FINANCIAL ENGINEERING / CAL POLY
LECTURE NOTES: HULL CH ~~11~~ 12
"THE BLACK-SCHOLES ANALYSIS"

USE FOR PHOTOCOPY

WEEK 5

WAS CH 11 IS NOW CH 12

WE'LL SEE THAT THE B/S PRICING EQ^N FOR OPTIONS, (WHICH YOU ARE ALREADY FAMILIAR WITH) IS REALLY THE SOLUTION TO ANOTHER EQUATION. THE 'OTHER EQN' IS A PARTIAL DIFFERENTIAL EQUATION (PDE).

THE ORIGINAL B/S PDE WAS FIRST WRITTEN DOWN IN 1969, BUT IT TOOK A FEW YEARS TO SOLVE IT, AND THIS SOLUTION WAS NOT PUBLISHED UNTIL MAY 1973.

§11.1 THE LOGNORMAL PROPERTY OF STOCK PRICES.

ASSUME $$\frac{dS}{S} = \mu\, dt + \sigma\, dz$$

IN CH 10 WE SAW, w/ $G = \ln S$

$$dG = \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma \cdot dz$$

$$d(\ln S) = \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma \cdot dz$$

SINCE $dz \sim N(0, \sqrt{dt})$

THEN

$$d(\ln S) = \ln \tilde{S}_T - \ln S_t \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t),\ \sigma\sqrt{T-t} \right) \qquad (11.1)$$

YOU SHOULD FULLY UNDERSTAND WHERE THIS COMES FROM.

SINCE $\ln S_t$ IS KNOWN NOW (@ TIME t), WE CAN WRITE THE DISTRIBUTION FOR $\ln \tilde{S}_T$ AS

$$(11.2) \quad \ln \tilde{S}_T \sim N\left(\ln S_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t),\ \sigma\sqrt{T-t}\right)$$

MOVED FROM LHS TO RHS

THUS $\ln S_T$ IS DIST^{B} NORMAL

$\boxed{\text{FACT:}}$ ANY VARIABLE WHO'S NATURAL LOGARITHIM IS DIST^{B} NORMAL, IS SAID TO BE DISTRIBUTED LOG-NORMAL.

ie S_T IS DIST^{B} LOG-NORMAL SINCE THE LOG OF S_T IS DIST^{B} NORMAL.

$\boxed{\text{EX}}$ SIMILAR TO $\boxed{\text{EX}}$ 11.1

ASSUME $S_t = \$50$, $\mu = .15$, $\sigma = .30/\text{YR}$

FIND 95% C.I OF S_T IN 3 MONTHS ($t = .25$)

$$\text{THUS } \ln S_T \sim N\Big(\underbrace{\ln(50) + \left(.15 - \tfrac{1}{2}(.3)^2\right)(.25)}_{\text{MEAN}},\ \underbrace{(.3)\sqrt{.25}}_{\text{STND DEV.}}\Big)$$

$$\ln S_T \sim N(3.93827, .15)$$

THUS A 95% CONFIDENCE INTERVAL (C.I) OF $\ln S_T$ WOULD BE

$$3.93827 - 1.96(.15) \le \ln S_T \le 3.93827 + 1.96(.15)$$

$$3.64427 \le \ln S_T \le 4.23227$$

SO WE HAVE A 95% C.I. OF $\ln S_T$, BUT WE WANT A 95% C.I. OF S_T.

SINCE $e^{\ln S_T} = S_T$ WE'LL TAKE LOGS

WE HAVE $3.64427 \le \ln S_T \le 4.23227$

RAISE TO e: $e^{3.64427} \le S_T \le e^{4.23227}$

$$\underbrace{\$38.25 \le S_T \le \$68.87}_{95\%\ \text{C.I. OF } S_T}$$

D — — — — — — — — —

STOCK PRICES ARE DIST^B LOG-NORMAL SINCE $\ln S_T$ IS DIST^B NORMAL.

A LOG-NORMAL RANDOM VARIABLE CANNOT BE NEGATIVE (ZERO IS THE MINIMUM POSSIBLE VALUE)

IF $S_T \sim$ LOG-NORMAL THEN

$$E[S_T] = S_t e^{\mu(T-t)}$$

AND

$$VAR(S_T) = S_t^2 e^{2\mu(T-t)} \cdot \left[e^{\sigma^2(T-t)} - 1 \right] \leftarrow \text{UGLY}$$

§ 11.2 THE DISTRIBUTION OF THE RATE OF RETURN.

WE JUST SAW THAT STOCK PRICES ARE DIST^{B} LOG-NORMAL, BUT HOW ARE RETURNS DISTRIBUTED?

LET'S USE CONTINUOUSLY COMPOUNDED RETURNS. RECALL FROM BUS-342:

$$F_T = P_t e^{r(T-t)}$$

w/ r = ANNUALIZED CONT. COMP RETURN

IN OUR SITUATION, THE ANALOGY IS.

$$S_T = S_t \, e^{r(T-t)}$$

CONTIN. COMPOUNDED RATE.

SOLVING FOR r GIVES $r = \frac{1}{T-t} \cdot \ln\left(\frac{S_T}{S_t}\right)$

FOCUS ON $\ln\left(S_T/S_t\right)$ FOR NOW:

THIS EQUALS $\ln(S_T) - \ln(S_t)$ WHICH PER EQ^{n} 11.1 (pg. 1) IS DIST^{B} NORMAL. THUS, PER (11.1) WE HAVE

$$\ln\left(S_T/S_t\right) \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t),\ \sigma\sqrt{T-t}\right)$$

$\downarrow$

RECALL $r = \underbrace{\frac{1}{T-t}}_{\text{A CONSTANT} = \text{"}K\text{"}} \cdot \ln\left(\frac{S_T}{S_t}\right)$

RECALL, FACT: IF $X \sim N(\mu, \sigma)$

THEN $K \cdot X \sim N(K \cdot \mu, K \cdot \sigma)$

w/ K = CONSTANT.

WHY?

