BVS-444 WEEK 4

SOLUTIONS, CH. 11 & 12, BIG HULL 5TH edition

CHAPTER 11

Model of the Behavior of Stock Prices

11.1. Imagine that you have to forecast the future temperature from a) the current temperature, b) the history of the temperature in the last week, and c) a knowledge of seasonal averages and seasonal trends. If temperature followed a Markov process, the history of the temperature in thr last week would be irrelevant.
To answer the second part of the question you might like to consider the following scenario for the first week in May:
(i) Monday to Thursday are warm days; today, Friday, is a very cold day.
(ii) Monday to Friday are all very cold days.
What is your forecast for the weekend? If you are more pessimistic in the case of the second scenario, temperatures do not follow a Markov process.

11.2. The first point to make is that any trading strategy can, just because of good luck, produce above average returns. The key question is whether a trading strategy *consistently* outperforms the market when adjustments are made for risk. It is certainly possible that a trading strategy could do this. However, when enough investors know about the strategy and trade on the basis of the strategy, the profit will disappear. As an illustration of this, consider a phenomenon known as the small firm effect. Portfolios of stocks in small firms appear to have outperformed portfolios of stocks in large firms when appropriate adjustments are made for risk. Papers were published about this in the early 1980s and mutual funds were set up to take advantage of the phenomenon. There is some evidence that this has resulted in the phenomenon disappearing.

11.3. Suppose that the company's initial cash position is x. The probability distribution of the cash position at the end of one year is

$$\phi(x + 4 \times 0.5, \sqrt{4} \times \sqrt{4}) = \phi(x + 2.0, 4)$$

where $\phi(m, s)$ is a normal probability distribution with mean m and standard deviation s.The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x + 2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x. From normal distribution tables

$$N\left(-\frac{x + 2.0}{4}\right) = 0.05$$

when:

$$-\frac{x + 2.0}{4} = -1.6449$$

i.e., when $x = 4.5796$. The initial cash position must therefore be \$4.56 million.

11.4. (a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution

$$\phi(a_1 + \mu_1 T, \sigma_1\sqrt{T})$$

and X_2 has a probability distribution

$$\phi(a_2 + \mu_2 T, \sigma_2\sqrt{T})$$

From the property of sums of independent normally distributed variables, $X_1 + X_2$ has the probability distribution

$$\phi\left(a_1 + \mu_1 T + a_2 + \mu_2 T, \sqrt{\sigma_1^2 T + \sigma_2^2 T}\right)$$

i.e.,

$$\phi\left[a_1 + a_2 + (\mu_1 + \mu_2)T, \sqrt{(\sigma_1^2 + \sigma_2^2)T}\right]$$

This shows that $X_1 + X_2$ follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2$.
(b) In this case the change in the value of $X_1 + X_2$ in a short interval of time δt has the probability distribution:

$$\phi\left[(\mu_1 + \mu_2)\delta t, \sqrt{(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)\delta t}\right]$$

If μ_1, μ_2, σ_1, σ_2 and ρ are all constant, arguments similar to those in Section 11.2 show that the change in a longer period of time T is

$$\phi\left[(\mu_1 + \mu_2)T, \sqrt{(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)T}\right]$$

The variable, $X_1 + X_2$, therefore follows a generalized Wiener process with drift rate $\mu_1 + \mu_2$ and variance rate $\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$.

11.5. The change in S during the first three years has the probability distribution

$$\phi(2 \times 3, 3 \times \sqrt{3}) = \phi(6, 5.20)$$

The change during the next three years has the probability distribution

$$\phi(3 \times 3, 4 \times \sqrt{3}) = \phi(9, 6.93)$$

The change during the six years is the sum of a variable with probability distribution $\phi(6, 5.20)$ and a variable with probability distribution $\phi(9, 6.93)$. The probability distribution of the change is therefore

$$\phi(6+9, \sqrt{5.20^2 + 6.93^2})$$

$$= \phi(15, 8.66)$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$\phi(20, 8.66)$$

11.6. From Ito's lemma

$$\sigma_G G = \frac{\partial G}{\partial S}\sigma_S S$$

Also the drift of G is

$$\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2$$

where μ is the expected return on the stock. When μ increases by $\lambda\sigma_S$, the drift of G increases by

$$\frac{\partial G}{\partial S}\lambda\sigma_S S$$

or

$$\lambda\sigma_G G$$

The growth rate of G, therefore, increases by $\lambda\sigma_G$.

