THEREFORE WE CAN DESCRIBE WHAT WE JUST DID AS:

$$\text{TOTAL DISTANCE} = \sum_{i=1}^{4} (\text{MEAN VELOCITY})_i \underbrace{(0.5 \text{ SECONDS})}_{\text{DELTA } t}$$

Δt

THE ANALOGY IS: $\rightarrow dt$

$$\Delta X = \text{TOTAL DISTANCE} = \int \underbrace{V(t)}_{} \cdot dt$$

ALLOW VELOCITY TO BE A FUNCTION OF TIME.

CAN ALSO BE THOUGHT OF AS

$$\int \frac{dx}{dt} \cdot dt = \int dx$$

$$\text{TOTAL DISTANCE} = \sum_i (\Delta X)$$

INTUITIVE!

= SUM OF ALL SMALLER DISTANCES

IN PHYSICS THERE IS A CALCULUS RELATION AMONGST POSITION, VELOCITY, AND ACCELERATION.

THE FOLLOWING IS TRUE:

$$V = \frac{dx}{dt}$$

$$A = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

(*) VELOCITY IS THE FIRST DERIVATIVE OF POSITION

(*) ACCEL. IS THE FIRST DERIV OF VELOCITY
" " " 2ND " " POSITION

THESE RELATIONS INVOLVING DERIVATIVES, ALSO IMPLY CALCULUS RELATIONS INVOLVING ANTI-DERIVATIVES (ie INTEGRALS) THUS,

LOOK @ $V = \dfrac{dx}{dt}$ (WE'LL ALLOW V TO BE A FUNCTION OF t, ie. $V(t)$.)

$$\Rightarrow \quad V(t) \cdot dt = dx$$

$$\Rightarrow \quad \int v(t) \cdot dt = \int dx \cong \sum \Delta x = \text{TOTAL DISTANCE (NET CHANGE IN X)}$$

"TOTAL DISTANCE TRAVELED IS THE INTEGRAL OF VELOCITY"

ALSO FROM $A = \dfrac{dV}{dt}$

$$\int_{t_1}^{t_2} A(t) \cdot dt = \int dV = \sum(\Delta V) = \text{NET CHANGE IN VELOCITY FROM } t_1 \text{ TO } t_2$$

THE DERIVATIVE $\xleftrightarrow{\text{VERSUS}}$ ANTI-DERIVATIVE (ie INTEGRALS)

CONCEPTS OF CALCULUS ARE EASILY DEMONSTRATED FOR PHYSICS, IN THE FORM OF A TABLE...

TABLE 1

DESCRIBING POSITION (3-D) AND TIME IN A CALCULUS MANNER.

	WRITE AS A DERIVATIVE ←	"BASE CASE"	→ WRITE AS AN INTEGRAL ($\int x\,dt$)
X	$X=\frac{d}{dt}\int x\,dt$	$X=X$	$X=\int v\,dt$ $=\int dx$
V	$V=\frac{dx}{dt}$	$V=V$	$V=\int A\,dt$ $=\int dv$
A	$A=\frac{dv}{dt}$ $=\frac{d^2x}{dt^2}$	$A=A$	$A=\int J\cdot dt$ $=\int dA$
JERK $=J$ ↓	$J=\frac{dA}{dt}$ $=\frac{d^3x}{dt^3}$	ETC...	
SNAP CRACKLE POP ↓	$\xi=\frac{d}{dt}\int \xi\,dt$	$\xi=\xi$	$\xi=\int d\xi$

SO FOCUS ON X FOR A MOMENT: 2 MAIN RELATIONS ARE IMPORTANT

$$V=\frac{dx}{dt} \qquad \text{THUS} \qquad \Delta X=\int V\cdot dt$$

↑ VELOCITY – ↑ DISTANCE TRAVELED

RELATION BETWEEN 2 & 3-DIM'S w/ CALCULUS:

CONSIDER A CIRCLE (A 2-DIM "BALL")

WE KNOW... AREA $= \pi R^2$

CIRCUM $= \pi \cdot$ DIAM

$= 2 \cdot \pi \cdot R$

NOTICE:

$$\frac{d}{dR} \text{AREA} = \text{CIRCUM}$$

$$\frac{d}{dR} \pi R^2 = 2\pi R$$

FOR A "3-D CIRCLE" (ie A SPHERE, A "BALL" IF YOU WILL) IT IS TRUE THAT:

