The Borel Cantor-Bernstein theorem for graphs

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Let $G = (X, E)$ be a graph, and let $A, B \subseteq V$. Suppose that G contains a set $\mathcal{P}$ of disjoint A-B paths covering A, and a set $\mathcal{Q}$ of disjoint A-B paths covering B. It was first proven by Pym in 1969 that G contains a set of disjoint A-B paths covering $A \cup B$ [3]. A simpler proof of this result, that does not rely on the axiom of choice, was found by Diestel and Thomassen in 2003 [1]. Suppose that X is Polish, and both $\mathcal{P}$ and $\mathcal{Q}$ are Borel in X^*. Is it then true that there exists a disjoint Borel set of A-B paths covering $A \cup B$? In 1994, Nahum and Zafrany proved that there is! Here, it is shown that the construction provided by Diestel and Thomassen is in fact Borel, if $\mathcal{P}$ and $\mathcal{Q}$ are as well.

Take two A-B paths $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ which share a common vertex, c. PcQ is the path formed by beginning at the start point of P, continuing along P until c, and then following Q to its endpoint. Consider a family of paths $(P_a)_{a \in A}$ of disjoint A-B paths. If for each $a \in A$, $a \in P_a$ and $P_a = PcQ$, then $(P_a)_{a \in A}$ is a called an A-*family*. The initial segment, Pc, is the path P up to, and including, c. The final segment cQ, is the path Q from c forward, including c. In an A-family $(P_a)_{a \in A}$, the initial segment up to, but not including c, is denoted $\bar{P}_a$.

Consider an A-family $(P_a)_{a \in A}$. Suppose there exists $Q \in \mathcal{Q}$ such that $\bar{P}_d \cap Q \neq \emptyset$ for some $d \in A$. Take an $x \in \bar{P}_d \cap Q$, and replace P_d with $P_d x Q$ to obtain a new family $(P'_a)_{a \in A}$. It is then said that the A-family (P'_a) is formed by a *switch* at x from the A-family (P_a). $\mathcal{P}$ can be realized as a family $(P_a)_{a \in A}$. For each $d \in A$, let $x_d \in P_d$ be the first point in P_d which ends a finite sequence of switches, turning (P_a) into the family (P^d_a). Diestel and Thomassen show that the set (P^a_a) is an A-family covering $A \cup B$. Using these switches, there is a new way to prove the central result of [2]:

Theorem (Nahum-Zafrany)**.** *Let $G = (X, E)$ be a graph such that X is a perfect Polish space. Let $A, B \subseteq X$ be disjoint, and let $f : A \rightsquigarrow B$ and*

$g : B \rightsquigarrow A$ be Borel injective linkings. Then there exists a Borel bijective linking $h : A \rightsquigarrow B$ such that $E(h) \subseteq E(f \cup g)$.

By the definitions of injective and bijective linkings [2], f is a Borel set of disjoint A-B paths covering A, and g is a Borel set of disjoint A-B paths covering B. Furthermore, it is not necessary for the space to be perfect. Thus, the theorem can be rewritten as:

Theorem (Nahum-Zafrany)**.** *Let $G = (X, E)$ be a graph such that X is a Polish space. Let $A, B \subseteq X$ be disjoint, $\mathcal{P}$ and $\mathcal{Q}$ be Borel sets of disjoint A-B paths covering A and B, respectively. Then G contains a Borel set of disjoint A-B paths covering $A \cup B$.*

Let us first establish the following lemma:

Lemma. *Take an $x \in X$ such that the shortest sequence of legal switches ending with a switch at x is of length n. Then the set of switching sequences of length n, ending at x, is finite.*

Proof. Consider a set $S \subset X$, the scope of S is defined as:

$$G(S) = \{x \in X | \exists s \in S, p \in \mathcal{P}, q \in \mathcal{Q} : s \in q \wedge p \cap q \neq \emptyset \wedge x \in p\}$$

This gives all points in any path X which could effect a switch at a point in S. Given an x which is the end of a switching sequence of minimal length n, define the set

$$(L_n)_x = \{(x_1, \cdots, x_{n-1}, x) \in X^n | (x_1, \cdots, x_{n-1}, x) \text{ is legal } \}.$$

It must be that $(L_n)_x \subset G^{n-1}(\{x\})^n$. For $n = 1$, this is true. Suppose $(L_n)_x \subset G^{n-1}(\{x\})^n$. Consider an x with a minimal switch length of $n + 1$, and suppose there is a legal minimal switch not in $G^n(\{x\})^{n+1}$. Some x_i in the seqeunce is not in $G^n(\{x\})$, which means that its removal has no effect on the legality of all the other switches in the sequence. Thus the sequence $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1}, x)$ is legal, contradicting minimality. Since $G^{n+1}(\{x\})$ is finite, each $(L_n)_x$ must be finite as well. □

Now to provide an alternate proof of the Nahum-Zafrany result.

