = \frac{1}{n+1}\omega_{n+1}(t)$$

$$\int_a^b \omega_n(t) f^n(t)dt = f^n(t)\omega_{n+1}(t)|_a^b - \frac{1}{n+1}\int_a^b \omega_{n+1}(t) f^{n+1}(t)dt$$

Since $\omega_{n+1}(t)$ is defined for all $n \geq 1$, $f^n(t)\omega_{n+1}(t)|_a^b$ is easy to compute. $\omega_{n+1}(a) = \omega_{n+1}(b) = B_{n+1}$ (by the first proposition), so:

$$\forall n \geq 1\text{: } \int_k^{k+1} \omega_n(t) f^n(t)dt = \frac{B_{n+1}}{n+1}[f^n(k+1) - f^n(k)] - \frac{1}{n+1}\int_k^{k+1} \omega_{n+1}(t) f^{n+1} dt$$

□

Now, to construct the differences between sum and integral to obtain the larger difference:

Proposition 4

$\forall\ a, b \in \mathbb{Z}$: $\displaystyle\int_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = \frac{1}{2}[f(b) - f(a)] - \int_a^b \omega_1(t) f'(t)dt$

Proof. Taking the unit difference between the integral and sum of f between k and $k+1$, and applying the first lemma:

$$\int_k^{k+1} f(t)dt - f(k) = \frac{1}{2}[f(k+1) - f(k)] - \int_k^{k+1} \omega_1(t) f'(t)dt - f(k)$$

Now summing all these peices from a to $b-1$, and combining the sums of $\frac{1}{2}[f(k+1) - f(k)]$ and $f(k)$:

$$\sum_{k=a}^{b-1}\int_k^{k+1} f(t)dt - \sum_{k=1}^{b-1} f(k) = \frac{1}{2}\sum_{k=a}^{b-1}[f(k+1) - f(k)] - \sum_{k=a}^{b-1}\int_k^{k+1} \omega_1(t) f'(t)dt - \sum_{k=a}^{b-1} f(k)$$

$$\sum_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = -\frac{1}{2}f(a) + \frac{1}{2}f(b) - \int_a^b \omega_1(t) f'(t)dt$$

$$\text{So } \sum_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = \frac{1}{2}[f(b) - f(a)] - \int_a^b \omega_1(t) f'(t)dt$$

□

The Euler-Maclaurin formula comes directly from the above relation, and an expression for the integral $\int_a^b \omega_1(t) f'(t)dt$. The next propostion will provide do just that, and the larger proof will follow.

Proposition 5

For any integers a and b, and any positive integer m:

$$\int_a^b \omega_1(t)f'(t)dt = \sum_{k=2}^{m}(-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+1}}{m!}\int_a^b \omega_m(t)f^m(t)dt$$

Proof. This follows directly from the second lemma by induction.

In the base (for $m = 1$), the above is true, since the hypothesis for $m = 1$ yeilds

$$\int_a^b \omega_1(t)f^1(t)dt = \frac{B_2}{2}[f^1(b) - f^1(a)] - \frac{1}{2}\int_a^b \omega_2(t)f^2 dt$$

Which is true by the second lemma, practically verbatim. So the base case of the induction holds true. For the inductive step, it must be shown that if the hypothesis is true for m, then it is also true for $m + 1$. Again, using the second lemma, this is not too hard to do.

If the hypothesis is true for m, then:

$$\int_a^b \omega_1(t)f'(t)dt = \sum_{k=2}^{m}(-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+1}}{m!}\int_a^b \omega_m(t)f^m(t)dt$$

By the second lemma:

$$\frac{(-1)^{m+1}}{m!}\int_a^b \omega_m(t)f^m(t)dt = \frac{(-1)^{m+1}}{m!}\left[\frac{B_{m+1}}{m+1}[f^m(b) - f^m(a)] - \frac{1}{m+1}\int_a^b \omega_{m+1}(t)f^{m+1}(t)dt\right]$$

Substituting the above and simplifying yeilds:

$$\int_a^b \omega_1(t)f'(t)dt = \sum_{k=2}^{m+1}(-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+2}}{(m+1)!}\int_a^b \omega_{m+1}(t)f^{m+1}(t)dt$$

Which shows the hypothesis is true for $m+1$ if it is true for m, and so the inductive hypothesis is proven. $\square$

Now that every tool needed to prove the Euler-Maclaurin formula has been built, here is the final proof:

Euler-Maclaurin Formula

Given an m-differentiable function f and any integers a and b:

$$\int_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = \sum_{n=1}^{m}(-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+1}}{m!}\int_a^b \omega_m(t) f^m(t)dt$$

Proof. The above is shown by simply combining propositions 4 and 5. By proposition 4, it is given that:

$$\forall\ a, b \in \mathbb{Z}\text{:}\ \int_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = \frac{1}{2}[f(b) - f(a)] - \int_a^b \omega_1(t) f'(t)dt$$

Also, by proposition 5, it is given that:

$$\int_a^b \omega_1(t) f'(t)dt = \sum_{k=2}^{m}(-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+1}}{m!}\int_a^b \omega_m(t) f^m(t)dt$$

By coincidence, since $B_1 = -\frac{1}{2}$ and $1! = 1$, it happens that:

$$\frac{1}{2}[f(b) - f(a)] = (-1)^1 \frac{B_1}{1}[f^{1-1}(b) - f^{1-1}(a)]$$

So, when the expression for $\int_a^b \omega_1(t) f(t)dt$ from proposition 5 is substituted into the expression for the difference of sums in proposition 4, the Euler-Maclaurin formula remains:

$$\int_a^b f(t)dt - \sum_{k=a}^{b-1} f(k) = \sum_{n=1}^{m}(-1)^n \frac{B_n}{n!}[f^{n-1}(b) - f^{n-1}(a)] + \frac{(-1)^{m+1}}{m!}\int_a^b \omega_m(t) f^m(t)dt$$

□