BUS-343/444

SCHAUM'S OUTLINE SERIES

THEORY AND PROBLEMS OF

* CALCULUS * WEEK 1

MATHEMATICS for ECONOMISTS

BUS-343/444:
WORK SECTIONS:
3.4, 3.5, 5.1, 5.2
DERIVATIVES
16.1 → 16.5, 17.1 → 17.6, 17.8
INTEGRATION

THIS IS SCANNED

SCHAUM'S OUTLINE SERIES IN ECONOMICS

McGRAW-HILL BOOK COMPANY

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3.4 RULES OF DIFFERENTIATION

Differentiation is the process of determining the derivative of a function, i.e., finding the change in y for a change in x when the change in $x(\Delta x)$ approaches zero. It involves nothing more complicated than applying a few basic formulas or rules to the function. In explaining these rules, it is common to use auxiliary functions such as u and v, where u is an unspecified function of x, $u(x)$, and v is an unspecified function of x, $v(x)$.

(*a*) *The Constant Function Rule.* The derivative of a constant function, $y = k$, where k is any constant, is zero.

Given $y = k$, $\dfrac{dy}{dx} = 0$

Example 6

Given $y = 5$, $\dfrac{dy}{dx} = 0$

Given $y = -10$, $\dfrac{dy}{dx} = 0$

Since y is constant, y will not change for any change in x. Hence $dy = 0$ and no matter what the change in x, $dy/dx = 0$. See Problem 3.1.

(*b*) *The Linear Function Rule.* The derivative of a linear function, $y = a + bx$, is equal to b, the coefficient of x.

Given $y = a + bx$, $\dfrac{dy}{dx} = b$

Example 7

Given $y = 2 + 3x$, $\dfrac{dy}{dx} = 3$

Given $y = 5 - \frac{1}{4}x$, $\dfrac{dy}{dx} = -\dfrac{1}{4}$

Given $y = 12x$, $\dfrac{dy}{dx} = 12$

The derivative dy/dx measures the instantaneous rate of change of the function, the slope. From Chapter 1, we know that the slope of a linear function is b, the coefficient of the independent variable, and that it is constant.

(*c*) *The Power Function Rule.* The derivative of a power function, $y = ax^p$, is equal to the exponent p times the coefficient a, multiplied by the variable x raised to the $(p - 1)$ power.

Given $y = ax^p$, $\dfrac{dy}{dx} = pax^{p-1}$

Example 8

Given $y = 4x^3$, $\dfrac{dy}{dx} = 3(4)x^{3-1} = 12x^2$

Given $y = 5x^2$, $\dfrac{dy}{dx} = 2(5)x^{2-1} = 10x$

Given $y = x^4$, $\dfrac{dy}{dx} = 4(1)x^{4-1} = 4x^3$ (*Note*: $a = 1$.)

(*d*) *The Rule for Sums and Differences.* The derivative of a sum, $y = u(x) + v(x)$, is equal to the sum of the derivatives of the *individual* functions. The derivative of a difference is equal to the difference of the derivatives of the *individual* functions.

Given $y = u(x) \pm v(x)$, $\qquad \dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

Example 9

Given $y = 12x^5 - 4x^4$, $\qquad \dfrac{dy}{dx} = 60x^4 - 16x^3$

Given $y = 9x^2 + 2x - 3$, $\qquad \dfrac{dy}{dx} = 18x + 2$

Note: To find the derivatives for the individual terms, apply whatever rule is appropriate.

(*e*) *The Product Rule.* The derivative of a product, $y = u(x) \cdot v(x)$, is equal to the first function multiplied by the derivative of the second plus the second function multiplied by the derivative of the first.

Given $y = u(x) \cdot v(x)$, $\qquad \dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

(Remember to add exponents in multiplication. For example, $3x^3 \cdot 4x^2 = 12x^5$, not $12x^6$. A review of exponents is found in Chapter 7.)

Example 10. Given $y = 3x^4(2x - 5)$, let $u = 3x^4$ and $v = (2x - 5)$. Then $du/dx = 12x^3$ and $dv/dx = 2$. Substituting these values in the product rule formula,

$$\frac{dy}{dx} = 3x^4(2) + (2x - 5)(12x^3)$$

Simplifying the equation algebraically,

$$\frac{dy}{dx} = 6x^4 + 24x^4 - 60x^3 = 30x^4 - 60x^3$$

(*f*) *The Quotient Rule.* The derivative of a quotient, $y = u/v$, is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the denominator squared.

Given $y = \dfrac{u(x)}{v(x)}$, $\qquad \dfrac{dy}{dx} = \dfrac{v(du/dx) - u(dv/dx)}{v^2}$

The order in the numerator of the formula for the derivative is important and cannot be reversed. (Remember, too, to subtract exponents in division. For example, $6x^6/3x^2 = 2x^4$, not $2x^3$.)

Example 11. Given

$$y = \frac{5x^3}{4x + 3}$$

where $u = 5x^3$ and $v = 4x + 3$, $du/dx = 15x^2$ and $dv/dx = 4$. Substituting these values in the quotient rule formula,

$$\frac{dy}{dx} = \frac{(4x + 3)(15x^2) - 5x^3(4)}{(4x + 3)^2}$$

Simplifying algebraically,

$$\frac{dy}{dx} = \frac{60x^3 + 45x^2 - 20x^3}{(4x + 3)^2} = \frac{40x^3 + 45x^2}{(4x + 3)^2}$$

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(g) *The Rule for a Function of a Function.* The derivative (dy/dx) of a function of a function, $y = f(u)$, where $u = g(x)$, is equal to the derivative of the first function with respect to u times the derivative of the second function with respect to x. This is called the *chain rule.*

Given $y = f(u)$, $u = g(x)$,
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 12. If $y = u^4$ and $u = 2x^2 + 3$, then $dy/du = 4u^3$, and $du/dx = 4x$. Substituting in the chain rule formula,
$$\frac{dy}{dx} = 4u^3(4x) = 16xu^3$$

But $u = 2x^2 + 3$, which when substituted above, gives
$$\frac{dy}{dx} = 16x(2x^2 + 3)^3$$

Similarly, if $y = (4x + 2)^3$, we can let $u = 4x + 2$, then $y = u^3$, $dy/du = 3u^2$, and $du/dx = 4$. Then using the chain rule,
$$\frac{dy}{dx} = 3u^2(4) = 12u^2$$

and substituting $u = 4x + 2$,
$$\frac{dy}{dx} = 12(4x + 2)^2$$

For more complicated functions, different combinations of the seven basic rules outlined above must be used. See Problems 3.17 to 3.24.

