6.6340 PS1 Q1

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1 Review of Physics of Electromagnetic Wave Propagation

To form an intuitive understanding of electromagnetic waves, let us first explore electromagnetic wave generation and models. The most well-accepted theory of electromagnetic wave generation comes from the dipole model, where the oscillations of the components of the dipole produces a radiating electric and magnetic field. Our observations of electromagnetic waves are thus simply the culminations of numerous dipoles oscillating together to produce an overall electromagnetic wave. Fig 1 demonstrates the radiation pattern of a single dipole:

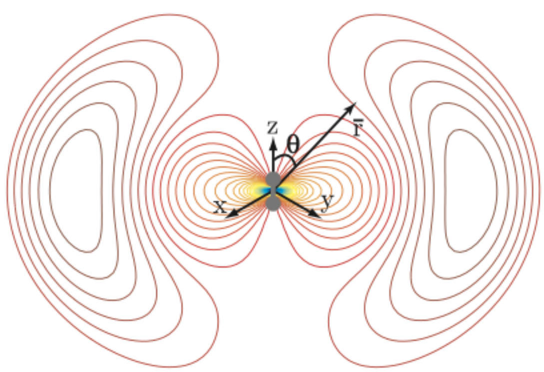

Figure 1: Dipole model of electromagnetic wave generation [6.2370 Lecture Notes]

While most intuition throughout this paper will be built from mathematical observations, it is important to understand a basic atomic-level view of electromagnetic wave generation, as it gives physical intuition to calculated mathematical results. This will be more prevalent as the impact of material properties on electromagnetic wave properties is discussed.

To determine the inner-workings of electromagnetic wave propagation, we must first recognize its governing equations. Maxwell's equations concisely capture these properties with the following formulas:

$$\overline{\nabla} \cdot \overline{E} = \rho \tag{1}$$

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{2}$$

$$\overline{\nabla} \times \overline{E} = -\frac{d\overline{B}}{dt} \tag{3}$$

$$\overline{\nabla} \times \overline{H} = \overline{J} + \frac{d\overline{D}}{dt} \tag{4}$$

Assuming we are observing a region of space with no net distribution of charge or current, we can set $\rho = 0$ and $\overline{J} = \overline{0}$. We will also assume that the time-derivative terms in the electric and magnetic fields are non-negligible. This simplifies Maxwell's formulas to the following formulas:

$$\overline{\nabla} \cdot \overline{E} = 0 \tag{5}$$

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{6}$$

$$\overline{\nabla} \times \overline{E} = -\frac{d\overline{B}}{dt} \tag{7}$$

$$\overline{\nabla} \times \overline{H} = \frac{d\overline{D}}{dt} \tag{8}$$

Noting that $\overline{B} = \mu\overline{H}$ and $\overline{D} = \epsilon\overline{E}$ in a lossless isotropic medium, we can express these equations in a more uniform notation, so as to be less confusing when noting mathematical relations between the electric and magnetic fields. It is valid to make this assumption since the question is asking for a general explanation of electromagnetic wave propagation. In "basic" or "introductory" electromagnetic wave propagation scenarios, waves propagate in air (essentially vacuum). To address this "introductory" scenario, I will proceed with this method of explanation:

$$\overline{\nabla} \cdot \overline{E} = 0 \tag{9}$$

$$\overline{\nabla} \cdot \mu\overline{H} = 0 \tag{10}$$

$$\overline{\nabla} \times \overline{E} = -\mu\frac{d\overline{H}}{dt} \tag{11}$$

$$\overline{\nabla} \times \overline{H} = \epsilon\frac{d\overline{E}}{dt} \tag{12}$$

We must now recognize the implications of the nabla (or del, ∇) operator on a general mathematical vector equation. The following relations can be observed:

1. The divergence ($\overline{\nabla}\cdot$) of the vector fields being zero in both relevant equations ((Equations (9) and (10))) demonstrates that the electric and magnetic fields have no parallel relation to each other.

