Burnside's Lemma

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Theorem

Let G act on X. Let r be the number of orbits in X, and N be the number of pairs $(g, x) \in G \times X$ such that $gx = x$. Then:

$$r = \frac{1}{|G|} \cdot N = \frac{1}{|G|} \sum_{x \in X} |\text{stab}(x)| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

Proof. Here is a proof of Burnsides's Lemma using the orbit-stabilizer theorem.

Let (g, x) be the pairs in $G \times X$ such that $gx = x$, and N be the number of such pairs. N can be found in two ways: by counting all the fixed points of all the group elements in G, or by counting all the stabilizers of all the set elements in X. In other words:

$$N = \sum_{x \in X} |\text{stab}(x)| \quad \text{and} \quad N = \sum_{g \in G} |\text{fix}(g)|$$

Let $\mathcal{O}$ denote and orbit of X. Since the orbits partition X, counting every element in X is the same as counting all the orbits in X, and then counting the elements in each orbit. Therefore, it follows that:

$$N = \sum_{x \in X} |\text{stab}(x)| = \sum_{\mathcal{O} \in X} \sum_{x \in \mathcal{O}} |\text{stab}(x)|$$

Since all stablizers in a given orbit have the same size, it must follow that, for some random element x in $\text{orb}(x)$:

$$\sum_{x \in \mathcal{O}} |\text{stab}(x)| = |\text{orb}(x)| \cdot |\text{stab}(x)| = |G| \quad \text{(by the orbit-stablizer theorem)}$$

Therefore:

$$N = \sum_{x \in X} |\text{stab}(x)| = \sum_{\mathcal{O} \in X} \sum_{x \in \mathcal{O}} |\text{stab}(x)| = \sum_{\mathcal{O} \in X} |\text{orb}(x)||\text{stab}(x)| = \sum_{\mathcal{O} \in X} |G| = r \cdot |G|$$

So $r = \frac{1}{|G|} N$. Since N has two different forms:

$$r = \frac{1}{|G|} \cdot N = \frac{1}{|G|} \sum_{x \in X} |\text{stab}(x)| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|$$

$\square$