On the computation of the infinite variety of Heegaard Floer homology

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Abstract

In [3], Manolescu, Ozsváth and Sarkar outline a method for the computation of HFK° using grid diagrams. Currently, there are programs that compute $\widehat{HFK}$ very quickly, using the results of [7]. However, there do not exist publicly available programs to compute HFK^{-}. Here one is provided, and used to find both HFK^{-} and the filtered CFK^{∞} for all nonalternating knots of crossing number eight and nine.

Contents

1 Introduction

Heegaard Floer homology has been an important tool in knot theory because it encodes information about tons of classical and contemporary invariants in a singular algebraic object. Inside the simplest variety of this homology, $\widehat{HFK}$, for example, sits the symmetrized Alexander polynomial, ([4], [5]) , the Seifert genus [8], and knot fiberedness [6].

Originally, knot Floer homology was computed using Heegaard diagrams, representing handlebody decompositions of knot complements in S^3. These diagrams can be quite difficult to produce by hand. However, in 2009, Manolescu, Ozsvath and Sarkar presented a combinatorial approach to compute all varieties of Heegaard Floer homologies using toroidal grid diagrams in time $\mathcal{O}(n!)$ [3]. In the past few years, several programs have been written to compute one variety, $\widehat{HFK}$. One of the most prominent of these programs is `gridlink`, made by Baldwin and Gillam.

Beliakova and Droz, who extended the toroidal grid complex to the far smaller rectangular grid complex, greatly simplified the computation of $\widehat{HFK}$. However the extension of their complex to the infinite varieties fails to speed up the process, due to the ambiguity of the differential. Here a program is provided to compute HFK^-, using the methods provided by [3]. It was written in C++ with the goal of making the computation as fast as possible with the algorithm provided. On a normal home computer, it can find HFK^- for knots with grid size seven or less relatively quickly, and HFK^- for knots of grid size eight and nine in a few hours. With a few tricks and a faster computer, you can find HFK^- for larger knots as well. Afterwards, it is often possible to find the filtered complex CFK^∞ if there are few enough generators.

Here, a brief explanation of Heegaard Floer homology will be provided, followed by a tutorial of how to download and use `hfkm`. Then an example of the computation of HFK^-, and the filtered complex CFK^∞ is provided. The same computations for all non-alternating knots with crossing number eight and nine are then provided in the appendix. `hfkm` can be downloaded from the following link:

`https://github.com/danrotsy/hfkm`

2 An overview of combinatorial Heegaard Floer homology

All closed four dimensional manifolds M admit a Heegaard decomposition $Y_1 \cup_\Sigma Y_2$ into two genus g handle bodies (Y_1, Y_2) that produce M when glued along their common boundary Σ. By drawing nice curves on Σ, and studying Lagrangian flows between intersections of symmetric tori created by those curves, we can compute the Heegaard Floer homology of the original manifold, M. In knot theory, we consider the knot Floer homology to be Heegaard Floer homology of the compact space $S^3 - K$, where K is the open tubular neighbourhood about the path of a given knot.

In general, we define a Heegaard diagram on a common boundary Σ of genus g as a collection of paths $\{\alpha_1, \cdots, \alpha_g\}$ and $\{\beta_1, \cdots, \beta_g\}$ such that the following conditions hold:

1. Each α_i and β_i bound disjoint embedded disks in Y_1 and Y_2, respectively.

2. $\alpha_i \cap \alpha_j = \emptyset$ and $\beta_i \cap \beta_j = \emptyset$ for all $i \neq j$.

3. $\Sigma - \alpha_1 - \cdots - \alpha_g$ and $\Sigma - \beta_1 - \cdots - \beta_g$ are both connected spaces.