'CUZ IF $E[X] = \mu$

THEN $E[KX] = K \cdot E[X] = K \cdot \mu$

AND

IF $VAR(X) = \sigma^2$

THEN $VAR(KX) = K^2 \cdot VAR(X) = K^2 \sigma^2$

THUS $STNDEV(KX) = \sqrt{K^2 \sigma^2} = K\sigma$

THEREFORE, w/ $\frac{1}{T-t} =$ "K" AND THE DIST^{B} OF $\ln(S_T/S_t)$ GIVEN AS NORMAL ON PREVIOUS PAGE, THEN WE HAVE

$$r \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t)\underbrace{\left(\frac{1}{T-t}\right)}_{\text{"}K\text{"}},\ \sigma\sqrt{T-t} \cdot \underbrace{\left(\frac{1}{T-t}\right)}_{\text{"}K\text{"}}\right)$$

THUS

$$r \sim N\left(\mu - \tfrac{1}{2}\sigma^2,\ \frac{\sigma}{\sqrt{T-t}}\right)$$

THUS, CONTINUOUSLY COMPOUNDED RETURNS ARE DIST^B NORMAL w/ MEAN $\mu - \frac{1}{2}\sigma^2$ AND STND DEV $= \sigma/\sqrt{T-t}$.

THAT IS, THE VOLATILITY OF THE RETURN DIST^B DECREASES AS WE LOOK FARTHER INTO THE FUTURE.

i.e THE VOLATILITY OF THE RETURN DIST^B FOR 3 YR RETURNS (ASSUMING ANNUALIZED σ' OF .30) WOULD BE $.3/\sqrt{3} = .173$, BUT THE VOLATILITY OF 5 YR RETURNS WOULD BE $.3/\sqrt{5} = .134$

* RETURNS ARE DIST^B NORMAL
* PRICES " " LOG NORMAL (i.e. THE LOG OF S IS DIST^B NORMAL)

DISCUSSION OF EXPECTED RETURN

Q: SO WHAT IS THE EXPECTED RETURN?

μ OR $\underbrace{\mu - \frac{1}{2}\sigma^2}_{\text{MEAN OF CONT. COMP. RETURNS.}}$?

RECALL

$$\frac{ds}{s} = \mu \cdot dt + \sigma \cdot dz$$

THUS $\mu \cdot dt$ IS THE EXPECTED PROPORTIONAL CHANGE IN S IN A PERIOD dt (dt IS VERY SMALL).

THUS, μ IS THE ANNUALIZED EXPECTED RETURN IN →

TIME dt EXPRESSED W/ A COMPOUNDING FREQUENCY OF dt. i.e. μ CORRESPONDS TO VERY SHORT PERIODS OF TIME. HOWEVER, PER ITO'S LEMMA APPLIED TO $\ln S_T$ (CHAPTER 10, pg 221), μ IS NOT THE MEAN OF THE CONTINUOUSLY COMPOUNDED RETURN. THE MEAN OF THE C.C. RETURN HAS BEEN SHOWN TO BE $\mu - \frac{1}{2}\sigma^2$ PER YEAR.

$\boxed{EX}$ ASSUME IN "SHORT" PERIODS WE'VE OBSERVED THE FOLLOWING RETURNS

$$.10, \ .20, \ .05, \ -.20, \ .15$$

GIVEN THIS TIME-SERIES, THE EXPECTED VALUE OF NEXT PERIODS RETURN IS THE MEAN OF THESE VALUES.

$$E[r] = \overline{r} = .06$$

SO WE EXPECT $.06 = \text{IRR}$ TO OCCUR IN EVERY PERIOD

AFTER 5 OF THE .06 PERIODS, A \$100/SH STOCK WOULD BE PRICED AT:

$$100(1.06)^5 = \$133.82$$

HOWEVER IF THE SAME SERIES OCCURED AGAIN, THE PRICE AFTER 5 PERIODS WOULD BE

$$100(1.10)(1.20)(1.05)(.80)(1.15) = 127.51$$

THUS THE ACTUAL RETURN PER PERIOD IS LESS THAN 6% (SINCE \$127.51 < \$133.82)

NEED 6%/PERIOD TO GET TO 133.82

IN THIS CASE, μ IS ANALAGOUS TO THE 6% RATE, AND $\mu - \frac{1}{2}\sigma^2$ IS ANALAGOUS TO THE RATE/PERIOD THAT GIVES \$127.51.

LETS FIND THAT RATE:

Q: WHAT r MAKES \$100 GROW TO 127.51

$$P = F/(1+r)^t$$

$$100 = \frac{127.51}{(1+r)^5} \Rightarrow r = \left(\frac{127.51}{100}\right)^{1/5} - 1$$

$$= .04981 < .06$$

.06 IS AN <u>ARITHMETIC</u> IRR (ADD UP 5 RETURNS & DIVIDE BY 5)

.04981 IS A <u>GEOMETRIC</u> IRR (COMPUTE BEGINING & ENDING VALUES, AND FIND RATE/PERIOD THAT WOULD ACOMPLISH THAT)

- FACT #1

ARITH RATES $\geq$ GEOM RATES

- FACT #2

ARITH RATES = GEOM RATES <u>ONLY</u> IF $\sigma^2_{RATES} = 0$

eg ASSUME ACTUAL RATES/PERIOD WERE

.06, .06, .06, .06, .06

IN THIS CASE $IRR^{ARITH} = IRR^{GEOM.}$

$$\sigma^2 = \frac{(10-6)^2 + (20-6)^2 + (5-6)^2 + (-20-6)^2 + (15-6)^2}{5}$$

$$= 16 + (14)^2 + 1 + 26^2 + 11^2 / 5$$

$$= 202$$

$$\sigma^2 = .0202$$

$$\sigma' = .142$$

$$.06 - \tfrac{1}{2}(.0202)$$

NOTICE THAT THE VARIANCE OF

.10, .20, .05, -.20, .15 IS .0202

THAT IS $\sigma^2 = .0202$

THUS

$$\mu - \tfrac{1}{2}\sigma^2 = .06 - \tfrac{1}{2}(.0202)$$

$$= .04990$$

VERY SIMILAR TO OUR IRR^{GEOM}

OF .04981

- CONCLUSION: IN A VERY SHORT PERIOD OF TIME dt THE EXPECTED RATE OF RETURN IS μ, HOWEVER THE EXPECTED CONTIN. COMPOUNDED RATE OF RETURN IS $\mu - \frac{1}{2}\sigma^2$

THUS, WHEN WE SAY "EXPECTED RETURN" WE NEED TO BE CAREFULL WHAT WE SAY. OUR CONVENTION WILL BE THAT "EXPECTED RETURN" WILL MEAN μ, NOT $\mu - \frac{1}{2}\sigma^2$.