11.7. Define S_A, μ_A and σ_A as the stock price, expected return and volatility for stock A. Define S_B, μ_B and σ_B as the stock price, expected return and volatility for stock B. Define δS_A and δS_B as the change in S_A and S_B in time δt. Since each of the two stocks follows geometric Brownian motion,

$$\delta S_A = \mu_A S_A \delta t + \sigma_A S_A \epsilon_A \sqrt{\delta t}$$

$$\delta S_B = \mu_B S_B \delta t + \sigma_B S_B \epsilon_B \sqrt{\delta t}$$

where ϵ_A and ϵ_B are independent random samples from a normal distribution.

$$\delta S_A + \delta S_B = (\mu_A S_A + \mu_B S_B)\delta t + (\sigma_A S_A \epsilon_A + \sigma_B S_B \epsilon_B)\sqrt{\delta t}$$

This *cannot* be written as

$$\delta S_A + \delta S_B = \mu(S_A + S_B)\delta t + \sigma(S_A + S_B)\epsilon\sqrt{\delta t}$$

for any constants μ and σ. (Neither the drift term nor the stochastic term correspond.) Hence the value of the portfolio does not follow geometric Brownian motion.

11.8. In:

$$\delta S = \mu S \delta t + \sigma S \epsilon \sqrt{\delta t}$$

the expected increase in the stock price and the variability of the stock price are constant when both are expressed as a proportion (or as a percentage) of the stock price
In:

$$\delta S = \mu \delta t + \sigma \epsilon \sqrt{\delta t}$$

the expected increase in the stock price and the variability of the stock price are constant in absolute terms. For example, if the expected growth rate is \$5 per annum when the stock price is \$25, it is also \$5 per annum when it is \$100. If the standard deviation of weekly stock price movements is \$1 when the price is \$25, it is also \$1 when the price is \$100.
In:

$$\delta S = \mu S \delta t + \sigma \epsilon \sqrt{\delta t}$$

the expected increase in the stock price is a constant proportion of the stock price while the variability is constant in absolute terms.
In:

$$\delta S = \mu \delta t + \sigma S \epsilon \sqrt{\delta t}$$

the expected increase in the stock price is constant in absolute terms while the variability of the proportional stock price change is constant.
The model:

$$\delta S = \mu S \delta t + \sigma S \epsilon \sqrt{\delta t}$$

is the most appropriate one since it is most realistic to assume that the expected *percentage return* and the variability of the *percentage return* in a short interval is constant.

11.9. The drift rate is $a(b-r)$. Thus, when the interest rate is above b the drift rate is negative and, when the interest rate is below b, the drift rate is positive. The interest rate is therefore continually pulled towards the level b. The rate at which it is pulled toward this level is a. A volatility equal to c is superimposed upon the "pull" or the drift.
Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10% per annum. This can be regarded as a long run average. The current drift is −4% per annum so that the expected rate at the end of one year is about 16% per annum. (In fact it is slightly greater than this, because as the interest rate decreases, the "pull" decreases.) Superimposed upon the drift is a volatility of 15% per annum.