3.5 HIGHER-ORDER DERIVATIVES

The second-order derivative (d^2y/dx^2) measures the rate of change of the first derivative, just as the first derivative (dy/dx) measures the rate of change of the original or *primitive function.* The third-order derivative (d^3y/dx^3) measures the rate of change of the second derivative, etc. Higher-order derivatives are found simply by applying the rules of differentiation to the derivative of the previous order.

Example 13. Besides d^2y/dx^2, common notation for the second derivative includes $f''(x)$, y'', D^2y, f_{xx}; for the third-order derivative, d^3y/dx^3, D^3y, f'''; for the fourth-order derivative, d^4y/dx^4, D^4y, $f^{(4)}$; etc.

Higher-order derivatives are found by applying the rules of differentiation to lower-order derivatives. Thus, if $y = 2x^4 + 5x^3 + 3x^2$,

$$\frac{dy}{dx} = 8x^3 + 15x^2 + 6x \qquad \frac{d^3y}{dx^3} = 48x + 30 \qquad \frac{d^5y}{dx^5} = 0$$
$$\frac{d^2y}{dx^2} = 24x^2 + 30x + 6 \qquad \frac{d^4y}{dx^4} = 48$$

Solved Problems

SLOPES AND DERIVATIVES

3.1. Explain, with the aid of a graph, why the derivative of a constant is equal to zero. Assume $y = 6$. Explain why $dy/dx = 0$.

The function $y = 6$ as graphed in Fig. 3-4 is a horizontal line, indicating that y equals 6 no matter what value x assumes. Hence a small change in x, or any change in x, produces no change in y. If $dy = 0$ for any change in x, then dy/dx (the derivative) must also equal zero.

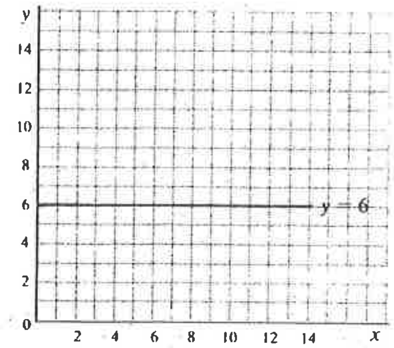

Fig. 3-4

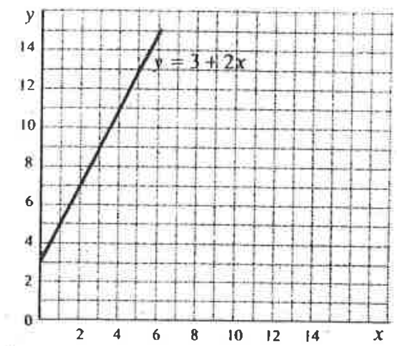

Fig. 3-5

3.2. Using a graph, explain why the derivative of a linear function is constant and equal to the coefficient of x. Assume $y = 3 + 2x$.

Figure 3-5 shows graphically that for every one unit change in x, y increases by 2, or $\Delta y/\Delta x = 2$. Because this graph is a straight line, any change in x, no matter how small, will produce a change in y that is two times larger than itself. Thus, the derivative, dy/dx, will in this case equal 2 at any point along the line, and is therefore constant.

Since the derivative of a function gives the slope of a line, and since the slope of a linear function is given by the coefficient of the independent variable, the derivative of a linear function must always equal the coefficient of the independent variable. See Section 3.4, item (b).

3.3. For each of the curvilinear functions in Fig. 3-6, show at which of the given points the absolute value of the slope of the curve is greatest.

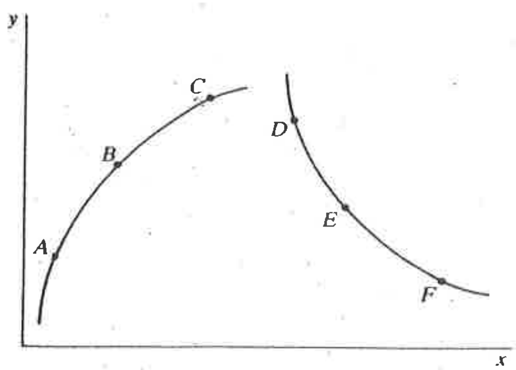

Fig. 3-6

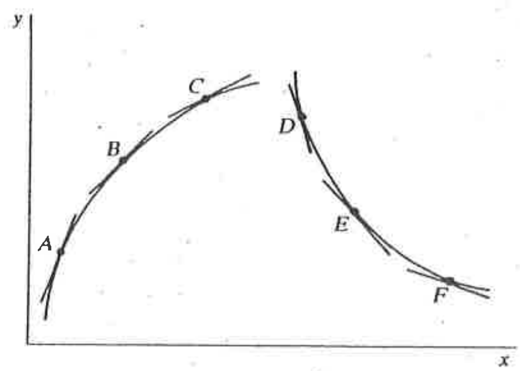

Fig. 3-7

The slope of a curvilinear function at a given point is given by the slope of a line drawn tangent to the curve at that point. By drawing tangents to the curves at the different points (see Fig. 3-7), it is clear that the absolute value of the slope $|\Delta y/\Delta x|$ is greater at A than B, and greater at B than C. Using the symbol $>$ (meaning greater than),

$$\left|\frac{\Delta y}{\Delta x}\right|_A > \left|\frac{\Delta y}{\Delta x}\right|_B > \left|\frac{\Delta y}{\Delta x}\right|_C$$

Similarly, the absolute value of the slope at D is greater than at E, and greater at E than F. Or,

$$\left|\frac{\Delta y}{\Delta x}\right|_D > \left|\frac{\Delta y}{\Delta x}\right|_E > \left|\frac{\Delta y}{\Delta x}\right|_F$$

In graphs of similar dimensions, the relative size of the slope in terms of absolute value is always indicated by the relative steepness of the tangent.

