Class Equation

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Proposition

Let $*: G \times G \to G$ be defined by $*((g,h)) = ghg^{-1}$. Then $G * G$ is a group action of G on itself.

Proof. $e * x = exe = x$ and $g * (h * x) = g * (hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = (gh) * x$ $\square$

Centralizer

The center of an element $x \in G$, denoted $Z(x)$, is: $Z(x) = \{g \in G : gxg^{-1} = x\} = \{g \in G : gx = xg\}$

Conjugacy Class

The conjugacy of an element $x \in G$, denoted $C(x)$, is: $C(x) = \{x' \in G : x' = gxg^{-1}\}$

Proposition

$|G| = |Z(x)||C(x)| \ \forall x \in G$

Proof. Under conjugation, $Z(x)$ is the stabilizer of x, and $C(x)$ is the orbit of x. Since conjugation is a group action, it follows by the orbit stabilizer theorem that $|G| = |Z(x)||C(x)|$ for all $x \in G$. $\square$

Class Equation

$$|G| = \sum |C_i| = 1 + \sum |C_j|$$

Proof. Since under conjugation $C(x)$ are orbits which partition G, it then follows that $G = \bigcup C_i$ where each C_i is a distinct conjugacy class. However, for the identity, $C(e) = \{e\}$, so $|G| = \sum C_i = 1 + \sum C_j$. $\square$