The chain complex CFK^∞ is defined as an $\mathbb{F}[U, U^{-1}]$ module, for some field $\mathbb{F}$, with generators corresponding to intersections of the product spaces $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$. Each generator is assigned an Alexander and Maslov grading, and multiplication by U decreases the Alexander grading by one, and the Maslov grading by two. The differential between these generators is by found counting holomorphic disks from one generator to another. However, a Heegaard diagram is often very difficult to produce for an arbitrary knot diagram. For computation, the Heegaard Floer homology is often found using $2k$ pointed Heegaard diagrams, as described in [3]. Given a genus g Heegaard diagram on a surface Σ with attaching curves $\{\alpha_i\}$ and $\{\beta_i\}$, we append $k-1$ disjoint paths, bounding embedded disks in Y_1 and Y_2, respectively, to produce curves $\{\alpha_1, \cdots, \alpha_{g+k-1}\}$ and $\{\beta_1, \cdots, \beta_{g+k-1}\}$. This produces k connected regions of $\Sigma - \alpha_1 - \cdots - \alpha_{g+k-1}$ and $\Sigma - \beta_1 - \cdots - \beta_{g+k-1}$: $\{A_i\}$ and $\{B_i\}$, respectively. Then we place $2k$ points $\{w_1, \cdots, w_k\}$ and $\{z_1, \cdots, z_k\}$ such that each w_i sits in a distinct A_i, and each z_i sits in a distinct B_i. Without loss of generality, A_i contains w_i and z_i, and each B_i contains w_i and $z_{\sigma(i)}$ for some $\sigma \in S_k$. Let ζ_i be an arc in Y_1, constructed by taking a path from z_i to w_i in A_i and pushing it into Y_1, such that it only intersects Σ at the endpoints. Similarly, let η_i be an arc constructed by taking a path from w_i to $z_{\sigma(i)}$ in B_i and pushing it into Y_2, such that it only intersects Σ at the endpoints, as well. If the loop created by adjoining all η_i and ζ_i is the path of the knot K, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ forms a $2k$ pointed Heegaard diagram for K.

As stated in [3], the chain complex, $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ of the pointed Heegaard diagram is generated by intersections of the tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \alpha_{g+k-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+k-1}$ over $\mathbb{F}[U_1, U_1^{-1}, \cdots, U_k, U_k^{-1}]$. Assume $\mathbb{F}$ is the integers modulo two. The differential is defined as follows:

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum \{\phi \in \pi_2(x,y) | \mu(\phi) = 1\} \#(\frac{\mathcal{M}(\phi)}{\mathbb{R}}) U_1^{n_{w_1}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \cdot y$$

Where $\pi_2(x,y)$ represents homology classes of Whitney disks connecting x to y, $\mu(\phi)$ the Maslov index of those disk representatives, $\mathcal{M}(\phi)$ the moduli space of pseudo-holomorphic representatives, and $n_{w_i}(\phi)$ is the intersection number of the disk with w_i. Generators have relative Alexander and Maslov gradings defined as follows:

$$A(x) - A(y) = \left(\sum_{i=1}^{n} n_{z_i}(\phi) \right) - \left(\sum_{i=1}^{n} n_{w_i}(\phi) \right)$$

$$M(x) - M(y) = \mu(\phi) - 2 \sum_{i=1}^{n} n_{w_i}(\phi)$$

It was shown in [3] that the above complex is filetered chain homotopic to the normal knot Floer homology, when the U_i are identified in the basis of $H_*(CFK^\circ(\sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z}))$. This is, of course, a gross simplification, but all is explained in depth in Manolescu, Ozváth and Sarkar's paper.

It is easy to show that, starting with the Heegaard diagram for the torus, a $2k$ pointed Heegaard diagram can be made representing any knot K, using grid diagrams. A grid diagram is a way to draw any link, and can be constructed as follows:

1. Take an $n \times n$ grid.

2. Place points, labelled O_i and X_i, on the grid such that each column contains exactly one O_i and X_i, each row contains exactly one O_i and X_i and no square contains more than one point.

Start at some point X_i, and draw a path horizontally until a point O_i is reached. Then from O_i travel vertically until another X_i is reached again. Rinse and repeat until the beginning of the path is reached. This defines one loop in the link of the grid diagram. At any path crossing, take the vertical path to travel over the horizontal. If all O_i and X_i are on the loop, stop. If not, find a point X_i not in the loop, and repeat the first step again. Continue until every labelled point is accounted for. The resulting link that has been drawn is the link that the grid diagram corresponds to. To create the $2k$ pointed Heegaard diagram on the torus, known as a toroidal grid diagram, simply identify the top and bottom boudaries of the grid diagram, and then the left and right boudaries. For a more detailed explanation of grid diagrams, and their existence for every knot K, see [9].

