Laplace's Method

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Gaussian Integration

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Proof. The above integral is evaluated by converting to polar coordinates. Since:

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int\int_A e^{-(x^2+y^2)} dA$$

And an integral in terms of area can be expressed in polar coordinates:

$$\int\int_A e^{-(x^2+y^2)} dA = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr d\theta$$

It follows that:

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr d\theta = \int_0^{2\pi} -\frac{1}{2} \cdot e^{-r^2}\Big|_0^{\infty} d\theta = \int_0^{2\pi} \frac{d\theta}{2} = \pi$$

Therefore:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

□

Corollary

$$\int_{-\infty}^{\infty} e^{-k(x-x_0)^2} dx = \sqrt{\frac{\pi}{k}}$$

Proof. The proof is essentially the same as above (just shifting the center of the polar coordinates to (x_0, x_0)):

$$\left(\int_{-\infty}^{\infty} e^{-k(x-x_0)^2} dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-k((x-x_0)^2+(y-x_0)^2)} dx dy = \int_0^{2\pi}\int_0^{\infty} r e^{-kr^2} dr d\theta = \frac{\pi}{k}$$

□

Proposition

Let M be arbitrarily large, and $f(x_0)$ be the global maximum of f on the interval:

$$\int_a^b e^{Mf(x)}dx \approx e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}$$

Proof. Conceptually, this approximation draws a bell curve under any function $e^{Mf(x)}$, so if the $e^{Mf(x)}$ does not resemble a bell curve, the approximation won't work very well. To describe an approximating bell curve, Taylor series is used to approximate $Mf(x)$ in the form $a - b(x - x_0)^2$ so that will resemble $e^{Mf(x)}$ will resemble the resulting bell curve: $e^a \cdot e^{-b(x-x_0)^2}$.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} \cdot f"(x)(x - x_0)^2 + R$$

Since $f'(x_0)$ is a global maximum, $f'(x_0) = 0$. Also, since f somewhat resembles a bell curve, it will be assumed for the sake of simplicity that $f"(x_0) < 0$. Therefore:

$$f(x) \approx f(x_0) - \frac{1}{2} \cdot |f"(x)|(x - x_0)^2$$

Turning this into an approximation for $e^{Mf(x)}$, operating under the assumption that the "tails" of $e^{Mf(x)}$ will be negligible just like the tails of a bell curve:

$$\int_a^b e^{Mf(x)}dx \approx e^{Mf(x_0)} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}\cdot|f"(x)|(x-x_0)^2}dx$$

Using Gaussian integration, it then follows that:

$$\int_a^b e^{Mf(x)}dx \approx e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}$$

□

On the next page, it will be proven that increasing M towards infinity makes $e^{Mf(x)}$ more "bell-curve-like", making the approximation more and more accurate.

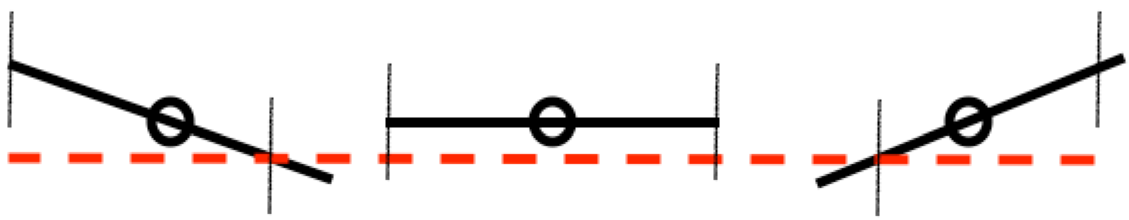
Laplace's Method

Let f be a continuous function with a continuous second derivative, with a local maximum on $[a, b]$ at x_0, such that $f"(x_0) = 0$ then:

$$\int_a^b e^{Mf(x)}dx \sim e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}} \quad \text{as} \quad M \to \infty$$

Proof. It must be shown that $\lim_{M\to\infty} \frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}} = 1.$

Since there is no visible and quick method to show that the above limit definitely converges to one, it must first itbe shown that the above limit is less or equal than one, and then that it is greater than or equal to one. First, to show that the above limit is less greater than one. Note throughout this proof that $f"(x_0)$ is assumed to be negative.



Considering the graphic above, of various plots of $f"(x)$ zoomed in to the point where $f"(x)$ looks like a line centered about $f"(x_0)$, it is clear that there is some δ for which $f"(x) \geq f"(x_0) - \epsilon$ for all x in the interval $(x_0 - \delta, x_0 + \delta)$. So by Taylor's theorem:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} \cdot (f"(c))(x - x_0)^2$$
(for all $x \in [x_0 - \delta, x_0 + \delta]$ and for some c in between x and x_0)

Remembering $f'(x_0) = 0$ and $f"(c) \geq f"(x_0) - \epsilon$ for all $x \in (x - \delta, x + \delta)$:

$$f(x) \geq f(x_0) + \tfrac{1}{2} \cdot (f"(x_0) - \epsilon)(x - x_0)^2 \ \forall \ x \in (x - \delta, x + \delta)$$

Therefore $\int_a^b e^{Mf(x)} \geq \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}dx \geq e^{Mf(x_0)} \int_{x_0-\delta}^{x_0+\delta} e^{\frac{M}{2}\cdot(f"(x_0)-\epsilon)(x-x_0)^2}dx$

