

Bézier Curves

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Bézier Curve Formulation

Given a series of lines $\overline{P_0P_1}, \dots, \overline{P_{n-1}P_n}$, the resulting Bézier curve, $B_{P_0 \dots P_n}(t)$ can be described as follows:

$$B_{P_0 \dots P_n}(t) = \sum_{i=0}^n \binom{n}{i} P_i (1-t)^i t^{n-i}$$

Where $0 \leq t \leq 1$.

Proof. Using the algorithm for the construction of Bézier curves and induction:

The trivial case, describing $B_{P_0P_1}$:

$$\begin{aligned} B_{P_0P_1} &= P_0 + (P_1 - P_0)t \text{ since the Bézier curve is an interpolation of } P_0 \text{ and } P_1 \text{ along } \overline{P_0P_1}. \\ \therefore B_{P_0P_1} &= P_0(1-t) + P_1t = \sum_{i=0}^1 \binom{1}{i} P_i (1-t)^i t^{1-i} \end{aligned}$$

So the hypothesis is true for the base case. Applying the Interpolation Lemma to the induction hypothesis, it can be shown that the hypothesis is true for any $(n+1)^{\text{th}}$ iteration if it is true for the n^{th} .

$$\begin{aligned} \text{It follows from the algorithm for the construction of Bézier curves that } B_{P_0 \dots P_{n+1}}(t) &= B_{B_{P_0 \dots P_n}(t) B_{P_1 \dots P_{n+1}}(t)}(t) \\ \therefore B_{P_0 \dots P_{n+1}}(t) &= [B_{P_0 \dots P_n}] \cdot (1-t) + [B_{P_1 \dots P_{n+1}}] \cdot (t) \end{aligned}$$

Which, by the induction hypothesis, is equal to the following:

$$\begin{aligned} & [\sum_{i=0}^n \binom{n}{i} P_i (1-t)^i t^{n-i}] \cdot (1-t) + [\sum_{i=0}^n \binom{n}{i} P_{i+1} (1-t)^i t^{n-i}] \cdot (t) \\ &= \sum_{i=0}^n \binom{n}{i} P_i (1-t)^{i+1} t^{n-i} + \sum_{i=0}^n \binom{n}{i} P_{i+1} (1-t)^i t^{(n+1)-i} \\ &= \sum_{i=0}^{n+1} [\binom{n}{i} + \binom{n}{i-1}] P_i (1-t)^i t^{n-i} \\ &= \sum_{i=0}^{n+1} [\binom{n+1}{i}] P_i (1-t)^i t^{(n+1)-i} \text{ by Pascal's Triangle Binomial Lemma} \end{aligned}$$

This proves that the following is true:

$$B_{P_0 \dots P_n}(t) = \sum_{i=0}^n \binom{n}{i} P_i (1-t)^i t^{n-i} \implies B_{P_0 \dots P_{n+1}}(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} P_i (1-t)^i t^{(n+1)-i}$$

$$\text{So } B_{P_0 \dots P_n}(t) = \sum_{i=0}^n \binom{n}{i} P_i (1-t)^i t^{n-i}$$

And this concludes the proof. □

(Pascal's Triangle Binomial Lemma is proved on the next page)

Pascal's Triangle Binomial Lemma

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$$

Proof. This is provable by direct computation:

Simply expand $\binom{n}{i}$:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

Then expand $\binom{n}{i-1}$:

$$\binom{n}{i-1} = \frac{n!}{(i-1)!(n-i+1)!}$$

Then add the two:

$$\binom{n}{i} + \binom{n}{i-1} = \frac{n!}{i!(n-i)!} \cdot \frac{(n-i+1)}{(n-i+1)} + \frac{n!}{(i-1)!(n-i+1)!} \cdot \frac{(i)}{(i)} = \frac{n!(n-i+1) + n!(i)}{i!(n+1-i)!}$$

$$= \frac{n!(n+1)}{i!(n+1-i)!} = \frac{(n+1)!}{i!(n+1-i)!} = \binom{n+1}{i}$$

□

Notes

Credit for showing Bézier curves to me goes to Vsevolod Rostovtsev

The animations and explanations of Jason Davies and Mike Kamermans were accessible, mesmerizing and intuitive. This is a rigorous proof that the explicit parametric formula given to describe Bézier curves is in fact correct. Hopefully this provides some helpful insight.

Jason Davies' page: <https://www.jasondavies.com/animated-bezier/>

Mike Kamerman's page: <https://pomax.github.io/bezierinfo/>