- Q: WHY DO WE CARE SO MUCH ABOUT $\ln S_T$ AND

$$\ln(S_T/S_t) = \ln S_T - \ln S_t \quad ?$$

ANSWER #1: WE CARE ABOUT $\ln S_T$ 'CUZ WE SHOWED IT TO BE DIST^B NORMAL (S_T IS LOG-NORMAL)

↓

ANSWER #2:

WE CARE ABOUT $\ln S_T - \ln S_t$ BECAUSE THIS GIVES THE CONTINUOUSLY COMPOUNDED RETURN OVER THE PERIOD t TO T. (WHICH IS OFTEN <u>NOT</u> EQUAL TO A 1 YEAR SPAN).

WHERE DOES THE IDEA OF $\ln S_T - \ln S_t$ EQUALLING A CONT. COMPOUNDED RETURN COME FROM?

IT COMES FROM BUS-342! (OF COURSE)

$$P = F e^{-rt}$$
$$F = P e^{rt}$$

r IS C.C. ANNUAL RETURN

t IS THE # OF YRS

$$F/P = e^{rt}$$

TAKE LOGS...

$$\ln(F/P) = rt$$

$$\ln F - \ln P = \underbrace{r \cdot t}$$

ANNUAL CC RETURN

YEARS BETWEEN P AND F

$r \cdot t$ = CONTIN COMP RETURN FOR THE <u>ENTIRE</u> PERIOD, WHICH IS OFTEN NOT ANNUAL (TYPICALLY DAILY OR MONTHLY)

$\boxed{eg}$

$S_t = 52$

$S_T = 59$

$\Delta t = 2$ MONTHS

$$\ln 59 - \ln 52 = .1263$$

THE C.C. <u>2 MONTH</u> RETURN IS 12.63%

THE ANNUALIZED CONT. COMP RETURN IS:

$$r_{cc}^{ANNUAL} \cdot t_{YEARS} = .1263$$

$$\Rightarrow r_{cc}^{ANN.} = .1263 / 1/6 = .7578 / YR$$

THUS, C.C. RETURNS ARE COMPUTED VIA LOGS.

§11.3 "ESTIMATING VOLATILITY FROM HISTORICAL DATA"

GIVEN A TIME SERIES OF, SAY, DAILY PRICES COMPUTE THE C.C. DAILY RETURNS VIA

$$\ln\left(S_t / S_{t-1}\right)$$

ie w/ 500 DAYS OF PRICE DATA, YOU CAN GENERATE 499 C.C. DAILY RETURNS,
CALL THESE 499 DAILY RETURNS $r_i \quad i=1,\ldots,499$

THE DAILY STND DEV OF RETURNS IS COMPUTED IN THE TYPICAL MANNER AS

$$S = \sqrt{\frac{1}{498}\sum_{i=1}^{499}(r_i - \bar{r})^2}$$

w/ $\bar{r}$ = ARITMETIC DAILY MEAN

SAMPLE STND DEVIATION

FROM EQ^n (11.1) $\ln\left(S_T / S_t\right) \sim N\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)(T-t),\ \sigma\sqrt{T-t}\right)$

S IS AN ESTIMATE OF $\sigma\sqrt{T-t}$

(RECALL THAT THE BEST ESTIMATE OF THE POPULATION VARIANCE IS THE SAMPLE VARIANCE)

THUS THE ANNUALIZED POPULATION STND DEV, σ, IS BEST ESTIMATED BY $\hat{\sigma}_{ANN} = \dfrac{S}{\sqrt{T-t}}$

WE CAN ALSO SPEAK OF HOW PRECISE THIS ESTIMATE OF THE POPULATION STND DEVIATION IS.

THAT IS, WE'D LIKE TO KNOW THE "STND DEVIATION OF THE ESTIMATE OF THE STND DEVIATION" WHICH IS TECHNICALLY REFERED TO AS THE STANDARD ERROR OF THE ESTIMATE.

IT CAN BE SHOWN THAT THE STND ERROR OF $\hat{\sigma}$ IS EQUAL TO $\frac{\hat{\sigma}}{\sqrt{2 \cdot N}}$ w/ N = # OF RETURNS (499 IN MY EXAMPLE)

$\boxed{\text{EX}}$

THUS, IF WE HAVE 500 DAILY PRICES, GIVING 499 DAILY C.C. RETURNS, WITH ARITMETIC DAILY MEAN RETURN OF, SAY, .0005 AND DAILY SAMPLE STND DEVIATION, S, EQUAL TO .0180, THEN WE'D ESTIMATE THE ANNUAL STND DEVIATION AS

$$\hat{\sigma}_{ANN} = \frac{S}{\sqrt{\Delta t}} = \frac{.0180}{\sqrt{1/252}} = .2857$$

IT'S BEST TO ASSUME 252 DAYS PER YEAR, NOT 365.

252 = # OF TRADING DAY'S PER YEAR.

THE STANDARD ERROR (se) OF THIS ~~ESTIMATE~~ IS COMPUTED AS

$$s.e. = \frac{\hat{\sigma}}{\sqrt{2n}} = \frac{.2857}{\sqrt{2 \cdot 499}} = .00904$$

THUS WE COULD 95% THAT THE TRUE POPULATION STND DEVIATION IS IN THE RANGE OF

$$\hat{\sigma} - 1.96(se) \le \sigma'_{ANN} \le \hat{\sigma} + 1.96(se)$$

$$.2857 - 1.96(.00904) \le \sigma'_{ANN} \le .2857 + 1.96(.00904)$$

$$.2680 \le \sigma'_{ANN} \le .3034$$

▷

Q: HOW MUCH DATA TO USE?