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Chapter 5

Calculus of Multivariable Functions

5.1 PARTIAL DERIVATIVES

Study of the derivative in Chapter 4 was limited to functions of a single independent variable such as $y = f(x)$. Many economic activities, however, involve functions of more than one variable, such as $z = f(x,y)$. To measure the effect of a change in a single independent variable on the dependent variable in a multivariable function, the partial derivative is needed. The *partial derivative* measures the instantaneous rate of change of the dependent variable (z) with respect to one of the independent variables (x), when the other independent variable or variables (y) are assumed held constant. Partial derivatives are particularly important in *comparative statics*. Comparative static analysis examines the effects of changes in autonomous variables on equilibrium positions by assuming that other variables do not change.

The partial derivative of z with respect to x for the multivariable function $z = f(x,y)$, can be written $\partial z/\partial x$, z_x, f_x, or f_1. Partial differentiation follows the same rules as for ordinary differentiation, but treats all the other independent variables as constants.

Example 1. The partial derivatives for $z = 5x^3 + 3xy + 4y^2$ are found as follows:

1. When differentiating with respect to x, only those terms containing x are differentiated. Ignore $4y^2$, as if an additive constant, and treat the y in $3xy$ as a multiplicative constant. Thus,

$$\frac{\partial z}{\partial x} = z_x = 15x^2 + 3y$$

2. When differentiating with respect to y, only those terms containing y are differentiated. Ignore $5x^3$, as if an additive constant, and treat the x in $3xy$ as a multiplicative constant. Thus,

$$\frac{\partial z}{\partial y} = z_y = 3x + 8y$$

Example 2. To find the partial derivatives for $z = 3x^2y^3$:

1. When differentiating with respect to x, treat y^3 as a multiplicative constant, as follows:

$$\frac{\partial z}{\partial x} = z_x = 6xy^3$$

2. When differentiating with respect to y, treat x^2 as a multiplicative constant:

$$\frac{\partial z}{\partial y} = z_y = 9x^2y^2$$

Example 3. Partial differentiation follows all the rules of differentiation. See Section 3.4.

1. Given: $z = (3x + 5)(2x + 6y)$, by the product rule,

$$\frac{\partial z}{\partial x} = z_x = (3x + 5)(2) + (2x + 6y)(3) = 12x + 10 + 18y$$

and,

$$\frac{\partial z}{\partial y} = z_y = (3x + 5)(6) + (2x + 6y)(0) = 18x + 30$$

2. Given: $z = (6x + 7y)/(5x + 3y)$, by the quotient rule,

$$\frac{\partial z}{\partial x} = z_x = \frac{(5x+3y)(6)-(6x+7y)(5)}{(5x+3y)^2} = \frac{30x+18y-30x-35y}{(5x+3y)^2} = \frac{-17y}{(5x+3y)^2}$$

and,

$$\frac{\partial z}{\partial y} = z_y = \frac{(5x+3y)(7)-(6x+7y)(3)}{(5x+3y)^2} = \frac{35x+21y-18x-21y}{(5x+3y)^2} = \frac{17x}{(5x+3y)^2}$$

5.2 SECOND-ORDER PARTIAL DERIVATIVES

The second-order partial derivative f_{xx} indicates that the function has been differentiated partially with respect to x twice and that the other independent variable(s) have been held constant. Thus, f_{xx} gives the rate of change of the first-order partial derivative f_x with respect to x, while y remains constant. Other notations for the second-order partial derivative include z_{xx}, $\partial^2 z/\partial x^2$, and $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$. The case for f_{yy} is exactly parallel.

The first partial derivative f_x (or f_y) can also be differentiated with respect to y (or x). This gives rise to the *cross* (or *mixed*) *partial derivative*, f_{xy} (or f_{yx}). The cross partial derivative measures the instantaneous rate of change of one of the first-order partial derivatives with respect to the other variable. In short, f_{xy} indicates that the primitive function $z = f(x, y)$ has been partially differentiated with respect to x and that the resulting partial derivative has been partially differentiated with respect to y. Other notations for f_{xy} include z_{xy}, $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$, and $\frac{\partial^2 z}{\partial y \partial x}$. The cross partial derivatives for a given function will always be equal (i.e. $f_{xy} = f_{yx}$), if both cross partials are continuous. This is known as *Young's Theorem*.