It is shown in [3] that toroidal grid diagrams have very nice properties. When isotopied, the horizontal gridlines can be identified as α_i, and the vertical gridlines as β_i. The w_i and z_i are naturally identified with the O_i and X_i, respectively. The generators of the Heegaard Floer complex correspond to the n-tuples of intersection points of the horizontal and vertical gridlines, such that no gridline has more than one intersection counted. There are $n!$ such generators. The Alexander and Maslov indices are far easier to find as well. The Alexander grading can be found for any tuple of intersections:

$$A(x) = \sum_{p \in x} a(p) - \frac{1}{8} \sum_{i,j} a(c_{i,j}) - \frac{n-1}{2}$$

Where a is minus one times the winding number of K about an intersection in x on the diagram, and $c_{i,j}$ is the set, with duplicates, of all corners of squares containing an O or an X. n is the grid size. The Maslov grading of x_0, the generator corresponding to the permutation which gives the position of each O_i in order, has absolute grading $1-n$. Next,a relative Maslov grading is assigned using methods that will be defined. Given two generators x and y, a flow only exists between them if x and y differ by exactly two intersection points. Take the points in x not in y to be x_1 and x_2, and the points in y not in x to be y_1 and y_2. A rectangle connects x and y if x_1, x_2, y_1, y_2 form the corners of that rectangle, and if its boundary path, oriented counterclockwise, travels horizontally from x_i to y_i, and vertically from y_i to x_i. There are always exactly two such rectangles, if they exist. They form a set, $\mathrm{rect}(x, y)$, which is either of size zero or two. Let p be a function of an intersection point and a rectangle, where given an intersection point c, $p_c(R)$ is the number of grid squares for which c is a corner, divided by four. The function P is defined as follows:

$$P_x(R) = \sum_{c \in x} p_c(R)$$

The relative Maslov grading is defined like so:

$$M(x) - M(y) = P_x(R) - P_y(R) - 2 \cdot W(R)$$

Where $R \in \mathrm{rect}(x, y)$ and $W(R) = \sum_{i=1}^{n} n_{w_i}(R)$. The differential is combinatorially computable as well:

$$\partial x = \sum_{y \in S_n} \sum_{\{R \in \mathrm{rect}(x,y) | P_x(R) + P_y(r) = 1\}} U_1^{n_{w_1}(R)} \cdots U_k^{n_{w_k}(R)} \cdot y$$

Again, for more detail, see [3].

3 hfkm

3.1 Implementation

When building `hfkm`, the goal was to find the fastest way to implement the algorithm set out in [3]. Since the filtered complex is chain homotopic to the standard Heegaard Floer complex, only the differential between elements of a given (a, m) bigrading to elements of bigrading $(a, m-1)$ needs to be considered. This speeds up the computation considerably. Using the Johnson-Trotter algorithm to sequentially compute the Alexander and Maslov gradings exactly once, starting from x_0, helps as well. Measures are taken to be efficient with memory and to make all the little parts as fast as possible, but nonetheless, the computation becomes so huge for larger knots that it can take hours on end to obtain a result.

3.2 General usage

To install `hfkm`, go to `https://github.com/danrotsy/hfkm.git`. Unzip the downloaded files if you have to. Then, install all the required dependencies on your system. All that is needed is boost, a general purpose C++ library, and NTL, a library for doing linear algebra in finite fields. Change directories to the program file, and run `make install` to create the executables. Put the executables wherever is most convenient, or add them to the system path. `hfkm` comes with two commands, `gen` and `hfkm`. `gen` is used to compute and save the Alexander and Maslov gradings of every generator of the knot complex. First, create a directory with the name of the desired knot to store the knot information in. Then run the following:

```
[usr]@[hostname] $ ./gen xlist olist dir
```

Where `xlist` is the permutation describing the position of the X's on the grid diagram, `olist` is the permutation describing the position of the O's on the grid diagram, and `dir` is the path to the directory you just created to save the knot information. `xlist` and `olist` use indices starting at zero, are comma separated, and contain no spaces. For small knots this process is almost instantaneous, for knots of grid size up to ten it is just a few minutes, for ten or larger it can be several hours. Once the generators are constructed, it is possible to run the `hfkm` command by typing the following:

```
[usr]@[hostname] $ ./hfkm dir a m
```

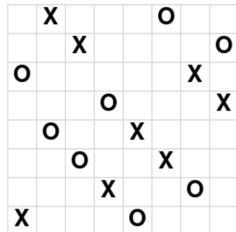
This computes the value of HFK^{-} at the Alexander-Maslov birgrading (a, m). Again, `dir` is the path to the directory created to save the knot diagram. Be careful not to delete or edit files in the knot save directory. To save the output of `hfkm`, simply create a dummy file `output.txt` and pipe. For Linux and MacOS:

```
[usr]@[hostname] $ ./hfkm dir a m >> output.txt
```

For more help, visit the github page listed, and see the readme file.

