ALMOST ι-COMPLEXES AS IMMERSED CURVES

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ABSTRACT. Here the existence of a new homomorphism $P_\omega : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ is proven and the existence of a $\mathbb{Z}^\infty$ summand in $\Theta^3_{\mathbb{Z}}$ is reproven. This is done by approximating the involutive Heegaard Floer complexes of homology 3-spheres with immersed curves on the twice punctured disk.

1. Introduction

The classification of the integer homology cobordism group $\Theta^3_{\mathbb{Z}}$ is a major open problem in low dimensional topology. It was initially conjectured that the Rokhlin homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}_2$ was an isomorphism before results in gauge theory identified an infinite subgroup [FS85] and then a $\mathbb{Z}^\infty$ subgroup [Fur90, FS90]. In 2002, it was shown with Yang-Mills theory that $\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}$ summand [Fy02]. Recently, Dai, Hom, Stoffregen and Truong used the involutive Heegaard Floer complexes defined in [HMZ16, HM17] to prove the existence of a $\mathbb{Z}^\infty$ summand in $\Theta^3_{\mathbb{Z}}$ [DHST18].

In [HMZ16], Hendricks, Manolescu and Zemke define the group of ι-complexes $\mathfrak{I}$ to be all complexes (C, ∂, ι) with a single U-tower in homology and $\iota^2 \simeq \mathrm{id}$ modulo local equivalence. The group operation of $\mathfrak{I}$ is the tensor product of complexes. It is shown that the map $h : \Theta^3_{\mathbb{Z}} \to \mathfrak{I}$ taking Y to $(CF^-(Y), \iota)$ is a homomorphism. Since it is unknown how to classify $\mathfrak{I}$, Dai, Hom, Stoffregen, and Truong classified the group of almost ι-complexes $\hat{\mathfrak{I}}$, an approximation of $\mathfrak{I}$ mod U, instead.

Theorem 1.1 ([DHST18]). *Every almost ι-complex in $\hat{\mathfrak{I}}$ is locally equivalent to a standard complex of the form $\mathcal{C}(a_1, b_2, \cdots, a_{2n-1}, b_{2n})$ where $a_i \in \{-, +\}$ and $b_i \in \mathbb{Z} \setminus \{0\}$.*

Dai, Hom, Stoffregen, and Truong defined maps $\phi_n : \hat{\mathfrak{I}} \to \mathbb{Z}$ on each local equivalence class by $\phi_n(\mathcal{C}) = \#\{b_i = n\} - \#\{b_i = -n\}$. They then proved that each ϕ_n is a homomorphism.

Theorem 1.2 ([DHST18]). *The maps $\phi_n : \hat{\mathfrak{I}} \to \mathbb{Z}$ are homomorphisms.*

These ϕ_n give a splitting of $\Theta^3_{\mathbb{Z}}$ and witness an infinite rank summand.

Corollary 1.3 ([DHST18]). *$\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ summand.*

Streamlined proofs of theorems 1.1 and 1.2 are given here using the Fukaya category of the twice punctured disk. These proofs rely on the precurves described in [Zib20] and [KWZ19]. From there, the proof of corollary 1.3 is the same as in [DHST18]. The precurve perspective also gives a new homomorphism $P_\omega : \hat{\mathfrak{I}} \to \mathbb{Z}$ defined by $P_\omega(\mathcal{C}) = \#\{a_i = +\} - \#\{a_i = -\}$. The existence of this homomorphism P_ω was conjectured by Dai, Hom, Stoffregen and Truong.

Theorem 1.4. *The map $P_\omega : \hat{\mathfrak{I}} \to \mathbb{Z}$ is a homomorphism.*

Theorem 1.5. *$P_\omega : \hat{\mathfrak{I}} \to \mathbb{Z}$ is not in the span of $\{\phi_n\}$.*

Once the classification is complete, it is shown that the homomorphisms defined in [DHST18] count the grading of a special generator in the precurve picture. Theorems 1.4 and 1.5 are proven by endowing precurves from [KWZ19] and [Zib20] with a natural bigrading.

The algebro-geometric correspondence in [KWZ19] and [Zib20] is motivated by a similar construction of Haiden, Katzarkov, and Kontsevich in [HKK14]. The proof given in [KWZ19] and [Zib20] that realizes chain complexes up to homotopy as decorated immersed curves draws heavily on the arrow sliding algorithm given by Hanselman, Rasmussen, and Watson in [HRW16].

Organization. The correspondence between almost ι-complexes, precurves, and immersed curves is described in section 2. In section 3, the classification of $\hat{\mathfrak{I}}$ is reproven by showing that a special immersed curve $\gamma_0(\mathcal{C})$ contains all the information about the local equivalence class of $\mathcal{C}$. Finally, all known homomorphisms from $\hat{\mathfrak{I}}$ to $\mathbb{Z}$ are described in section 4 by counting the grading of a special generator in $\gamma_0(\mathcal{C})$.

Acknowledgements. I would most of all like to thank Ian Zemke for taking the time to mentor me this summer, and for his help, encouragement, and advice. I would also like to thank Yi Ni for being a fantastic teacher, and for the allowing me to take several reading courses with him. I would also like to thank the Caltech Summer Faculty Program for sponsoring this research, Claudius Zibrowius for helping me understand the technicalities of his paper, and for working through several instructive examples, and Irving Dai, Jennifer Hom, Matthew Stoffregen and Linh Truong for the suggestion that P_ω might be a homomorphism.

2. Preliminaries

This section begins with brief review of almost ι-complexes and concludes with a crash-course in precurves. For a comprehensive description of almost ι-complexes, see [DHST18]. For a comprehensive description of precurves, see [KWZ19] and [Zib20].

2.1. Almost ι-complexes.

Motivated by the involution of the Heegaard Floer complex given in [HM17], Hendricks, Manolescu and Zemke define the group $\mathfrak{I}$ of ι-complexes in [HMZ16]. More precisely,

Definition 2.1 ([HMZ16])**.** An ι-complex $\mathcal{C} = (C, \iota)$ is given by:

- a free, finitely generated $\mathbb{Z}$-graded chain complex $\mathcal{C}$ over $\mathbb{F}[U]$ with
$$U^{-1}H_*(C) \cong \mathbb{F}[U, U^{-1}]$$
where U has degree -2 and $U^{-1}H_*(C)$ is supported in even gradings.
- a grading preserving, U-equivariant chain homomorphism $\iota : C \to C$ such that $\iota^2 \simeq \mathrm{id}$.

The group $\mathfrak{I}$ is the set of ι-complexes modulo local equivalence, which is defined below, with the group operation being tensor product.

Definition 2.2 ([HMZ16])**.** Two ι-complexes $\mathcal{C}, \mathcal{C}'$ are *locally equivalent* if there exists grading preserving chain maps $f : C \to C'$, $g : C' \to C$ such that

$$f \circ \iota \simeq \iota' \circ f, \qquad g \circ \iota' \simeq \iota \circ g$$

and f, g induce isomorphisms between $U^{-1}H_*(C)$ and $U^{-1}H_*(C')$.

It can be shown that $\otimes : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}$ forms a group operation that respects local equivalence. In essence, $\mathfrak{I}$ contains all possible complexes that could arise as the involutive Heegaard Floer complex CFI^- of some 3-manifold M. The classification of $\mathfrak{I}$ is not yet complete, although it would be a significant step towards the classification of the homology cobordism group. It is known that $\mathfrak{I}$ has torsion free elements and elements which are 2-torsion, but it is not known if $\mathfrak{I}$ has n-torsion for any $n > 2$.

To further understand $\mathfrak{I}$, the authors of [DHST18] define a group $\hat{\mathfrak{I}}$ of almost ι-complexes, which is an approximation of $\mathfrak{I}$ modulo U up to local equvalence. More precisely, $\hat{\mathfrak{I}}$ is the set of almost ι-complexes (defined below) modulo local equivalence.

Definition 2.3 ([DHST18])**.** An almost ι-complex $\mathcal{C} = (C, \bar{\iota})$ is given by:

- a free, finitely generated $\mathbb{Z}$-graded chain complex $\mathcal{C}$ over $\mathbb{F}[U]$ with
$$U^{-1}H_*(C) \cong \mathbb{F}[U, U^{-1}]$$
where U has degree -2 and $U^{-1}H_*(C)$ is supported in even gradings.
- a grading preserving $\mathbb{F}[U]$ module homomorphism $\bar{\iota} : C \to C$ such that
$$\bar{\iota} \circ \partial + \partial \circ \bar{\iota} \in \mathrm{im} U \qquad \text{and} \qquad \bar{\iota}^2 \simeq \mathrm{id} \mod U$$

Definition 2.4. Two almost ι-complexes $\mathcal{C}_1 = (C_1, \bar{\iota}_1)$ and $\mathcal{C}_2 = (C_2, \bar{\iota}_2)$ are *locally equivalent* if there are grading-preserving, U-equivariant chain maps $f : C \to C'$, $g : C' \to C$ such that

$$f \circ \bar{\iota} \simeq \bar{\iota}' f \mod U, \qquad g \circ \bar{\iota}' \simeq \bar{\iota} \circ g \mod U$$

andf, g induce isomorphisms between $U^{-1}H_*(C)$ and $U^{-1}H_*(C')$.

It can be shown that $\hat{\mathfrak{I}}$ is a group with respect to tensor product by using the same argument as in [HMZ16] for $\mathfrak{I}$.

Local equivalence classes in $\hat{\mathfrak{I}}$ have special representatives that Dai, Hom, Stoffregen, and Truong call standard standard complexes. These are defined below. Note the convention that $\omega = 1 + \bar{\iota}$.

Definition 2.5. A *standard complex* $\mathcal{C}(a_1, b_2, \cdots, a_{2n-1}, b_{2n})$ is an almost ι-complex parametrized by $a_i \in \{-, +\}$ and $b_i \in \mathbb{Z} - \{0\}$. It is generated by $T_0, \cdots, T_{2n}$. The b_i specify the differential. Let $\partial T_{2i} = U^{b_{2i}} T_{2i-1}$ if $b_{2i} > 0$ and let $\partial T_{2i-1} = U^{b_{2i}} T_{2i}$ if $b_{2i} < 0$, . The a_i specify the involution. Let $\omega T_i = T_{i-1}$ if $a_i = +$ and let $\omega T_{i-1} = T_i$ if $a_i = -$.

Recall that Dai, Hom, Stoffregen and Truong proved that every almost ι-complex is locally equivalent to a standard complex. This is the content of theorem 1.1.