Using substitution from x to $y = \sqrt{M(-f"(x_0) + \epsilon)}(x - x_0)$ to simplify $\int_{x_0-\delta}^{x_0+\delta} e^{\frac{M}{2}\cdot(f"(x_0)-\epsilon)(x-x_0)^2}dx$:

$$\frac{dy}{dx} = \sqrt{M(-f"(x_0)+\epsilon)} \equiv dx = dy\sqrt{\frac{1}{M(-f"(x_0)+\epsilon)}}$$

$$y(x_0-\delta) = -\delta\sqrt{M(-f"(x_0)+\epsilon)}\ ;\ y(x_0+\delta) = \delta\sqrt{M(-f"(x_0)+\epsilon)}$$

$$\int_{x_0-\delta}^{x_0+\delta} e^{\frac{M}{2}\cdot(f"(x_0)-\epsilon)(x-x_0)^2}dx = \int_{x_0-\delta}^{x_0+\delta} e^{-\frac{M}{2}\cdot(-f"(x_0)+\epsilon)(x-x_0)^2}dx = \sqrt{\frac{1}{M(-f"(x_0)+\epsilon)}}\int_{-\delta\sqrt{M(-f"(x_0)+\epsilon)}}^{\delta\sqrt{M(-f"(x_0)+\epsilon)}} e^{-\frac{1}{2}y}dy$$

Taking the limit as $M \to \infty$:

$$\lim_{M\to\infty} \frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}} \geq \lim_{M\to\infty} \frac{e^{Mf(x_0)}\sqrt{\frac{1}{M(-f"(x_0)+\epsilon)}}\int_{-\delta\sqrt{M(-f"(x_0)+\epsilon)}}^{\delta\sqrt{M(-f"(x_0)+\epsilon)}} e^{-\frac{1}{2}y}dy}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M(-f"(x_0))}}}$$

Therefore:

$$\lim_{M\to\infty} \frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}} \geq \lim_{M\to\infty}\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}dy \cdot \sqrt{\frac{-f"(x_0)}{-f(x_0)+\epsilon}} = \frac{1}{\sqrt{2\pi}}\cdot\sqrt{2\pi}\cdot\sqrt{\frac{-f"(x_0)}{-f(x_0)+\epsilon}}$$

So $\lim_{M\to\infty} \frac{\int_a^b e^{Mf(x)}dx}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}} \geq 1$

Now it only remains to show that the above limit is less than one. The method used will be very similar. By the same logic used above, there is some δ for which $f(x) \leq f(x_0)+\epsilon$ for all $x \in (x_0-\delta, x_0+\delta)$. Using Taylor's theorem with remainder:

$$f(x) \leq f(x_0) + \tfrac{1}{2}\cdot(f"(x_0)+\epsilon)(x-x_0)^2\ \forall\ x \in (x-\delta, x+\delta)$$

This time it is not true that $\displaystyle\int_a^b e^{Mf(x)}dx \leq \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)}$, but the following is certainly true:

$$\int_a^b e^{Mf(x)}dx \leq (b-a)e^{M(f(x_0)-\Delta)} + \int_{x_0-\delta}^{x_0+\delta} e^{Mf(x)} \text{ where } \Delta = f(x_0) - \min_{[a,b]}f(x)$$

And, by extension $\displaystyle\int_a^b e^{Mf(x)} \leq (b-a)e^{M(f(x_0)-\Delta)} + e^{Mf(x_0)} \int_{x_0-\delta}^{x_0+\delta} e^{\frac{M}{2}\cdot(f"(x_0)+\epsilon)(x-x_0)^2} dx$

Yet $\displaystyle\int_{x_0-\delta}^{x_0+\delta} e^{\frac{M}{2}\cdot(f"(x_0)+\epsilon)(x-x_0)^2} dx \leq \int_{-\infty}^{\infty} e^{\frac{M}{2}\cdot(f"(x_0)+\epsilon)(x-x_0)^2} dx$

Integrating the indefinite integral above using Gaussian integration:

$$\int_{\infty}^{\infty} e^{\frac{M}{2}\cdot(f"(x_0)+\epsilon)(x-x_0)^2} dx = \int_{-\infty}^{\infty} e^{-\frac{M}{2}\cdot(-f"(x_0)-\epsilon)(x-x_0)^2} dx = \sqrt{\frac{2\pi}{n(-f"(x_0)-\epsilon)}}$$

Thus $\displaystyle\int_a^b e^{Mf(x)} \leq (b-a)e^{M(f(x_0)-\Delta)} + e^{Mf(x_0)}\sqrt{\frac{2\pi}{n(-f"(x_0)-\epsilon)}}$

Finally, finding the limit of the estimate and true integral as $M \to \infty$:

$$\lim_{M\to\infty} \frac{\int_a^b e^{Mf(x)} dx}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}} \leq \lim_{M\to\infty} \frac{(b-a)e^{M(f(x_0)-\Delta)}}{e^{Mf(x_0)}} \cdot \sqrt{\frac{M|f"(x_0)|}{2\pi}} + \sqrt{\frac{M(-f(x_0))}{M(-f"(x_0)-\epsilon)}}$$

$$\lim_{M\to\infty} \frac{\int_a^b e^{Mf(x)} dx}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}} \leq \lim_{M\to\infty} (b-a)e^{-M\Delta)} \cdot \sqrt{\frac{M|f"(x_0)|}{2\pi}} + \sqrt{\frac{(-f(x_0))}{(-f"(x_0)-\epsilon)}} \leq 0+1 \leq 1$$

Therefore $\displaystyle\int_a^b e^{Mf(x)} dx \sim e^{Mf(x_0)}\sqrt{\frac{2\pi}{M|f"(x_0)|}}$ as $M \to \infty$ $\square$

While it is fois-pas to cite wikipedia in most subjects, I feel that it is fine to do in math. Nobody can cheat you if you are vigilant, because it is possible to catch flaws in logic. Plus, nobody comes in to edit Laplace's method as a joke, because nobody will see it. That being said, wikipedia has a really good proof and explanation of Laplace's Method, so I will give it credit:

`https://en.wikipedia.org/wiki/Laplace%27s_method`