3.3 Example with 9_{42}

The knot 9_{42} has a minimal grid size of 7 with the following grid diagram:



From `gridlink`, we know that $\widehat{HFK}$ has nontrivial rank only in Alexander-Maslov bigradings $(2, 1)$, $(1, 0)$, $(0, 0)$, $(0, -1)$, $(-1, -2)$ and $(-2, -3)$. Using a result in Rasmussen's thesis [5], which states that $CFK^{\infty}(K) \cong \mathbb{F}[U, U^{-1}] \otimes \widehat{HFK}$, it will be possible to find the filtered differential of CFK^{∞} with HFK^{-} if there are few enough generators. First, create a directory to hold the knot data for 9_{42}:

```
[usr]@[hostname] $ mkdir 9_42
[usr]@[hostname] $ ./gen 0,7,6,1,3,2,5,4 5,3,2,4,0,7,1,6 9_42
```

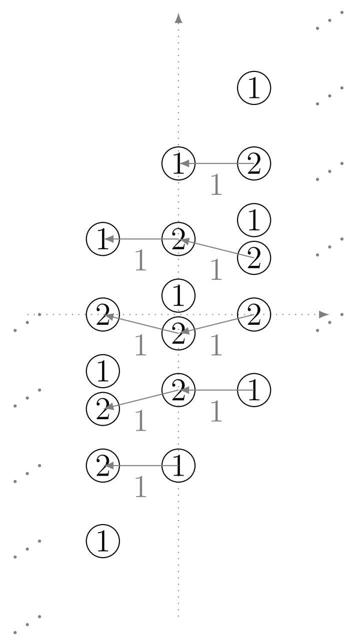
Once the program is finished running, all that remains to do is to find $HFK^{-}(9_{42})[a, m]$ for all bigradings for which were nontrivial in $\widehat{HFK}(9_{42})$, until there is enough information to compute $HFK^{-}(9_{42})$ in general. The highest valued bigradings normally have the fewest generators, and are simpler to compute.

```
[usr]@[hostname] $ ./hfkm 9_42 2 1
[usr]@[hostname] $ ./hfkm 9_42 1 0
[usr]@[hostname] $ ./hfkm 9_42 0 0
[usr]@[hostname] $ ./hfkm 9_42 0 -1
[usr]@[hostname] $ ./hfkm 9_42 -1 -2
[usr]@[hostname] $ ./hfkm 9_42 -2 -3
```

The first few bigradings should only take few minutes to find, but $HFK^{-}(9_{42})[-1, -2]$ and $HFK^{-}(9_{42})[-2, -3]$ may take a little longer. With this, the following is known about 9_{42}:

a	m	rk$\widehat{HFK}$	rkHFK^{-}
2	1	1	1
1	0	2	1
0	0	1	1
0	-1	2	1
-1	-2	2	2
-2	-3	1	0

Let x_3 denote the generator with bigrading $(0,0)$, note that Ux_3 has bigrading $(-1,-2)$. Being careful to remember this fact, we see that the filtered complex $CFK^\infty(K)$ must be as follows:



Note that often it will not be necessary to run `hfkm` for all generating bigradings. Since the infinite varieties of the homology will only have one infinite tower, the differential of the hardest generators can often be deduced.

References

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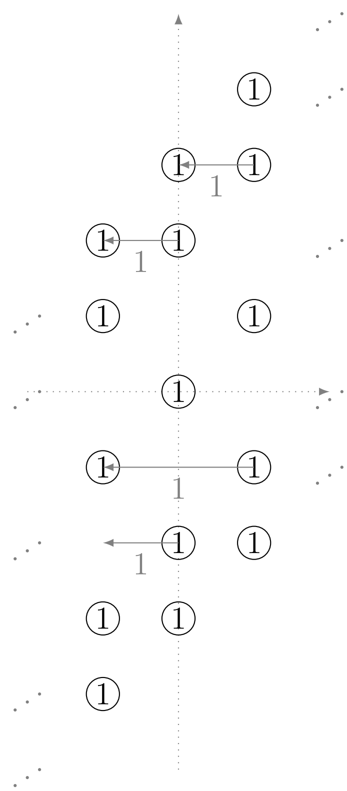
5 Appendices

