Euler-Goldbach and Legendre

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The concept of a factorial has been around almost as long as algebra. Under its classical definition, the factorial is a function from $(n)! : \mathbb{N} \to \mathbb{N}$, mapping counting numbers strictly to counting numbers. Its scope is limited. Yet, with the invention of the limit and the beginnings of real analysis, a way to compute factorials using non-integers emerged. And, as a result, a way to compute the "factorial" any real or complex number.

This new "factorial" is called the gamma function: $\Gamma(x) : \mathbb{C} \to \mathbb{C}$. Like many other named functions in mathematics, the gamma function takes many forms. Its study has lead to great advancements in real and complex analysis, and it all started with a small discovery made in the 1730s by Euler and Goldbach:

$$n! = \lim_{m\to\infty} \frac{m!(m+1)^n}{\prod_{i=1}^{m}(n+i)}$$

Lemma

Given any $n \in \mathbb{N}$: $\displaystyle\lim_{m\to\infty} \frac{m!(m+1)^n}{(n+m)!} = 1$

Proof. It is not as bad as it looks.

$$\lim_{m\to\infty} \frac{m!(m+1)^n}{(n+m)!} = \lim_{m\to\infty} \frac{m!}{m!} \frac{(m+1)^n}{\prod_{i=1}^{n}(m+i)} = \lim_{m\to\infty} \frac{(m+1)^n}{\prod_{i=1}^{n}(m+i)}$$

There are n terms in both the numerator and the denominator, so:

$$\lim_{m\to\infty} \frac{(m+1)^n}{\Pi_{i=1}^{n}(m+i)} = \lim_{m\to\infty} \prod_{i=1}^{n} \frac{(m+1)}{(m+i)} = \prod_{i=1}^{n} \lim_{m\to\infty} \frac{(m+1)}{(m+i)}$$

But for each limit $\dfrac{(m+1)}{(m+i)}$:

$$\lim_{m\to\infty} \frac{(m+1)}{(m+i)} = \lim_{m\to\infty} \frac{(\frac{m}{m}+\frac{1}{m})}{(\frac{m}{m}+\frac{i}{m})} = \lim_{m\to\infty} \frac{(1+\frac{1}{m})}{(1+\frac{i}{m})} = \lim_{m\to\infty} \frac{1}{1} = 1$$

It then follows from the above that:

$$\lim_{m\to\infty} \frac{m!(m+1)^n}{(n+m)!} = \prod_{i=1}^{n} \lim_{m\to\infty} \frac{(m+1)}{(m+i)} = 1$$

$\square$

Euler-Goldbach Limit

Given any $n \in \mathbb{N}$: $n! = \lim_{m\to\infty} \dfrac{m!(m+1)^n}{\prod_{i=1}^{m}(n+i)}$

Proof. Manipulating the limit so that the lemma above is applicable:

$$\frac{1}{n!} \cdot \lim_{m\to\infty} \frac{m!(m+1)^n}{\prod_{i=1}^{m}(n+i)} = \lim_{m\to\infty} \frac{m!(m+1)^n}{(n+m)!} = 1$$

So it then follows that:

$$\lim_{m\to\infty} \frac{m!(m+1)^n}{\prod_{i=1}^{m}(n+i)} = n! \qquad \square$$

It is worth while to note how incredibly computationally difficult this sequence is to compute. Define the function $\Psi(n, m)$ to estimate the use the Euler Goldbach Limit, to m, so that:

$$\Psi(n, m) = \frac{m!(m+1)^n}{\prod_{i=1}^{m}(n+i)}$$

Just to find 3!, or 6, to an accuracy of 5 percent, it takes an m of 57, because $\Psi(3, 57) \approx 5.70169$. And, to get to 5.9 it takes at least and m of 167, since $\Psi(3, 167) = 5.89432$–after stagnating at 5.89 for about ten values of m. (Personally, I wouldn't know how long it actually takes to get all the way to 5.91, because my computers returns an overload error after $m = 167$)

Yet it is incredible that n did not have to be an integer to compute $n!$ in either proof. So, in principle, n could be absolutely anything! Here are some test values of the Euler Goldbach Limit for various $z \in \mathbb{C}$:

$$\Psi(1.4, 150) = 1.16521$$
$$\Psi(2.6, 150) = 3.66641$$
$$\Psi(-2.2, 150) = 4.73851$$
$$\Psi(i, 150) = 0.50018 - 0.15381i$$

The precise value of the Euler Goldbach Limit at a given z, with m truly taken to infinity, would essentially be the gamma function. Except the gamma function, for the sake of tradition, is defined such that $\Gamma(n) = (n-1)!$, so everything is offset by one. As a result, the gamma function is defined as followed:

$$\Gamma : \mathbb{C} \to \mathbb{C}$$

$$\Gamma(z) = \lim_{m\to\infty} \frac{m!(m+1)^{z-1}}{\prod_{i=1}^{m}(z+i-1)}$$

However, there are many other ways to definitions to the gamma function, and not all are trivially equivalent. Afterall, as long as the curve is smooth, and passes through all $n!$, then one definition is just as good as any other. Legendre found an integral form for the gamma function about a century later,

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$$

To show that this curve passes through all $n!$ is just a little bit of work.

Proposition

If $\Gamma(1) = 1$ and $\Gamma(n+1) = \Gamma(n) \cdot n$, then $\Gamma(n) = (n-1)! \; \forall \; n \in \mathbb{N}$.

Proof. $n!$ is defined recursively as $n! = (n-1)! \cdot n$ and $0! = 1$. So $\Gamma(1) = 1$ and $\Gamma(n+1) = \Gamma(n) \cdot n$, then $\Gamma(n) = (n-1)! \; \forall \; n \in \mathbb{N}$, by definition. □

Legendre Gamma Function

$\Gamma(z) = \displaystyle\int_0^\infty x^{z-1} e^{-x} dx = n! \; \forall \; n \in \mathbb{N}$

Proof. It suffices to show that $\Gamma(1) = 1$ and $\Gamma(n+1) = \Gamma(n) \cdot n$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = -e^{-x}|_0^\infty = e^{-\infty} + e^0 = 0 + 1 = 1$$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$$

Using integration by parts where $u = x^n$ and $dv = e^{-x}dx$, it follows that $du = nx^{n-1}$ and $v = -e^{-}x$. So:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = -x^n e^{-x}|_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx$$

But uv evaluates simply to 0 since:

$$-x^n e^{-x}|_0^\infty = -\lim_{x \to \infty} \frac{x}{e^x} + 0^n e^0 = 0$$

Also $n \int_0^\infty x^{n-1} e^{-x} dx = \Gamma(n) \cdot n$. So $\Gamma(n+1) = \Gamma(n) \cdot n$. □

This paper about the gamma funciton by Philip J. Davis is the best resource I ran into. It gives a great mathematical and historical overview.

$http://www.maa.org/sites/default/files/pdf/upload_library/22/Chauvenet/Davis.pdf$