2. The curl ($\overline{\nabla}\times$) of a vector generates a set of vectors orthogonal to the original. Equations (11) and (12) demonstrate that the electric and magnetic fields have an orthogonal relation to each other.

From these results, we can infer that the electric and magnetic fields have a strictly-orthogonal relation. Another way to think about this result is to consider the dipole model. The oscillations of the dipole can be modeled as a small piece of wire, thus producing a magnetic field due to the "current" produced by the oscillations. The oscillations will also produce an electric field that, by the symmetry of the dipole model, must be orthogonal to the magnetic field.

To understand why the electric and magnetic fields are modeled as sinusoids, we must solve our modified set of equations for $\overline{E}$ and $\overline{H}$. Using the double cross product identity ($\overline{\nabla} \times \overline{\nabla} \times \overline{A} = \nabla(\overline{\nabla} \cdot \overline{A}) - \nabla^2 \overline{A}$) on Faraday's Law (eq.11) to solve for $\overline{E}$,...

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = \nabla(\overline{\nabla} \cdot \overline{E}) - \nabla^2 \overline{E} = \nabla(0) - \nabla^2 \overline{E} = -\nabla^2 \overline{E} \tag{13}$$

Another valid way to express the double cross product of the electric field is to substitute $\overline{\nabla} \times \overline{E}$ with its magnetic field equivalent according to Faraday's Law (eq.11). Performing this calculation:

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = \overline{\nabla} \times (-\mu \frac{d\overline{H}}{dt}) = -\mu \frac{d}{dt}(\overline{\nabla} \times \overline{H}) \tag{14}$$

Substituting the magnetic field term with our wave equation assumption with the Ampere-Maxwell Law:

$$\overline{\nabla} \times \overline{\nabla} \times \overline{E} = -\mu\epsilon \frac{d^2\overline{E}}{dt^2} \tag{15}$$

Equating equations (13) and (15),...

$$-\nabla^2 \overline{E} = -\mu\epsilon \frac{d^2\overline{E}}{dt^2} \implies \nabla^2 \overline{E} - \mu\epsilon \frac{d^2\overline{E}}{dt^2} = 0 \tag{16}$$

The general solution to this differential equation is well-known and can be generalized to the following:

$$\overline{E}(x, y, z, t) = Re\{Ce^{\pm jk_x x} e^{\pm jk_y y} e^{\pm jk_z z} e^{\pm j\omega t}\}, C \in \Re \tag{17}$$

A physical way to understand this is to once again consider the dipole model of electromagnetic wave radiation. Since the dipole is oscillating as a spring-mass system, the generated electric and magnetic fields must share some sinusoidal characteristic.

This solution does not give us immediate insight as to how the k and ω terms are related, or how they are physically defined. As of now, they are mathematically defined as arbitrary constants that successfully solve our proposed partial differential equation. To explore the physics of these terms further, we must understand how $\overline{k}$ affects wave propagation.

Suppose we define a vector $\overline{k}$ as the propagation, or movement, direction of our wave. Let us give it the equation $\overline{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$. This vector will impact what regions of space our electromagnetic wave inhabits. We can thus write (Eq.17) in a much more compact notation:

$$\overline{E}(x, y, z, t) = Re\{Ce^{j\overline{k}\cdot\overline{r}}e^{j\omega t}\}, C \in \Re \tag{18}$$

We can now plug this into Faraday's Law (Eq.11) to yield:

$$\overline{\nabla} \times \overline{E} = jE(\overline{k} \times \hat{E})e^{j\overline{k}\cdot\overline{r}}e^{j\omega t} = -\mu\frac{d\overline{H}}{dt} \tag{19}$$

We previously asserted that $\overline{E}$ and $\overline{H}$ had a strictly-orthogonal relation, and since $\overline{k} \times \hat{E}$ yields $\frac{d\overline{H}}{dt}$, we know that $\overline{k}$ must be strictly orthogonal to $\overline{E}$ and $\overline{H}$. Therefore, the propagation direction of our electromagnetic wave occurs in the direction of $\overline{E} \times \overline{H}$.