With each knot, K, is a graph representing a chain complex, group isomorphic and chain homotopic to $CFK^{\infty}(K)$. The chain complex, up to isomorphism, is found with `gridlink`, using Rasmussen's result. (`http://homepages.math.uic.edu/~culler/gridlink`) The differential of that complex is then found using `hfkm`, written by me this summer. (`https://github.com/danrotsy/hfkm`) $HFK^{-}(K)$ is then written as a direct sum of blocks and towers. A table of the exact output of hfkm is provided as well. Note that the generators on the vectical axis of the plot correspond to the generators in the table. The towers are written as $\mathcal{T}^{-}_{(a,m)}$ where (a, m) is the Alexander Maslov bigrading of the maximally graded element in the tower. The blocks are written as $\mathbb{F}_{(a,m)}$ in a similar fashion. The explicit module structure is given as well, with a few conventions. x_i^j is the j^{th} generator of the i^{th} generating bigrading of HFK^{-}. x_i^{1-j} the first j such generators. x_i^* means all generators of the i^{th} generating bigrading of HFK^{-}.

8_{19}

a	m	rk$\widehat{HFK}$	rkHFK^-
3	0	1	1
2	-1	1	0
0	-2	1	1
-2	-5	1	0
-3	-6	1	1

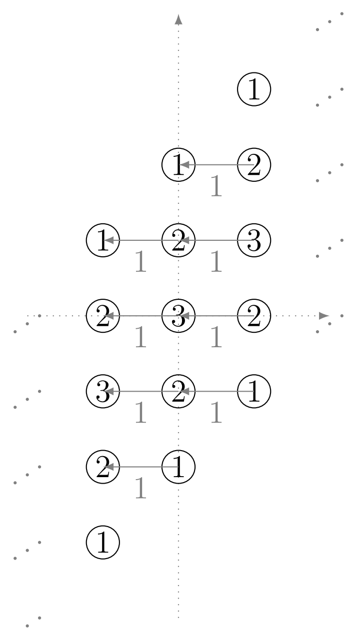


$$HFK^-(8_{19}) \cong \mathcal{T}^-_{(-3,-6)} \oplus \mathbb{F}_{(-1,-4)} \oplus \mathbb{F}_{(0,-2)} \oplus \mathbb{F}_{(3,0)}$$

$$HFK^-(8_{19}) = \mathbb{F}[U]\langle x_1, x_3, x_5 | Ux_1, U^2x_3 \rangle$$

8_{20}

a	m	rk$\widehat{HFK}$	rkHFK^-
2	2	1	1
1	1	2	1
0	0	3	2
-1	-1	2	1
-2	-2	1	0

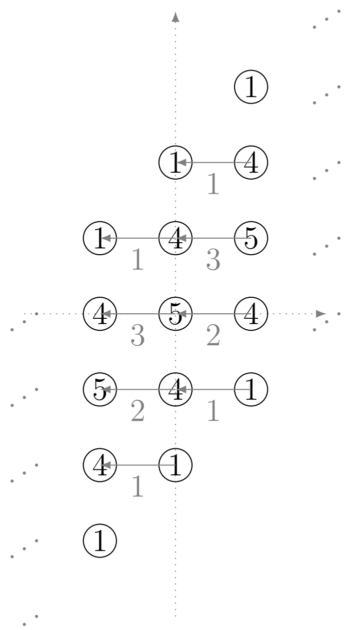


$$HFK^-(8_{20}) \cong \mathcal{T}^-_{(0,0)} \oplus \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(1,1)} \oplus \mathbb{F}_{(2,2)}$$

$$HFK^-(8_{20}) = \mathbb{F}[U]\langle x_1, x_2, x_3, x_4 | Ux_1, Ux_2, Ux_4 \rangle$$