Standard complexes are convenient to visualize as chains alternating between ∂ and ω arrows, which always start with an ω arrow and end with a ∂ arrow. An example is given in figure 8a for the standard complex $\mathcal{C}(+, -2)$.

The standard complex representative of an almost ι-complex is difficult to find in general. A significant portion of [DHST18] is dedicated to overcoming this obstacle. This motivates the following definition:

Definition 2.6. An almost ι-complex $\mathcal{C}$ is *reduced* if $\partial \equiv 0 \mod U$.

A key lesson in [DHST18] is that reduced almost ι-complexes are easier to work with than standard complexes. It is not difficult to show that an almost ι-complex can be assumed to be reduced without loss of generality. This relies on the classification of chain complexes over a PID, and mirrors the classification of modules over a PID. The interested reader may consult [HMZ16] or [DHST18].

Lemma 2.7. *Every almost ι-complex is locally equivalent to a reduced almost ι-complex.*

The following property of reduced almost ι-complexes will be useful in realizing almost ι-complexes as precurves.

Lemma 2.8 ([DHST18])**.** *Every reduced almost ι-complex has that $\omega^2 \equiv 0 \mod U$.*

The beginning of section 3 will describe how lemma 2.8 can be used to realize almost ι-complexes as precurves in full detail.

2.2. **Precurves.** Precurves provide a way of describing chain complexes geometrically as decorated immersed curves on a given surface, up to homotopy. The general construction is described in full detail in [KWZ19] and [Zib20]. An *arc system* is a pair (Σ, A) such that

- Σ is an oriented surface with boundary and without closed components,
- A is a set of pairwise disjoint oriented arcs properly embedded in Σ,
- and $\Sigma - \bigcup_{a\in A} N(a)$ is a set of pairwise disjoint annuli such that each annulus bounds exactly one component of $\partial\Sigma$. Here, $N(a)$ is a small tubular neighborhood of a.

The disjoint annuli in $\Sigma - \bigcup_{a\in A} N(a)$ are called *faces* and the set of faces is denoted $F(\Sigma, A)$. The fact that each face bounds exactly one component of $\partial\Sigma$ allows boundary components to be partitioned into inner and outer boundaries. The *inner boundary* $\partial_i\Sigma$ is the union of all boundary components which are bounded by a face $f \in F(\Sigma, A)$. The *outer boundary* $\partial_o\Sigma$ is $\partial\Sigma \setminus \partial_i\Sigma$. Let $s_2(a)$ and $s_1(a)$ be left and right sided boundaries of $N(a) - \partial\Sigma$ respectively, according to the orientation of a.

A picture is worth a thousand words, of course. See figure 1 for an example of an arc surface on the twice punctured disk. In this figure, the outer boundary $\partial_o\Sigma$ is shown in green and the inner boundary $\partial_i\Sigma$ is shown in blue.

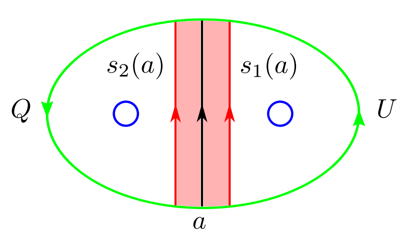


FIGURE 1. An arc system on the twice punctured disk.

There is an associated quiver algebra $Q(\Sigma, A)$ for any arc system (Σ, A). $Q(\Sigma, A)$ is generated by the graph with vertices being the arcs $a \in A$ and the directed edges being the oriented components of $\partial_o \Sigma \setminus \bigcup_{a \in A} a$ which connect arc endpoints. See figure 2 for an example.

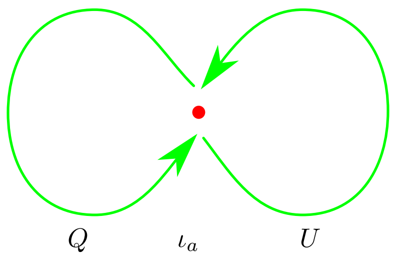


FIGURE 2. The quiver algebra corresponding to the arc system shown in figure 1.

The algebra corresponding to an arc system is the *arc algebra*

$$\mathcal{A}(\Sigma, A) := Q(\Sigma, A)/\mathcal{R}$$

where $\mathcal{R}$ is the relation set

$$\mathcal{R} := \{\rho_2\rho_1 = \rho_3\rho_4 = 0 : a \in A\}$$

and the ρ_i are defined according to the local relationship shown in figure 3.

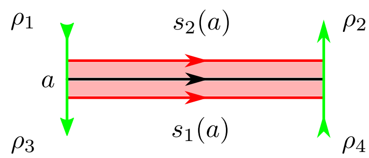


FIGURE 3. To obtain $\mathcal{A}(\Sigma, A)$, $Q(\Sigma, A)$ is quotiented by $\rho_2\rho_4 = \rho_3\rho_1 = 0$ for each $a \in A$.

The arc algebra corresponding to the arc system in figure 1 is $\mathbb{F}[U, Q]/(UQ)$. This is the arc system on which all the proofs in sections 3 and 4 will rely. By the end of this section, the reader will have the tools to describe a chain complex over $\mathbb{F}[U, Q]/(UQ)$ as an immersed curve on the twice punctured disk. For a demonstration, see example 2.26.

The key object in the translation of chain complexes to immersed curves is the precurve. Precurves are complicated objects, and their explanation will require some algebraic preliminaries. First it is necessary to define the linear extension $\mathrm{Mat}(\mathcal{C})$ of a differential graded category $\mathcal{C}$. Compare the following to [Zib20] and [BN05].

Definition 2.9. Given a differential category $\mathcal{C}$, define the linear extension of $\mathcal{C}$, $\mathrm{Mat}(\mathcal{C})$ by setting

$$\mathrm{Ob}(\mathrm{Mat}(\mathcal{C})) = \{\bigoplus_{k=0}^{n} x_k : x_k \in \mathrm{Ob}(\mathcal{C})\}$$

and

$$\mathrm{Mor}_{(\mathrm{Mat}(\mathcal{C}))}(\bigoplus_{k=0}^{m} x_k, \bigoplus_{l=0}^{n} x_l) = \bigoplus_{(k,l)\in[0,m]\times[0,n]} \mathrm{Mor}_{\mathcal{C}}(x_k, x_l)$$

where composition is given by matrix multiplication and

$$\partial(f_{lk} : X_k \to X_l)_{l,k} := (-1)^{i_l} \cdot \partial(f_{lk} : X_k \to X_l)_{l,k}$$

This allows for a definition of chain complexes over $\mathcal{C}$.

Definition 2.10. Given a differential category $\mathcal{C}$, the category of complexes over $\mathcal{C}$, $\mathrm{Cx}(\mathcal{C})$, is defined as follows:

$$\mathrm{ObCx}(\mathcal{C}) = \{(X, d) : X \in \mathrm{Ob}(\mathcal{C}), d \in \mathrm{Mor}(X, X; 1, 0), d^2 + \partial(d) = 0\}$$

and

$$\mathrm{Mor}_{\mathrm{Cx}(\mathcal{C})}((X, d), (X', d')) = \mathrm{Mor}_{\mathcal{C}}(X, X').$$

Define endomorphisms D on these morphisms by

$$D(f) = d' \circ f - (-1)^{i(f)} f \circ d + \partial f.$$

This provides a new, albeit more convoluted, way in which to understand chain complexes over a differential category. Before continuing, note $\mathcal{A}(\Sigma, A)$ can be made into a differential category, with

- $\mathrm{Ob}(\mathcal{A}(\Sigma, A)) = \mathcal{I} = \{\iota_a : a \in A\}$,
- $\mathrm{Mor}_{\mathcal{A}(\Sigma,A)}(a, b) = \iota_b.\mathcal{A}(\Sigma, A).\iota_a$
- $\partial = 0$.

It follows immediately that $\mathrm{Cx}(\mathrm{Mat}(\mathcal{A}(\Sigma, A)))$ is the category of finitely and freely generated chain complexes over $\mathcal{A}(\Sigma, A)$, or type-D structures over $\mathcal{A}(\Sigma, A)$.

Consider for a moment the difficulty in simplifying chain complexes over $\mathbb{F}[U, Q]/(UQ)$ to some canonical form. The devil is in the fact that changing basis effects both the U-arrows and the Q-arrows, which makes it impossible to simplify the U-arrows and Q-arrows separately. The advantage of this complicated categorical construction is that it will provide an equivalent category in which the simplification of the U-arrows and Q-arrows can be separated. This equivalent category is the category of precurves.

Before separating the category of chain complexes $\mathrm{Cx}(\mathrm{Mat}(\mathbb{F}[U, Q]/(UQ)))$ into U and Q components, the algebra $\mathbb{F}[U, Q]/(UQ)$ must first be separated. There is a general method for this over any arc system which takes the algebra $\mathcal{A}(\Sigma, A)$ to the expanded algebra $\bar{\mathcal{A}}(\Sigma, A)$.

For each face $f \in F(\Sigma, A)$, there is an associated cyclic graph where the vertices are arcs $a \in \partial f$, and the directed edges are the oriented boundary components of $\Sigma - \bigcup_{a\in A} a$ connecting the arcs. Let $\mathcal{A}_f$ be the path algebra of this cyclic graph. Let the *length* of an element of $\mathcal{A}_f$ be the length of the path in the cyclic graph corresponding to that element. Then, the expanded algebra is constructed by setting

$$\bar{\mathcal{A}}(\Sigma, A) := \bigoplus_{f\in F(\Sigma,A)} \mathcal{A}_f.$$

Denote $\bar{\mathcal{I}}$ to be the idempotents in $\bar{\mathcal{A}}(\Sigma, A)$. Note there are twice as many idempotents in $\bar{\mathcal{A}}(\Sigma, A)$ as there are in $\mathcal{A}(\Sigma, A)$. It is important later to remember that $\mathcal{I}$ embeds into $\bar{\mathcal{I}}$ by $\iota_a \mapsto \iota_{s_1(a)} \oplus \iota_{s_2(a)}$. It is also be important to define the subalgebras with elements of length greater than zero in $\bar{\mathcal{A}}(\Sigma, A)$:

$$\mathcal{A}^+(\Sigma, A) := \mathcal{A}/\{\iota = 0 : \iota \in \mathcal{I}\}$$
$$\bar{\mathcal{A}}^+(\Sigma, A) := \mathcal{A}/\{\iota = 0 : \iota \in \bar{\mathcal{I}}\}.$$

The case of the twice punctured disk is covered in example 2.11.

