Homeomorphisms

Daniel Rostovtsev
Date: May 31, 2017

As the idea of the real numbers was generalized to a topological space, the idea of continuity was extended to the concept of open set continuity. Open set continuity allows for the study of functions in different topological spaces that are entirely foreign to the real numbers. It allows the topologist to look for continuity and preservation of structure in finite spaces, spaces of functions, or any possible space imaginable. As will be proved later, the classic definition of ϵ-δ continuity is synonymous with saying that the function maps the open sets in $\mathbb{R}$ to open sets in $\mathbb{R}$, so naturally the idea of continuity in foreign spaces is considered in terms of open sets, not an ϵ-δ definition.

Definition: Cauchy Limit Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point c if and only if $\lim_{x \to c} f(x) = f(c)$

Cauchy continuity's criterion was the first attempt to formally define continuity. He created it in the late seventeen hundreds, as real analysis was just beginning to form. However, Bolzano and Weierstrass (the fathers of modern real analysis) considered this definition to be too restrictive–since it required the function to be defined for it to be continuous at a given point.

In particular, Bolzano and Weierstrass were concerned about special cases like the following:

$$y = \frac{(x-1)}{x(x-1)}$$

Where y mimics the smooth, "continuous" curve of $1/x$, yet remains undefined for $x = 1$. So, in the 1830s, Bolzano and Weierstrass developed the ϵ-δ definition which behaves identically to Cauchy's definition, but allows for undefined points to be continuous. This definition has been found, over time, to be less restrictive and more consistent.

Definition: ϵ-δ Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is ϵ-δ continuous at a point x_0 if one of the following is true:

$$\forall \ \epsilon > 0 \ , \ \exists \ \delta > 0 : \ f(x_0) - \epsilon < f(x) < f(x) + \epsilon \text{ where } x_0 - \delta < x < x_0 + \delta$$

This definition works for real numbers, but obviously it fails in the case of more general topologies which may be discrete, or may not contain concepts of addition and subtraction. Thus, an even more general definition of continuity was formulated:

Definition: Open Set Continuity

A function $f : (X, \tau) \to (Y, \sigma)$ is "open set" continuous if and only if $\forall \ V \in \sigma \ \exists \ U \in \tau : \ f(U) = V$

Continuity Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is open set continuous if it is ϵ-δ continuous.

Proof. Take the open set $D_{x_0} = (x_0 - \delta, x_0 + \delta)$ in the domain of f. By definition, f is defined for all x in D_{x_0}. If the image of D_{x_0}, $f[D_{x_0}]$, is also open, and all open sets in the range of f can be describe as unions of various $f[D_{x_0}]$, then all open sets in the range of f can be describe as $f[\bigcup D_{x_0}]$. Since $f[\bigcup D_{x_0}]$ is open, it would follow that f is open set continuous.

The image D_{x_0} can be described as $f[D_{x_0}] = f[(x_0 - \delta, x_0 + \delta)]$. The following is true if and only if $f[D_{x_0}]$ is open:

$$f[D_{x_0}] = (f(x_0) - \epsilon, f(x_0) + \epsilon) \text{ for some } \epsilon > 0$$

Suppose that f is ϵ-δ continuous, and the interval $f[D_{x_0}]$ is not open. Then there would be some interval (a, b), $[a, b)$ or $(a, b]$ contained in $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ not overlapping $f[D_{x_0}]$. To illustrate this point, suppose there was some interval (a, b) in $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ not overlapping $f[D_{x_0}]$. Then $f[D_{x_0}] = (f(x_0) - \epsilon, a] \cup [b, f(x_0) + \epsilon)$, which is not open. Similar reasoning explains why the other intervals yeild images of D_x which are not open. Considering each case in which one of these intervals exist, it can be shown that f is not continuous, thus yeilding a contradiction.