FACT VOLUME $= \frac{4}{3}\pi R^3$ R = RADIUS

SURFACE AREA $= 4\pi R^2$

THUS $\frac{d}{dR}$ VOLUME = SURFACE AREA

3-DIM

COMPARE TO:

2-DIM

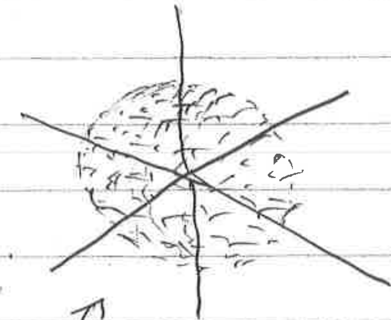
$$\frac{d}{dR} \text{AREA} = \text{CIRCUM.}$$

SINCE

$$\frac{d\,\text{VOLUME}}{dR} = \text{SURFACE AREA}$$

THIS IMPLIES:

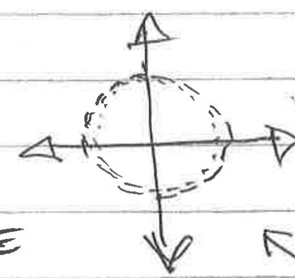
(3-D) VOLUME $= \int$ SURF AREA



"A BALL IN 3-D"

(2-D) AREA $= \int$ CIRCUM.

(2-D) => A PLOT OF THE X-Y POINTS ON THE CIRCUMFERENCE



A BALL IN 2-D

WHAT MIGHT WE CARE ABOUT IN FINANCE? (NOT PHYSICS)

THINGS LIKE
- EXPECTED RATES OF RETURN
- EXPECTED VARIANCES OF RETURNS (AND VAR-COV MATRICIES)
- BASIC PRESENT VALUE
- ESTIMATION OF REGRESSION "BETAS"
- SIMPLYFYING DIFFICULT EQUATIONS (ie TAYLOR SERIES)
- DIAGNOSTICS & PRICING OF DERIVATIVE FINANCIAL SECURITIES (eg OPTIONS/SWAPS/ FUTURES)

ALL THESE THINGS BECOME MORE POWERFUL ONCE YOU CAN WRITE THEM IN A CALCULUS MANNER

Δ

IMPORTANT CONCEPT!

EXPECTED RETURNS:

THIS RELATES TO CALCULUS VIA

$\boxed{\text{FACT:}}$ $E[f(x)] = \int f(x) \cdot \underbrace{pdf(x)}_{\text{PROBABILITY DISTRIBUTION FUNCTION FOR } x} \cdot dx$

$\boxed{\text{NOTE: IF } f(x)=x \text{ THEN } E[x] = \int x \cdot pdf(x) \cdot dx.}$

EXPECTED "RETURNS" (w/ $x=r$).

WE'LL OFTEN ASSUME THE pdf FOR RETURNS IS NORMAL. THE NORMAL pdf HAS TERMS THAT LOOK LIKE

$$e^{-\frac{1}{2}Z^2} \longrightarrow \text{w/ } Z = \frac{x-\bar{x}}{\sigma}$$

THE "Z-STAT" FROM ~~HYPOTHESIS~~ HYPOTHESIS TESTING ETC.

THUS WE NEED TO EVALUATE INTEGRALS THAT LOOK SOMETHING LIKE

$$\int r \cdot e^{-\frac{1}{2}r^2} \, dr$$

[WHICH BY THE WAY IS NOT OF THE FORM $\int r \cdot e^{A \cdot r} \, dr$
(SO INT-BY-PARTS IS NOT OF USE HERE)
BUT INT-BY-SUB IS OF HELP.

WE WRITE:

$$E[r] = \int r \cdot pdf(r) \cdot dr$$

WE ALSO CARE ABOUT VARIANCE (TOTAL RISK). VARIANCE CAN BE CALCULATED A FEW DIFFERENT WAYS.

MOST COMMON: $$VAR(x) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}$$

BUT VARIANCE CAN ALSO BE WRITTEN IN TERMS OF "EXPECTED VALUES" (OF X AND X^2).