Example 2.11. Consider the arc system from figure 1. It has already been established that the arc algebra $\mathcal{A}(\Sigma, A)$ is $\mathbb{F}[U, Q]/(UQ)$. Here $\mathcal{I}$ is the single idempotent ι_a, which for all intents and purposes can be considered to be 1. $\bar{\mathcal{A}}(\Sigma)$ is $\mathbb{F}[U] \oplus \mathbb{F}[Q]$ and $\bar{\mathcal{I}} = \{\iota_{s_1(a)}, \iota_{s_2(a)}\}$. Here $\iota_{s_1(a)}$ corresponds to $0 \oplus 1$ and $\iota_{s_2(a)}$ corresponds to $1 \oplus 0$, if it is taken that the first summand in $\bar{\mathcal{A}}(\Sigma, A)$ is generated by U and the second by Q. $\mathcal{I}$ embeds in $\bar{\mathcal{I}}$ by $1 \mapsto 1 \oplus 1$. $\bar{\mathcal{A}}^+(\Sigma, A)$ is the algebra of nonzero length paths generated by U and Q.

The subalgebra $\bar{\mathcal{A}}^+$ is important because it gives rise to an exact sequence:

$$0 \to \bar{\mathcal{A}}^+ \to \bar{\mathcal{A}} \to \bar{\mathcal{I}} \to 0$$

that splits the morphisms of the linear extensions:

$$0 \to \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{A}}^+}(C, C') \to \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{A}}}(C, C') \to \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{I}}}(C, C') \to 0$$

thereby allowing any $\varphi \in \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{A}}}(C, C')$ to be split into $\varphi^+ + \varphi^\times$ where $\varphi^+ \in \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{A}}^+}(C, C')$ and $\varphi^\times \in \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{I}}}(C, C')$. This splitting allows for the construction of an expanded linearization $\mathrm{Mat}_i\bar{\mathcal{A}}$.

Definition 2.12. Let $\mathrm{Mat}_i\bar{\mathcal{A}}$ be the category defined by

- $\mathrm{ObMat}_i\bar{\mathcal{A}} := \{(C, (P_a)_{a \in A})\}$ where C is a bigraded right module over $\bar{\mathcal{I}}$ and
$$P_a : C.\iota_{s_1(a)} \to C.\iota_{s_2(a)}$$
is a bigrading preserving vector space isomorphism for every arc $a \in A$,
- $\mathrm{Mor}_{\mathrm{Mat}_i\bar{\mathcal{A}}}((C, (P_a)), (C', (P'_a))) :=$
$$\{\varphi \in \mathrm{Mor}_{\mathrm{Mat}\bar{\mathcal{A}}}(C, C') : (\iota_{s_2(a)}.\varphi^\times.\iota_{s_2(a)}) \circ P_a = P'_a \circ (\iota_{s_1(a)}.\varphi^\times.\iota_{s_1(a)}) \forall a \in A\}$$
where $\varphi^\times$ is the restricition of φ to the identity component, as described in [Zib20], and
- composition in $\mathrm{Mat}_i\bar{\mathcal{A}}$ is inherited from $\mathrm{Mat}\bar{\mathcal{A}}$.

With all this book-keeping in order, it is finally time to define the category of precurves.

Definition 2.13 ([Zib20])**.** $\mathrm{Cx}(\mathrm{Mat}_i\bar{\mathcal{A}})$ is the category of *precurves.*

The following results were proven by Zibrowius in [Zib20]:

Lemma 2.14 ([Zib20])**.** $\mathrm{Mat}\mathcal{A}$ *and* $\mathrm{Mat}_i\bar{\mathcal{A}}$ *are equivalent categories.*

Proof. This is witnessed by the equivalences $\mathcal{F} : \mathrm{Mat}\mathcal{A} \to \mathrm{Mat}_i\bar{\mathcal{A}}$ and $\mathcal{G} : \mathrm{Mat}_i\bar{\mathcal{A}} \to \mathrm{Mat}\mathcal{A}$. $\mathcal{F} : \mathrm{Mat}\mathcal{A} \to \mathrm{Mat}_i\bar{\mathcal{A}}$

- takes $\iota_a \in \mathrm{Ob}(\mathcal{A}) = \mathcal{I}$ (idempotents) to $\iota_{s_1(a)} \oplus \iota_{s_2(a)}$ in $\mathrm{Ob}(\bar{\mathcal{A}}) = \bar{\mathcal{I}}$,
- takes morphisms in $\varphi \in \mathrm{Mor}_{\mathcal{A}}(\iota_a, \iota_b)$ to

$$\iota_{s_1(a)} \oplus \iota_{s_2(a)} \xrightarrow{\begin{pmatrix} \iota_{s_1(b)}.\varphi.\iota_{s_1(a)} & \iota_{s_1(b)}.\varphi.\iota_{s_2(a)} \\ \iota_{s_2(b)}.\varphi.\iota_{s_1(a)} & \iota_{s_2(b)}.\varphi.\iota_{s_2(a)} \end{pmatrix}} \iota_{s_1(b)} \oplus \iota_{s_2(b)}$$

in $\mathrm{Mor}_{\mathrm{Mat}_i\bar{\mathcal{A}}}(\iota_{s_1(a)} \oplus \iota_{s_2(a)}, \iota_{s_1(b)} \oplus \iota_{s_2(b)})$
- and for each $x.\iota_a \in C.\iota_a$, $P_a(x.\iota_{s_1(a)}) = x.\iota_{s_2(a)}$.

$\mathcal{G} : \mathrm{Mat}_i\bar{\mathcal{A}} \to \mathrm{Mat}\mathcal{A}$ is roughly the projection of $C.\iota_{s_2(a)} \oplus C.\iota_{s_1(a)}$ to $C.\iota_a \cong C.\iota_{s_1(a)}$. The morphisms must begin at an element of $C.\iota_{s_1(a)}$ and end at an element $C.\iota_{s_1(b)}$ for any b to make this projection into an equivalence. In full formality, $\mathcal{G}$

- takes an object $X = (C, \{P_a\}_{a\in A})$ to $\mathcal{G}(X)$ by
- takes a morphism $\varphi \in \mathrm{Mor}_{\mathrm{Mat}_i\bar{\mathcal{A}}}((C, \{P_a\}), (C', \{P'_a\}))$ to $\mathcal{G}(\varphi)$ in $\mathrm{Mat}\mathcal{A}$ defined by

$$\begin{aligned}\iota_b.\mathcal{G}(\varphi).\iota_a =& (\iota_{s_1(b)}.\varphi^\times.\iota_{s_1(a)}) + (P'_b)^{-1} \circ (\iota_{s_2(b)}.\varphi^+.\iota_{s_1(a)}) \\ &+ (\iota_{s_1(b)}.\varphi^+.\iota_{s_2(a)}) \circ (P_a) + (P'_b)^{-1} \circ (\iota_{s_2(b)}.\varphi^+.\iota_{s_2(a)}) \circ P_a\end{aligned}$$

for each $a, b \in A$.

It is a quick calculation to show that $\mathcal{F}$ and $\mathcal{G}$ compose to the identity. ☐

An immediate corollary of lemma 2.14 is that the category of chain complexes over $\mathcal{A}$ is equivalent to the category of precurves.

Corollary 2.15 ([Zib20])**.** $\mathrm{Cx}(\mathrm{Mat}\mathcal{A})$ *are* $\mathrm{Cx}(\mathrm{Mat}_i\bar{\mathcal{A}})$ *equivalent differential categories.*

Proof. Consider the functors $\mathcal{F}$ and $\mathcal{G}$ from lemma 2.14. Then $\mathcal{F}$ and $\mathcal{G}$ are mutually inverse and witness the equivalence of $\mathrm{Cx}(\mathrm{Mat}\mathcal{A})$ and $\mathrm{Cx}(\mathrm{Mat}_i\bar{\mathcal{A}})$. ☐

The benefit of considering chain complexes over $\mathcal{A}(\Sigma, A)$ as precurves is that precurves can be represented geometrically as decorated immersed curves on Σ up to homotopy. This geometric description shall be described shortly. When considering the geometric presentation of a precurve, it will be the easiest to work with *reduced* precurves.

Definition 2.16. A *reduced* precurve $(C, \{P_a\}, \partial)$ is a precurve such that $\partial^+ = \partial$.

In other words, a reduced precurve is a complex in which the differential has no idempotents. Zibrowius proves in [Zib20] that precurves are reduced up to homotopy.

Proposition 2.17. *All precurves are chain homotopic to a reduced one.*

Now to describe the geometric presentatation of an arbitrary precurve $(C, \{P_a\}, \partial) \in \mathrm{CxMat}_i\bar{\mathcal{A}}(\Sigma, A)$ on Σ. The first step is to draw C. For each a, fix a basis of $C.\iota_{s_1(a)}$ and $C.\iota_{s_2(a)}$,

$$\{e_1^{s_1(a)}, \cdots, e_{n_a}^{s_1(a)}\} \qquad \text{and} \qquad \{e_1^{s_2(a)}, \cdots, e_{n_a}^{s_2(a)}\}.$$

Start by mark a point on $s_1(a)$ and $s_2(a)$ for the first generators of $C.\iota_{s_1(a)}$ and $C.\iota_{s_2(a)}$, respectively. Following the orientation of a and the order of each basis, mark a point on $s_1(a)$ and $s_2(a)$.

The next step is to draw the differential. Consider each nonzero component of the differential:

$$e_i^{s_k(a)} \xrightarrow{p^+} e_j^{s_l(a)}.$$

since ∂ is reduced, $p^+ \in \bar{\mathcal{A}}^+$ is a nontrivial path about some face $f \in F(\Sigma, A)$. For each such nonzero component, draw a curve from $e_i^{s_k(a)}$ to $e_j^{s_l(a)}$, following the path p^+. If, at the end of this process, there is some generator not connected to a path, connect it to the puncture on the face which it neighbors.

The final step is to draw $\{P_a\}$. Every matrix $P_a \in \mathrm{GL}_n(\mathbb{F} = \mathbb{Z}_2)$ can be written in the form $P_a = P_a^{l_a} \cdots P_a^1$ where each matrix P_a is a transposition T_{ij} (transposing columns i and j), or a single row addition E_{ij} (adding row i to row j). P_a can then be represented within $N(a)$ in the spirit of [HRW16], in which T_{ij} gives a *crossing* and E_{ij} gives a *crossover-arrow*. This method is shown in figure 4.

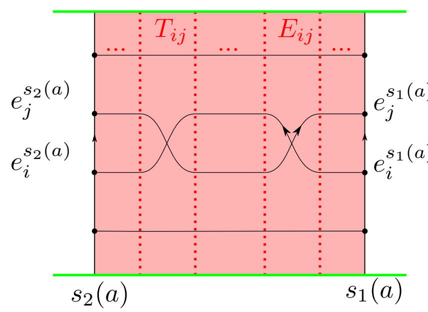


FIGURE 4. Representing P_a in $N(a)$.