MORE DATA INCREASES PRECISION, BUT AT THE RISK OF USING OLD AND IRRELEVANT DATA.

OFTEN 180 DAYS OF DAILY DATA IS USED, AND SOMETIMES THE AMT OF HISTORICAL DATA USED IS SET EQUAL TO THE HORIZON IN QUESTION (ie IF USING THE DATA TO PRICE A 3 YR OPTION, THEN USE 3 YRS OF DATA, PREFERABLY DAILY)

▷ STUDENTS:
SEE END OF SECTION 11.3 ON RETURNS ON EX-DIVIDEND DATES.

§11.4 "CONCEPTS UNDERLYING THE B/S DIFF EQ^n"

B/S COMES FROM THE SOLUTION TO A DIFFERENTIAL EQUATION.

THE DIFF EQ^n IS CONSTRUCTED VIA A NO ARBITRAGE ARGUMENT.

THE IDEA BEHIND THE CONSTRUCTION IS

THAT A COMBO OF THE DERIVATIVE, f, AND THE STOCK, S, CAN BE MADE WHICH MUST EARN THE RISK-FREE RATE.

NOTICE THAT THE UNCERTAINTY IN BOTH S AND f IS $dz = \varepsilon\sqrt{dt}$, HENCE IF WE CAN COMBINE S AND f SUCH THAT THE dz's CANCEL OUT, OUR PORTFOLIO OF f AND S WILL BE RISK FREE, AND THUS BY NO-ARB IT MUST EARN THE RISK-FREE RATE.

THIS ARGUMENT WHEN WRITTEN DOWN GIVES A DIFFERENTIAL EQUATION WHICH WHEN SOLVED GIVES THE SOLUTION OF

$$C = S \cdot N(d_1) - K e^{-rt} \cdot N(d_2)$$

IN ORDER TO DERIVE THE B/S SOLUTION WE ASSUME THE FOLLOWING:

1. S FOLLOWS GBM w/ CONST μ & σ
2. SHORT SELLING w/ FULL USE OF PROCEEDS IS ALLOWE
3. NO FRICTIONS (NO TRX COSTS, NO TAXES, NO INDIVISIBILITIES)
4. NO DIVIDENTS OVER THE LIFE OF THE OPTION
5. NO ARBITRAGE
6. TRADING IS CONTINUOUS
7. THE YIELD CURVE IS FLAT.

SEE CH 19 TO RELAX SOME OF THESE ASSUMPTIONS.

§11.5 DERIVATION OF B/S DIFF. EQ.

(11.8) $$dS = \mu S\,dt + \sigma S\,dz$$ ASSUMED

LET f BE THE PRICE OF A EURO CALL (OR OTHER DERIV THAT DEPENDS ON S AND t). VIA ITO'S LEMMA, CH 10, WE HAVE:

(11.9) $$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dz$$

NOTE THAT THE UNCERTAINTY IN BOTH S AND f IS DUE ENTIRELY TO dz. THE dz'S IN THE 2 EQ^n'S ARE THE SAME dz.

CONSIDER THE FOLLOWING PORFOLIO CONSISTING OF S AND f:

-1 UNITS OF THE DERIVATIVE

$+\dfrac{\partial f}{\partial S}$ SHARES OF STOCK

IE THE PORT IS SHORT 1 DERIVATIVE (ON 1 SHARE OF STOCK) AND LONG $\dfrac{\partial f}{\partial S}$ SHARES OF STOCK.

LET Π STAND FOR THE VALUE OF THE PORTFOLIO, THUS

$$\Pi = (-1)f + \left(\frac{\partial f}{\partial S}\right)\cdot S$$

THE CHANGE IN PORT VALUE IN TIME dt IS

(11.13) $$d\Pi = (-1)(df) + \left(\frac{\partial f}{\partial S}\right)\cdot(dS)$$

NOW SUB df & dS FROM EQ^n'S 11.8 AND 11.9 INTO 11.13. THIS GIVES

$$d\Pi = (-1)\left\{\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dz\right\} +$$

$$+ \frac{\partial f}{\partial S}\left\{\mu S\,dt + \sigma S\,dz\right\}$$

ALL RISK GOES AWAY!

THE TERMS INVOLVING dz CANCEL OUT (DO YOU SEE WHY WE PICKED -1 UNITS OF f AND $+\frac{\partial f}{\partial S}$ UNITS OF STOCK? YOU SHOULD!)

WE HAVE ONLY TERMS INVOLVING dt. AFTER A SMALL AMT OF COMBINING TERMS WE GET

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right) dt$$

SINCE THERE IS NO RISK, THIS PORTFOLIO MUST EARN r_f. THUS,

$$d\Pi = r_f \cdot \Pi \cdot dt \quad \text{w/} \quad \Pi = -f + \frac{\partial f}{\partial S}\cdot S$$

THUS WE HAVE

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt = r_f\left(-f + \frac{\partial f}{\partial S}\cdot S\right)dt$$

OR

$$\frac{\partial f}{\partial t} + r_f \cdot S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r \cdot f \qquad (11.15)$$

B/S DIFF EQUATION!

THERE ARE MANY POSSIBLE SOLUTIONS TO THE B/S DIFF EQ^n. ONE MUST SPECIFY EXACTLY HOW f BEHAVES AS A FUNCTION OF S AND t AT MATURITY IN ORDER TO GET A SPECIFIC SOLUTION. THIS SPECIFICATION OF THE PRICE OF f @ AT THE "BOUDARIES" OF S AND t IS KNOWN AS "THE BONDARY CONDITIONS." DIFFERENT BOUNDARY CONDITIONS GIVE DIFFERENT SOLUTIONS.

