12.16. A call option on a non-dividend-paying stock has a market price of $\$2\frac{1}{2}$. The stock price is \$15, the exercise price is \$13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

12.17. With the notation used in this chapter

a. What is $N'(x)$?

b. Show that $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, where S is the stock price at time t and

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

c. Calculate $\partial d_1/\partial S$ and $\partial d_2/\partial S$.

d. Show that, when $c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$,

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c is the price of a call option on a non-dividend-paying stock.

e. Show that $\partial c/\partial S = N(d_1)$.

f. Show that c satisfies the Black–Scholes differential equation.

g. Show that c satisfies the boundary condition for a European call option, i.e., that $c = \max(S - K, 0)$ as $t \to T$.

12.18. Show that the Black–Scholes formulas for call and put options satisfy put–call parity.

12.19. A stock price is currently \$50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends:

	Maturity (months)		
Strike price (\$)	3	6	12
45	7.0	8.3	10.5
50	3.7	5.2	7.5
55	1.6	2.9	5.1

Are the option prices consistent with Black–Scholes?

12.20. Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

12.21. Consider an American call option on a stock. The stock price is \$50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is \$55, and the volatility is 25%. Dividends of \$1.50 are expected after 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

12.22. Show that the probability that a European call option will be exercised in a risk-neutral world is $N(d_2)$, using the notation introduced in this chapter. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K?

12.23. Show that S^{-2r/σ^2} could be the price of a traded security.

ASSIGNMENT QUESTIONS

12.24. A stock price is currently \$50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine 95% confidence intervals.

12.25. Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0, 32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

12.26. A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 at time T.

a. Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price, S, at time t. (*Hint*: The expected value of S_T^2 can be calculated from the mean and variance of S_T given in Section 12.1.)

b. Confirm that your price satisfies the differential equation (12.15).

12.27. Consider an option on a non-dividend-paying stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is four months.

a. What is the price of the option if it is a European call?

b. What is the price of the option if it is an American call?

c. What is the price of the option if it is a European put?

d. Verify that put–call parity holds.

12.28. Assume that the stock in Problem 12.27 is due to go ex-dividend in $1\frac{1}{2}$ months. The expected dividend is 50 cents.

a. What is the price of the option if it is a European call?

b. What is the price of the option if it is a European put?

c. If the option is an American call, are there any circumstances under which it will be exercised early?

12.29. Consider an American call option when the stock price is \$18, the exercise price is \$20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option with ex-dividend dates at the end of two months and five months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. How high can the dividends be without the American option being worth more than the corresponding European option?

APPENDIX 12A

Proof of the Black–Scholes–Merton Formula

We will prove the Black–Scholes result by first proving another key result that will also be useful in future chapters.

Key Result

If V is lognormally distributed and the standard deviation of $\ln V$ is s, then

$$E[\max(V - K,\ 0)] = E(V)N(d_1) - KN(d_2) \tag{12A.1}$$

where

$$d_1 = \frac{\ln[E(V)/K] + s^2/2}{s}$$

$$d_2 = \frac{\ln[E(V)/K] - s^2/2}{s}$$

and E denotes the expected value.

Proof of Key Result

Define $g(V)$ as the probability density function of V. It follows that

$$E[\max(V - K),\ 0] = \int_K^{\infty} (V - K)g(V)\,dV \tag{12A.2}$$

The variable $\ln V$ is normally distributed with standard deviation s. From the properties of the lognormal distribution the mean of $\ln V$ is m, where

$$m = \ln[E(V)] - s^2/2 \tag{12A.3}$$

Define a new variable

$$Q = \frac{\ln V - m}{s} \tag{12A.4}$$

This variable is normally distributed with a mean of zero and a standard deviation of 1.0. Denote the density function for Q by $h(Q)$, so that

$$h(Q) = \frac{1}{\sqrt{2\pi}} e^{-Q^2/2}$$

Using equation (12A.4) to convert the expression on the right-hand side of equation (12A.2) from an integral over V to an integral over Q, we get

$$E[\max(V - K,\ 0)] = \int_{(\ln K - m)/s}^{\infty} (e^{Qs+m} - K)h(Q)\,dQ$$

or

$$E[\max(V - K,\ 0)] = \int_{(\ln K - m)/s}^{\infty} e^{Qs+m}h(Q)\,dQ - K\int_{(\ln K - m)/s}^{\infty} h(Q)\,dQ \tag{12A.5}$$