There is no longer any need to distinguish between the geometric and algebraic representations of a reduced precurve. Note that it is unreasonable to expect a reduced precurve to be an immersed curve in general. At the very least, it is possible there are many components of ∂ associated with a single generator. Denote a precurve in which each generator is associated with exactly one component *simply faced*. The following lemma is required to manipulate an arbitrary immersed curve into a simply faced curve:

Lemma 2.18 (Clean-Up Lemma, [Zib20])**.** *Let (C, d_C) be an object in $\mathrm{Cx}(\mathrm{Mat}\mathcal{C})$ for some differential category $\mathcal{C}$. Then for any morphism $h \in \mathrm{Mor}((C, d_C), (C, d_C)$ for which*

$$h^2, \qquad hD(h), \qquad \text{and} \qquad D(h)h$$

are all zero, (C, d_C) is chain isomorphic to $(C, d_C + D(h))$.

It is straightforward to check that $(C, d_C + D(h))$ is in $\mathrm{Cx}(\mathrm{Mat}(\mathcal{C}))$ and that $(1+h) : (C, d_C) \to (C, d_C + D(h))$ and $(1+h) : (C, d_C + D(h)) \to (C, d_C)$ are chain isomorphisms. For a detailed presentation, see [Zib20]. Using the clean-up lemma it is possible to prove the following:

Proposition 2.19 ([Zib20])**.** *Each reduced precurve is chain isomorphic to a simply faced precurve.*

This is shown by inductively applying the *arc-reduction* moves shown in figure 5. In figure 5, the integers m and n are the length of the arc. The clean-up lemma can be applied to show that each arc-reduction move gives a chain isomorphism. This is the content of lemma 2.20.

Lemma 2.20. *$\mu_{1a}, \mu_{1b}, \mu_{1c}$ and μ_{1d} are all chain isomorphisms of precurves.*

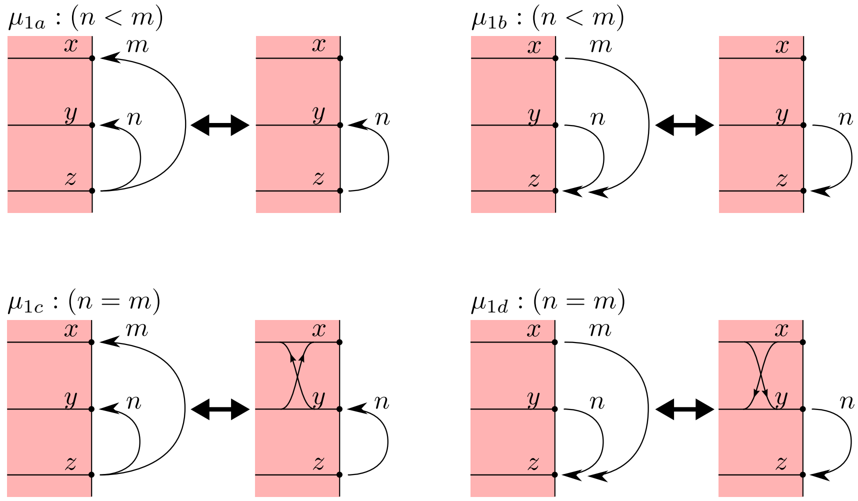


FIGURE 5. Arc reductions, $\mu_{1a}, \mu_{1b}, \mu_{1c}$, and μ_{1d}. Here it be can assumed that $m, n > 0$. The arcs are shown with orientation so that faces do not have to be drawn. Note that the orientation is generated by the direction of the path algebra element in the differential.

Proof. First consider μ_{1a}. If the precurve on the L.H.S. of μ_{1a} in figure 5 is $((C, \{P_a\}), d)$, then the complex on the R.H.S. of μ_{1a} in figure 5 is $((C, \{P_a\}), d + D(h))$ where

$$h(p) = \begin{cases} U^{m-n}x & \text{if } p = y \\ 0 & \text{else} \end{cases}$$

It must be that $h \in \mathrm{Mor}_{\mathrm{Mat}_i \bar{\mathcal{A}}}((C, \{P_a\}), (C, \{P_a\}))$ since $h^\times = 0$ gives that

$$(\iota_{s_2(a)}.h^\times.\iota_{s_2(a)}) \circ P_a = P'_a \circ (\iota_{s_1(a)}.h^\times.\iota_{s_1(a)}).$$

Lemma 2.18 gives that

$$(1 + h) : ((C, \{P_a\}), d) \to ((C, \{P_a\}), d + D(h)) \text{ and}$$

$$(1 + h) : ((C, \{P_a\}), d + D(h)) \to ((C, \{P_a\}), d)$$

are both chain isomorphisms since $h^2 = hD(h) = D(h)h = 0$. The proof for μ_{1b} is identical, except h is defined by

$$h(p) = \begin{cases} U^{m-n}y & \text{if } p = x \\ 0 & \text{else} \end{cases}.$$

Now consider μ_{1c}. Here

$$h(p) = h(p) = \begin{cases} x & \text{if } p = y \\ 0 & \text{else} \end{cases}.$$

It could be that $h \notin \mathrm{Mor}_{\mathrm{Mat}_i \bar{\mathcal{A}}}((C, \{P_a\}), (C, \{P_a\}))$ since $h = h^\times \neq 0$ and it may not be that

$$(\iota_{s_2(a)}.h^\times.\iota_{s_2(a)}) \circ P_a = P'_a \circ (\iota_{s_1(a)}.h^\times.\iota_{s_1(a)}).$$

Let a be the arc on which x, y, and z lie. If P_a is pre- or post-compose with E_{yx} to get P'_a then $h \in \mathrm{Mor}_{\mathrm{Mat}_i \bar{\mathcal{A}}}((C, \{P_a\}), (C, \{P'_a\}))$. Thus the clean-up lemma gives that

$$(1+h) : ((C, \{P_a\}), d) \to ((C, \{P'_a\}), d + D(h)) \text{ and}$$
$$(1+h) : ((C, \{P'_a\}), d + D(h)) \to ((C, \{P_a\}), d)$$

are both chain isomorphisms. The proof for μ_{1d} is identical, except P_a is pre- or post-composed with E_{xy} and

$$h(p) = \begin{cases} y & \text{if } p = x \\ 0 & \text{else} \end{cases}.$$

□

With lemma 2.20, it is now possible to prove proposition 2.19 in full detail.

Proof of Proposition 2.19. If a precurve $(C, \{P_a\}, d)$ is not simply faced, there is a generator with two incoming or outgoing edges. By applying $\mu_{1a}, \mu_{1b}, \mu_{1c}$ or μ_{1d}, it is possible to remove one of these edges. Lemma 2.20 guarantees that this process can be done while preserving the chain isomorphism type of $(C, \{P_a\}, d)$. This process can be continued indefinitely until the resulting precurve is simply faced. □

Note that even if a precurve is simply faced, it still may not be a collection of decorated immerersed curves, since it is unclear how to treat the crossover-arrows and crossings in each arc neighborhood.

Definition 2.21. A *decorated immersed curve* is a pair (γ, X) where γ is either closed or $\partial\gamma \subset \partial_{\text{inner}}(\Sigma, A)$, and $X \in GL_n(\mathbb{F}_2)$.

A decorated immersed curve (γ, X) is realized as a precurve by the following process:

1. Place $\dim X$ parallel copies of γ, $\{\gamma_1, \cdots, \gamma_{\dim X}\}$, in a small neighborhood of eachother, then
2. Pick any arc $a \in A$ intersecting γ, and index the generators $\gamma_i \cap s_j(a) = e_i^{s_j(a)}$, and finally
3. Represent X as crossover-arrows and crossings between the γ_i, according to the same method which P_a is represented on $N(a)$.

Now for the punchline,:

Theorem 2.22. *Any reduced precurve is chain isomorphic to a disjoint set of decorated immersed curves* $(\gamma_1, X_1), \cdots, (\gamma_n, X_n)$.

This last result is mostly due to the arrow sliding algorithm detailed in [HRW16], with the addition of the *crossover-arrow slide*, *crossing slide*, and *decomposition change* moves shown in figure 6. These moves, and the arrow-sliding algorithm, do not change the chain isomorphism type of the precurve. It will be particularly helpful in the proof of theorem 4.4 and 1.4 to know the expicit chain isomorphisms involved, so they are demonstrated here.

Lemma 2.23. μ_{2a} *and* μ_{2b} *are chain isomorphisms of precurves.*

Proof. Let us first begin with μ_{2a}. Let the L.H.S. of μ_{2a} in figure 6 be $((C, \{P_a\}), d)$ and the R.H.S. of μ_{2a} in figure 6 be $(C, \{P'_a\}, d)$. Note that

$$1 \in \mathrm{Mor}_{\mathrm{Mat}_i \bar{\mathcal{A}}}((C, \{P_a\}), (C, \{P'_a\}))$$

since

$$(\iota_{s_2(a)}.1.\iota_{s_2(a)}) \circ P_a = P'_a \circ (\iota_{s_1(a)}.1.\iota_{s_1(a)}).$$

Checking that 1 is a chain map gives that

$$1 : ((C, \{P_a\}), d) \to (C, \{P'_a\}), d) \text{ and}$$
$$1 : ((C, \{P'_a\}, d) \to (C, \{P_a\}), d)$$

are chain isomorphisms. The proof the chain isomorphism induced by μ_{2b} is $1 : C \to C$ and the proof is almost identical. □

Lemma 2.24. *μ_3 is a chain isomorphism of precurves.*

Proof. Let the L.H.S. of μ_3 in figure 6 be $((C, \{P_a\}), d)$ and the R.H.S. of μ_3 in figure 6 be $((C, \{P'_a\}), d')$. Let $f : C \to C$ be the map interchanging x and y. Then

$$f \in \mathrm{Mor}_{\mathrm{Mat}_i \bar{\mathcal{A}}}((C, \{P_a\}), (C, \{P'_a\}))$$

since

$$(\iota_{s_2(a)}.f.\iota_{s_2(a)}) \circ P_a = P'_a \circ (\iota_{s_1(a)}.f.\iota_{s_1(a)}).$$

Since $fd = d'f$ it follows that f is a chain map and

$$f : ((C, \{P_a\}), d) \to (C, \{P'_a\}), d') \text{ and}$$
$$f : ((C, \{P'_a\}, d') \to (C, \{P_a\}), d)$$

are chain isomorphisms. □

Lemma 2.25. *τ is a chain isomorphism of precurves.*

Proof. τ doesn't actually change the precurve. It only changes how P_a is drawn in $N(a)$. $1 : C \to C$ witnesses the isomorphism $((C, \{P_a\}), d) \to ((C, \{P_a\}), d)$. □

Example 2.26. Now to give an example of the precurve simplification process in action. Consider the $\mathbb{F}[U, Q]/(UQ)$ complex shown below:

$$\begin{tikzcd}
& \bullet_y \arrow[dr, "Q"] & \\
\bullet_x \arrow[ur, "U"] \arrow[dr, "U"'] & & \bullet_z \\
& \bullet_w \arrow[ur, "Q"'] &
\end{tikzcd}$$

The precurve representation of this complex is shown in figure 7. The simplification of this precurve to an immersed curve is shown in figure 7 as well. Note that for any single row addition matrix E_{ij} over $\mathbb{F}$, $E_{ij}E_{ij} = I$. The τ move in the last step of figure 7 simply represents the cancellation of the two crossover arrows.

