Isomorphism Theorems

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Kernel Subgroup Lemma

Let $\varphi : G \to H$ be a homomorphism. $\ker\varphi$ is a subgroup of G.

Proof. $\ker\varphi$ is a subgroup of G if $\ker\varphi$ contains the identity, and is closed under both multiplication and inverses. $e \in \ker\varphi$ since $\varphi(e) = \varphi(e)\varphi(e) \equiv \varphi(e) = e'$. $\ker\varphi$ is closed under multiplication since if $g, h \in \ker\varphi$, then $\varphi(gh) = \varphi(g)\varphi(h) = e'$. Lastly, if $g \in \ker\varphi$, then $g^{-1} \in \ker\varphi$ since if $g \in \ker\varphi$, then $\varphi(g^{-1}g) = \varphi(g^{-1})\varphi(g) = e'$ and $\varphi(g) = e'$, so $\varphi(g^{-1}) = e$. ☐

Kernel Bijection Lemma

Let $\varphi : G \to H$ be a homomorphism. If $\ker\varphi = \{e\}$, then φ is an isomorphism between G and $\varphi[G] \subseteq H$.

Proof. To show that φ is one to one, it must be shown that $\forall\ h \in \varphi[G]$, $\varphi(g_1) = \varphi(g_2) = h \implies g_1 = g_2$.

$$\varphi(g_1) = \varphi(g_2) \implies \varphi(g_1)\varphi(g_2)^{-1} = e' \implies \varphi(g_1 g_2^{-1}) = e' \implies g_1 g_2^{-1} \in \ker\varphi$$

Since $\ker\varphi = \{e\}$, it must follow that $g_1 g_2^{-1} = e$, and thus $g_1 = g_2$. φ is onto $\varphi[G]$ by definition so φ is an isomorphism between G and $\varphi[G]$ when $\ker\varphi = \{e\}$. ☐

First Isomorphism Theorem

Let $\varphi : G \to G'$ be a homomorphism, and $H \leq G$. If $\ker\varphi = H$, then $G/H \cong G'$.

Proof. Let the map $\mu : G/H \to G'$ be defined by $\mu(gH) = \varphi(g)$. μ is a homomorphism:

$$\mu(g_1 H)\mu(g_2 H) = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2) = \mu(g_1 g_2 H) \because \varphi \text{ is a homomorphism}$$

If $\ker\mu$ is equal to the identity in G/H, or eH, then μ is an isomorphism since it is defined for all $gH \in G/H$. By definition, $\ker\mu = \{gH \in G/H : \mu(gH) = \mu(eH) = \varphi(e)\}$, so $\ker\mu = \{gH \in G/H : g \in \ker\varphi\}$. Since it is given that $\ker\varphi = H$, $\ker\mu = gH \in G/H : g \in H = \{eH\}$. Thus G' and G/H are isomorphic. ☐

Second Isomorphism Theorem

If $H, K \leq G$, and $K \trianglelefteq G$, then $\dfrac{HK}{K} \cong \dfrac{H}{H \cap K}$.

Proof. Let $\varphi : H \to HK/K$ be a map defined by $\varphi(h) = hK$. Then, because K is normal in G, φ is a homomorphism, as $\varphi(h_1 h_2) = h_1 h_2 K = (h_1 K)(h_2 K) = \varphi(h_1)\varphi(h_2)$. Furthermore, if $\ker\varphi = H \cap K$, then by the first isomorphism theorem, $HK/K \cong H/(H \cap K)$.

$$\ker\varphi = \{h \in H : \varphi(h) = \varphi(e)\} = \{h \in H : hK = H\} = \{h \in H : h \in K\} = H \cap K$$

☐

Third Isomorphism Theorem

Let $H, K \trianglelefteq G$ and $K \leq H$. Then $\dfrac{G}{H} \cong \dfrac{G/K}{H/K}$.