Now

$$\begin{aligned} e^{Qs+m}h(Q) &= \frac{1}{\sqrt{2\pi}}e^{(-Q^2+2Qs+2m)/2} \\ &= \frac{1}{\sqrt{2\pi}}e^{[-(Q-s)^2+2m+s^2]/2} \\ &= \frac{e^{m+s^2/2}}{\sqrt{2\pi}}e^{[-(Q-s)^2]/2} \\ &= e^{m+s^2/2}h(Q-s) \end{aligned}$$

This means that equation (12A.5) becomes

$$E[\max(V-K,\,0)] = e^{m+s^2/2}\int_{(\ln K-m)/s}^{\infty} h(Q-s)\,dQ - K\int_{(\ln K-m)/s}^{\infty} h(Q)\,dQ \qquad (12A.6)$$

If we define $N(x)$ as the probability that a variable with a mean of zero and a standard deviation of 1.0 is less than x, the first integral in equation (12A.6) is

$$1 - N[(\ln K - m)/s - s]$$

or

$$N[(-\ln K + m)/s + s]$$

Substituting for m from equation (12A.3) gives

$$N\left(\frac{\ln[E(V)/K] + s^2/2}{s}\right) = N(d_1)$$

Similarly the second integral in equation (12A.6) is $N(d_2)$. Equation (12A.6) therefore becomes

$$E[\max(V-K,\,0)] = e^{m+s^2/2}N(d_1) - KN(d_2)$$

Substituting for m from equation (12A.3) gives the key result.

The Black–Scholes Result

We now consider a call option on a non-dividend-paying stock maturing at time T. The strike price is K, the risk-free rate is r, the current stock price is S_0, and the volatility is σ. As shown in equation (12.22), the call price, c, is given by

$$c = e^{-rT}\hat{E}[\max(S_T - K,\,0)] \qquad (12A.7)$$

where S_T is the stock price at time T and $\hat{E}$ denotes the expectation in a risk-neutral world. Under the stochastic process assumed by Black–Scholes, S_T is lognormal. Also from equations (12.3) and (12.4), $\hat{E}(S_T) = S_0e^{rT}$ and the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$.

From the key result just proved, equation (12A.7) implies that

$$c = e^{-rT}[S_0e^{rT}N(d_1) - KN(d_2)]$$

or

$$c = S_0N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{\ln[\hat{E}(S_T)/K] + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

and

$$d_2 = \frac{\ln[\hat{E}(S_T)/K] - \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$= \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

APPENDIX 12B
Exact Procedure for Calculating Values of American Calls on Dividend-Paying Stocks

The Roll, Geske, and Whaley formula for the value of an American call option on a stock paying a single dividend D_1 at time t_1 is

$$C = (S_0 - D_1 e^{-rt_1})N(b_1) + (S_0 - D_1 e^{-rt_1})M\left(a_1, -b_1; -\sqrt{\frac{t_1}{T}}\right)$$
$$- Ke^{-rT}M\left(a_2, -b_2; -\sqrt{\frac{t_1}{T}}\right) - (K - D_1)e^{-rt_1}N(b_2) \quad (12B.1)$$

where

$$a_1 = \frac{\ln[(S_0 - D_1 e^{-rt_1})/K] + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$a_2 = a_1 - \sigma\sqrt{T}$$
$$b_1 = \frac{\ln[(S_0 - D_1 e^{-rt_1})/S^*] + (r + \sigma^2/2)t_1}{\sigma\sqrt{t_1}}$$
$$b_2 = b_1 - \sigma\sqrt{t_1}$$