Thus

$$\begin{tikzcd}
& \bullet_y \arrow[dr, "Q"] & & & \bullet_y & \\
\bullet_x \arrow[ur, "U"] \arrow[dr, "U"'] & & \bullet_z \arrow[r, phantom, "\underset{\text{chain}}{\cong}"] & \bullet_x \arrow[ur, "U"] & & \bullet_z \\
& \bullet_w \arrow[ur, "Q"'] & & & \bullet_w \arrow[ur, "Q"'] &
\end{tikzcd}$$

as $\mathbb{F}[U, Q]/(UQ)$ complexes.

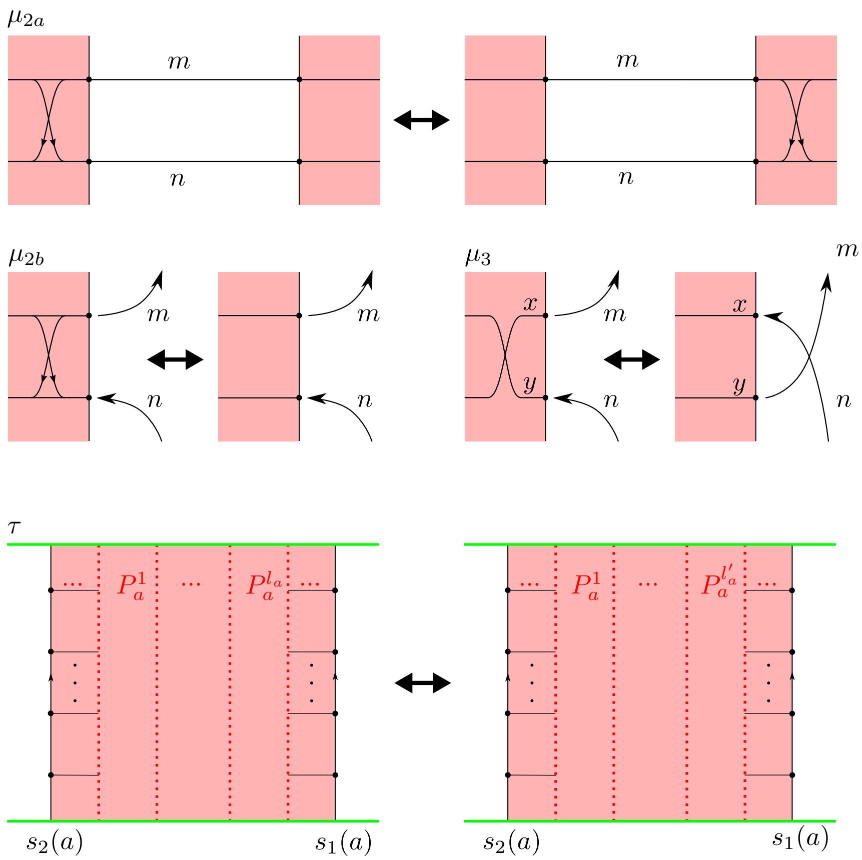


FIGURE 6. The crossover-arrow slides μ_{2a}, μ_{2b}, crossing slide μ_3 and decomposition change τ. Here m and n are allowed to be 0, except in μ_{2a}. In τ, the left and right hand sides are two equivalent decompositions of P_a into matrices P_a^k which are either transpositions T_{ij} (crossings) or row additions E_{ij} (crossover-arrows).

3. CLASSIFYING ALMOST ι-COMPLEXES

3.1. **Realizing almost ι-complexes as immersed curves.** Take an almost ι-complex $\mathcal{C} = (C, \bar{\iota})$, with generators $\{x_1, \cdots, x_n\}$. Denote the $\mathbb{F}$ span of these generators by X. $\mathcal{C}$ can be considered as a type D structure over $\mathbb{F}[U,Q]/(UQ,Q^2)$ by defining $\delta^1 : X \to (\mathbb{F}[U,Q]/(UQ,Q^2)) \otimes X$ to be

$$\delta^1(x) = (\partial + Q\omega)(x).$$

This defines the *type D representative of* $\mathcal{C}$ *over* $\mathbb{F}[U,Q]/(UQ,Q^2)$. Take any a type D structure M over $\mathbb{F}[U,Q]/(UQ)$. By setting Q^2 to zero, M becomes a type D structure over $\mathbb{F}[U,Q]/(UQ,Q^2)$.

Definition 3.1. The Q^2-*reduction* of a type D structure over $\mathbb{F}[U,Q]/(UQ)$ is the type D structure over $\mathbb{F}[U,Q]/(UQ,Q^2)$ obtained by setting Q^2 to zero.

It is said that an almost ι-complex $\mathcal{C}$ is the Q^2-reduction of M if the Q^2-reduction of M is the type D representative of $\mathcal{C}$ over $\mathbb{F}[U,Q]/(UQ,Q^2)$.

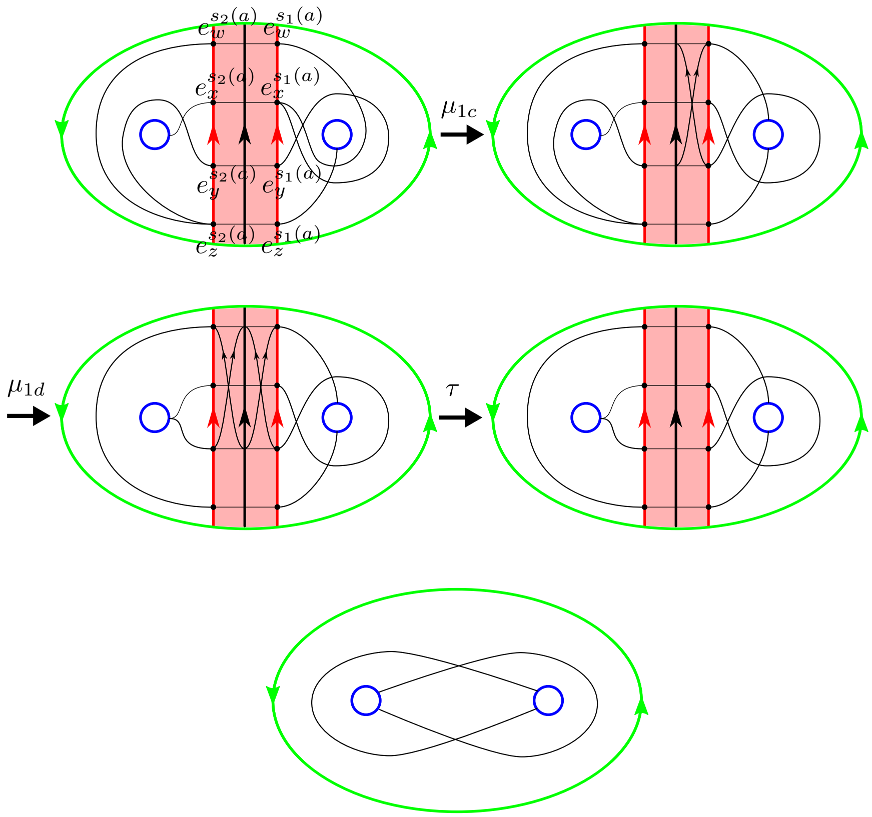


FIGURE 7. An example of a precurve being simplified to two immersed curves.

Proposition 3.2. *Every almost ι-complex is locally equivalent to the Q^2-reduction of a type D structure over $\mathbb{F}[U,Q]/(UQ)$.*

Proof. Take an almost ι-complex $\mathcal{C} = (C, \bar{\iota})$. By lemma 2.7 it can be assumed that $\mathcal{C}$ is locally equivalent to a reduced complex. Lemma 2.8 gives that it can be assumed $\omega^2 \equiv 0 \mod U$. Let the $\mathbb{F}$ span of the generators of $\mathcal{C}$ be denoted X. Define $\delta^1 : X \to (\mathbb{F}[U,Q]/(UQ,Q^2)) \otimes X$ to be

$$\delta^1(x) = (\partial + Q\omega)(x).$$

The fact that $(\delta^1)^2 = 0$ is a consequence of the fact that $\omega^2 \equiv 0 \mod U$ for reduced complexes.

$$(\delta^1)^2 x = \delta^1(\partial x + Q\omega x) = \partial^2 x + Q\omega\partial x + \partial Q\omega x + Q^2\omega^2 x.$$

$\partial^2 x = 0$, $\partial x \in \operatorname{im} U$ and ω is U equivariant, so $Q\omega\partial x = \partial Q\omega x = 0$. Furthermore, $\omega^2 x \in \operatorname{im} U$, so $Q\omega^2 x = 0$. It is now immediate that the Q^2-reduction of $(\mathbb{F}[U,Q]/(UQ) \otimes X, \delta^1)$ is locally equivalent to $\mathcal{C}$. $\square$

To start, a precurve on the twice-punctured disk will be associated to each almost ι-complex. While this is not necessarily well defined up to local equivalence, it is possible to assign a unique precurve $M(\mathcal{C})$ to each reduced ι complex $\mathcal{C}$.

Recall the functors $\mathcal{F} : \mathrm{CxMat}(\mathcal{A}) \to \mathrm{CxMat}_i(\bar{\mathcal{A}})$ and $\mathcal{G} : \mathrm{CxMat}_i(\bar{\mathcal{A}}) \to \mathrm{CxMat}\mathcal{A}$ used in the proof of lemma 2.14 and corollary 2.15. These will be helpful in relating precurves to chain complexes.