Example 4. The first and second partial derivatives (including cross partials) for $z = 7x^3 + 9xy + 2y^5$ are taken as shown below.

$$\frac{\partial z}{\partial x} = z_x = 21x^2 + 9y \qquad \text{and} \qquad \frac{\partial z}{\partial y} = z_y = 9x + 10y^4$$

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = 42x \qquad \frac{\partial^2 z}{\partial y^2} = z_{yy} = 40y^3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(21x^2 + 9y) = z_{xy} = 9 \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(9x + 10y^4) = z_{yx} = 9$$

Example 5. In functions such as $z = 3xy^2$, it is important to note that $3xy^2 \neq (3xy)^2$. The first and second partial derivatives (including cross partials) would be

$$\frac{\partial z}{\partial x} = z_x = 3y^2 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = z_y = 6xy$$

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = 0 \qquad \frac{\partial^2 z}{\partial y^2} = z_{yy} = 6x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(3y^2) = z_{xy} = 6y \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(6xy) = z_{yx} = 6y$$

5.3 DIFFERENTIALS

In Section 3.2, the derivative dy/dx was given as a single symbol denoting the limit of $\Delta y/\Delta x$ as $\Delta x \to 0$ as a limit. dy/dx may also be treated as a ratio of differentials in which dy is the differential of y and dx the differential of x. The *differential* of y measures the change in y resulting from a small change in x. Thus, if $y = 2x^2 + 5x + 4$, the derivative is

$$\frac{dy}{dx} = 4x + 5$$

and multiplying both sides by dx, the differential is

$$dy = (4x + 5)\,dx$$

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Chapter 16

Integral Calculus: The Indefinite Integral

16.1 INTEGRATION

Chapters 3 to 6 were devoted to differential calculus, which measures the rate of change of functions. Frequently in economics it is necessary to reverse the process of differentiation and find the function $F(x)$ whose rate of change [i.e. derivative $f(x)$] has been given. This is called *integration*. The function $F(x)$ is termed an *integral* or *antiderivative* of the function $f(x)$.

Example 1. The integral of a function $f(x)$ is expressed mathematically as

$$\int f(x)\,dx = F(x) + c$$

Here the left-hand side of the equation is read "the integral of f of x with respect to x." The symbol $\int$ is an *integral sign*, $f(x)$ is the *integrand*, c is the *constant of integration*, and $F(x)+c$ is an *indefinite integral*, so-called because, as a function of x, which is here unspecified, it can assume many values.

16.2 RULES OF INTEGRATION

The following rules of integration are obtained by reversing the corresponding rules of differentiation. Their accuracy is easily checked, since the derivative of the integral must equal the integrand. Each rule is illustrated in Example 2. The important role of the constant of integration c is demonstrated in Example 3.

Rule 1. The integral of a constant k is

$$\int k\,dx = kx + c$$

Rule 2. The integral of 1, written simply as dx, not $1\,dx$, is

$$\int dx = x + c$$

Rule 3. The integral of a power function x^n, where $n \neq -1$, is given by the *power rule*:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad n \neq -1$$

Rule 4. The integral of x^{-1} (or $1/x$) is

$$\int x^{-1}\,dx = \ln x + c \qquad x > 0$$

The condition $x > 0$ is added because only positive numbers have logarithms. For negative numbers,

$$\int x^{-1}\,dx = \ln|x| + c \qquad x \neq 0$$

Rule 5. The integral of an exponential function is

$$\int a^{kx}\,dx = \frac{a^{kx}}{k \ln a} + c$$

Rule 6. The integral of a natural exponential function is

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c \qquad \text{since} \quad \ln e = 1$$

Rule 7. The integral of a constant times a function equals the constant times the integral of the function.

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Rule 8. The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Rule 9. The integral of the negative of a function equals the negative of the integral of that function.

$$\int -f(x)\,dx = -\int f(x)\,dx$$

Example 2. The rules of integration are illustrated below. Check each answer on your own by making sure that the derivative of the integral equals the integrand.

(*i*) $\int 3\,dx = 3x + c$ (Rule 1)

(*ii*) $\int x^2\,dx = \frac{1}{2+1}x^{2+1} + c = \frac{1}{3}x^3 + c$ (Rule 3)

(*iii*) $\int 5x^4\,dx = 5\int x^4\,dx$ (Rule 7)

$= 5\left(\frac{1}{5}x^5 + c_1\right)$ (Rule 3)

$= x^5 + c$

where c_1 and c are arbitrary constants and $5c_1 = c$. Since c is an arbitrary constant, it can be ignored in the preliminary calculation and included only in the final solution.

(*iv*) $\int (3x^3 - x + 1)\,dx = 3\int x^3\,dx - \int x\,dx + \int dx$ (Rules 7, 8, and 9)

$= 3\left(\frac{1}{4}x^4\right) - \frac{1}{2}x^2 + x + c$ (Rules 2 and 3)

$= \frac{3}{4}x^4 - \frac{1}{2}x^2 + x + c$

(*v*) $\int 3x^{-1}\,dx = 3\int x^{-1}\,dx$ (Rule 7)

$= 3\ln x + c$ (Rule 4)

(*vi*) $\int 2^{3x}\,dx = \frac{2^{3x}}{3\ln 2} + c$ (Rule 5)

(*vii*) $\int 9e^{-3x}\,dx = \frac{9e^{-3x}}{-3} + c$ (Rule 6)

$= -3e^{-3x} + c$

Example 3. Functions which differ only by a constant have the same derivative. The function $F(x) = 2x + k$ has the same derivative, $f(x) = 2$, for any infinite number of possible values for k. If the process is reversed, it is clear that $\int 2\,dx$ must be the antiderivative or indefinite integral for an infinite number of functions differing from each other only by a constant. The constant of integration c thus serves to represent the value of any constant which was part of the primitive function but precluded from the derivative by the rules of differentiation.

The graph of an indefinite integral $\int f(x)\,dx = F(x) + c$, where c is unspecified, is a family of curves parallel in the sense that the slope of the tangent to any of them at x is $f(x)$. Specifying c specifies the curve; changing c shifts the curve. This is illustrated in Fig. 16-1 for the indefinite integral $\int 2\,dx = 2x + c$ where $c = -7$, -3, 1, and 5, respectively. If $c = 0$, the curve begins at the origin.

