Bernoulli Numbers

Daniel Rostovtsev
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In the 1600s, the German mathematician Johann Faulhaber found explicit formulas for finite nth power sums. (known as Faulhaber sums, denoted $S_m(n)$)

$$S_0(n) = 0^0 + 1^0 + 2^0 + \cdots + (n-1)^0 = \sum_{k=0}^{n-1} k^0 = n$$

$$S_1(n) = 0^1 + 1^1 + 2^1 + \cdots + (n-1)^1 = \sum_{k=0}^{n-1} k^1 = \frac{n(n-1)}{2}$$

$$S_2(n) = 0^2 + 1^2 + 2^2 + \cdots + (n-1)^2 = \sum_{k=0}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}$$

$$S_3(n) = 0^3 + 1^3 + 2^3 + \cdots + (n-1)^3 = \sum_{k=0}^{n-1} k^3 = \frac{n^2(n-1)^2}{4}$$

$$S_4(n) = 0^4 + 1^4 + 2^4 + \cdots + (n-1)^4 = \sum_{k=0}^{n-1} k^4 = \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30}$$

$\ldots$

All the way up to $n = 17$. Yet, he could find no pattern computing $S_m(n)$, in general. It wasn't until the 1700s that the Swiss mathematician Jacob Bernoulli discovered that Faulhaber sums were best considered as the sums $c_{m+1}n^{m+1} + c_m n^m + \cdots + c_2 n^2 + c_1 n$, because the coefficients c_i stayed the same for any value of n. For example:

$S_0(n) = n$

$S_1(n) = \frac{1}{2}n^2 - \frac{1}{2}n$

$S_2(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$

$S_3(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2$

$S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

$\ldots$

It was then that Bernoulli hypothesized that each $c_k n^k$ for a given $S_m(n)$ was equal to $\frac{1}{m+1}\binom{m+1}{k}B_k$ where B_k was the kth Bernoulli number, and that these B_k had a generating rule. More formally:

$$\sum_{k=0}^{n-1} k^m = S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k \cdot n^{m+1-k} \text{ such that } \sum_{k=0}^{m}\binom{m+1}{k}B_k = 0 \text{ where } B_0 = 1$$

Here is a proof that these statements are true:

Lemma

The $(m+1)$th power of n is equal to the binomial weighted sum of all preceding Faulhaber sums of n:

$$n^{m+1} = \sum_{j=0}^{m} \binom{m+1}{j} S_j(n) = 0$$

Proof. Relating the binomial expansion theorem, which states that $n^k = \sum_{i=0}^{k} \binom{k}{i} n^i$, and Faulhaber's sum yields the above equality.

$$S_{m+1}(n) + n^{m+1} = \sum_{k=0}^{n-1} k^{m+1} + n^{m+1} = \sum_{k=0}^{n} k^{m+1} = 0^{m+1} + \sum_{k=1}^{n} k^{m+1} = \sum_{k=0}^{n-1} (k+1)^{m+1}$$

By the binomial expansion theorem, $(k+1)^{m+1} = \sum_{j=0}^{m+1} \binom{m+1}{j} k^j$. Expressing the above sum in terms of binomial expansions is the final step.

$$\sum_{k=0}^{n-1} (k+1)^{m+1} = \sum_{k=0}^{n-1} \sum_{j=0}^{m+1} \binom{m+1}{j} k^j = \sum_{j=0}^{m+1} \binom{m+1}{j} \sum_{k=0}^{n-1} k^j = \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n)$$

Therefore:

$$S_{m+1}(n) + n^{m+1} = \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n)$$

Thus

$$n^{m+1} = \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n) - S_{m+}(n) = \sum_{j=0}^{m} \binom{m+1}{j} S_j(n)$$

□

Now, to prove the main theorem of Bernoulli numbers: (next page)

Main Theorem of Bernoulli Numbers

The mth Faulhaber sum of n is proportional to the binomial-Bernoulli number weighted sum of the powers of n.

$$S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k \cdot n^{m+1-k} \text{ such that } \sum_{k=0}^{m}\binom{m+1}{k}B_k = 0 \text{ where } B_0 = 1$$

Proof. Let $H_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k \cdot n^{m+1-k}$ and $\Delta_m(n) = S_m(n) - H_m(n)$

Here $H_m(n)$ is the hypothesis for $S_m(n)$ and $\Delta_m(n)$ is the difference between the hypothesis and the Faulhaber sum. $\Delta_m(n) = 0 \ \forall\ m, n$, then the theorem is proven. Since $S_m(n) = H_m(n) + \Delta_m(n)$:

$$\sum_{j=0}^{m}\binom{m+1}{j}S_j(n) = \sum_{j=0}^{m}\binom{m+1}{j}H_m(n) + (m+1)\Delta_m(n)$$

Since $H_m(n)$ is clearly defined, the right hand side is as follows:

$$R.H.S. = \sum_{j=0}^{m}\binom{m+1}{j}\frac{1}{j+1}\sum_{k=0}^{j}\binom{j+1}{k}B_k \cdot n^{j+1-k} + (m+1)\Delta_m(n)$$

Which can be rearranged to the following since several terms are independent of k:

$$R.H.S. = \sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m+1}{j}\binom{j+1}{k}\frac{B_k}{j+1} \cdot n^{j+1-k} + (m+1)\Delta_m(n)$$