To make further calculations simpler, let us assume that the direction of propagation is purely in the $\hat{z}$-direction, therefore making $\overline{k} = |k|\hat{z}$. We must now revisit (Eq.16) to determine the signs of our temporal and spatial terms. Mathematically, either solution works. Physically, however, we must recognize how choosing the signs of these terms affects the wave's propagation direction.

To explore the mathematical implications of these sign choices, let us consider scenarios where the temporal term is strictly positive and the spatial term signs vary.

<u>Case 1</u>: spatial term is negative:

$$\overline{E}(x, y, z, t) = Re\{e^{-jkz}e^{+j\omega t}\} = cos(\omega t - kz) \tag{20}$$

Let us now consider how the signage propagates the solution through space. Letting time increase, the sinusoid "shifts right." This correlates to a propagation direction of $+\hat{z}$

<u>Case 2</u>: spatial term is positive:

$$\overline{E}(x, y, z, t) = Re\{e^{+jkz}e^{+j\omega t}\} = cos(\omega t + kz) \tag{21}$$

Let us now consider how the signage propagates the solution through space. Letting time increase, the sinusoid "shifts left." This correlates to a propagation direction of $-\hat{z}$

We can thus conclude that, making the temporal term strictly positive, the propagation direction of our wave can mathematically and physically be accounted for by placing a negative sign on a spatial term to demonstrate positive movement in the spatial region and a positive sign on the spatial term to demonstrate negative movement in the spatial region.

Let us now revisit our wave equation formula. We know the general solution to $\overline{E}$ from (eq.17), so plugging it in, we get...

$$j^2(k_x^2+k_y^2+k_z^2)-\mu\epsilon j^2(\omega^2) = 0 \implies |k|^2 = \mu\epsilon\omega^2 \implies \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu\epsilon}} = v_{light} \quad (22)$$

We now have a well-defined relation between our k-vector, the ω term, and the speed of light in a medium. Let us explore the mathematical intricacies of the k and ω terms further.

A sinusoidal function will complete a period after every 2π[rad]. Thus, the ω term has a relation to the temporal frequency of our electromagnetic wave through the following formula:

$$\omega = \frac{2\pi}{T} = 2\pi f \quad (23)$$

Using the same period completion after every 2π[rad], we can argue that the k-vector is directly related to the wavelength of our electromagnetic wave through the following formula:

$$k = \frac{2\pi}{\lambda} \quad (24)$$

Thus, we can rewrite (Eq.22) as:

$$\frac{\omega}{|k|} = \lambda f = v_{light} \quad (25)$$

We have now established a relation between the wavelength, frequency, and speed of light. There are a couple of other interesting conclusions we can draw from our work so far:

1. Assuming our wave is propagating in the $+\hat{z}$ direction, we can solve for the magnetic field $\overline{H}$ using Faraday's Law (Eq.11). From this, we find that the electric and magnetic fields *must* oscillate in phase. If their phases were offset by some phase $Re\{e^{j\phi}\}$, Maxwell's Laws would not be maintained.

2. An electromagnetic wave's speed is constant and depends solely on medium properties. Waves will propagate through a medium at the same speed REGARDLESS of frequency or wavelength.

To explore point 2 further, let us more clearly define the influence of μ and ϵ on a system.

In general, μ (permeability) is a proportionality constant that scales magnetic field strength based on a material's response to input magnetic fields. On the atomic scale, this response is based on how well a material's internal magnetic dipole moments can line up with an input magnetic field. The easier it is for these dipoles to line up with the magnetic field, the stronger the overall magnetic field becomes, thus yielding a higher permeability value for the material.