11.10. If $G(S,t) = S^n$ then $\partial G/\partial t = 0$, $\partial G/\partial S = nS^{n-1}$, and $\partial^2 G/\partial S^2 = n(n-1)S^{n-2}$. Using Ito's lemma:

$$dG = [\mu n G + \frac{1}{2}n(n-1)\sigma^2 G]\,dt + \sigma n G\,dz$$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is

$$\mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is

$$S_0^n e^{[\mu n + \frac{1}{2}n(n-1)\sigma^2]T}$$

11.11. The process followed by B, the bond price, is from Ito's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

Since:

$$B = e^{-x(T-t)}$$

the required partial derivatives are

$$\frac{\partial B}{\partial t} = xe^{-x(T-t)} = xB$$
$$\frac{\partial B}{\partial x} = -(T-t)e^{-x(T-t)} = -(T-t)B$$
$$\frac{\partial^2 B}{\partial x^2} = (T-t)^2 e^{-x(T-t)} = (T-t)^2 B$$

Hence:

$$dB = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right]Bdt - sx(T-t)Bdz$$

CHAPTER 12
The Black–Scholes Model

12.1. The Black–Scholes option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock on the stock during the year is normally distributed.

12.2. The standard deviation of the percentage price change in time δt is $\sigma\sqrt{\delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

12.3. The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

12.4. In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times 0.25} - 50N(-0.2417)$$
$$= 50 \times 0.4634e^{-0.1\times 0.25} - 50 \times 0.4045 = 2.37$$

or \$2.37.

12.5. In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes. Hence the appropriate value of S_0 is

$$S_0 = 50 - 1.50e^{-0.1667\times 0.1} = 48.52$$

As before $K = 50$, $r = 0.1$, $\sigma = 0.3$, and $T = 0.25$. In this case

$$d_1 = \frac{\ln(48.52/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.0414$$
$$d_2 = d_1 - 0.3\sqrt{0.25} = -0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25} - 48.52N(-0.0414)$$

$$= 50 \times 0.5432e^{-0.1\times 0.25} - 48.52 \times 0.4835 = 3.03$$

or \$3.03.

12.6. The implied volatility is the volatility that makes the Black–Scholes price of an option equal to its market price. It is calculated using an iterative procedure.

12.7. In this case $\mu = 0.15$ and $\sigma = 0.25$. From equation (12.7) the probability distribution for the rate of return over a 2-year period with continuous compounding is:

$$\phi\left(0.15 - \frac{0.25^2}{2}, \frac{0.25}{\sqrt{2}}\right)$$

i.e.,

$$\phi(0.11875, 0.1768)$$

The expected value of the return is 11.875% per annum and the standard deviation is 17.68% per annum.

12.8. (a) The required probability is the probability of the stock price being above \$40 in six months' time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \phi(\ln 38 + (0.16 - \frac{0.35^2}{2})0.5,\ 0.35\sqrt{0.5})$$

i.e.,

$$\ln S_T \sim \phi(3.687, 0.247)$$

Since $\ln 40 = 3.689$, the required probability is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

From normal distribution tables N(0.008) = 0.5032 so that the required probability is 0.4968. In general the required probability is $N(d_2)$. (See Problem 12.22).
(b) In this case the required probability is the probability of the stock price being less than \$40 in six months' time. It is

$$1 - 0.4968 = 0.5032$$

12.9. From equation 12.2:

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0 + (\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$e^{\ln S_0 + (\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

i.e.

$$S_0 e^{(\mu - \sigma^2/2)T - 1.96\sigma\sqrt{T}}$$

and

$$S_0 e^{(\mu - \sigma^2/2)T + 1.96\sigma\sqrt{T}}$$

12.10. The statement is misleading in that a certain sum of money, say \$1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.
The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

12.11. (a) At time t, the expected value of $\ln S_T$ is, from equation (12.2)

$$\ln S + (\mu - \frac{\sigma^2}{2})(T - t)$$

In a risk-neutral world the expected value of $\ln S_T$ is therefore:

$$\ln S + (r - \frac{\sigma^2}{2})(T - t)$$

Using risk-neutral valuation the value of the security at time t is:

$$e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T - t)\right]$$

(b) If:

$$f = e^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right] - e^{-r(T-t)}(r - \frac{\sigma^2}{2})$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black Scholes differential equation is

$$e^{-r(T-t)}\left[r\ln S + r(r - \frac{\sigma^2}{2})(T-t) - (r - \frac{\sigma^2}{2}) + r - \frac{\sigma^2}{2}\right]$$

$$= re^{-r(T-t)}\left[\ln S + (r - \frac{\sigma^2}{2})(T-t)\right]$$

$$= rf$$

Hence equation (12.15) is satisfied.