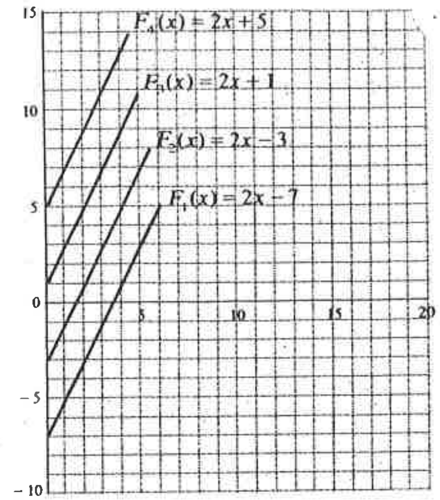

Fig. 16-1

16.3 INITIAL CONDITIONS AND BOUNDARY CONDITIONS

In many problems an *initial condition* ($y = y_0$ when $x = 0$) or a *boundary condition* ($y = y_0$ when $x = x_0$) is given which uniquely determines the constant of integration. By permitting a unique determination of c, the initial or boundary condition singles out a specific curve from the family of curves illustrated in Example 3.

Example 4. Given the boundary condition $y = 11$ when $x = 3$, the integral $y = \int 2\,dx$ is evaluated as follows:

$$y = \int 2\,dx = 2x + c$$

Substituting $y = 11$ when $x = 3$,

$$11 = 2(3) + c \qquad c = 5$$

Therefore, $y = 2x + 5$. Note that even though c is specified, $\int 2\,dx$ remains an *indefinite* integral because x is unspecified. Thus, the integral $2x + 5$ can assume an infinite number of possible values.

16.4 INTEGRATION BY SUBSTITUTION

Integration of a product or quotient of two differentiable functions of x, such as

$$\int 12x^2(x^3 + 2)\,dx$$

cannot be done directly using the simple rules above. However, if the integrand can be expressed as a *constant* multiple of another function u and its derivative du/dx, integration by substitution is possible. By expressing the integrand $f(x)$ as a function of u and its derivative du/dx, and integrating with respect to x,

$$\int f(x)\,dx = \int \left(u\,\frac{du}{dx}\right) dx$$

Canceling dx's,

$$\int f(x)\,dx = \int u\,du = F(u) + c$$

The substitution method is the counterpart of the chain rule in differential calculus.

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Example 5. The substitution method is used below to determine the integral $\int 12x^2(x^3+2)\,dx$. Check the answer on your own, noticing how the chain rule is used.

1. Be sure that the integrand can be expressed as a constant multiple of u and du/dx. For u, pick the function in which the independent variable is raised to the higher power. Letting $u = x^3+2$, therefore, $du/dx = 3x^2$. Solving for dx, $dx = du/3x^2$. Then, substituting $u = x^3+2$ and $dx = du/3x^2$ in the original integrand,

$$\int 12x^2(x^3+2)\,dx = \int 12x^2 u \frac{du}{3x^2} = \int 4u\,du = 4\int u\,du \qquad \text{a constant multiple of } u$$

2. Integrate, using Rule 3, and ignoring c in the initial calculation.

$$4\int u\,du = 4\left(\frac{1}{2}u^2\right) = 2u^2 + c$$

3. Substitute $u = x^3+2$. Thus,

$$\int 12x^2(x^3+2)\,dx = 2u^2 + c = 2(x^3+2)^2 + c$$

See also Problems 16.7–16.18.

Example 6. Determine the integral $\int 4x(x+1)^3\,dx$.

Let $u = x+1$. Then $du/dx = 1$ and $dx = du/1 = du$. Substitute $u = x+1$ and $dx = du$ in the original integrand.

$$\int 4x(x+1)^3\,dx = \int 4xu^3\,du = 4\int xu^3\,du$$

Since x is a *variable* multiple which cannot be factored out, the original integrand cannot be transformed into a *constant* multiple of $u\,du/dx$. Hence the substitution method is ineffectual. Integration by parts is necessary (see Section 16.5).

16.5 INTEGRATION BY PARTS

If an integrand is a product or quotient of differentiable functions of x and cannot be expressed as a constant multiple of $u\,du/dx$, integration by parts is frequently useful. The method is derived by reversing the process of differentiating a product. If from the product rule in Section 3.4,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

taking the integral of the derivative gives

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

Then solving for the first integral on the right-hand side,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \qquad (16.1)$$

For more complicated functions, *integration tables* are generally used. Integration tables provide formulas for the integrals of as many as 500 different functions, and can be found in mathematical handbooks.

Example 7. Integration by parts is used below to determine $\int 4x(x+1)^3\,dx$.

1. Separate the integrand into two parts amenable to the formula in (*16.1*). Let $f(x) = 4x$ and $g'(x) = (x+1)^3$. If $f(x) = 4x$, then $f'(x) = 4$. And if $g'(x) = (x+1)^3$, $g(x) = \int (x+1)^3\,dx$, which can be integrated using the power rule (Rule 3):

$$g(x) = \tfrac{1}{4}(x+1)^4 + c$$

where c can be omitted until the final stage.

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2. Substitute the values for $f(x)$; $g(x)$, and $f'(x)$, in (*16.1*); and note that $g'(x)$ is not used in the formula.

$$\int 4x(x+1)^3\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$= 4x\left[\frac{1}{4}(x+1)^4\right] - \int \frac{1}{4}(x+1)^4 4\,dx = x(x+1)^4 - \int (x+1)^4\,dx$$

3. Use Rule 3 to determine the remaining integral.

$$\int 4x(x+1)^3\,dx = x(x+1)^4 - \frac{1}{5}(x+1)^5 + c$$

Example 8. The integral $\int 2xe^x\,dx$ is determined as follows:
Let $f(x) = 2x$, $f'(x) = 2$, $g'(x) = e^x$, and $g(x) = \int e^x\,dx = e^x$. Substituting in (*16.1*),

$$\int 2xe^x\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$= 2xe^x - \int e^x 2\,dx = 2xe^x - 2\int e^x\,dx$$

Applying Rule 6,

$$\int 2xe^x\,dx = 2xe^x - 2e^x + c = 2e^x(x-1) + c$$

STOP

16.6 ECONOMIC APPLICATIONS

Net investment I is defined as the rate of change in capital stock formation K over time t. If the process of capital formation is continuous over time, $I(t) = dK(t)/dt = K'(t)$. From the rate of investment, the level of capital stock can be estimated. Capital stock is the integral with respect to time of net investment:

$$K_t = \int I(t)\,dt = K(t) + c = K(t) + K_0$$

where $c =$ the initial capital stock K_0.