Definition 3.3. $M(\mathcal{C})$, the *precurve representative of a reduced almost ι-complex* $\mathcal{C} = (C, \bar{\iota})$, is defined as:

$$M(\mathcal{C}) = \mathcal{F}(\mathbb{F}[U,Q]/(UQ) \otimes X, \delta^1)$$

where

$$\delta^1(x) = (\partial + Q\omega)(x),$$

and X is the $\mathbb{F}$-span of the generators of C.

It is evident that $\mathcal{G}(M(\mathcal{C}))$ Q^2-reduces to $\mathcal{C}$ since $\mathcal{G}(M(\mathcal{C}))$ is simply the type D structure $(\mathbb{F}[U,Q]/(UQ) \otimes X, \delta^1)$ defined by $\mathcal{C}$. Furthermore, $M(\mathcal{C})$ can be expressed as a reduced precurve on the twice punctured disk, and is chain isomorphic to a set of decorated immersed curves, by theorem 2.22.

Definition 3.4. A *decorated immersed representative* of a reduced almost ι-complex $\mathcal{C}$ is a set of decorated immersed curves $\gamma(\mathcal{C}) = \{(\gamma_i, X_i)\}$ which is chain isomorphic to $M(\mathcal{C})$.

By definition, $\gamma(\mathcal{C})$ is also a precurve. It will be useful to define the Q^2-reduction of $\gamma(\mathcal{C})$, or any other precurve, in general.

Definition 3.5. The Q^2-*reduction* of a precurve X on the twice punctured disk is the Q^2-reduction of $\mathcal{G}(X)$.

Lemma 3.6. *For a reduced almost ι complex $\mathcal{C}$, any differential component of $\gamma(\mathcal{C})$ which passes around the Q-puncture does so exactly once.*

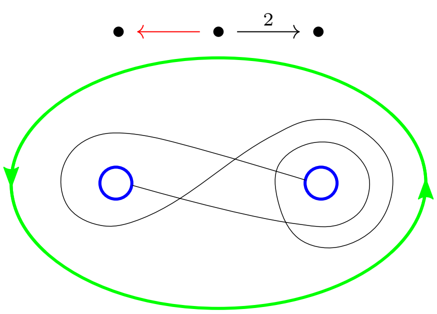
Proof. In $M(\mathcal{C})$, the only power of Q appearing in δ^1 is 1. However, $M(\mathcal{C})$ is not necessarily a decorated immersed curve. Consider the moves μ_1, μ_2, μ_3, and τ shown in figures 5 and 6, which simplify a precurve to a set of decorated immersed curves. These moves do not alter the number of times a segment of $M(\mathcal{C})$ wraps around the Q-puncture, unless it removes the segment entirely. Thus any segment of $\gamma(\mathcal{C})$ that passes around the Q-puncture does so exactly once. □

Example 3.7. It is not true that if $\mathcal{C}_1$ is locally equivalent to $\mathcal{C}_2$, then $\gamma(\mathcal{C}_1)$ is homotopic to $\gamma(\mathcal{C}_2)$, as shown in figure 8. In this example $\mathcal{C}_1$ and $\mathcal{C}_2$ are locally equivalent, but the extra summand of $\mathcal{C}_2$ adds a second immersed curve.

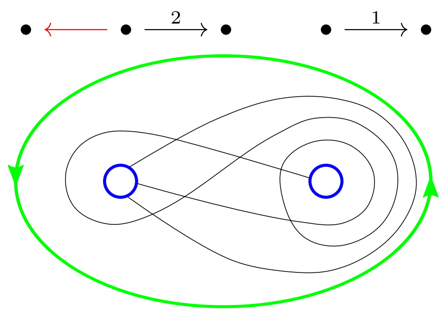
3.2. **Primitive representatives.** In this section, it will be shown that any decorated immersed representative of a reduced almost ι-complex $\mathcal{C}$ contains a unique immersed curve $\gamma_0(\mathcal{C})$ with no decoration that begins at the U-puncture and ends at the Q-puncture. This primitive component $\gamma_0(\mathcal{C})$ determines the local equivalence class of $\mathcal{C}$.

The following lemma dispels the need to distinguish between $M(\mathcal{C})$ and $\gamma(\mathcal{C})$ for reduced complexes.

Lemma 3.8. *For a reduced almost ι complex $\mathcal{C}$, the Q^2-reduction of $\gamma(\mathcal{C})$ is chain isomorphic to $\mathcal{C}$.*



(A) $\mathcal{C}_1 = \mathcal{C}(+,-2)$ as an immersed curve.



(B) $\mathcal{C}_2 = \mathcal{C}(+,-2) \oplus (\bullet \xrightarrow{U} \bullet)$ as an immersed curve.

FIGURE 8. $\mathcal{C}_1$ is locally equivalent to $\mathcal{C}_2$, but they are not homotopic as immersed curves.

Proof. It is known that $M(\mathcal{C})$ Q^2-reduces to $\mathcal{C}$. Furthermore, it is known that the simplification to a set of decorated immersed curves do not change the chain isomorphism type of $M(\mathcal{C})$. Thus it only needs to be shown that the chain isomorphism f given by the moves taking $M(\mathcal{C})$ to $\gamma(\mathcal{C})$ preserves the local equivalence class of $\mathcal{C}$. However, if f is a chain isomorphism of $\mathbb{F}[U,Q]/(UQ)$ complexes, then it is a chain isomorphism of the $\mathbb{F}[U,Q]/(UQ,Q^2)$ as well. Thus $M(\mathcal{C})$ and $\gamma(\mathcal{C})$ are chain isomorphic as $\mathbb{F}[U,Q]/(UQ,Q^2)$ complexes, and it follows that $\mathcal{C}$ is locally equivalent to the Q^2-reduction of $\gamma(\mathcal{C})$. $\square$

This allows $\mathcal{C}$ to be imagined as $\gamma(\mathcal{C})$ for reduced complexes up to chain isomorphism. Now it remains to remove excess immersed curves, and reduce to the primitive case.

Definition 3.9. A *primitive representative* of an almost ι-complex $\mathcal{C}$ is an immersed curve $\gamma_0(\mathcal{C})$ with no decoration that Q^2-reduces to an almost ι-complex which is locally equivalent to $\mathcal{C}$.

It will be shown in theorem 3.14 that any almost ι-complex has a primitive representative which Q^2-reduces to the standard representative of $\mathcal{C}$. At first, it will only be necessary to work with reduced almost ι-complexes.

Definition 3.10. An almost ι-complex $\mathcal{C} = (C, \bar{\iota})$ splits if C can be written as $C' \oplus B$ with $\bar{\iota}C' \subset C', \bar{\iota}B \subset B, \partial C' \subset C'$, and $\partial B \subset B$, where without loss of generality, the generator of the infinite tower in $H_*(C)$ lies in C'. Then it is said $\mathcal{C} = \mathcal{C}' \oplus \mathcal{B}$, where $\mathcal{C}' = (C', \partial|_{C'}, \bar{\iota}|_{C'})$ and $\mathcal{B} = (B, \partial|_B, \bar{\iota}|_B)$.

Lemma 3.11. *If an ι-complex $\mathcal{C}$ splits into $\mathcal{C}' \oplus \mathcal{B}$, then $\mathcal{C}$ is locally equivalent to $\mathcal{C}'$.*

Proof. Let $i : C' \to C$ and $\pi : C \to C'$ be the natural inclusion and projection. It will be shown that i and π give a local equivalence. Clearly i and π are grading preserving chain maps. Since the generator of $H_*(C)$ lies in C' and $\partial C' \subset C'$, i and π give isomorphims on homology after localization by U. It only remains to show that i and π commute with $\bar{\iota}$ up to chain homotopy. For i, $i \circ \bar{\iota} = \bar{\iota} \circ i$ since $\bar{\iota}C' \subset C'$. Similarly for π, $\pi \circ \bar{\iota} = \bar{\iota} \circ \pi$ since $\bar{\iota}C' \subset C'$. $\square$

Lemma 3.12. *Any reduced almost ι-complex $\mathcal{C}$ has a primitive representative.*

Proof. Enumerate the components of $\gamma(\mathcal{C})$ by $\{(\gamma_0, X_0), \cdots, (\gamma_n, X_n)\}$ and assume without loss of generality that γ_0 is the unique curve with an endpoint on the U-puncture. Since there is a unique strand joining the U-puncture and marking the tower element in $M(\mathcal{C})$, and the simplification moves μ_1, μ_2, μ_3 and τ are chain isomorphisms, such a γ_0 must exist.

Now to show that (γ_0, X_0) is locally equivalent to $\mathcal{C}$, and that X_0 is trivial. By lemma 3.8, the Q^2-reduction of $\gamma(\mathcal{C})$ is locally equivalent to $\mathcal{C}$. Furthermore, the (γ_i, X_i) induce a splitting on the Q^2-reduction of $\gamma(\mathcal{C})$ into $\mathcal{C}_0 \oplus \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$, where $\mathcal{C}_0$ is the Q^2-reduction of (γ_0, X_0), and $\mathcal{B}_{i\neq 0}$ are the Q^2-reductions of (γ_i, X_i) with $i \neq 0$. Thus by lemma 3.11, the Q^2-reduction of $\gamma(\mathcal{C})$ is locally equivalent to Q^2-reduction of (γ_0, X_0).

Lastly, it must be shown that the decoration on γ_0 is trivial. Suppose to the contrary that X_0 is nontrivial. Then $\dim X_0 > 1$. Suggestively denote the intersection of γ_0 and $s_1(a)$ to be e_1^1, and let the other $(\dim X_0 - 1)$ generators be denoted e_i^1. By lemma 3.6, the segment of γ_0 traveling from e_1^1 into the Q-face can wrap about the Q-puncture at most once. This gives three cases:

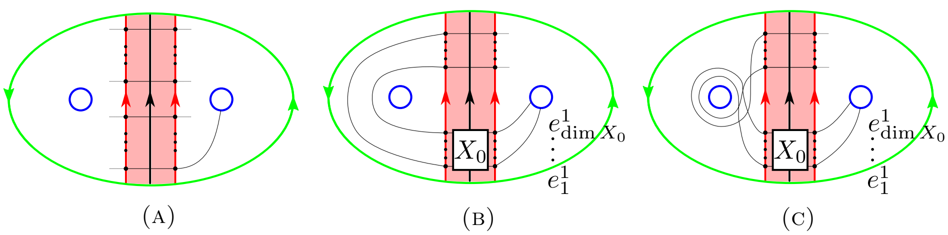


FIGURE 9. (γ_0, X_0) if $\dim X_0$ could be greater than one.