FOR EURO CALLS THE BOUNDARY CONDITION IS

$$f = \max(0, S-K) \quad @\ t = T$$

W/ THIS BOUNDARY CONDITION, THE SOLUTION TO (11.15) IS

$$C = f = S\,N(d_1) - K e^{-rt} N(d_2)$$

§ 11.6. "RISK NEUTRAL VALUATION"

RISK NEUTRAL VALUATION IS <u>THE</u> MOST IMPORTANT CONCEPT FOR VALUING (PRICING) DERIVATIVES. WE USED IT IN BINOMIAL OPTION PRICING, AND IT GETS USED AGAIN W/ B/S.

WHY IS IT OK TO ASSUME INVESTORS ARE RISK NEUTRAL WHEN WORKING W/ B/S?

BECAUSE IN THE B/S DIFF EQ^n THERE ARE NO VARIABLES THAT ARE RELATED TO <u>SYSTEMATIC</u> RISK.

THE ONLY VARIABLES IN THE B/S DIFF EQ^{n} ARE t, r_f, S, AND σ. σ IS NOT REWARDED (VIA AN INCREASE IN $E[R]$ BECAUSE IT IS THE IDIOSYNCRATIC PORTION OF GBM (RECALL THAT σdz HAS MEAN ZERO, THUS INCREASING σ DOES NOT AFFECT $E[dS/S]$).

WE'RE LUCKY THAT μ DOES NOT APPEAR IN THE B/S DIFF EQ^{n}. IF IT DID, THEN WE COULD NOT EMPLOY RISK NEUTRAL VALUATION.

(*) FACT: IF AN INVESTOR IS RISK NEUTRAL THEN HE/SHE PRICES ALL SECURITIES VIA DISCOUNTING @ r_f (i.e. ALL SECURITIES IN A RISK NEUTRAL WORLD EARN r_f).

SINCE μ DOES NOT EXIST IN THE B/S DIFF EQ^{n} THEN WE CAN USE ANY TYPE OF INVESTOR RISK PREFERENCES TO VALUE THE DERIVATIVE. SOME (MOST) INVESTORS ARE RISK AVERSE (REQUIRE ↑ $E[R]$ AS COMPENSATION FOR ↑ SYS. RISK), OTHERS MAY BE RISK NEUTRAL. BOTH TYPES OF INVESTORS WILL AGREE ON THE PRICE OF THE DERIVATIVE, f, SINCE SYSTEMATIC RISK PLAYS NO ROLE.

THUS WE COULD USE SOME COMPLICATED RISK PREFERENCE STRUCTURE (ASSOC. W/ A RISK AVERSE INVESTOR) OR WE COULD USE A VERY SIMPLE RISK PREFERENCE STRUCTURE (RISK NEUTRAL) TO FIND THE PRICE OF THE DERIVATIVE.

IT'S MUCH EASIER TO USE 'RISK NEUTRAL VALUATION.'

NOTE: WE DO NOT EVEN REQUIRE THAT A SINGLE PERSON IS RISK NEUTRAL. WE NEED ONLY CONSIDER THAT THIS TYPE OF PERSON COULD EXIST, AND IF HE/SHE DID, THEN WE KNOW THE PRICE OF f HE/SHE WOULD COMPUTE WOULD AGREE W/ THAT COMPUTED (SOMEHOW) BY RISK AVERSE INVESTORS.

△ HOW TO USE R. NEUTRAL VALUATION W/ B/S:

CONSIDER A DERIVATIVE WITH PAYOFF @ TIME T (@ MATURITY) THAT DEPENDS ON S_T.

WE FIRST COMPUTE THE EXPECTED PRICE OF S @ TIME T USING THE RISK-FREE RATE r_f.

i.e. $E^*[S_T] = S_t e^{r_f(T-t)}$

DO NOT USE μ!

THE * ON THE E MEANS USE r_f, NOT μ.

THIS GIVES THE EXPECTED VALUE OF THE DERIVATIVE @ TIME T.

FOR EXAMPLE, W/ A LONG FORWARD CONTRACT THE PAYOFF @ TIME T IS $S_T - K$

FORWARD PRICE YOU LOCKED IN. (YOU BOUGHT @)

LET f NOW DENOTE THE VALUE OF THE FORWARD (AS IN HULL CH 3)

THE EXPECTED PAYOFF IS $E^*(S_T - K)$

THEREFORE THE EXPECTED VALUE @ TIME t (USING R. NEUTRAL VALUATION) IS

$$f = PV^* \cdot E^*(S_T - K)$$

$$= e^{-r_f(T-t)} \cdot \{E^*(S_T - K)\}$$

$$= e^{-r_f(T-t)} \{E^*(S_T) - K\}$$

$$= e^{-r(T-t)} \{S_t e^{r(T-t)} - K\}$$

$$f = S_t - K e^{-r_f(T-t)}$$ ← AGREE'S w/ EQ^n (3.6)

IS THIS A SOLUTION TO THE DIFF EQ^n (11.15)?

LETS SEE (11.15) IS

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

COMPUTE DERIVATIVES AND PLUG IN:

$$\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}$$

$$\frac{\partial f}{\partial S} = 1$$

$$\frac{\partial^2 f}{\partial S^2} = 0$$

SUB IN:

$$-rKe^{-r(T-t)} + r \cdot S \cdot (1) + \tfrac{1}{2}\sigma^2 S^2 (0) = rf$$

$$-\cancel{r}Ke^{-r(T-t)} + \cancel{r}S = \cancel{r}f$$

(3.6)
(11.19)
$$\Rightarrow \underbrace{f = S - Ke^{-r(T-t)}}$$

THUS OUR ORIGINAL GUESS OF $f = S - Ke^{-r(T-t)}$ DOES SATISFY THE DIFF EQ$\hat{=}$, CONFIRMING (AGAIN) THAT (11.19) IS THE SOLUTION TO THE DIFF. EQ$\hat{=}$.

§11.7 "B/S PRICING FORMULA'S"

NOW LET'S SEE HOW RISK NEUTRAL VALUATION IS APPLIED TO EURO CALLS.

