Stirling's Approximation

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Stirling's Approximation is a way of approximating the gamma function for positive real numbers. It can be proven easily using Laplace's Method.

Stirling's Approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \approx \Gamma(n+1)$$

Proof. Using Lagrange's expression of the gamma function:

$$n! = \Gamma(n+1) = \int_0^\infty x^{(n+1)-1}e^{-x}dx = \int_0^\infty x^n e^{-x}dx = \int_0^\infty e^{n\ln x}e^{-x}dx = \int_0^\infty e^{n\ln x - x}dx$$

Now to apply Laplace's method. But first, to make $f(x)$ a little cleaner, substitute x for ny, so that $dx = ndy$:

$$\int_0^\infty e^{n\ln x - x}dx = n\int_0^\infty e^{n\ln ny - ny}dy = n\int_0^\infty e^{n\ln n + n\ln y - ny}dy = n^n n\int_0^\infty e^{n(\ln y - y)}dy = n^{n+1}\int_0^\infty e^{n(\ln y - y)}dy$$

So $f(y) = n(\ln y - y)$. Therefore $f'(y) = n(\frac{1}{y} - 1)$ and $f"(y) = -\frac{n}{y^2}$. Since $f'(y)$ is only equal to zero at $y = 1$, at which there is a global maximum $f(y_0) = f(1) = -n$. Also, the second derivative $f"(y)$ is equal to $-n$ at $y = 1$. Therefore, by Laplace's method, for $M = 1$

$$n! = n^{n+1}\int_0^\infty e^{n(\ln y - y)}dy \sim n^{n+1}e^{Mf(y_0)}\sqrt{\frac{2\pi}{M|f"(y_0)|}} \approx n^{n+1}e^{-n}\sqrt{\frac{2\pi}{n}}$$

Finally, it only remains to simplify the end result.

$$n^{n+1}e^{-n}\sqrt{\frac{2\pi}{n}} = n^n\left(\frac{1}{e}\right)^n n\sqrt{\frac{2\pi}{n}} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

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