Similarly, the integral can be used to estimate total cost from marginal cost. Since marginal cost is the change in total cost from an incremental change in output, $\text{MC} = d\text{TC}/dQ$, and only variable costs change with the level of output,

$$\text{TC} = \int \text{MC}\,dQ = \text{VC} + c = \text{VC} + \text{FC}$$

since $c =$ the fixed or initial cost FC. Economic analysis which traces the time path of variables or attempts to determine whether variables will converge toward equilibrium over time is called *dynamics*. For similar applications, see Problems 16.25–16.35.

Example 9. The rate of net investment is given by $I(t) = 140t^{3/4}$ and the initial stock of capital at $t = 0$ is 150. Determining the function for capital K, the time path $K(t)$,

$$K = \int 140t^{3/4}\,dt = 140\int t^{3/4}\,dt$$

By the power rule,

$$K = 140(\tfrac{4}{7}t^{7/4}) + c = 80t^{7/4} + c$$

But $c = K_0 = 150$. Therefore, $K = 80t^{7/4} + 150$.

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Chapter 17

Integral Calculus: The Definite Integral

17.1 AREA UNDER A CURVE

There is no geometrical formula for the area under an irregularly shaped curve, such as $y = f(x)$ between $x = a$ and $x = b$ in Fig. 17-1(a). If the interval $[a,b]$ is divided into n subintervals $[x_1,x_2]$, $[x_2,x_3]$, etc., and rectangles are constructed such that the height of each is equal to the smallest value of the function in the subinterval, as in Fig. 17-1(b), the sum of the areas of the rectangles $\Sigma_{i=1}^{n} f(x_i)\,\Delta x_i$ will approximate, but underestimate, the actual area under the curve. The smaller the subintervals (the smaller the Δx_i), the more rectangles are created and the closer the combined area of the rectangles $\Sigma_{i=1}^{n} f(x_i)\,\Delta x_i$ approaches the actual area under the curve. If the number of subintervals is increased so that $n \to \infty$, each subinterval becomes infinitesimal ($\Delta x_i = dx_i = dx$) and the area of the curve A can be expressed mathematically as

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

where the integral symbol $\int$ replaces Σ because Σ properly represents the sum of a finite number of terms.

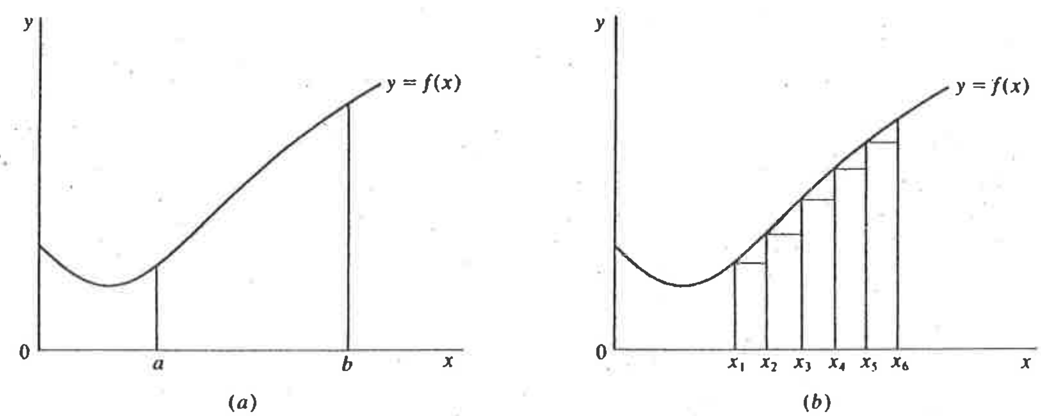

Fig. 17-1

17.2 THE DEFINITE INTEGRAL

The definite integral of a continuous function $f(x)$ over the interval a to b ($a < b$) is expressed mathematically as $\int_a^b f(x)\,dx$ which reads "the integral from a to b of f of x dx." a is termed the *lower limit* of integration, b the *upper limit* of integration. Limit in this context means the value of the variable at the given end of its range. Unlike the indefinite integral, which is a function of a variable with no specified value and hence possesses no definite numerical value, the *definite integral* of $f(x)$ with x ranging from a to b has a numerical value which can be calculated from the indefinite integral by using the fundamental theorem of calculus. See Section 17.3.

17.3 THE FUNDAMENTAL THEOREM OF CALCULUS

The *fundamental theorem of calculus* states that the numerical value of the definite integral of the continuous function $f(x)$ over the interval from a to b is given by the indefinite integral $F(x)+c$ evaluated at the upper limit of integration b, minus the same indefinite integral $F(x)+c$ evaluated at the lower limit of integration a. Since c is common to both, the constant of integration is eliminated in subtraction. Expressed mathematically,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$$

where the symbol $|_a^b$, $]_a^b$, or $[\cdots]_a^b$ indicates that b and a are to be substituted successively for x.