THE PAYOFF OF C @ TIME T IS

$$\max(0, S_T - K)$$

THE EXPECTED PAYOFF (USING R. NEUTRAL VALUATION) IS

$$E^*[\max(0, S_T - K)]$$

IMPORTANT NOTE, THIS IS NOT THE SAME AS

$$\max(0, E^*(S_T) - K)$$

IF THIS ↑ WERE TRUE WE'D GET A DIFFERENT SOLUTION!

THE PRICE OF THE EURO CALL AT TIME t IS

$$C_t = PV^* \cdot E^*[\max(0, S_T - K)]$$

$$= e^{-r_f(T-t)} \cdot \underbrace{E^*[\max(0, S_T - K)]}_{\text{HOW TO EVALUATE?}}$$

[WE KNOW, IN A RISK NEUTRAL WORLD (W/ μ REPLACED BY r_f)

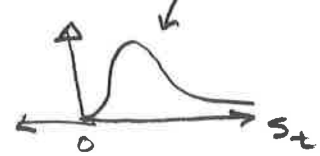
$$\ln S_T \sim N\left(\ln S_t + (r - \tfrac{1}{2}\sigma^2)(T-t),\ \sigma\sqrt{T-t}\right)$$

WAS μ

$$C_t = e^{-r_f(T-t)} \cdot \int_0^\infty \max(0, S_T - K) \cdot \underbrace{g(S_T)}_{\text{pdf of } S_T} \cdot dS_T$$

pdf OF S_T

(RECALL S_T IS DIST$^{\underline{B}}$ LOG-NORMAL)



RECALL "MATH PRELIMINARIES" HANDOUT.

$$E[x] = \int_{-\infty}^{\infty} x \cdot \text{pdf}(x)\, dx$$

$$\text{\&} \quad E[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot \text{pdf}(x)\, dx$$

THIS IS ZERO SINCE S CANNOT BE BELOW 0

HOW TO EVALUATE THE INTEGRAL?

WE KNOW $\displaystyle\int_0^K \max(0, S_T - K) \cdot g(S_T)\, dS_T = 0$

SINCE IF S_T IS IN THE RANGE OF 0 TO K THE

PAYOFF IS ZERO, AND THUS ZERO · pdf IS ZERO, AND THE AREA UNDER A CURVE w/ CONSTANT VALUE OF ZERO IS ZERO.

THUS

$$\int_0^{\infty} \max(0, S_T - K) \cdot g(S_T)\, dS_T = \int_0^{K} \max(0, S_T - K) \cdot g(S_T) \cdot dS_T +$$

$$+ \int_K^{\infty} \max(0, S_T - K) \cdot g(S_T) \cdot dS_T$$

PER PREVIOUS PAGE

$$= 0 + $$

$$+ \int_K^{\infty} \max(0, S_T - K) \cdot g(S_T) \cdot dS_T$$

Q: SO WHAT IS THIS EQUAL TO?

A: NOTICE THAT IN THE RANGE OF $K < S_T < \infty$ $\max(0, S_T - K) = S_T - K$ SO WE CAN WRITE

$$\int_K^{\infty} \max(0, S_T - K) \cdot g(S_T)\, dS_T$$

$$= \int_K^{\infty} (S_T - K) \cdot \underbrace{g(S_T)}\, dS_T$$

SINCE S_T IS LOG-NORMAL THIS IS AN UGLY pdf.

SEE HULL 4^{TH} EDITION, APPENDIX 11-A FOR A STEP BY STEP EVALUATION OF THIS INTEGRAL

THE SOLUTION TO THIS INTEGRAL IS NOT EASY TO OBTAIN, BUT IT IS

$$c = S\,N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$= \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

SINCE IT CAN BE SHOWN THAT IT IS NEVER OPTIMAL TO EXERCISE AN AMERICAN CALL ON A NON-DIVIDEND PAYING STOCK, IT MUST BE TRUE THAT FOR NON-DIVIDEND PAYING STOCKS, THE B/S EQUATION ALSO HOLDS FOR AN AMERICAN CALL (NOT TRUE OF FX AMER CALLS, ONLY TRUE FOR STOCK OPTION AMER CALLS — SINCE FOREIGN CURRENCIES PAY "DIVIDENDS" IN THE FORM OF INTEREST).

TO VALUE AN AMER. PUT ON A NON-DIV PAYING STOCK B/S DOES NOT WORK (SINCE IT IS SOMETIMES OPTIMAL TO EXERCISE THE OPTION BEFORE MATURITY). IN THIS CASE THE BINOMIAL METHOD IS USED.

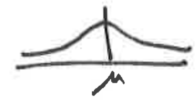
IF DIVIDENDS ARE PAID CONTINUALLY (IN A SMOOTH STREAM, "EVERY SECOND") THEN B/S CAN BE MODIFIED AND APPLIED TO EURO PUTS & CALLS. SEE EQ^N'S (12.4) & (12.5).

IF DIVIDENDS ARE "LUMPY" (IE EVERY QUARTER) THEN S IN B/S IS EVERYWHERE REPLACED BY S - PV(DIV'S) AND B/S CAN BE USED. SEE SECTION 11.12 OF HULL.

FOR AMERICAN OPTIONS — SEE SECTION 11.12 OF HULL ⇒ EARLY EXERCISE, IF IT OCCURS, WILL ONLY OCCUR JUST PRIOR TO THE LAST LUMPY DIVIDEND.