8_{21}

a	m	rk$\widehat{HFK}$	rkHFK^-
2	1	1	1
1	0	4	3
0	-1	5	2
-1	-2	4	2
-2	-3	1	0

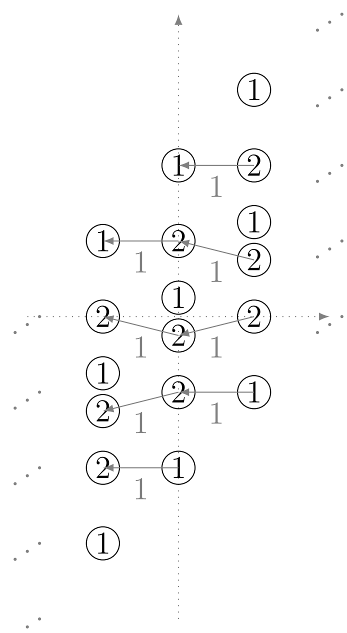


$$HFK^-(8_{21}) \cong \mathcal{T}^-_{(-1,-2)} \oplus \mathbb{F}_{(-1,-2)} \oplus \mathbb{F}^2_{(0,-1)} \oplus \mathbb{F}^3_{(1,0)} \oplus \mathbb{F}_{(2,1)}$$

$$HFK^-(8_{21}) = \mathbb{F}[U]\langle x_1, x_2^{1-3}, x_3^{1-2}, x_4^{1-2} | Ux_1, Ux_2^*, Ux_3^*, Ux_4^1\rangle$$

9_{42}

a	m	rk$\widehat{HFK}$	rkHFK^-
2	1	1	1
1	0	2	1
0	0	1	1
0	-1	2	1
-1	-2	2	2
-2	-3	1	0

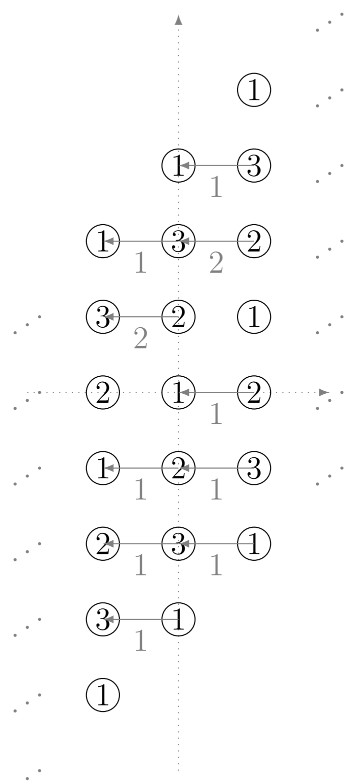


$$HFK^-(9_{42}) \cong \mathcal{T}^-_{(0,0)} \oplus \mathbb{F}_{(-1,-2)} \oplus \mathbb{F}_{(0,-1)} \oplus \mathbb{F}_{(1,0)} \oplus \mathbb{F}_{(2,1)}$$

$$HFK^-(9_{42}) = \mathbb{F}[U]\langle x_1, x_2, x_3, x_4, x_5 | Ux_1, Ux_2, Ux_4, Ux_5\rangle$$

9_{43}

a	m	rk$\widehat{HFK}$	rkHFK^-
3	1	1	1
2	0	3	2
1	-1	2	0
0	-2	1	1
-1	-3	2	1
-2	-4	3	2
-3	-5	1	0

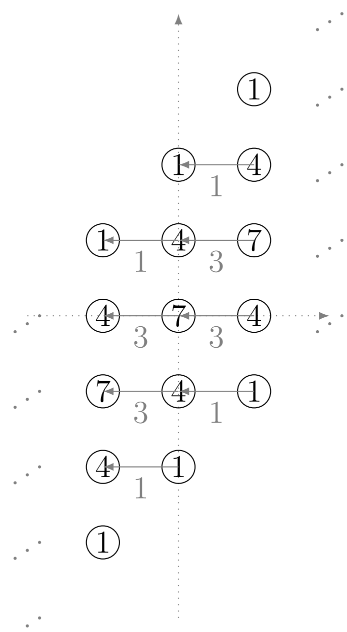


$$HFK^-(9_{43}) \cong \mathcal{T}^-_{(-2,-4)} \oplus \mathbb{F}_{(-2,-4)} \oplus \mathbb{F}_{(-1,-3)} \oplus \mathbb{F}_{(0,-2)} \oplus \mathbb{F}^2_{(2,0)} \oplus \mathbb{F}_{(3,1)}$$

$$HFK^-(9_{43}) = \mathbb{F}[U]\langle x_1, x_2^{1-2}, x_4, x_5, x_6^{1-2} | Ux_1, Ux_2^*, Ux_5, Ux_6^1\rangle$$