Proof. Let $\varphi : G/K \to G/H$ be the map $\varphi(gK) = gH$. By the first isomorphism theorem, if φ is a homomorphism and $\ker\varphi$ is equal to H/K, then $G/H \cong \frac{G/K}{H/K}$.

$$\varphi(g_1Kg_2K) = \varphi(g_1g_2K) = g_1g_2H = g_1Hg_2H = \varphi(g_1K)\varphi(g_2K$$

$$\ker\varphi = \{gK : \varphi(gK) = H\} = \{gK \in G/K : h \in H\} = \{hK : h \in H\} = H/K$$ □

Homomorphism Subgroup Lemma

Let $\varphi : G \to G'$ be a homomorphism, $H \leq G$ and $H' \leq G'$. Then $\varphi[H] \leq G'$ and $\varphi^{-1}[H'] \leq G'$.

Proof. For a subset to be a subgroup, that subgroup must contain the identity, and be closed under both multiplication and inverses. To show that $\varphi[H] \leq G'$:

$e' \in \varphi[H] \because \varphi[H] \ni \varphi(e) = e'$

$g', h' \in \varphi[H] \implies g'h' \in \varphi[H]$:

$$\begin{aligned} g', h' \in \varphi[H] \equiv \varphi(g), \varphi(h) \in \varphi[H] &\implies g, h \in H \implies gh \in H \because H \text{ is a group} \\ &\implies \varphi(gh) \in \varphi[H] \equiv \varphi(g)\varphi(h) \in \varphi[H] \because \varphi \text{ is a homomorphism} \\ &\implies g'h' \in \varphi[H] \end{aligned}$$

$g' \in \varphi[H] \implies g'^{-1} \in \varphi[H]$:

$$\begin{aligned} g' \in \varphi[H] \equiv \varphi(g) \in \varphi[H] &\implies g \in H \implies g^{-1} \in H \\ &\implies \varphi(g^{-1}) \in \varphi[H] \implies \varphi(g)^{-1} \in \varphi[H] \\ &\implies g'^{-1} \in \varphi[H] \end{aligned}$$

Now to show that $\varphi^{-1}[H'] \leq G$. Note that if $\varphi(g) = g'$, then $\varphi^{-1}(g') = g\ker\varphi$.

$e \in \varphi^{-1}[H'] \because \varphi^{-1}[H'] \ni \varphi^{-1}(e') = \ker\varphi$

$g, h \in \varphi^{-1}[H'] \implies gh \in \varphi^{-1}[H']$

$$\begin{aligned} g, h \in \varphi^{-1}[H'] &\implies \varphi(g), \varphi(h) \in H' \implies \varphi(g)\varphi(h) \in H' \implies \varphi(gh) \in H' \\ &\implies gh \in \varphi^{-1}[H'] \end{aligned}$$

$g \in \varphi^{-1}[H'] \implies g^{-1} \in \varphi[H']$:

$$\begin{aligned} g \in \varphi^{-1}[H'] &\implies \varphi(g) \in H' \implies \varphi(g)^{-1} \in H' \implies \varphi(g^{-1}) \in H' \\ &\implies g^{-1} \in \varphi^{-1}[H'] \end{aligned}$$

□

Fourth Isomorphism Theorem

Let G be a group and let N be a normal subgroup of G. Then there is a bijection from the set of subgroups A of G which contain N onto the set of subgroups $\overline{A} = A/N$ of G/N. In particular, every subgroup of $\overline{G}$ is of the form A/N for some subgroup A of G containing N. (statement but not proof from Dummit and Foote III)

Proof. Let $\varphi : G \to G/N$ such that $\varphi(g) = gN$ and $N \trianglelefteq G$. Consider the set of subgroups of G containing N, denoted $\mathbb{A}$, and the subgroups of G/N, $\overline{\mathbb{A}}$, such that:

$$\mathbb{A} = \{A \in G : N \leq A \leq G\} \text{ and } \overline{\mathbb{A}} = \{\overline{A} \in G/N : A \leq G/N\}$$

Then φ induces a bijection between $\mathbb{A}$ and $\overline{\mathbb{A}}$. However simply showing that φ is one-to-one and onto is not enough to show that φ is a bijection, because it is not clear that $\varphi[A] \in \overline{\mathbb{A}}$ for all $A \in \mathbb{A}$.