In general, ϵ (permittivity) is a proportionality constant that scales electric field strength based on a material's response to input electric fields. On the atomic scale, this response is based on how well a material's internal electric dipole moments line up with the input field. The easier is is for these dipoles to line up with the electric field, the stronger the overall electric field becomes, thus yielding a higher permittivity value for the material.

Because a material can react differently to input electric and magnetic fields, it can alter the overall speed and shape of the wave. For instance, the material's electric and magnetic dipole moments might align in such a way that the overall electric field seems to go "slower" than the speed of light in vacuum. Since this effect is only spatial, the wavelength of the input wave can also vary once inside materials with ϵ and μ not equivalent to those in vacuum. For mediums that apply to the assumptions made for this paper, including vacuum space, the permittivity and permeability are constant in space and time, allowing for constant speeds of light and, thus, constant propagation speeds for our electromagnetic waves.

Let us now examine the flow of energy of our electromagnetic wave. We will start by defining a relation between work, power, and energy:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V u dV \tag{26}$$

Where,...
$\frac{dW}{dt}$ = Power (change in work over time) [W]
u = Potential Energy Density $[\frac{J}{m^3}]$
V = Inhabited Volume $[m^3]$

Let us now define the flow of energy in our system with a vector $\overline{S}$, known as the Poynting vector. While it is well-known that $\overline{S} = \overline{E} \times \overline{H}$, students often lack the intuition as to why power flows orthogonal to both fields. Thus, it is important to delve into the physics of this conclusion. We can define a surface area 2-D flux integral to investigate energy flow in our system:

$$W = \int_{dV} \overline{S} \cdot \hat{n} dA \tag{27}$$

We can now turn this into a volume integral using Gauss' Theorem for Vector Calculus and equate it to our previously-defined volume integral (Eq.26):

$$W = \int_{dV} \overline{S} \cdot \hat{n} dA = \int_V \overline{\nabla} \cdot \overline{S} dV \implies \overline{\nabla} \cdot \overline{S} = -\frac{du}{dt} \tag{28}$$

From experimental observation and theoretical techniques of electric and magnetic field energy storage, we know the energy density of the electric and magnetic fields of an electromagnetic wave as the following:

$$u = \frac{1}{2}\mu|H|^2 + \frac{1}{2}\epsilon|E|^2 \tag{29}$$

Using this information, we can reach the following conclusion:

$$-\overline{\nabla} \cdot \overline{S} = -\frac{d}{dt}(\frac{1}{2}\mu|H|^2 + \frac{1}{2}\epsilon|E|^2) = \mu\overline{H} \cdot \frac{d\overline{H}}{dt} + \epsilon\overline{E} \cdot \frac{d\overline{E}}{dt} \tag{30}$$

Using equations (11) and (12), we can substitute the time-derivative terms with space-derivative terms. We can also use the vector identity $\overline{A} \cdot (\overline{\nabla} \times \overline{B}) - \overline{B} \cdot (\overline{\nabla} \times \overline{A}) = \overline{\nabla} \cdot (\overline{B} \times \overline{A})$:

$$\overline{\nabla} \cdot \overline{S} = \overline{H} \cdot (\overline{\nabla} \times \overline{E}) - \overline{E} \cdot (\overline{\nabla} \times \overline{H}) = \overline{\nabla} \cdot (\overline{E} \times \overline{H}) \implies \overline{S} = \overline{E} \times \overline{H} \tag{31}$$

Thus, for isotropic media, we have shown that the energy flow and wave propagation point in the same direction. It is important to note that the two need not point in the same direction, as explored in class with anisotropic media. I have explored the derivations of both $\overline{S}$ and $\overline{k}$ to distinguish the logical methods of reaching each vector's directions in space. The key assumption to distinguishing the two is assuming that μ and ϵ are constant scalars, which we have recently explored in class is not always the case (i.e. they can be modeled as matrices that make $\overline{D}$ and $\overline{E}$ point in different directions.