9_{44}

a	m	rk$\widehat{HFK}$	rkHFK^-
2	2	1	1
1	1	4	3
0	0	7	4
-1	-1	4	1
-2	-2	1	0

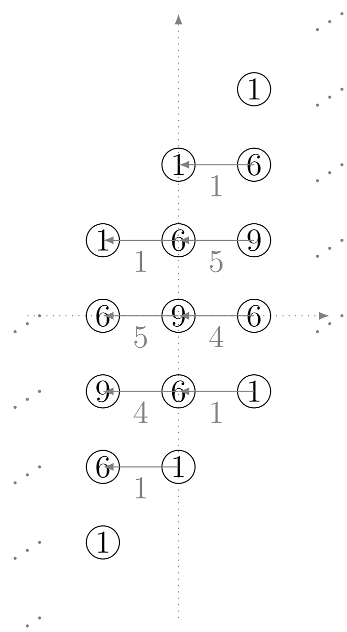


$$HFK^-(9_{44}) \cong \mathcal{T}^-_{(0,0)} \oplus \mathbb{F}_{(-1,-1)} \oplus \mathbb{F}^3_{(0,0)} \oplus \mathbb{F}^3_{(1,1)} \oplus \mathbb{F}_{(2,2)}$$

$$HFK^-(9_{44}) = \mathbb{F}[U]\langle x_1, x_2^{1-3}, x_3^{1-4}, x_4 | Ux_1, Ux_2^*, Ux_3^{1-3}, Ux_4\rangle$$

9_{45}

a	m	rk$\widehat{HFK}$	rkHFK^-
2	1	1	1
1	0	6	5
0	-1	9	4
-1	-2	6	2
-2	-3	1	0

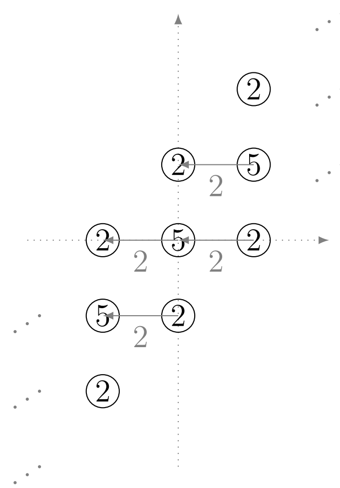


$$HFK^-(9_{45}) \cong \mathcal{T}^-_{(-1,-2)} \oplus \mathbb{F}_{(-1,-2)} \oplus \mathbb{F}^4_{(0,-1)} \oplus \mathbb{F}^5_{(1,0)} \oplus \mathbb{F}_{(2,1)}$$

$$HFK^-(9_{45}) = \mathbb{F}[U]\langle x_1, x_2^{1-5}, x_3^{1-4}, x_4^{1-2} | Ux_1, Ux_2^*, Ux_3^*, Ux_4^1\rangle$$

9_{46}

a	m	rk$\widehat{HFK}$	rkHFK^-
1	1	2	2
0	0	5	3
-1	-1	2	0

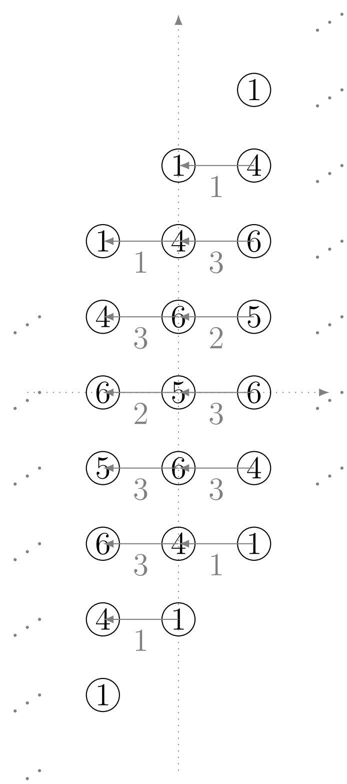


$$HFK^-(9_{46}) \cong \mathcal{T}^-_{(0,0)} \oplus \mathbb{F}^2_{(0,0)} \oplus \mathbb{F}^2_{(1,1)}$$

$$HFK^-(9_{46}) = \mathbb{F}[U]\langle x_1^{1-2}, x_2^{1-3} | Ux_1^*, Ux_2^{1-2}\rangle$$