Example 1. The definite integrals given below

$$(1)\quad \int_1^4 10x\,dx \qquad (2)\quad \int_1^3 (4x^3+6x)\,dx$$

are evaluated as follows:

(1) $$\int_1^4 10x\,dx = 5x^2\Big|_1^4 = 5(4)^2-5(1)^2 = 75$$

(2) $$\int_1^3 (4x^3+6x)\,dx = [x^4+3x^2]_1^3 = [(3)^4+3(3)^2]-[(1)^4+3(1)^2] = 108-4 = 104$$

Example 2. The definite integral is used below to determine the area under the curve in Fig. 17-2 over the interval 0 to 20 as follows:

$$A = \int_0^{20} \frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_0^{20}$$

$$= \frac{1}{4}(20)^2 - \frac{1}{4}(0)^2 = 100$$

The answer can be checked by using the geometrical formula $A = \frac{1}{2}xy$:

$$A = \frac{1}{2}xy = \frac{1}{2}(20)(10) = 100 \qquad \text{Compare}$$

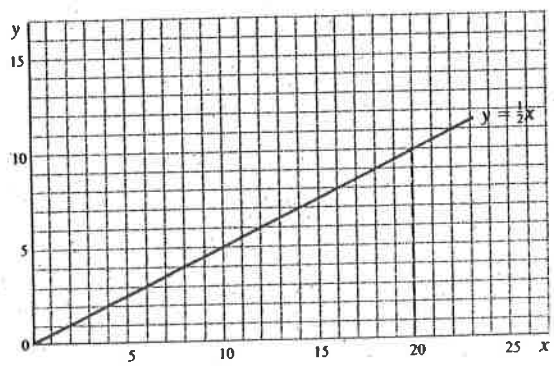

Fig. 17-2

17.4 PROPERTIES OF DEFINITE INTEGRALS

1. Reversing the order of the limits changes the sign of the definite integral.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{17.1}$$

2. If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero.

$$\int_a^a f(x)\,dx = F(a)-F(a) = 0 \tag{17.2}$$

3. The definite integral can be expressed as the sum of component subintegrals.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad a \le b \le c \tag{17.3}$$

Example 3. To illustrate the properties presented above, the following definite integrals are evaluated.

1. $\int_1^3 2x^3\,dx = -\int_3^1 2x^3\,dx$

$$\int_1^3 2x^3\,dx = \frac{1}{2}x^4\Big|_1^3 = \frac{1}{2}(3)^4 - \frac{1}{2}(1)^4 = 40$$

Checking this answer,

$$\int_3^1 2x^3\,dx = \frac{1}{2}x^4\Big|_3^1 = \frac{1}{2}(1)^4 - \frac{1}{2}(3)^4 = -40$$

2. $\int_5^5 (2x+3)\,dx = 0$

Checking this answer,

$$\int_5^5 (2x+3)\,dx = [x^2+3x]_5^5 = [(5)^2+3(5)] - [(5)^2+3(5)] = 0$$

3. $\int_0^4 6x\,dx = \int_0^3 6x\,dx + \int_3^4 6x\,dx$

$$\int_0^4 6x\,dx = 3x^2\Big|_0^4 = 3(4)^2 - 3(0)^2 = 48$$

$$\int_0^3 6x\,dx = 3x^2\Big|_0^3 = 3(3)^2 - 3(0)^2 = 27$$

$$\int_3^4 6x\,dx = 3x^2\Big|_3^4 = 3(4)^2 - 3(3)^2 = 21$$

Checking this answer, $48 = 27 + 21$

17.5 IMPROPER INTEGRALS

A definite integral with infinity for either an upper or lower limit of integration is called an *improper integral*.

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^b f(x)\,dx$$

are improper integrals because ∞ is not a number and cannot be substituted for x in $F(x)$. However, they can be defined as the limits of other integrals, as shown below.

$$\int_a^\infty f(x)\,dx \equiv \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^b f(x)\,dx \equiv \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

If the limit in either case exists, the improper integral is said to *converge*, and the integral has a definite value. If the limit does not exist, the improper integral *diverges* and is meaningless.

Example 4. The improper integrals given below

$$(1)\quad \int_1^\infty 3x^{-2}\,dx \qquad (2)\quad \int_1^\infty \frac{5}{x}\,dx$$

are evaluated as follows:

(1) $$\int_1^\infty 3x^{-2}\,dx = \lim_{b\to\infty}\int_1^b 3x^{-2}\,dx = \lim_{b\to\infty}\left[\frac{-3}{x}\right]_1^b = \lim_{b\to\infty}\left[\frac{-3}{b} - \frac{(-3)}{1}\right] = \lim_{b\to\infty}\left[\frac{-3}{b}+3\right] = 3$$

because $-3/b \to 0$ as $b \to \infty$. Hence the improper integral is convergent and equal to 3. For simplicity, the limit notation is sometimes omitted.

(2) $$\int_1^\infty \frac{5}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{5}{x}\,dx = [5\ln|x|]_1^b = 5\ln|b| - 5\ln|1| = 5\ln|b|$$

since $\ln|1| = 0$. As $b\to\infty$, $5\ln|b|\to\infty$. The improper integral diverges and has no definite value.

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Example 5. Integrals with finite limits of integration can also be improper if within the interval of integration the integrand becomes infinite. Such integrals also call for the use of limits. For example, the integral $\int_0^{27} x^{-2/3}\, dx$ is improper because as x approaches 0 from the right ($x \to 0^+$), $x^{-2/3} \to \infty$. Evaluated,

$$\int_0^{27} x^{-2/3}\, dx = \lim_{a \to 0} \int_a^{27} x^{-2/3}\, dx = 3x^{1/3}\Big]_a^{27} = 3\sqrt[3]{27} - 3\sqrt[3]{a} = 9 - 3\sqrt[3]{a}$$

As $a \to 0^+$, the integral converges to 9. See Problems 17.20–17.21.

