Orbit-Stabilizer Theorem

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Let some finite G act on X with the group action $\circ$. Then $|\mathrm{orb}(x)||\mathrm{stab}(x)| = |G|$.

Proof. By Lagrange's theorem, $|G| = |\mathrm{stab}(x)|(G : \mathrm{stab}(x))$ for all $x \in X$, because $\mathrm{stab}(x) \leq G$. Therefore, if $|\mathrm{orb}(x)| = (G : \mathrm{stab}(x)) = |G/\mathrm{stab}(x)|$, then $|\mathrm{orb}(x)||\mathrm{stab}(x)| = |G|$. Let $\mu : \mathrm{orb}(x) \to G/\mathrm{stab}(x)$. Since, by definition, all $y \in \mathrm{orb}(x)$ can be expressed as $g \circ x$ for some $g \in G$, define $\mu(y) = \mu(g \circ x) = g[\mathrm{stab}(x)]$.

$$\forall\ g_1 \circ x, g_2 \circ x \in \mathrm{orb}(x),\ \mu(g_1 \circ x) = \mu(g_2 \circ x) \implies g_1 \circ x = g_2 \circ x \because$$

$$\begin{aligned}
\mu(g_1 \circ x) = \mu(g_2 \circ x) &\implies g_1[\mathrm{stab}(x)] = g_2[\mathrm{stab}(x)] \\
&\implies g_1 \in g_2[\mathrm{stab}(x)] \\
&\implies g_1 = g_2 s : s \in \mathrm{stab}(x) \\
&\implies g_1 \circ x = (g_2 s) \circ x = g_2 \circ (s \circ x) = g_2 \circ x
\end{aligned}$$

$\forall\ g[\mathrm{stab}(x)] \in G/\mathrm{stab}(x),\ \exists\ g \circ x \in \mathrm{orb}(x) : \mu(g \circ x) = g[\mathrm{stab}(x)]$ by the definition of $\mathrm{orb}(x)$. Therefore μ is a bijection, and $|\mathrm{orb}(x)||\mathrm{stab}(x)| = |G|$. $\square$