In each case, the Q^2-reduction of (γ_0, X_0) has $\dim X_0$ generators which are both in $\ker \partial$ and not in $\mathrm{im}\partial$ for any power of U. This would imply the existence of $\dim X_0$ infinite U towers in $H_*(\mathcal{C})$, which is impossible. $\square$

3.3. **Proving the classification theorem.** Now that a primitive representative can be assigned to any reduced almost ι-complex $\mathcal{C}$, the classification of almost ι-complexes is nearly complete. What remains to be done is to classify the possible $\gamma_0(\mathcal{C})$, and then to show that two almost ι-complexes $\mathcal{C}$ and $\mathcal{C}'$ are locally equivalent if and only if $\gamma_0(\mathcal{C}) = \gamma_0(\mathcal{C}')$.

Lemma 3.13. *Given a reduced almost ι-complex $\mathcal{C}$, the Q^2-reduction of $\gamma_0(\mathcal{C})$ is a standard complex.*

Proof. The argument to show that $\gamma_0(\mathcal{C})$ represents a standard complex is purely combinatorial. Lemma 3.12 guarantees that $\gamma_0(\mathcal{C})$ is an immersed curve with no decoration. Therefore $\gamma_0(\mathcal{C})$ must start from the U-puncture and continue to $N(a)$, as shown in figure 10.

This proof will traverse inductively up the spine of a, determining each possible segment of $\gamma_0(\mathcal{C})$ until it is shown the Q^2-reduction of $\gamma_0(\mathcal{C})$ is a standard complex. Note that by homotoping $\gamma_0(\mathcal{C})$, it can be assumed that each arc connects a generator on $s_i(a)$ to the generator directly above it. The generators on $s_1(a)$ are be labelled by $x_0^1, \cdots, x_n^1$ and the generators on $s_2(a)$ are labelled $x_0^2, \cdots, x_n^2$, from

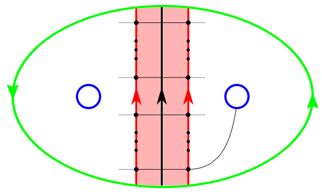


FIGURE 10. Shown above is the connection of the unique tower element with the U-puncture. The other segments of $\gamma_0(\mathcal{C})$ are yet to be constructed.

bottom to top. In figure 10, x_0^1 is the only completed generator. Now to continue from x_0^2. The segment connecting x_0^2 can wrap around the Q-puncture at most once by lemma 3.6. Thus there are three possibilities, shown in figure 11.

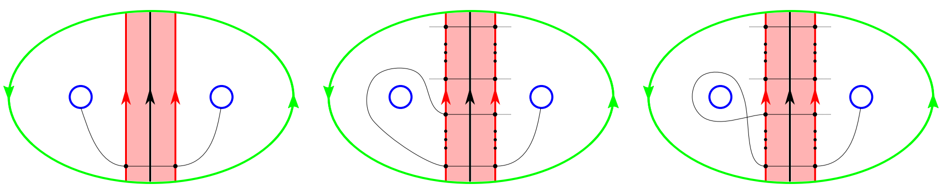


FIGURE 11. (γ_0, X_0) if $\dim X_0$ could be greater than one.

In the first case, x_0^2 is connected$Q$-puncture, terminating $\gamma_0(\mathcal{C})$. In this case, the Q^2-reduction of $\gamma_0(\mathcal{C})$ is the trivial standard complex $\mathcal{C}(0)$. In the second case, x_0^2 is connected to x_1^2 by a segment that wraps once around the Q-puncture counter clockwise. This sets $\omega x_1 = x_0$ in $\gamma_0(\mathcal{C})$ the Q^2-reduction of $\gamma_0(\mathcal{C})$. In the last case, x_0^2 is connected x_0^1 by a segment that wraps once around the Q-puncture clockwise. This sets $\omega x_0 = x_1$ in the Q^2-reduction of $\gamma_0(\mathcal{C})$.

Suppose all generators $x_0^1, \cdots, x_{i-2}^1$ and $x_0^2, \cdots, x_{i-1}^2$ have been completed, and the segment connecting x_i^1 is yet to be determined. In this inductive process, this will only happen for even i. Then there are two possible ways to connect x_i^1, shown in figure 12. Notice neither possiblility terminates $\gamma_0(\mathcal{C})$, since the Q^2-reduction of $\gamma_0(\mathcal{C})$ would have two infinite U-towers in homology if the curve went to the U-puncture again. In the first case, x_{i-1}^1 is connected to x_i^1 by a segment that travels n times around the U-puncture in the counter clockwise direction. This sets $\partial x_{i-1} = U^n x_i$ in the Q^2-reduction of $\gamma_0(\mathcal{C})$. In the second case, x_{i-1}^1 is connected to x_i^1 by a segment that travels n times around the U-puncture in the clockwise direction. This sets $\partial x_i = U^n x_{i-1}$ in the Q^2-reduction of $\gamma_0(\mathcal{C})$.

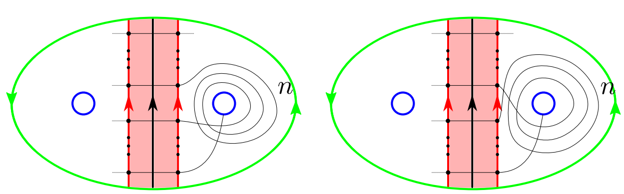


FIGURE 12. The two possible ways to complete x_{i-1}^1 and x_i^1 in $\gamma_0(\mathcal{C})$.

Now suppose that all the generators $x_0^1, \cdots, x_{i-1}^1$ and $x_0^2, \cdots, x_{i-2}^2$ have been completed, and the segment connecting x_{i-1}^2 is not yet determined. According to the inductive process, this will only happen for odd i. Then lemma 3.6 again requires that any segment from x_{i-1}^2 wraps at most one time about the Q-puncture. This gives three cases, shown in figure 13. The first possibility sends x_{i-1}^2 to the Q-puncture, and terminates $\gamma_0(C)$. In this case, $\gamma_0(\mathcal{C})$ Q^2-reduces to the standard complex parametrized by the previous steps. In the second case, x_{i-1}^2 is connected to x_i^2 by a segment wrapping around the Q-puncture once counter clockwise. This sets $\omega x_{i-1} = x_i$ in the Q^2-reduction of $\gamma_0(\mathcal{C})$. In the third case, x_{i-1}^2 is connected to x_i^2 by a segment wrapping around the Q-puncture once clockwise. This sets $\omega x_i = x_{i-1}$ in the Q^2-reduction of $\gamma_0(\mathcal{C})$.

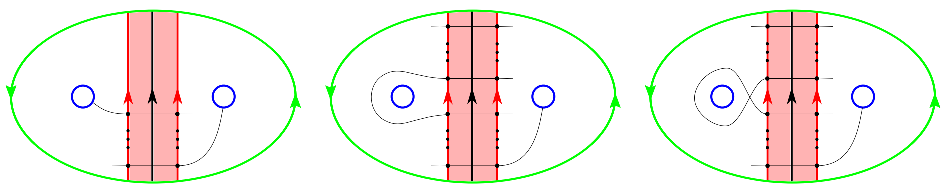


FIGURE 13. The three possible ways to complete x_{i-1}^2 and x_i^2 for odd i in $\gamma_0(\mathcal{C})$.

Inductively, this proves that the Q^2 reduction of $\gamma_0(\mathcal{C})$ is a standard complex. □

Theorem 3.14. *For any almost ι complex $\mathcal{C}$, there is a unique primitive representative $\gamma_0(\mathcal{C})$, and $\gamma_0(\mathcal{C})$ Q^2-reduces to the standard complex representative of $\mathcal{C}$.*

Proof. Suppose $\mathcal{C}$ is locally equivalent to two different reduced standard complexes $\mathcal{C}'$ and $\mathcal{C}''$. Then $\mathcal{C}'$ is locally equivalent to the Q^2-reduction of $\gamma_0(\mathcal{C}')$ and $\mathcal{C}''$ is locally equivalent to the Q^2-reduction of $\gamma_0(\mathcal{C}'')$. This gives is a local equivalence between the Q^2-reductions of $\gamma_0(\mathcal{C}')$ and $\gamma_0(\mathcal{C}'')$, which are both standard complexes. By theorem 4.6 of [DHST18], this proves that $\gamma_0(\mathcal{C}') = \gamma_0(\mathcal{C}')$. □

Proof of theorem 1.1. Every almost ι-complex $\mathcal{C}$ is locally equivalent to the Q^2-reduction of $\gamma_0(\mathcal{C})$, which is a standard complex. □

4. HOMOMORPHISMS FROM $\hat{\mathfrak{I}}$ TO $\mathbb{Z}$

This section will describe all known homomorphisms from $\hat{\mathfrak{I}}$ to $\mathbb{Z}$ using precurves. First, it will be proven that P and P_ω are homomorphisms, and then it will be shown that the maps $\phi_n : \hat{\mathfrak{I}} \to \mathbb{Z}$ are homomorphisms as well.

The heart of the proofs in this section come down to studying the following element of an almost ι-complex:

Definition 4.1. For an almost ι-complex $\mathcal{C}$, $x_{\text{final}}(\mathcal{C})$ is the unique generator in the Q^2-reduction of $\gamma_0(\mathcal{C})$ trailing to the Q-puncture.

In standard complexes $\mathcal{C}$ generated by $T_0, \cdots, T_{2n}$, $x_{\text{final}}(\mathcal{C})$ is T_{2n}. Any standard complex can be given a bigrading $\text{gr} = (\text{gr}_U, \text{gr}_Q)$, where $\text{gr}U = (-2, 0)$ and $\text{gr}Q = (0, -1)$. gr_U agrees with the normal grading. gr_Q can be defined by setting $\text{gr}_Q\omega = -1$ and $\text{gr}_Q\partial = 0$, while requiring $\text{gr}_Q T_0 = 0$. Since the Q^2-reduction of $\gamma_0(\mathcal{C})$ is a standard complex, P and P_ω can be defined as follows:

Definition 4.2. For an almost ι-complex $\mathcal{C}$, $P(\mathcal{C}) = \mathrm{gr}_U(x_{\mathrm{final}})$.