Suppose that interval is (a, b), then the image must be $f[D_{x_0}] = (f(x_0) - \epsilon, a] \cup [b, f(x_0) + \epsilon)$, as stated above. f is not ϵ-δ continuous at point $f^{-1}(a)$ because

$$\forall\ \epsilon > 0\ ,\ \nexists\ \delta > 0 :\ f(x_0) - \epsilon < f(x) < f(x) + \epsilon \text{ where } x_0 - \delta < x < x_0 + \delta$$

Suppose that interval is $(a, b]$, then the image must be $f[D_{x_0}] = (f(x_0) - \epsilon, a] \cup (b, f(x_0) + \epsilon)$. Just as before, f is not ϵ-δ continuous at point $f^{-1}(a)$ because

$$\forall\ \epsilon > 0\ ,\ \nexists\ \delta > 0 :\ f(x_0) - \epsilon < f(x) < f(x) + \epsilon \text{ where } x_0 - \delta < x < x_0 + \delta$$

Suppose that interval is $[a, b)$, then the image must be $f[D_{x_0}] = (f(x_0) - \epsilon, a) \cup [b, f(x_0) + \epsilon)$. Yet again, f is not ϵ-δ continuous at point $f^{-1}(b)$ because

$$\forall\ \epsilon > 0\ ,\ \nexists\ \delta > 0 :\ f(x_0) - \epsilon < f(x) < f(x) + \epsilon \text{ where } x_0 - \delta < x < x_0 + \delta$$

In all cases, continuity breaks for any epsilon smaller than the distance between a and b. Note that both ϵ-δ and open set continuity fail not when there is an undefined point on f, but only when there is a semi-open or open chunk of points which are undefined in the image of an open set.

□

Open-set continuous functions are central in topology. They provide a way of "morphing" the open sets, or structure, of one space to yeild a structure resembling another. If this function, or morphing, is bijective (one-to-one and onto), then one topology can be morphed into another without the loss of any fundamental structure. These one to one open set continuous functions are called homeomorphisms.

Definition: Homeomorphism

A function $f : (X, \tau) \to (Y, \sigma)$ is a homemorphism if f is open set continuous and bijective.

Definition: Homeomorphic

(X, τ) is homeomorphic to (Y, σ), denoted $(X, \tau) \sim (Y, \sigma)$, if there exists a homeomorpish $f : (X, \tau) \to (Y, \sigma)$.

Homeomorphisms of Euclidean Spaces

Bijective ϵ-δ continuous functions from $f : \mathbb{R}^n \to \mathbb{R}^n$ are homeomorphisms of Euclidean spaces.

Proof. This will be more of a safe conjecture than an explicit proof, as the explicit proof would be too tedious. As shown in the Continuity Theorem, open set continuity for sets of real numbers will only fail if open sets in the domain have images which are not open, which cannot happen for ϵ-δ continuous functions for any dimension $\mathbb{R}^n$. Writing out the proof for infinite n would be very tedious. So, given that the function is bijective and ϵ-δ continuous, it follows the function is bijective and open set continuous. Thus it is a homeomorphism as well. $\square$

As explained above, homeomorphisms show which topologies are structurally identical. To show what it means for a stucture to be topologically identical, here are some cases of homeomorphisms in $\mathbb{R}^n$

Proposition: $(a, b) \subset \mathbb{R} \sim (c, d) \subset \mathbb{R}$

Proof. Define the homeomorphism between (a, b) and (c, d) as follows:

$$f(x) = \frac{d(x-a)}{b-a} - \frac{c(x-b)}{b-a} = \frac{d-c}{b-a} \cdot x + \frac{bc-ad}{b-a}$$

$f(x)$ is clearly continuous, as it is a line. Since $f(a) = c$ and $f(b) = d$ if f were defined on a and b (and not just on (a, b)), it is clear that $f(x)$ is also bijective.

$\square$

Proposition: $(a, b) \subset \mathbb{R} \sim \mathbb{R}$

Proof. Define the homeomorphism from (a, b) to $\mathbb{R}$ as follows:

$$f(x) = \cot(\pi \frac{x-a}{b-a})$$

f is continuous and invertible, defined on (a, b) and its image is $\mathbb{R}$. So (a, b) is homeomorphic to $\mathbb{R}$.

$\square$

Proposition: $\{(x, y) : |x| + |y| = 1\} \subset \mathbb{R}^2 \sim \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$

(i.e.) A square is homeomorphic to a circle.

Proof. Define the homeomorphism from the square in $\mathbb{R}^2$ to the circle in $\mathbb{R}^2$ as follows:

$$f((x, y)) = (\cos \tan^{-1} \tfrac{y}{x}, \sin \tan^{-1} \tfrac{y}{x})$$

$\square$