Bolzano's Theorem and the Bolzano Weierstrass Theorem

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Continuity

Let f be a function $f : \mathbb{R} \to \mathbb{R}$, then f is continuous at point $x_0 \in \mathbb{R}$ if $f(x_0) \approx_\epsilon f(x)$ for some $x_0 \approx_\delta x$ where ϵ and δ are arbitrarily close to 0.

Limit Mutation Theorem

Let f be a function $f : \mathbb{R} \to \mathbb{R}$ and $\{a_i\}$ be a sequence such that $\lim a_i = L$, then $\lim f(a_i) = f(L)$ if f is continuous at L.

$\lim f(a_i) = f(\lim a_i) \leftrightarrow f$ is continuous at L.

Proof. This follows directly from the definitions of limits and continuous functions.

Since $\lim a_i = L$, it must follow that $a_i \approx_\delta L$. Also, since f is continuous at L, it must also follow that $f(x) \approx_\epsilon f(L)$ for some $x \approx_\delta L$. So it follows that $f(a_i) \approx_\epsilon f(L)$ since $a_i \approx_\delta L$. Therefore, if f is continuous at L, $\lim f(a_i) = f(L)$.

□

Nested Intervals

An interval $U_1 = [a_1, b_1]$ is nested inside of the interval $U_0 = [a_0, b_0]$ if and only if $a_1 \geq a_0$ and $b_1 \leq b_0$.

Nested Interval Theorem

Let $\{U_i\} = \{[a_i, b_i]\}$ be an infinite sequence of nested intervals where $U_{i+1} \neq U_i$. Then there exists a unique $c \in \mathbb{R}$ such that $\lim a_i = \lim b_i = c$.

Proof. This theorem can be proven using the completeness theorem, which states that if a sequence is monotone (always increasing or always decreasing) and bounded, then it has a limit.

The sequence a_i is, by definition, always increasing, so it is monotone. Also, since $a_i < b_i$ for all i, $\{a_i\}$ must certainly be bounded above b_0. Likewise the sequence $\{b_i\}$ is, by definition, always increasing, so it is also monotone. And, since $b_i > a_i$ for all i, $\{b_i\}$ must certainly be bounded below by a_0. So, by the completeness theorem, both $\{a_i\}$ and $\{b_i\}$ must converge.

Let their limits be defined like this:
$\lim a_i = A$
$\lim b_i = B$.

To show that $A = B$, consider the limit of $\{c_i\} = \{b_i - a_i\}$. Since $\{U_i\}$ is an infinite sequence of nested intervals, a_i and b_i must move infinitely close together, so $\lim c_i = 0$. But the limit of c_i can also be expressed as follows:

$$\lim c_i = \lim(b_i - a_i) = \lim b_i - \lim a_i = B - A = 0$$

Therefore $A = B$ and there exists a unique $c \in \mathbb{R}$ such that $\lim a_i = \lim b_i = c$.

□

Bolzano's Theorem

Let f be continuous on $[a, b]$, and $f(a)$ and $f(b)$ are of opposite polarity. Then there exists a number $C \in \mathbb{R}$ such that $f(C) = 0$.

Proof. At the heart of this proof is an algorithm for constructing an infinite sequence of nested intervals that center around some root in the interval $[a, b]$,. For the sake of clarity, the algorithm will be described in two different cases, which depend on the sign of $f(a)$ and $f(b)$.

Suppose $f(a) < 0$ and $f(b) > 0$. The first nested interval in the sequence is, trivially, $U_0 = [a, b]$. Take the midpoint of a and b, say x_1. If $f(x_1) < 0$, let the second nested interval be $U_1 = [x_1, b]$. Since f must switch sign in that interval, there might be a root on that interval. If $f(x_1) > 0$, let the second nested interval be $U_1 = [a, x_1]$ for the same reason. If $f(x_1) = 0$, then clearly there is some C such that $f(C) = 0$. Continue this process, of slicing the interval at the midpoint, to generate the sequence of nested intervals $\{U_i\}$ so that f switches sign on each interval. Denote each U_i in the sequence to be the interval $[l_i, r_i]$, for left and right.

By the nested interval theorem, both $\{l_i\}$ and $\{r_i\}$ converge to the same unique C. By the construction of the intervals, it is also evident that $\{f(l_i)\}$ and $\{f(r_i)\}$ converge to 0. So, by the limit mutation theorem:

$f(C) = f(\lim r_i) = \lim f(r_i) = 0$ (and)
$f(C) = f(\lim l_i) = \lim f(l_i) = 0$

So $f(C) = 0$ and there is a zero $[a, b]$. The same construction of nested intervals applies if $f(a) > 0$ and $f(b) < 0$, the signs are just reversed. This proves Bolzano's Theorem.

□

The Bolzano Weierstrass Theorem

Let a_i be an infinite bounded sequence. Then $\{a_i\}$ has a convergent subsequence.

Proof. This is, in essence, a corollary to the nested interval theorem.

To construct the nested intervals for the proof, start with the trivial interval $U_0 = [L, H]$, where L is the lower bound and H is the upper bound.

Take any point x_1 in this interval. If there are infinitely many points greater than x_1, let U_1 be the interval $[x_1, H]$, and if there are infinitely many points less than x_1, let U_1 be the interval $[L, x_1]$. If there are infinitely many points to either side, pick any side randomly, and if there are infinitely many points equal to x_1 then clearly some subsequence of $\{a_i\}$ converges to x_1. Continue generating new U_i infinitely to create the infinite sequence of nested intervals $\{U_i\}$.

By the nested interval theorem, the lower bound of the intervals converges to some C in $[L, H]$, and since the lower bound of these intervals is a subsequence of a_i, this proves the Bolzano Weierstrass Theorem.

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