12.12. This problem is related to Problem 11.10.
(a)If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

(b)The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$
(c)The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [-r(n-1) - \frac{1}{2}\sigma^2 n(n-1)]t + k$$

where k is a constant. Since $\ln h = 0$ when $t = T$ it follows that

$$k = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)]T$$

so that

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1) + r(n-1)](T-t)}$$

12.13. In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times 0.25}N(0.3865)$$

$$= 52 \times 0.7042 - 50e^{-0.03} \times 0.6504$$

$$= 5.06$$

or \$5.06.

12.14. In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times 0.5}N(0.0809) - 69N(-0.1666)$$

$$= 70e^{-0.025} \times 0.5323 - 69 \times 0.4338$$

$$= 6.40$$

or \$6.40.

12.15. Using the notation of Section 12.13, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times 0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times 0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2-t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

12.16. In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be calculated using an iterative procedure.
A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.397 or 39.7% per annum.

12.17. (a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

(b)

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$
$$= N'(d_2)\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$d_1 = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t}$$
$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial}{\partial t}(\sigma\sqrt{T-t})$$
$$= -\frac{\sigma}{2\sqrt{T-t}}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\begin{aligned}\frac{\partial^2 c}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S}\\ &= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}\end{aligned}$$

From the results in d) and e)

$$\begin{aligned}\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} &= -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}\\ &\quad + rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}\\ &= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)]\\ &= rc\end{aligned}$$

This shows that the Black–Scholes formula for a call option does indeed satisfy the Black–Scholes differential equation

12.18. From the Black–Scholes equations

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

Because $1 - N(-d_1) = N(d_1)$ this is

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

Also:

$$c + Ke^{-rT} = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

Because $1 - N(d_2) = N(-d_2)$, this is also

$$Ke^{-rT}N(-d_2) + S_0N(d_1)$$

The Black–Scholes equations are therefore consistent with put–call parity.

12.19. This problem naturally leads on to the material in Chapter 15 on volatility smiles. Using DerivaGem we obtain the following table of implied volatilities:

	Maturity (months)		
Strike Price ($)	3	6	12
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

The option prices are not exactly consistent with Black–Scholes. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock.

12.20. Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.
It appears that Black's approach understates the true option value. This is because the holder of the option has more alternative strategies for deciding when to exercise the option than the two alternatives implicitly assumed by the approach. These alternatives add value to the option. In fact Black's approach sometimes gives a higher value than the approach in Appendix 12B. This is because Appendix 12B applies the volatility to the stock price less the present value of the dividend whereas Black's approach when considering exercise just prior to the dividend date applies the volatility to the stock price itself. Thus part of the Black calculation assumes more stock price variability than Roll–Geske–Whaley. This issue is also discussed in Example 12.9.

12.21. With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.25, \quad t_2 = 0.3333, \quad T = 0.8333, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times 0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1 - e^{-r(t_2-t_1)}\right] = 55(1 - e^{-0.08\times 0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1 - e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 12.13 that the option should never be exercised early.
The present value of the dividends is

$$1.5e^{-0.3333\times 0.08} + 1.5e^{-0.8333\times 0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times 1.25} \times 0.3692 = 4.17$$

or \$4.17.

12.22. The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \phi[\ln S_0 + (r - \sigma^2/2)T, \sigma\sqrt{T}]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$\begin{aligned} 1 - N&\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right] \\ &= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right] \\ &= N(d_2) \end{aligned}$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time t is

$$100e^{-rT}N(d_2)$$

12.23. If $f = S^{-2r/\sigma^2}$ then

$$\frac{\partial f}{\partial S} = -\frac{2r}{\sigma^2}S^{-2r/\sigma^2-1}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right)S^{-2r/\sigma^2-2}$$

$$\frac{\partial f}{\partial t} = 0$$

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rS^{-2r/\sigma^2} = rf$$

This shows that the Black–Scholes equation is satisfied. S^{-2r/σ^2} could therefore be the price of a traded security.