Definition 4.3. For an ι-complex $\mathcal{C}$, $P_\omega(\mathcal{C}) = \mathrm{gr}_Q(x_{\mathrm{final}})$.

gr_U agrees with the usual grading, so the definition of the *pivotal map* P here agrees with that in [DHST18]. Note that for a standard complex $\mathcal{C}$ generated by $T_0, \cdots, T_{2n}$,

$$\mathrm{gr}_Q T_i = \sum_{k>0|2k+1\leq i} a_{2k+1} \qquad \text{and} \qquad P_\omega(\mathcal{C}) = \mathrm{gr}_Q T_{2n} = \sum_{k=0}^{n-1} a_{2k+1}.$$

While it was shown using a homological argument in [DHST18] that

Theorem 4.4. [DHST18] *P is a homomorphism from $\hat{\mathfrak{I}}$ to $\mathbb{Z}$.*

there is another way to prove this with precurves. The fact that P and P_ω are homomorphisms comes down to the following:

Lemma 4.5. *For any standard complexes $\mathcal{C}, \mathcal{C}'$,*

$$\mathrm{gr}_*(x_{\mathit{final}}(\mathcal{C} \otimes \mathcal{C}')) = \mathrm{gr}_*(x_{\mathit{final}}(\mathcal{C})) + \mathrm{gr}_*(x_{\mathit{final}}(\mathcal{C}'))$$

for $* \in \{U, Q\}$.

The following definitions will be necessary to prove lemma 4.5:

Definition 4.6. Two elements $x, y \in C$ on an arc a are *immediately connected* in a representation of P_a by $P_a = P_a^1 \cdots P_a^{l_a}$ if there is a crossover arrow or crossing in the decomposition connecting x and y.

Definition 4.7. Two elements $x, y \in C$ on an arc a are *connected* in a representation of P_a by $P_a = P_a^1 \cdots P_a^{l_a}$ if there is a sequence of generators $\{x = z_1, \cdots, z_n = y\}$ such that z_i and z_{i+1} are immediately connected.

Proof of lemma 4.5. It will shown that the chain isomorphism from $M(\mathcal{C} \otimes \mathcal{C}')$ to $\gamma(\mathcal{C} \otimes \mathcal{C}')$ preserves bigrading. Each step of the precurve simplification either manipulates arcs with a μ_i move, or rearranges P_a with τ. It will be shown the induced chain isomorphism is bigrading preserving in two steps. First $M(\mathcal{C} \otimes \mathcal{C}')$ will be reduced to a simply-faced curve, and then the simply faced precurve will be reduced to a decorated immersed curve.

To arrive at a simply faced precurve, $\mu_{1a}, \cdots, \mu_{1d}$ are applied successively. To show that $\mu_{1a}, \cdots, \mu_{1d}$ are bigrading preserving it will be shown that the following properties are preserved throughout the process:

(I) two generators are connected only if they have the same bigrading,
(II) the differential respects the bigrading, and
(III) there is a component $x.\iota_{s_1(a)} \xrightarrow{U^n} y.\iota_{s_1(a)}$ only if x and y have the same Q-grading, and there is a component $x.\iota_{s_2(a)} \xrightarrow{Q} y.\iota_{s_2(a)}$ only if x and y have the same U-grading,

Condition (I) is true in the case of $M(\mathcal{C} \otimes \mathcal{C}')$ as no generators are connected yet. (II) and (III) true in $M(\mathcal{C} \otimes \mathcal{C}')$ by construction.

Suppose that (I)-(III) hold at a given step of this simplification and μ_{1a} is about to be applied. Properties (II) and (III) guarantee that the chain isomorphism $(1 + h)$ induced by μ_{1a} in lemma 2.20 is bigrading preserving. After $(1 + h)$ is applied, (I)-(III) still hold true. A similar argument applies in the case of μ_{1b}.

Now suppose that at a given step of the simplification, (I)-(III) hold and μ_{1c} is about to be applied. If μ_{1c} is being applied in the direction from the L.H.S. to the R.H.S. with respect to figure 5, then properties (II) and (III) guarantee that the chain isomorphism $(1+h)$ induced by μ_{1c} in lemma 2.20 is bigrading preserving. If μ_{1c} is being applied in the direction from the R.H.S. to the L.H.S. then the induced chain isomophism $(1+h)$ is bigrading preserving since (I) holds. In both directions the resulting precurve satisfies properties (I)-(III). Note that μ_{1c} will not need to be applied in the latter direction during the simplification a simply-faced precurve, but could be necessary later. The argument for μ_{1d} is entirely parallel.

Once the precurve is simply faced, it remains to apply the arrow sliding algorithm given in [HRW16] to reduce to a decorated immersed curve. Only the moves $\mu_{2a}, \mu_{2b}, \mu_3$, and τ must be employed to acheive this. The arrow sliding algorithm only manipulates crossover-arrows and crossings between generators which are connected. Thus the isomorphisms 1 and f induced by $\mu_{2a}, \mu_{2b}, \mu_3$, and τ are bigrading preserving.

Thus the total chain isomorphism from $M(\mathcal{C} \otimes \mathcal{C}')$ to $\gamma_0(\mathcal{C} \otimes \mathcal{C}')$ is bigrading preserving. Now to argue that the grading of the final generator is additive with respect to tensor product.

The product $\mathcal{C} \otimes \mathcal{C}'$ will have only one generator in $\ker\omega/\mathrm{im}\omega$, and thus the homology of $\mathcal{G}(M(\mathcal{C} \otimes \mathcal{C}'))$ over $\mathbb{F}[U,Q]/(UQ)$ will have only one infinite Q-tower. Note that since precurve moves preserve the homology over $\mathbb{F}[U,Q]/(UQ)$, the only segment in $\gamma(\mathcal{C}_1 \otimes \mathcal{C}_2)$ connecting to the Q-puncture must be on $\gamma_0(\mathcal{C})$. Any other segment connecting to the Q-puncture would result in the homology over $\mathbb{F}[U,Q]/(UQ)$ having a second Q-tower in homology. Furthermore, since the homology of $\mathcal{G}(M(\mathcal{C}\otimes\mathcal{C}'))$ over $\mathbb{F}[U,Q]/(UQ)$ is an invariant of chain isomorphism, and the last generator in $\gamma_0(\mathcal{C}_1 \otimes \mathcal{C}_2)$ emits the only arc connected to the Q-puncture. Thus the isomorphism between the Q^2-reductions of $M(\mathcal{C} \otimes \mathcal{C}')$ and $\gamma(\mathcal{C} \otimes \mathcal{C}')$ takes $x_{\text{final}}(\mathcal{C}) \otimes x_{\text{final}}(\mathcal{C}')$ to $x_{\text{final}}(\mathcal{C} \otimes \mathcal{C}')$. This proves that $\mathrm{gr}_*(x_{\text{final}}(\mathcal{C} \otimes \mathcal{C}')) = \mathrm{gr}_*(x_{\text{final}}(\mathcal{C})) + \mathrm{gr}_*(x_{\text{final}}(\mathcal{C}'))$ for $* \in \{U, Q\}$ because it is known that the simplification preserves bigrading. □

It can now be proven that P and P_ω are homomorphisms as direct corollaries of lemma 4.5

Proof of theorems 1.4 and 4.4. By lemma 4.5,

$$\mathrm{gr}_U(x_{\text{final}}(\mathcal{C} \otimes \mathcal{C}')) = \mathrm{gr}_U(x_{\text{final}}(\mathcal{C})) + \mathrm{gr}_U(x_{\text{final}}(\mathcal{C}'))$$

and

$$\mathrm{gr}_Q(x_{\text{final}}(\mathcal{C} \otimes \mathcal{C}')) = \mathrm{gr}_Q(x_{\text{final}}(\mathcal{C})) + \mathrm{gr}_Q(x_{\text{final}}(\mathcal{C}'))$$

by definition, this gives that

$$P(\mathcal{C} \otimes \mathcal{C}') = P(\mathcal{C}) + P(\mathcal{C}') \qquad \text{and} \qquad P_\omega(\mathcal{C} \otimes \mathcal{C}') = P_\omega(\mathcal{C}) + P_\omega(\mathcal{C}')$$

□

Now it will be reproven that each ϕ_n is a homomorphism. All that is left to do is to show that the shift maps from [DHST18] are endomorphisms of $\hat{\mathfrak{I}}$. An equivalent definition of the shift maps sh_n can be given as follows:

Definition 4.8. Given any almost ι-complex $\mathcal{C}$ considered as a precuve, $\mathrm{sh}_n(\mathcal{C})$ replaces every arc around the U puncture of length $m \geq n$ with an arc of length $m+1$.

Theorem 4.9 ([DHST18]). sh_n *is an endomorphism of* $\hat{\mathfrak{I}}$ *for* $n \geq 1$.

Proof. Consider the sequence of moves μ_i, τ taking $M(\mathcal{C}_1 \otimes \mathcal{C}_2)$ to $\gamma(\mathcal{C}_1 \otimes \mathcal{C}_2)$. Since the relative lengths of the generators and arcs do not change, the exact same moves clearly take $M(\mathrm{sh}_n(\mathcal{C}_1) \otimes \mathrm{sh}_n(\mathcal{C}_2)$ to $\mathrm{sh}_n(\gamma(\mathcal{C}_1 \otimes \mathcal{C}_2))$. Thus $\gamma_0(\mathrm{sh}_n(\mathcal{C}_1) \otimes \mathrm{sh}_n(\mathcal{C}_2)) = \mathrm{sh}_n(\gamma_0(\mathcal{C}_1 \otimes \mathcal{C}_2))$. This proves that $\mathrm{sh}_n(\mathcal{C}_1) \otimes \mathrm{sh}_n(\mathcal{C}_2)$ is locally equivalent to $\mathrm{sh}_n(\mathcal{C}_1 \otimes \mathcal{C}_2)$ which shows that sh_n is a homomorphism. □

Proof of theorem 1.2. Using theorems 4.4 and 4.9, the rest is exactly as in [DHST18]. By counting the length of the differentials, it can be shown that

$$P(\mathcal{C}) = \sum_{i=1}^{\infty} (-2i+1)\phi_i(\mathcal{C}),$$

and

$$P(\mathrm{sh}_n(\mathcal{C})) = P(\mathcal{C}) - 2\sum_{i=n}^{\infty} \phi_i(\mathcal{C}),$$

from which it can quickly be shown with the inductive argument given in [DHST18] that $\phi_n : \hat{\mathfrak{I}} \to \mathbb{Z}$ is a homomorphism for all n. □

Proof of theorem 1.5. Suppose P_ω were in the span of $\{\phi_n\}$. Then $P_\omega = \sum_{n=1}^{\infty} a_n \phi_n$. Since $P_\omega(\mathcal{C}(+,k)) = 1$ for all k, and $\phi_n(\mathcal{C}(+,k))$ is nonzero if and only if $n = k$, it follows $a_n = 1$ for all n. If this is the case, then $P_\omega(\mathcal{C}(-,k)) = \sum_{n=1}^{\infty} \phi_n(\mathcal{C}(-,k)) = 1$, which is a contradiction since $P_\omega(\mathcal{C}(-,k)) = -1$. □

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