9_{47}

a	m	rk$\widehat{HFK}$	rkHFK^-
3	4	1	1
2	3	4	3
1	2	6	3
0	1	5	3
-1	0	6	3
-2	-2	4	1
-3	-2	1	0

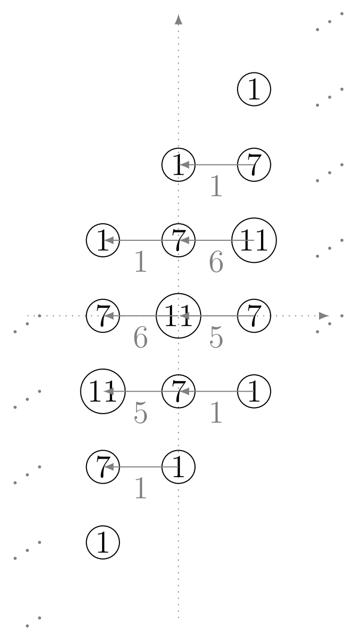


$$HFK^-(9_{47}) \cong \mathcal{T}^-_{(1,2)} \oplus \mathbb{F}_{(-2,-1)} \oplus \mathbb{F}^3_{(-1,0)} \oplus \mathbb{F}^3_{(0,1)} \oplus \mathbb{F}^2_{(1,2)} \oplus \mathbb{F}^3_{(2,3)} \oplus \mathbb{F}_{(3,4)}$$

$$HFK^-(9_{47}) = \mathbb{F}[U]\langle x_1, x_2^{1-3}, x_3^{1-3}, x_4^{1-3}, x_5^{1-3}, x_6 | Ux_1, Ux_2^*, Ux_3^{1-2}, Ux_4^*, Ux_5^*, Ux_6\rangle$$

9_{48}

a	m	rk$\widehat{HFK}$	rkHFK^{-}
2	1	1	1
1	0	7	6
0	-1	11	5
-1	-2	7	2
-2	-3	1	0

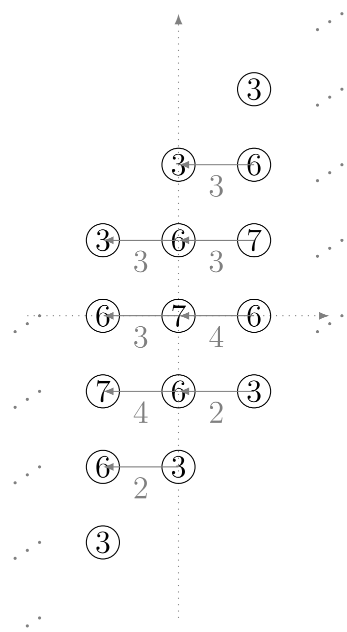


$$HFK^{-}(9_{48}) \cong \mathcal{T}^{-}_{(-1,-2)} \oplus \mathbb{F}_{(-1,-2)} \oplus \mathbb{F}^{5}_{(0,-1)} \oplus \mathbb{F}^{6}_{(1,0)} \oplus \mathbb{F}_{(2,1)}$$

$$HFK^{-}(9_{48}) = \mathbb{F}[U]\langle x_1, x_2^{1-6}, x_3^{1-5}, x_4^{1-2} | Ux_1, Ux_2^{*}, Ux_3^{*}, Ux_4^{1}\rangle$$

9_{49}

a	m	rk$\widehat{HFK}$	rkHFK^-
2	0	3	3
1	-1	6	3
0	-2	7	4
-1	-3	6	2
-2	-4	3	1



$$HFK^-(9_{49}) \cong \mathcal{T}^-_{(-2,-4)} \oplus \mathbb{F}^2_{(-1,-3)} \oplus \mathbb{F}^4_{(0,-2)} \oplus \mathbb{F}^3_{(1,-1)} \oplus \mathbb{F}^3_{(2,0)}$$

$$HFK^-(9_{48}) = \mathbb{F}[U]\langle x_1^{1-3}, x_2^{1-3}, x_3^{1-4}, x_4^{1-2}, x_5 | Ux_1^*, Ux_2^*, Ux_3^*, Ux_4^*\rangle$$