Topological Spaces

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The term ”open set” was originally used only to describe the following type of set in the real numbers: (a, b). However, as people began to work with spaces other than $\mathbb{R}$, the idea of an open set was generalized. In topology, two numbers are considered to be in the same ”neighborhood” if they are in the same open set, the same way 2.3 and 2.6 are in the same ”neighborhood” since they are both in the open set $(2, 3)$.

By defining the open sets of a space, that space is given structure–and it thus becomes a topology. However, the definition of these open sets must satisfy some basic axioms, so that neighborhoods, and thus topologies, behave in a way that allow consistent structure.

Definition: Open Set

Let τ be all the open sets of a set X. The following must be true:

(i) $\emptyset \in \tau$ and $X \in \tau$

(ii) $S_1 \cup S_2 \in \tau \ \forall \ S_1, S_2 \in \tau$

(iii) $S_1 \cap S_2 \in \tau \ \forall \ S_1, S_2 \in \tau$

Open Set Topological Axioms

Let τ contain only subsets of X, then the pair (X, τ) defines a topological space if the following axioms are satisfied:

(i) $\emptyset \in \tau$ and $X \in \tau$

(ii) $S_1 \cup S_2 \in \tau \ \forall \ S_1, S_2 \in \tau$

(iii) $S_1 \cap S_2 \in \tau \ \forall \ S_1, S_2 \in \tau$

As is clear above, topological spaces were designed specifically so that all $S \in \tau$ are open sets. However, there are other ways to define a topology that guarantees all elements of τ are open sets. It is sometimes easier to show that a given pair (X, τ) is a topology using either closed sets or neighborhoods.

Definition: Closed Set

Given the open sets τ for some set X, the set C is closed if and only there exists some open sets $S \in \tau$ such that the following is true:

(i) $C \cup S = X$

(ii) $C \cap S = \emptyset$

The process of finding all elements of a set not in a subset is called finding the complement of that subset. In other words, C is the complement of S, so $C = \bar{S}$.

The definition of a closed set leads to the following axioms for the set of closed sets κ of X. These axioms turn out to be the exact same axioms for that of the set of open sets–so both the closed sets and the open sets can be considered the ”open” sets of a topology.

Proposition: Dichotomy of Open Set/Closed Set Axioms

Let κ be all the closed sets of a set X. The following must be true:

(i) $\emptyset \in \kappa$ and $X \in \kappa$

Proof. From the first axiom of open sets:
$X \in \tau \therefore \emptyset \in \kappa$ and $\emptyset \in \tau \therefore X \in \kappa$ □

(ii) $C_1 \cup C_2 \in \kappa \; \forall \; C_1, C_2 \in \kappa$

Proof. From the third axiom of open sets:
$\exists \; S_1 \in \tau : \; C_1 = \bar{S_1}$ and $\exists \; S_2 \in \tau : \; C_2 = \bar{S_2}$
$\therefore C_1 \cup C_2 = \bar{S_1} \cup \bar{S_2} = \overline{(S_1 \cap S_2)} = \bar{S}'$ for some $S' \in \tau \because S_1 \cap S_2 \in \tau$
$\therefore C_1 \cup C_2 \in \kappa \because S' \in \tau$ □

(iii) $C_1 \cap C_2 \in \kappa \; \forall \; C_1, C_2 \in \kappa$

Proof. From the second axiom of open sets:
$\exists \; S_1 \in \tau : \; C_1 = \bar{S_1}$ and $\exists \; S_2 \in \tau : \; C_2 = \bar{S_2}$
$\therefore C_1 \cap C_2 = \bar{S_1} \cap \bar{S_2} = \overline{(S_1 \cup S_2)} = \bar{S}'$ for some $S' \in \tau \because S_1 \cup S_2 \in \tau$
$\therefore C_1 \cap C_2 \in \kappa \because S' \in \tau$ □

Closed Set Topological Axioms

Let τ be the open sets of X and κ be the closed sets of X. The pair (X, κ) defines a topology on X.

Proof. If κ is the set of closed sets, then κ satisfies all the requirements to be an open set, too. Therefore, the topology (X, κ) defines a topology on X. □

Oftentimes defining open sets for an arbitrary topology can be difficult. Neighborhoods are like open sets, but are looser in their definition. Nonetheless, it is possible to use neighborhoods to define topologies–since if neighborhoods satisfy certain axioms, it follows that some topology must exist.