* 17.6 PRESENT VALUE OF CASH FLOWS

In Section 8.3, the present value of a sum of money to be received in the future, when interest is compounded continuously, was given by $P = Se^{-rt}$. The present value of a *stream of future income* (money to be received *each year* for n years), therefore, is given by the integral

$$P_n = \int_0^n Se^{-rt}\, dt = S\int_0^n e^{-rt}\, dt = S\left[-\frac{1}{r}e^{-rt}\right]_0^n = -\frac{S}{r}\left[e^{-rt}\right]_0^n$$

$$= -\frac{S}{r}(e^{-rn} - e^{-r(0)}) = -\frac{S}{r}(e^{-rn} - 1)$$

$$= \frac{S}{r}(1 - e^{-rn}) \qquad (17.4)$$

Notice the similarity to the formula in Section 8.4 for the present value of a future stream of income under conditions of annual compounding.

Example 6. The present value of \$1000 to be paid each year for 3 years when the interest rate is 5% compounded continuously is calculated below using (17.4).

$$P_n = \frac{1000}{0.05}[1 - e^{-(0.05)(3)}] = 20{,}000(1 - e^{-0.15}) = 20{,}000(1 - 0.8607) = \$2786$$

17.7 CONSUMERS' AND PRODUCERS' SURPLUS

SKIP 17.7

A demand function $P_1 = f_1(Q)$, as in Fig. 17-3(a), represents the different prices consumers are willing to pay for different quantities of a good. If equilibrium in the market is at (Q_0, P_0), then the consumers who would be willing to pay more than P_0 benefit. Total benefit to consumers is represented by the shaded area and is called *consumers' surplus*. Mathematically,

$$\text{Consumers' surplus} = \int_0^{Q_0} f_1(Q)\, dQ - Q_0P_0 \qquad (17.5)$$

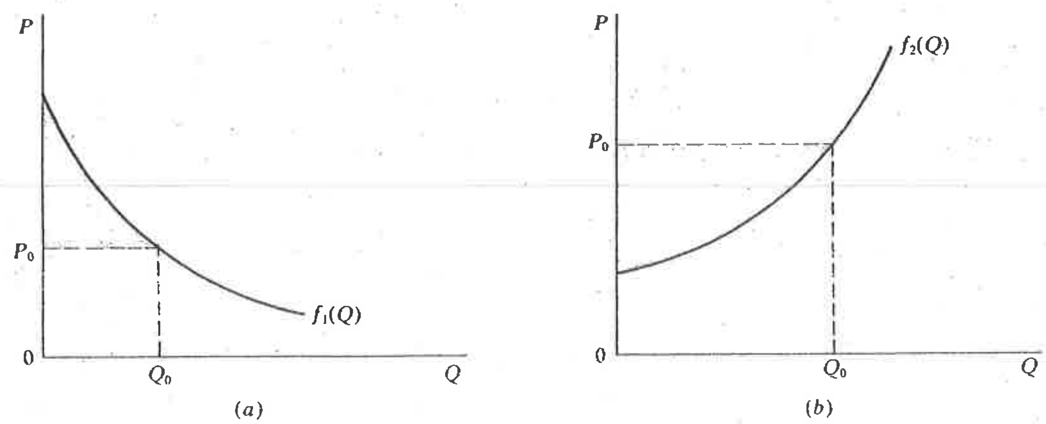

Fig. 17-3

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A supply function $P_2 = f_2(Q)$, as in Fig. 17-3(*b*), represents the prices at which different quantities of a good will be supplied. If market equilibrium occurs at (Q_0, P_0), the producers who would supply at a lower price than P_0 benefit. Total gain to producers is called *producers' surplus* and is designated by the shaded area. Mathematically,

$$\text{Producers' surplus} = Q_0P_0 - \int_0^{Q_0} f_2(Q)\,dQ \tag{17.6}$$

Example 7. Given the demand function $P = 42 - 5Q - Q^2$. Assuming that the equilibrium price is 6, the consumers' surplus is evaluated as follows:

At $P_0 = 6$,

$$42 - 5Q - Q^2 = 6$$
$$36 - 5Q - Q^2 = 0$$
$$(Q+9)(-Q+4) = 0$$

$Q_0 = 4$, because $Q = -9$ is not feasible. Substituting in (17.5),

$$\text{Consumers' surplus} = \int_0^4 (42 - 5Q - Q^2)\,dQ - (4)(6)$$
$$= \left[42Q - 2.5Q^2 - \frac{1}{3}Q^3\right]_0^4 - 24$$
$$= (168 - 40 - 21\tfrac{1}{3}) - (0) - 24 = 82\tfrac{2}{3}$$

* 17.8 THE DEFINITE INTEGRAL AND PROBABILITY

The probability P that an event will occur can be measured by the corresponding area under a probability density function. A *probability density* or *frequency function* is a continuous function $f(x)$ such that:

1. $f(x) \geq 0$. Or, probability cannot be negative.
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Or, the probability of the event occurring over the entire range of x is 1.
3. $P(a < x < b) = \int_a^b f(x)\,dx$. Or, the probability of the value of x falling within the interval $[a,b]$ is the value of the definite integral from a to b.

Example 8. The time in minutes between cars passing on a highway is given by the frequency function $f(t) = 2e^{-2t}$ for $t \geq 0$. The probability of a car passing in 0.25 minutes is calculated as follows:

$$P = \int_0^{0.25} 2e^{-2t}\,dt = -e^{-2t}\Big|_0^{0.25} = (-e^{-0.5}) - (-e^0) = -0.606531 + 1 = 0.393469$$

Solved Problems

DEFINITE INTEGRALS

17.1. Evaluate each of the following definite integrals:

(*a*) $\int_0^6 5x\,dx$

$$\int_0^6 5x\,dx = 2.5x^2\Big|_0^6 = 2.5(6)^2 - 2.5(0)^2 = 90$$

(*b*) $\int_1^{10} 3x^2\,dx$

$$\int_1^{10} 3x^2\,dx = x^3\Big|_1^{10} = (10)^3 - (1)^3 = 999$$