§11.8 CUMMULATIVE NORMAL DIST^{B}

RECALL FROM "MATH PRELIM" HANDOUT THE pdf OF THE NORMAL LOOKS LIKE



THE AREA UNDER THE CURVE TO THE LEFT OF ANY POINT ON THE X-AXIS IS $\int_{-\infty}^{x} \text{pdf}(x)\, dx$

THIS CAN BE PLOTTED. THE ABOVE INTEGRAL CAN NEVER FALL BELOW ZERO OR RISE ABOVE 1. THAT IS,

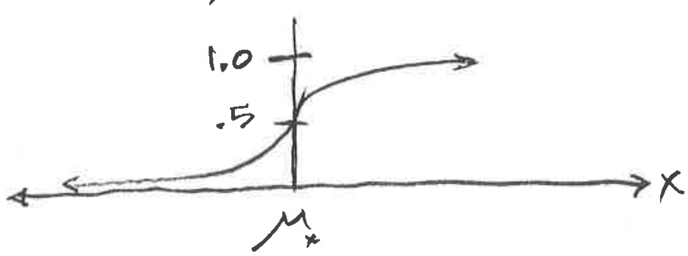
$$0 \leq \int_{-\infty}^{\infty} \text{pdf}(x)\, dx \leq 1.00$$

WE CAN DEFINE $\int_{-\infty}^{x} \text{pdf}(x)\, dx = \text{CDF}(x)$

CUMMULATIVE DISTRIBUTION FUNCTION

FOR THE NORMAL, THE CDF LOOKS LIKE

CDF:



THE CDF GIVES THE AREA UNDER THE pdf TO THE LEFT OF ANY POINT ON THE X AXIS.

eg THE AREA UNDER THE pdf TO THE LEFT OF $\mu_x = E[x]$ IS 0.500

THE B/S NOTATION OF $N(d_1)$ AND $N(d_2)$ MEANS GET THE VALUE OF THE CDF @ POINTS d_1 AND d_2, OR EQUIVALENTLY THE AREA UNDER THE pdf CURVE TO THE LEFT OF d_1 AND d_2.

§11.9 "WARRANTS ISSUED BY A COMPANY ON ITS OWN STOCK."

WARRANTS ARE SIMPLY INTERNALLY ISSUED EURO CALLS. THERE IS AN ISSUE W/ 'DILUTION' (AN INCREASE IN THE # OF SH'S OUTSTANDING OCCURS WHEN WARRANTS ARE EXERCISED AND STOCK IS THEREFORE ISSUED)

SEE §11.9 FOR DETAILS OF THE ARGUMENTS... THE BOTTOM LINE IS THAT B/S CAN BE USED TO VALUE THE WARRANTS, BUT ONLY AFTER SOME MODIFICATIONS TO B/S ARE MADE.

THE MOD'S ARE:

1. EVERYWHERE REPLACE S W/ $S+\left(\frac{M}{N}\right)W$

   W = WARRANT VALUE

   N = # OUTSTANDING SH'S (NOW)

   M = # " WARRANTS (AFTER ISSUANCE)

   (EACH WARRANT GIVES THE HOLDER THE RIGHT TO γ SHARES)

2. σ SHOULD MEASURE VOLATILITY OF THE EQUITY W/ WARRANTS IN PLACE. — WHICH IS PROBABLY NOT MUCH DIFFERENT FROM σ W/ STOCK ALONE.

③ MULTIPLY THE ENTIRE FORMULA BY

$$\left(N\gamma / N + M\gamma\right)$$

THUS

$$C = S\,N(d_1) - K e^{-r(T-t)} N(d_2)$$

BECOMES

$$\circledast \quad W = \left\{ \left(S + \tfrac{M}{N} W\right) N(d_1) - K e^{-r(T-t)} N(d_2) \right\} \cdot \left(\frac{N\gamma}{N + M\gamma} \right)$$

$$d_1 = \frac{\ln\left(\frac{S + \frac{M}{N} W}{K}\right) + \left(r + \frac{1}{2} \sigma^{*2}\right)(T-t)}{\sigma^* \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma^* \sqrt{T-t}$$

w/ σ^* = VOLATILITY OF EQUITY w/ OPTIONS IN PLACE

NOTICE THAT, UNLIKE REGULAR B/S, W NOW APPEARS ON <u>BOTH</u> SIDES OF THE EQUATION (IN B/S C WAS ON THE LHS ONLY)

THE ABOVE EQ^n ⊛ CAN<u>NOT</u> BE SOLVED ANALYTICALLY (VIA MATH). INSTEAD IT MUST BE SOLVED NUMERICALLY.

HOW TO DO THAT?

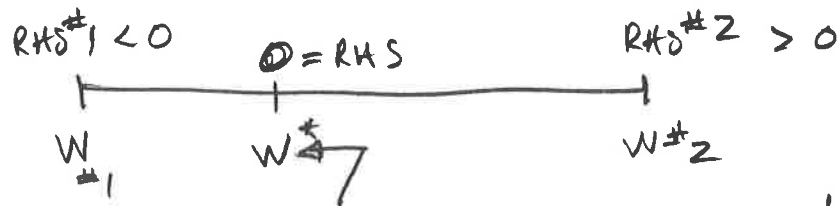
↓

① SET THE ENTIRE EQUATION EQUAL TO ZERO VIA SUBTRACTING W FROM BOTH SIDES. THIS GIVES

$$0 = \Big\{ \underbrace{\quad \sim\sim\sim \quad}_{\text{previous pg}} \Big\} \cdot \left(\frac{N\gamma}{N + M\gamma} \right) - W = RHS$$

② GUESS @ W & COMPUTE VALUE OF RHS.
IF YOU ARE FORTUNATE, THE RHS WILL = 0 AND THUS YOUR GUESS OF W WAS CORRECT, AND YOU'RE DONE.
A MUCH MORE LIKELY SCENARIO IS THAT THE RHS ≠ 0. IN THIS CASE GUESS ANOTHER VALUE FOR W. IF ON GUESS #1 RHS WAS POSITIVE, GUESS A W SO THAT THE SECOND TRY VALUE OF THE RHS IS NEGATIVE.

③ USE LINEAR INTERPOLATION TO MAKE A THIRD GUESS @ W

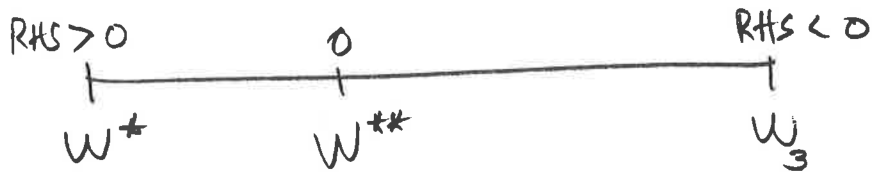


WE WANT THIS VALUE!
THE VALUE OF W THAT GIVES RHS = 0

4 CHECK TO SEE IF W^* FROM THE LINEAR INTERPOLATION IS "CLOSE ENOUGH TO ZERO"

SINCE RHS IS <u>NOT</u> A LINEAR EQUATION RHS EVALUATED @ W^* <u>WILL</u> <u>NOT</u> = 0 !