MATH FACT: $$VAR(x) = E[x^2] - (E[x])^2$$

IF THE RANDOM VARIABLE IS ALREADY DE-MEANED (ie HAS MEAN = ZERO) THEN THE TERM $(E[x])^2$ IS ZERO AND THE VAR(X) IS SIMPLY $E[x^2]$. THIS MEANS WE CARE ABOUT INTERGALS SUCH AS

$$E[x^2] = \int x^2 \cdot pdf(x) \cdot dx$$

VALUATION & CALCULUS:

$$PV = \sum_i PV(\text{INDIVIDUAL CF'S})$$

$$= \sum_i PV\big(R(t)\big)$$

w/ $R(t)$ A FUNCTION THAT DESCRIBES HOW MONEY ENTERS THE ACCOUNT.

IT MAY BE DISCRETELY (ie IN "CHUNKS", OR CONTINUOUSLY LIKE A FLOW OF LIQUID)

NOTE: THE TOTAL FLOW OF MONEY (IGNORING DISCOUNTING) IS $= \sum R(t)$

- **DISCRETE EXAMPLE:**

$$R(t) = \$10 \text{ @ } t=1,2,3$$
$$= \$0 \text{ OTHERWISE}$$

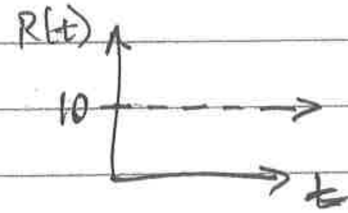
THEN

$$PV = pv(\$10 \text{ @ } t=1) + pv(\$10 \text{ @ } t=2) + pv(\$10 \text{ @ } t=3)$$

- **CONTINUOUS EXAMPLE**

$$R(t) = \$10/\text{min}$$

← FLOWING IN CONTINUOUSLY (AND AT A CONSTANT RATE)



$$PV = \sum pv\big(R(t)\cdot \Delta t\big)$$

$$= \int e^{-r\cdot t}\cdot R(t)\cdot dt$$

$\boxed{\text{NOTE:}}$ IN BOTH EXAMPLES, \$30 IN TOTAL FLOWS INTO THE ACCOUNT, BUT THE CONT. COMPOUNDED (IE CONTINUOUS FLOW) CASE WILL BE MORE VALUABLE SINCE IT PROVIDES THE \$30 FASTER.

THUS, PV INVOLVES INTEGRALS, AND INTEGRATION IS QUITE HELPFUL IF WE HAVE THE RATE OF FLOW (OF MONEY) <u>VARYING</u> IN SOME DYNAMIC WAY. — IN WHICH CASE WE CAN WRITE $R(t)$ IN A "DYNAMIC MANNER"

eg.

$$R(t) \equiv \begin{cases} = \$\tfrac{3}{2}\pi \cdot t^2 / \text{YR} & 0 \le t < 5 \\ = \$0 / \text{YR} & \text{— IF —}\ 5 \le t < 8 \\ = \$10 / \text{YR} & \text{— IF —}\ 8 \le t \end{cases}$$

UNITS: DOLLARS PER YEAR

Q: IGNORING PV ISSUES, HOW MUCH TOTAL MONEY FLOWS INTO THE ACCT IN THE FIRST 4.5 YEARS?

$$\text{QUANTITY} = \int \text{RATE} \cdot dt$$

$$\$_{\text{TOTAL AMOUNT}} = \int_0^{4.5} \tfrac{3}{2}\pi \cdot t^2 \cdot dt$$

EASY TO EVALUATE

EX #2

FIND PV OF ALL MONEY FLOWING FROM $t = 1.5$ TO $t = 9.5$

$$PV = \underbrace{\int_{1.5}^{5} e^{-r \cdot t} \cdot (3/2\pi) \cdot t^2 \cdot dt}_{= PV_0 \text{ OF THE MONEY FLOW OCCURING FROM } t=1.5 \text{ TO } t=5.0} + 0 + \underbrace{\int_{8}^{9.5} e^{-r \cdot t} \cdot 10 \cdot dt}_{\text{PV OF THE MONEY FLOWING FROM } t=8 \text{ TO } t=9.5}$$

INT-BY-PARTS, TWICE.

EASY TO EVALUATE THIS INTEGRAL

▷

TAYLOR SERIES EXPANSIONS (ie APPROXIMATIONS OF FUNCTIONS) ALSO REQUIRES CALCULUS...

