Taylor Remainder Theorem

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Lemma: Rolle's Theorem

Let f be a continuous, differentiable function on the interval $[a, b]$. If $f(a) = f(b)$ and $a \neq b$, then there exists some c between a and b such that $f'(c) = 0$.

Proof. Since f is defined on $[a, b]$, f has at least one local maximum and or one local minimum on the interval $[a, b]$.

Say f has a local maximum, $f(x_m)$, on $[a, b]$. Then $f'(x) \geq 0$ if x both near x_m and less than x_m. Additionally, $f'(x) \leq$ if x is both near x_m and more than x_m. Therefore, by Bolzano's theorem, $f'(x)$ must equal zero for some c on $[a, b]$.

Say f has a local minimum, $f(x_m)$, on $[a, b]$. Then $f'(x) \leq 0$ if x both near x_m and less than x_m. Additionally, $f'(x) \geq 0$ if x is both near x_m and more than x_m. Therefore, by Bolzano's theorem, $f'(x)$ must equal zero for some c on $[a, b]$.

Bolzano's Theorem is necessary in this proof because it guarantees that a function passes through zero if it flips sign and is continuous. Otherwise, it could just skip from say $-\epsilon$ to δ where ϵ and δ are unimaginably small numbers.

$\square$

Lemma: Extended Rolle's Theorem

Let f be a continuous, $n+1$ differentiable function on the interval $[a, b]$. If $f(a) = f'(a) = f"(a) = f^{n+1}(a) = f(b) = 0$, then $f^{n+1}(c) = 0$ for some $c \in [a, b]$.

Proof. This is proved simply by applying Rolle's Theorem over and over again.

$f(a) = f(b) = 0 \to f'(c_1) = 0$ by Rolle's Theorem
$f'(a) = f'(c_1) = 0 \to f"(c_2) = 0$ by Rolle's Theorem
$\cdots$
$f^n(a) = f^n(c_{n-1}) = 0 \to f^{n+1}(c) = 0$ by Rolle's Theorem

$\square$

Note that by applying Rolle's Theorem again and again, all c_i are related by the following inequality:

$a \leq c \leq c_n \leq c_{n-1} \leq \cdots \leq c_2 \leq c_1 \leq b$

Taylor's Remainder Theorem

Let $P_n(x)$ be the nth order Taylor polynomial for f at a. Then for some c on the interval between a and x:

$$f(x) = P_n(x) + \frac{f^{n+1}(c)}{n!}(x-a)^{n+1} = \sum_{i=0}^{n} \frac{f(a)}{i!}(x-a)^i + \frac{f^{n+1}(c)}{n!}(x-a)^{n+1}$$

Proof. Define a function $P_{n,b}$ with the following properties:

$P_{n,b}(x) = P_n(x) + C(x-a)^{n+1}$
$P_{n,b}(b) = f(b)$ (this can be guaranteed by choosing an appropriate value of C (i.e. $C = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}$))

Now consider the difference function $\Delta(x) = f(x) - P_{n,b}(x)$. The following must be true:

$\Delta(a) = f(a) - P_{n,b}(a) = f(a) - f(a) = 0$
$\Delta'(a) = f'(a) - P'_{n,b}(a) = f'(a) - f'(a) = 0$
...
$\Delta^{n-1}(a) = f^{n-1}(a) - P^{n-1}_{n,b}(a) = f^{n-1}(a) - f^{n-1}(a) = 0$
$\Delta^{n}(a) = f^{n}(a) - P^{n}_{n,b}(a) = f^{n}(a) - f^{n}(a) = 0$
and
$\Delta(b) = 0$ (because $P_{n,b}(b) = f(b)$

Since the nth derivative of any Taylor polynomial at a is the same as the nth derivative of f at a by design. So, by the Extended Rolle's Theorem, there is some c between a and b such that $\Delta^{n+1}(c) = 0$. Therefore:

$$0 = \Delta^{n+1}(c) = f^{n+1}(c) - P^{n+1}_{n,b}(c) = f^{n+1}(c) - (P^{n+1}_{n}(c) + [\frac{d^{n+1}}{dx^{n+1}}C(x-a)^{n+1}](c))$$
$$= f^{n+1}(c) - (0 + (n+1)!C) = f^{n+1}(c) - (n+1)!C$$

Therefore $C = \dfrac{f^{n+1}(c)}{(n+1)!}$ for some $c \in [a, b]$, thus proving Taylor's Theorem.

□