Now, if the m+1 binomial and B_i can be expressed in the same terms, then, for most terms $\sum_{i=0}^{m}\binom{m+1}{i}B_i = 0$, so the problem will be greatly simplified. Algebraically, here is an outline of how the $R.H.S.$ will be simplified in this proof:

Starting with $\sum_{j=0}^{m}\sum_{k=0}^{j} f(j,k)$:

$$\sum_{j=0}^{m}\sum_{k=0}^{j} f(j,k) = \sum_{k=0}^{m}\sum_{j=k}^{m} f(j,k) = \sum_{k=0}^{m}\sum_{j=k}^{m} f(j,j-k) \text{ since } 0 \le k \le j \le m$$

It will follow that $f(j, j-k)$ allows the sum to be written as:

$$\sum_{k=0}^{m}\sum_{j=k}^{m} f(j,j-k) = \sum_{k=0}^{m}\sum_{j=0}^{m-k}\binom{m-k+1}{j}g(j,k)$$

Which will have all terms equal to zero, except for when $k = m$ and $\sum_{i=0}^{0}\binom{1}{i}B_i = B_0 = 1$

So the whole $R.H.S$ will reduce to $g(0,k) + \Delta_n(m)$.

Since $0 \leq k \leq j \leq m$ for all terms, every combination (k, j) such that $k \leq j$ is used exactly once. Therefore, the order of the sums can be reversed if every combination is still used exactly once, since the overall sum will still be the same. It follows that the right hand side is now equivalent to:

$$R.H.S. = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m+1}{j} \binom{j+1}{k} \frac{B_k}{j+1} \cdot n^{j+1-k} + (m+1)\Delta_m(n)$$

This is great progress, now it only remains to manipulate the larger sum so that the inner sum is from 0 to j, which will eliminate most terms according to the definition of the Bernoulli numbers. To "shift" the inner sum, simply set k to $j-k$ and simplify.

In other words, since $\sum_{k=0}^{m} \sum_{j=k}^{m} f(j,k) = \sum_{k=0}^{m} \sum_{j=k}^{m} f(j, k-j)$:

$$R.H.S. = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m+1}{j} \binom{j+1}{j-k} \frac{B_{j-k}}{j+1} \cdot n^{k+1} + (m+1)\Delta_m(n)$$

Now to simplify. $\binom{j+1}{j-k} = \frac{(j+1)!}{(j-k)!(k+1)!} = \binom{j+1}{k+1}$ so:

$$R.H.S. = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m+1}{j} \binom{j+1}{k+1} \frac{B_{j-k}}{j+1} \cdot n^{k+1} + (m+1)\Delta_m(n)$$

Furthermore $\binom{j+1}{k+1} = \frac{(j+1)!}{(j-k)!(k+1)!} = \frac{j+1}{k+1} \cdot \frac{j!}{(j-k)!k!} = \frac{j+1}{k+1}\binom{j}{k} so:$

$$R.H.S. = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m+1}{j} \binom{j}{k} B_{j-k} \cdot \frac{n^{k+1}}{k+1} + (m+1)\Delta_m(n)$$

The remaining binomials can be combined in a very convenient way:

$$\binom{m+1}{j}\binom{j}{k} = \frac{(m+1)!}{j!(m-j+1)!} \cdot \frac{j!}{k!(j-k)!} = \frac{(m+1)!}{k!(m-j+1)(j-k)!}$$

$$= \frac{(m+1)!}{k!(m-k+1)!} \cdot \frac{(m-k+1)!}{(j-k)!(m-j+1)!} = \binom{m+1}{k}\binom{m-k+1}{j-k}$$

So it follows that:

$$R.H.S. = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{m+1}{k} \binom{m-k+1}{j-k} B_{j-k} \cdot \frac{n^{k+1}}{k+1} + (m+1)\Delta_m(n)$$

Which can be rearranged and simplified to:

$$R.H.S. = \sum_{k=0}^{m} \sum_{j=0}^{m-k} \binom{m-k+1}{j-k} B_j \cdot \binom{m+1}{k} \frac{n^{k+1}}{k+1} + (m+1)\Delta_m(n)$$

By the definition of Bernoulli numbers:

$$\sum_{j=0}^{m-k} \binom{m-k+1}{j-k} B_j = 0 \text{ for all } k \neq m$$
$$\sum_{j=0}^{m-k} \binom{m-k+1}{j-k} B_j = 1 \text{ for } k = m$$

So the double sum is reduced to just one term:

$$R.H.S. = \binom{m+1}{m} \frac{n^{m+1}}{m+1} + (m+1)\Delta_m(n)$$

Since $\binom{m+1}{m} = m+1$:

$$R.H.S. = n^{m+1} + (m+1)\Delta_m(n)$$

By the previous lemma, it is also clear that $L.H.S. = \sum_{j=0}^{m} \binom{m+1}{j} S_j(n) = n^{m+1}$, so:

$$n^{m+1} = n^{m+1} + (m+1)\Delta$$

Therefore $\Delta = 0$, and the theorem is proven.

$\square$

Credits

Credit must always be given where credit is due. Larry Freeman's blog has a great explanation of the Bernoulli numbers and a proof of their existence. I followed his proof, and tried to make it more readable and understandable for myself. In his proof, however, he implies that the argument above is inductive, which I am pretty sure it is not. (If it is, let me know) Also, some of his markup is difficult to read. Yet as far as content and rigor is concerned, it was certainly the best source I found. Here is his page:

$http://fermatslasttheorem.blogspot.com/2006/10/bernoulli-numbers.html$