5 DO 1 MORE LINEAR INTERPOLATION. IF W^* GAVE RHS > 0 THEN PICK ANOTHER W (<u>CLOSE</u> TO W^*) THAT GIVES RHS < 0 (AND VICE VERSA). CALL THIS GUESS W_3. THEN DO 1 MORE LINEAR INTERPOLATION (THE SECOND ITERATION)



WILL STILL NOT GIVE RHS = 0 BUT IT WILL BE A LOT CLOSER TO ZERO THAN W^* GAVE.

6 IF W^{**} IS STILL NOT "CLOSE ENOUGH" TO ZERO YOU NEED TO KEEP DOING THE LINEAR INTERPOLATION (IN A LOOP) UNTIL YOU GET

$$-1 * swtEpsilon < RHS < swtEpsilon$$

YOU SET swtEpsilon TO BE SOME SMALL # THAT DEFINES "CLOSE ENOUGH TO ZERO."

§11.10 IMPLIED VOLATILITIES

THE B/S SOLUTION TO THE B/S DIFF EQ^n CANNOT BE SOLVED EXPLICITLY FOR σ. HOWEVER IF EVERY INPUT VARIABLE (INCLUDING C) EXCEPT σ IS SPECIFIED, THEN THE ONLY UNKNOWN IS σ. THERE WILL BE SOME VALUE OF σ THAT MAKES THE B/S EQ^n TRUE. THIS VALUE IS KNOWN AS THE IMPLIED VOLATILITY.

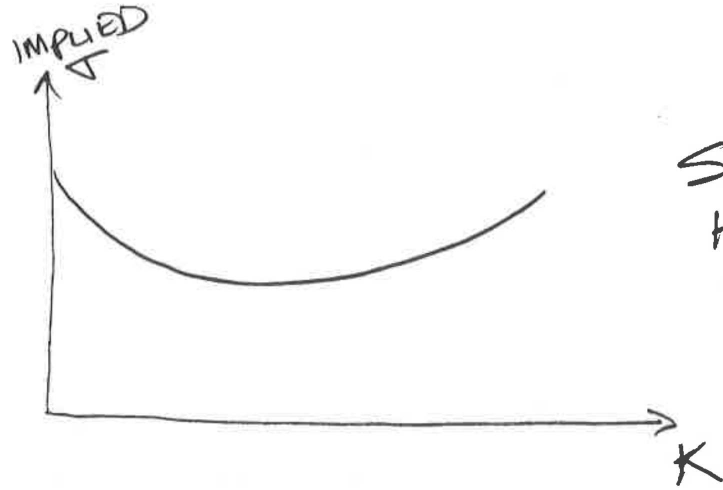
SINCE IT IS IMPOSSIBLE TO SOLVE B/S FOR σ EXPLICITLY, IT MUST BE FOUND NUMERICALLY VIA SOME TYPE OF 'SEARCH ALGORITHM'.

LINEAR INTERPOLATION, AS OUTLINED IN §11.09 WORKS FINE - BUT A FEW ITERATIONS (>3) SHOULD BE USED.

IF YOU HAVE BUILT A B/S SPREADSHEET YOU CAN USE TOOLS, GOAL SEEK (WHICH EMPLOYS SEVERAL ITERATIONS OF LINEAR INTERPOLATION) TO FIND THE IMPLIED VOL.

YOU CAN USE IMPLIED σ'S TO ASSESS THE MKT'S OPINION REGARDING THE CURRENT VOLATILITY OF THE UNDERLYING ASSET. BEWARE THAT IF YOU HAVE SEVERAL OPTION PRICES (FOR OPTIONS W/ DIFFERING K, BUT SAME TIME TO MATURITY) THEY WILL NOT HAVE THE SAME IMPLIED σ. ! TYPICALLY A GRAPH OF IMPLIED σ AS A FN OF K (FOR FIXED S AND T) LOOKS

LIKE



SEE HULL §19.7

THIS IS CALLED THE VOLATILITY SMILE ☺.
THE LOWEST IMP. σ TYPICALLY OCCURS W/ $K = S$.

THERE ARE VARIOUS METHODS OF EXPLAINING THE VOLATILITY SMILE. SEE HULL CH 19, ESPECIALLY SECTION 19.7 FOR MORE ON THIS.

GIVEN THAT DIFFERENT K'S GIVE DIFFERENT IMPLIED σ'S, WHICH σ'S SHOULD BE USED TO ASSESS THE MKT'S VIEW OF σ?

FACT: OPTIONS ARE MOST SENSITIVE TO CHANGES IN σ WHEN $K=S$, THUS OPTIONS 'AT THE MONEY' PROVIDE THE MOST INFORMATION WITH RESPECT TO IMPLIED σ.

THUS USE IMP. σ FOR THE OPTION THAT HAS K CLOSES TO S IN ORDER GAUGE THE MKT'S VIEW OF σ.

↓

§11.11 CAUSES OF VOLATILITY

STUDENTS: READ THIS SECTION ON YOUR OWN.

THE BOTTOM LINE IS THAT VOLATILY ARISES ON TRADING DAYS, NOT CALENDAR DAYS.

THUS WE ASSUME 252 DAYS/YR, NOT 365.

§11.12 DIVIDENDS

- EURO OPTIONS

  SIMPLY SUBTRACT PV OF DIVIDENDS FROM S, CALL THAT S^*, AND USE B/S w/ S^*.

- AMERICAN OPTIONS

  GOOD STUFF, BUT WE'LL SKIP IN THE INTEREST OF TIME.

§11.13 SUMMARY.

— IMPORTANT — READ IT.