Jordan Holder Theorem

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Composition Series

A composition series of G is a series $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{r-1} \subset G_r = G$ such that $G_i \triangleleft G_{i+1}$ and G_{i+1}/G_i is simple for all $0 \leq i \leq r-1$.

Composition Series Similarity

Two composition series, $\{G_i\}$ and $\{H_j\}$ are similar if they have the same length, and there exists some j for which $G_{i+1}/G_i = H_{j+1}/H_j$ for each G_{i+1}/G_i.

Existence of Composition Series

All finite G have a composition series.

Proof. For any G, the series $\{e\} = G_0 \subset G_1 = G$ satisfies the criteria $G_i \triangleleft G_{i+1}$ for all $0 \leq i \leq r-1$. It is not guaranteed, however, that G_{i+1}/G_i is simple for all i. Suppose G_{i+1}/G_i has a normal subroup H. Let $\pi : G_{i+1} \to G_{i+1}/G_i$. Then, by the lattice theorem, $\pi^{-1}[H]$ is a subgroup of G_{i+1}. To show that $\pi^{-1}[H]$ is normal, let $h, h' \in \pi^{-1}[H]$ (and thus $hG_i, h'G_i \in H$)

$$\begin{aligned}
ghg^{-1} &= \pi^{-1}(gG_i)\pi^{-1}(hG_i)\pi^{-1}(g^{-1}G_i) \\
&= \pi^{-1}((gG_i)(hG_i)(g^{-1}G_i)) \\
&= \pi^{-1}((gG_i)(hG_i)(gG_i)^{-1}) \\
&= \pi^{-1}(h'G_i) \text{ since } H \text{ is normal} \\
&\in \pi^{-1}[H]
\end{aligned}$$

Thus the series can be expanded by $\pi^{-1}[H]$ to form a new series

$$\{e\} = G_0 \subset \cdots \subset G_i \subset \pi^{-1}[H] \subset \cdots G_{i+1} \subset \cdots \subset G_r = G$$

until the resulting series is a composition series, in finitely many steps. ☐

Lemma 1

Let $K \trianglelefteq N \trianglelefteq G$. Then $N/K \trianglelefteq G/K$.

Proof. Let $g \in G$ and $n, n' \in N$. $(gK)(nK)(gK)^{-1} = (gK)(nK)(g^{-1}K) \subset gng^{-1}K = n'K \in N/K$ ☐

Lemma 2

Let G be a group, and let M and N be normal subgroups, $M \neq N$ with G/M and G/N simple. Then

$$M/(M \cap N) \cong G/N \text{ and } N/(M \cap N) \cong G/M$$

Proof. Let $K = M \cap N$. Since K is the intersection of two normal subgroups, it is normal as well:

$$\begin{aligned}
g(M \cap N)g^{-1} &\subset gMg^{-1} \subset M \because M \trianglelefteq G \\
&\subset gNg^{-1} \subset N \because N \trianglelefteq G \\
\therefore &\subset (M \cap N) \\
\therefore &(M \cap N) \trianglelefteq G
\end{aligned}$$

If $MN = G$, it follows from the second isomorphism theorem that

$$M/K = M/(M \cap N) = MN/N = G/N$$
$$N/K = N/(M \cap N) = MN/M = G/M$$

Now to show that $G = MN$. Since M and N are normal, MN is normal in G. By the above lemma, MN/M is normal in G/M. Since G/M is simple, $MN/M = G/M$ or $MN/M = \{e\}$. Since $M \neq N$, MN/M cannot be $\{e\}$:

$$MN/M = \{e\} \implies MN = M \implies M = N$$

Therefore $MN/M = G/M$ and

$$\begin{aligned} MN/M = G/M &\implies \pi^{-1}[MN/M] = \pi^{-1}[G/M] \text{ where } \pi : G \to G/M \text{ is defined by } \pi(g) = gM \\ &\implies G = MN \end{aligned}$$

Therefore $M/(M \cap N) \cong G/N$ and $N/(M \cap N) \cong G/M$ □

Jordan Holder Theorem

Let $\{e\} = G_0 \subset \cdots \subset G_r = G$ and $\{e\} = H_0 \subset \cdots \subset H_s = G$ be two composition series for G. Then $r = s$ and for each the two composition series are similar.

Proof. This is a proof by induction on groups in the composition series of G. The inductive hypothesis is as follows: suppose all composition series for H are similar and of equal length (i.e. obeying the Jordan Holder theorem). If $H_0 \subset \cdots \subset H_r$ is a composition series for H, and there exists some composition series for G which is simply $H_0 \subset \cdots \subset H_r \subset G$, then all composition series for G are similar and have length $r + 1$, and thus obey the Jordan Holder theorem as well.

In the base case, for $\{e\}$, all composition series are similar and of length one. Now let $\{H_i\}$ and $\{G_i\}$ be composition series for G, where $\{e\} = H_0 \subset \cdots \subset H_r \subset H_{r+1} = G$ and $\{e\} = G_0 \subset \cdots \subset G_s \subset G_{s+1} = G$. If $H_r = G_s$, then by induction, $\{H_i\}$ and $\{G_i\}$ satisfy Jordan Holder. Suppose $H_r \neq G_s$, then denote $M = H_r$, $N = G_s$, and $K = M \cap N$. By the existence of composition series, K has a composition series:

$$\{e\} = K_0 \subset \cdots \subset K_t = K$$

Appending M to the composition series for K gives a composition series for M, since $M/K = G/N$ is simple. Appending G to the composition series for M gives a composition series for G by definition. Let $\sim$ signify the fact that two series are rearrangements of eachother. Since, by the inductive hypothesis, $M = H_r$ obeys Jordan Holder, it follows $r = t + 1$ and:

$$(H_1/H_0, \cdots, H_{r-1}/H_{r-2}, H_r/H_{r-1}) \sim (K_1/K_0, \cdots, K_t/K_{t-1}, M/K) \implies$$
$$(H_1/H_0, \cdots, H_{r-1}/H_{r-2}, H_r/H_{r-1}, G/H_r) \sim (K_1/K_0, \cdots, K_t/K_{t-1}, M/K, G/M)$$

Similarly, appending N to the composition series for K gives a composition series for N, since $N/K = G/M$ is simple. Appending G to the composition series for N gives a composition series for G by definition. Since $N = G_s$ obeys Jordan Holder by the inductive hypothesis, it follows $s = t + 1$ and:

$$(G_1/G_0, \cdots, G_{s-1}/G_{s-2}, G_s/G_{s-1}) \sim (K_1/K_0, \cdots, K_t/K_{t-1}, N/K) \implies$$
$$(G_1/G_0, \cdots, G_{s-1}/G_{s-2}, G_s/G_{s-1}, G/G_s) \sim (K_1/K_0, \cdots, K_t/K_{t-1}, N/K, G/N)$$

It then follows that $r = s$, and:

$$(H_1/H_0, \cdots, H_{r-1}/H_{r-2}, H_r/H_{r-1}, G/H_r) \sim (G_1/G_0, \cdots, G_{s-1}/G_{s-2}, G_s/G_{s-1}, G/G_s)$$

If $(M/K, G/M) \sim (N/K, G/N)$, which they are, since by lemma 2, $M/K = M/(M \cap N) = G/N$ and $N/K = N/(M \cap N) = G/M$. Therefore any two $\{H_i\}$ and $\{G_i\}$ are similar with length $r + 1$, thus proving the Jordan Holder theorem. □