Fixed Point Theorem

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Let G be a p-group, and let S be a finite set on which G operates. If the order of S is not divisible by p, then there is a fixed point for the for the operation of G on S - an element whose stabilizer is the whole group.

Proof. Just as in with class equation, the order of G was found by summing the orders of the conjugacy classes, for any group action it follows that:

$$|S| = \sum |O_i|, \text{ where each } |O_i| \text{ divides } |G| \text{ (by the orbit stabilizer theorem)}$$

If the order of G is p^e, then each $|O_i| = p^k$. Since S is not divisible by p, there must be at least one $|O_i| = 1$, and the element in that orbit will have a stabilizer equal to the entirety of G, by the orbit stabilizer theorem. $\square$