Showing that φ is one to one, or that $\varphi[A] = \varphi[A'] \iff A = A'$:

$$\begin{aligned}
\varphi^{-1}\varphi[A] = \bigcup_{a \in A} \varphi^{-1}\varphi(a) &= \bigcup_{a \in A} a \ker\varphi = A\ker\varphi = A \because \ker\varphi = N \leq A \\
&\implies \varphi^{-1}\varphi[A] = A' \iff A = A' \\
&\implies \varphi[A] = \varphi[A] \iff A = A'
\end{aligned}$$

For φ to be onto, there must be some $A \in \mathbb{A}$ such that $\varphi[A] = \overline{A}$ for all $\overline{A} \in \overline{\mathbb{A}}$. By the homomorphism subgroup lemma below, this A is just $\varphi^{-1}[\overline{A}]$ for any $\overline{A} \in \overline{\mathbb{A}}$. Lastly, the homomorphism subgroup lemma below also shows that $\varphi[A] \in \overline{\mathbb{A}}$ for all $A \in \mathbb{A}$. $\square$

This bijection has the following properties: for all $A, B \leq G$, with $N \leq A$ and $N \leq B$:

(i) $A \leq B \iff \overline{A} \leq \overline{B}$

Proof. Showing that $A \leq B \iff \overline{A} \leq \overline{B}$ is the same as showing that $A \leq B \iff \varphi[A] \leq \varphi[B]$.

$$\begin{aligned}
A \leq B &\iff A \subseteq B \because A \text{ and } B \text{ are subgroups of } G \\
&\iff \{aN : a \in A\} \subseteq \{bN : b \in B\} \\
&\iff \varphi[A] \subseteq \varphi[B] \\
&\iff \varphi[A] \leq \varphi[B] \text{because } \varphi[A] \text{ and } \varphi[B] \text{ are subgroups of } G/N \\
&\iff \overline{A} \leq \overline{B}
\end{aligned}$$

$\square$

(ii) $A \leq B \implies [B : A] = [\overline{B} : \overline{A}]$

Proof. By explicitly counting the cosets in G and G/N:

$$\begin{aligned}
[\overline{B} : \overline{A}] &= |\{(\bar{b})\overline{A} : \bar{b} \in \overline{B}\}| \\
&= |\{bN \bigcup_{a \in A} aN : b \in B\}| \\
&= |\{bNAN : b \in B\}| = |\{bAN : b \in B\}| = |\{bA : b \in B\}| \\
&= [B : A]
\end{aligned}$$

$\square$

(iii) $\overline{\langle A, B \rangle} = \langle \overline{A}, \overline{B} \rangle$

Proof. By direct computation:
$\overline{\langle A, B\rangle} = \overline{AB} = \{abN : a, b \in A, B\} = \{(aN)(bN) : a, b \in A, B\} = \overline{A}\,\overline{B} = \langle \overline{A}, \overline{B}\rangle$ $\square$

(iv) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

Proof. By direct computation:
$\overline{A} \cap \overline{B} = \{aN : a \in A\} \cap \{bN : b \in B\} = \{xN : x \in A, x \in B\} = \{xN : x \in A \cap B\} = \overline{A \cap B}$ $\square$

(v) $A \trianglelefteq G \iff \overline{A} \trianglelefteq \overline{G}$

Proof. Showing $A \trianglelefteq G \implies \overline{A} \trianglelefteq \overline{G}$ and $\overline{A} \trianglelefteq \overline{G} \implies A \trianglelefteq G$ separately:

$$\begin{aligned} A \trianglelefteq G \implies \overline{A} \trianglelefteq \overline{G} \because A \trianglelefteq G &\implies gag^{-1} \in A \\ &\implies (gN)(aN)(g^{-1}N) \subseteq gag^{-1}N \in \overline{A} \\ &\implies \overline{A} \trianglelefteq \overline{B} \end{aligned}$$

$$\begin{aligned} \overline{A} \trianglelefteq \overline{G} \implies A \trianglelefteq G \because \overline{A} \trianglelefteq \overline{G} &\implies (gN)(aN)(g^{-1}N) \in \overline{A} \\ &\implies gag^{-1} \in A \\ &\implies A \trianglelefteq G \end{aligned}$$

$\square$