EX A 3RD ORDER TAYLOR SERIES EXPANSION OF A FUNCTION, $f(x)$ AROUND A POINT $X = X_0$ IS WRITTEN AS:

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} \cdot (x - x_0)^1 + \frac{1}{2!} \left.\frac{d^2f}{dx^2}\right|_{x_0} \cdot (x - x_0)^2 +$$

$$+ \frac{1}{3!} \cdot \left.\frac{d^3f}{dx^3}\right|_{x_0} \cdot (x - x_0)^3$$

REQUIRES CALC!

- CALCULUS AND OPTIONS/FUTURES/SWAPS

MANY PARTIAL DERIVATIVES OF B-SCHOLES ARE OF INTEREST

B-SCHOLES ITSELF IS A SOLUTION TO ANOTHER EQUATION WHICH IS WRITTEN IN A CALCULUS (i.e DIFFERENTIAL CALCULUS) MANNER. THE EQ^N IS KNOWN AS A "DIFFERENTIAL EQ^N" (~~IT DESCRIBES HOW MONEY IN A NO-ARBITRAGE WORLD MUST FLOW~~)

FLOW RATES... CALCULUS...

THE SOLUTION TO THE DIFFERENTIAL EQUATION IS BLACK SCHOLES.
(1973 – MIT)

ALSO NOTICE THAT A LOT OF DERIVATIVE SECURITY PRICING IS VERY MATHEMATICALLY INVOLVED...
BUT EVEN THOUGH WE MAY NOT THEORETICALLY PRICE DERIVATES IN OUR DAILY LIFES (RATHER, WE "LOOK-UP" PRICES IN THE WSJ) – WE STILL WANT TO UNDERSTAND HOW OPTION PRICES VARY WHEN INPUT VARIABLES CHANGE

$\Rightarrow$ PARTIAL DERIVATIVES.

EXPECTED VALUES WHEN THERE ARE MORE THAN 1 INPUT (x) TO THE FUNCTION f:

ASSUME $\begin{matrix} X \searrow \\ Y \nearrow \end{matrix} f(X,Y)$ THAT IS, f DEPENDS UPON BOTH X AND Y

FACT:

$$E\left[f(x,y)\right] = \underbrace{\int\int}_{\text{DOUBLE INTEGRAL}} f(x,y) \cdot pdf(x,y)\ \underbrace{dx \cdot dy}$$

DOUBLE INTEGRAL

(NEEDED FOR 2 INPUTS X AND Y

A QUICK OVERVIEW OF DOUBLE, TRIPPLE, (ETC) INTEGRALS. LET'S JUST DO ONE...

$\boxed{EX}$ $$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} \pi x^2 y^3 z^4\ dx\ dy\ dx$$

DO INNER MOST INTEGRAL FIRST

$$\underline{\pi y^3 z^4} \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int x^2\, dx\, dy\, dz$$

TREAT EVERYTHING NOT INVOLVING X AS A CONSTANT.

$$= \pi y^3 z^4 \int_{z_1}^{z_2}\int_{y_1}^{y_2} \tfrac{1}{3}x^3 \; dy\, dz$$

$$= \frac{\pi}{3} x^3 z^4 \int\underbrace{\int y^3 \, dy}_{\text{EVALUATE THIS}} \cdot dz$$

(TREAT X & Z'S AS CONSTANTS)

$$= \frac{\pi}{3} x^3 z^4 \int_{z_1}^{z_2} \tfrac{1}{4} y^4 \; dz$$

$$= \frac{\pi}{12} x^3 y^4 \int z^4 \, dz = \frac{\pi}{12} x^3 y^4 \left(\tfrac{1}{5} z^5\right)$$

TREAT X & Y AS CONSTANTS.

$$= \underbrace{\frac{\pi}{60} \cdot x^3 y^4 z^5}_{\text{THE ANSWER}}$$

THUS, TAKING PARTIAL DERIVATIVES IS SOMEWHAT SIMILAR TO TAKING MULTIPLE INTEGRALS (IN THAT YOU WORK W/ ONE VARIABLE AT A TIME AND TREAT <u>ALL</u> OTHERS AS <u>CONSTANTS</u>)