The variable σ is the volatility of the stock price less the present value of the dividend. The function $M(a, b; \rho)$, is the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b, when the coefficient of correlation between the variables is ρ. We give a procedure for calculating the M function in Appendix 12C. The variable S^* is the solution to

$$c(S^*) = S^* + D_1 - K$$

where $c(S^*)$ is the Black–Scholes option price given by equation (12.20) when the stock price is S^* and the time to maturity is $T - t_1$. When early exercise is never optimal, $S^* = \infty$. In this case, $b_1 = b_2 = -\infty$ and equation (12B.1) reduces to the Black–Scholes equation with S_0 replaced by $S_0 - D_1 e^{-rt_1}$. In other situations, $S^* < \infty$ and the option should be exercised at time t_1 when $S(t_1) > S^* + D_1$.

When several dividends are anticipated, early exercise is normally optimal only on the final ex-dividend date (see Section 12.13). It follows that the Roll, Geske, and Whaley formula can be used with S_0 reduced by the present value of all dividends except the final one. The variable D_1 should be set equal to the final dividend and t_1 should be set equal to the final ex-dividend date.

APPENDIX 12C

Calculation of Cumulative Probability in Bivariate Normal Distribution

As in Appendix 12B, we define $M(a, b; \rho)$ as the cumulative probability in a standardized bivariate normal distribution that the first variable is less than a and the second variable is less than b, when the coefficient of correlation between the variables is ρ. Drezner provides a way of calculating $M(a, b; \rho)$ to an accuracy of four decimal places.[19] If $a \leqslant 0$, $b \leqslant 0$, and $\rho \leqslant 0$, then

$$M(a, b; \rho) = \frac{\sqrt{1-\rho^2}}{\pi} \sum_{i,j=1}^{4} A_i A_j f(B_i, B_j)$$

where

$$f(x, y) = \exp[a'(2x - a') + b'(2y - b') + 2\rho(x - a')(y - b')]$$

$$a' = \frac{a}{\sqrt{2(1-\rho^2)}}, \qquad b' = \frac{b}{\sqrt{2(1-\rho^2)}}$$

$$A_1 = 0.3253030, \quad A_2 = 0.4211071, \quad A_3 = 0.1334425, \quad A_4 = 0.006374323$$

$$B_1 = 0.1337764, \quad B_2 = 0.6243247, \quad B_3 = 1.3425378, \quad B_4 = 2.2626645$$

In other circumstances where the product of a, b, and ρ is negative or zero, one of the following identities can be used:

$$M(a, b; \rho) = N(a) - M(a, -b; -\rho)$$

$$M(a, b; \rho) = N(b) - M(-a, b; -\rho)$$

$$M(a, b; \rho) = N(a) + N(b) - 1 + M(-a, -b; \rho)$$

In circumstances where the product of a, b, and ρ is positive, the identity

$$M(a, b; \rho) = M(a, 0; \rho_1) + M(b, 0; \rho_2) - \delta$$

can be used in conjunction with the previous results, where

$$\rho_1 = \frac{(\rho a - b)\,\mathrm{sgn}(a)}{\sqrt{a^2 - 2\rho ab + b^2}}, \qquad \rho_2 = \frac{(\rho b - a)\,\mathrm{sgn}(b)}{\sqrt{a^2 - 2\rho ab + b^2}}, \qquad \delta = \frac{1 - \mathrm{sgn}(a)\,\mathrm{sgn}(b)}{4}$$

with

$$\mathrm{sgn}(x) = \begin{cases} +1 & \text{if } x \geqslant 0 \\ -1 & \text{if } x < 0 \end{cases}$$

[19] Z. Drezner, "Computation of the Bivariate Normal Integral," *Mathematics of Computation*, 32 (January 1978), 277–79. Note that the presentation here corrects a typo in Drezner's paper.