Proof. It will be shown that the A-family (P^a_a) constructed in the proof of Pym's theorem is Borel. We will first define Borel sets $\tilde{L}_n, \tilde{M}_n \subseteq (X^* \times X^* \times X \times X)^n$ and $M_n \subseteq X$, which will describe legal, and minimal legal, switching sequences of length n. Define the set $R = \{(x, (P, Q) | P \in \mathcal{P}, Q \in \mathcal{Q}$ and $x \in P \cap Q\}$. R is Borel. For $n = 1$:

$$\tilde{L}_1(U_1, P_1, x_1, Q_1) \iff U_1 = P_1 \wedge R(P_1, x_1, Q_1) \wedge (\mathcal{P} - \{U_1\}) \cap P_1 x_1 Q_1 = \emptyset$$

$$M_1 = \pi_3(\tilde{L}_1)$$

$\tilde{L}_1$ is Borel by construction, and each x_1-section of $\tilde{L}_1$ is either empty or a singleton, so by KAL, $\pi_3(\tilde{L}_1)$ is Borel. To construct $\tilde{L}_2$ we find all legal 2-switches, and then remove the ones with final switches that could be reached in a 1-switch to obtain $\tilde{M}_2$, and then project to get M_2.

$$\begin{aligned}
&\tilde{L}_2(U_1, P_1, x_1, Q_1, U_2, P_2, x_2, Q_2) \iff \tilde{L}_1(U_1, P_1, x_1, Q_1) \wedge R(P_2, x_2, Q_2) \\
&\quad \wedge P_2 x_2 \subset U_2 \\
&\quad \wedge U_2 \in (\mathcal{P} - \{U_1\}) \cup \{P_1 x_1 Q_1\} \\
&\quad \wedge P_2 x_2 Q_2 \cap (((\mathcal{P} - \{U_1\}) \cup \{P_1 x_1 Q_1\}) - \{U_2\}) = \emptyset \\
&\tilde{M}_2(U_1, P_1, x_1, Q_1, U_2, P_2, x_2, Q_2) \iff \tilde{L}_2(U_1, P_1, x_1, Q_1, U_2, P_2, x_2, Q_2) \\
&\quad \wedge \neg M_1(x_2) \\
&M_2 = \pi_7(\tilde{M}_2)
\end{aligned}$$

In general, $\tilde{L}_n$ is all legal n-switches, $\tilde{M}_n$ is all minimal legal n-switches, and M_n is the projection onto X. Note that the x_n-section of $\tilde{M}_n$ is finite by the lemma, so by KAL, each M_n is Borel:

$$\begin{aligned}
&\tilde{L}_n(U_1, P_1, x_1, Q_1, \cdots, U_n, P_n, x_n, Q_n) \iff \tilde{L}_1(U_1, P_1, x_1, Q_1) \wedge \cdots \\
&\quad \wedge \tilde{L}_{n-1}(U_1, P_1, x_1, Q_1, \cdots, U_{n-1}, P_{n-1}, X_{n-1}, Q_{n-1}) \\
&\quad \wedge R(P_n, x_n, Q_n) \wedge P_n x_n \subset U_n \\
&\quad \wedge U_n \in (((((\mathcal{P} - \{U_1\}) \cup \{P_1 x_1 Q_1\}) - \cdots) - \{U_{n-1}\}) \cup \{P_{n-1} x_{n-1} Q_{n-1}\}) \\
&\quad \wedge P_n x_n Q_n \cap ((((((\mathcal{P} - \{U_1\}) \cup \{P_1 x_1 Q_1\}) - \cdots) - \{U_{n-1}\}) \\
&\qquad \cup \{P_{n-1} x_{n-1} Q_{n-1}) - \{U_n\}) = \emptyset \\
&\tilde{M}_n(U_1, P_1, x_1, Q_1, \cdots, U_n, P_n, x_n, Q_n) \iff \\
&\quad \tilde{L}_n(U_1, P_1, x_1, Q_1, \cdots, U_n, P_n, x_n, Q_n) \wedge \bigwedge_{i=1}^{n-1} \neg M_i(x_n) \\
&M_n = \pi_{4n-1}(\tilde{M}_n)
\end{aligned}$$

Using the M_n, we can show that the family (P_a^a) is Borel. Consider

$$\begin{aligned}
H = \{(U, P, x, Q) | P \in \mathcal{P} \wedge Q \in \mathcal{Q} \wedge x \in Q \wedge U = PxQ \\
\exists n \exists i : (P(n) = x \wedge M_i(x) \wedge \forall m < n \forall j : \neg M_j(P(m)))\}
\end{aligned}$$

H is Borel, as each U-section is finite, so $\pi_1(H) = (P_a^a)$ is Borel. $\square$

References

[1] Reinhard Diestel and Carsten Thomassen, *A Cantor-Bernstein theorem for paths in graphs*

[2] Ronny Nahum and Samy Zafrany, *A Topological Linking Theorem in Simple Graphs*

[3] J.S. Pym, The linking of sets in graphs, *J. Lond. Math. Soc.* **44**. (1969), 542-550