Definition: Neighborhood

If (X, τ) is a topological space, then N is a neigborhood of x if N contains an open set an open set which contains x. In other words:

$x \in S \subseteq N$

Neighborhood Topological Axioms

Let $N : X \to N[X]$ be a function mapping $x \in X$ to the set of neighborhoods containing x, denoted $N(x)$. Additionally, let $\tau = \{U \subseteq X \; : \; U \in N(u) \; \forall \; u \in U\}$. If the following axioms are satisfied, then the pair (X, τ) is a topology.

(i) (the empty set is the only neighborhood of the empty set)
$N(\emptyset) = \{\emptyset\}$

(ii) (all sets containing a neighborhood of x are themselves neighborhoods of x)
$N \in N(x)$ and $N \subseteq U \implies U \in N(x) \; \forall \; U \subseteq X$

(iii) (the intersection of two neighborhoods of x is a neighborhood of x)
$N_1 \cap N_2 \in N(x)\ \forall\ N_1, N_2 \in N(x)$

Proof. Showing that the above axioms imply that (X, τ) is a topology:

(i) $\emptyset \in \tau$ and $X \in \tau$

Clearly, by the first axiom, the empty set is in τ.

$\emptyset \in N(\emptyset)\ \forall\ x \in \emptyset \therefore \emptyset \in \tau$

Since all neighborhoods are subsets of X, X is a neighborhood for all $x \in X$ by the second axiom. So:

$X \in N(x)\ \forall\ x \in X \therefore X \in \tau$

(ii) $N_1 \cup N_2 \in \tau\ \forall\ N_1, N_2 \in \tau$

Let $x \in N_1 \cup N_2$. If $x \in N_1$, then $N_1 \in N(x)$, otherwise $N_1 \notin \tau$. Likewise, if $x \in N_2$, then $N_2 \in N(x)$, otherwise $N_2 \notin \tau$.

Suppose x truly is in N_1. Then $N_1 \cup N_2 \in N(x)$ by the second axiom, since $N_1 \subseteq N_1 \cup N_2$. Likewise, suppose x truly is in N_2. Then $N_1 \cup N_2 \in N(x)$ by the second axiom, since $N_2 \subseteq N_1 \cup N_2$.

Therefore $N_1 \cup N_2 \in N(x)\ \forall\ x \in N_1 \cup N_2$, so $N_1 \cup N_2 \in \tau$.

(iii) $N_1 \cap N_2 \in \tau\ \forall\ N_1, N_2 \in \tau$

For all $x \in N_1 \cap N_2$, $N_1 \in N(x)$ and $N_2 \in N(x)$. Therefore, for all $x \in N_1 \cap N_2$, $N_1 \cap N_2 \in N(x)$ by the third axiom.

Therefore $N_1 \cap N_2 \in \tau$.

□

Here are some examples of common topologies:

Euclidean Spaces

A Euclidean space is a topology of the form $(\mathbb{R}^n, \tau)$ where τ consists of the unions and intersections of all sets, $\{\vec{x}\}$, such that:

$x_1 \in (a_1, b_1)$
$x_2 \in (a_2, b_2)$
$\ldots$
$x_n \in (a_n, b_n)$ such that $a_i, b_i \in \mathbb{R}$

Proof. This is easy to do using the open set topological axioms. Clearly the empty set is contained in τ, if $a_i = b_i\ \forall\ i$. Likewise, $\mathbb{R}^n$ is in τ when $a_i = -\infty$ and $b_i = \infty\ \forall\ i$. The unions and intersections, are by definition, also in τ. □

Metric Spaces

A metric space is any set of numbers X, with a defined distance function (a metric) between any two points $d : X \times X \to \{\mathbb{R} > 0\}$. It has "ball"-like neighborhoods about any point x denoted $B_r(x)$ such that:

$$B_r(x) = \{p \in X : d(x, p) < r\}$$

$$N(x) = \{N \in \bigcup B_r(p) : x \in N\}$$

Proof. The empty set is clearly present, if r is set to 0. The union of two neighborhoods is clearly a neighborhood by definition, and the intersection of two neighborhoods is present since any region can be drawn using the unions of "ball" functions. □