B-04-1

BUS 444 WEEK 4

~~THE SAME IDEA CAN BE APPLIED TO A GROWING PERPETUITY (growing AT RATE g) WHICH WILL GIVE~~

$$\cancel{P = \frac{1}{r-g}}$$

~~(AS AN STUDENTS EXERCISE, DERIVE THIS B)~~

WEEK 4 USED w/ HULL MAT'L CH 11, 12 ETC

Δ TAYLOR SERIES EXPANSIONS:

SUPPOSE THAT YOU HAVE A FUNCTION $f(x)$ AND THIS FUNCTION IS RATHER COMPLEX – AND HENCE YOU'D LIKE TO FIND A DIFFERENT FUNCTION THAT BEHAVES IN A VERY SIMILAR MANNER, YET HAS A SIMPLER EXPRESSION. ~~THAT~~ IS, YOU'D LIKE TO SIMPLIFY THE ORIGINAL EXPRESSION, YET RETAIN THE ESSENTIAL CHARACTERISTICS.

Q: CAN THIS BE DONE?

A: YES – IN ORDER TO DO IT, TAKE A TAYLOR SERIES EXPANSION (AKA A TAYLOR SERIES APPROXIMATION)

SO WHAT IS A TAYLOR SERIES EXPANSION?

FYI: THESE NOTES ARE FROM THE BUS-343 CLASS/QUANT. METHODS

WELL, FIRST OF ALL, A TAYLOR SERIES EXPANSION IS ALWAYS TAKEN "AROUND A POINT" (OF INTEREST). THUS THE T.S. EXPANSION OF $y = x^4$ AROUND $x = 0$ IS DIFFERENT THAN THE T.S. EXPANSION OF $y = x^4$ AROUND $x = 2$.

THE "POINTS OF INTEREST" ARE ALSO REFERED TO AS "POINTS OF EXPANSION."

LET C BE THE VALUE OF X AROUND WHICH WE A TAKING THE T.S. EXPANSION.

ALSO, WE CAN TAKE A 1ST, OR 2ND, OR 3RD ETC ORDER TS EXPANSION AROUND C (WHICH REQUIRES DERIVATIVES OF THE 1ST, 2ND, 3RD ORDER ETC).

GIVEN ALL THAT, A 4TH ORDER T.S. EXPANSION OF $f(x)$ AROUND $X=C$ IS:

$$f(x) \cong f(c) + \left.\frac{df}{dx}\right|_{x=c}(x-c)^1 + \frac{1}{2!}\cdot\left.\frac{d^2f}{dx^2}\right|_{x=3}\cdot(x-c)^2 +$$

$$\frac{1}{3!}\cdot\left.\frac{d^3f}{dx^3}\right|_{x=c}\cdot(x-c)^3 + \frac{1}{4!}\cdot\left.\frac{d^4f}{dx^4}\right|_{x=c}\cdot(x-c)^4$$

AN APPROXIMATION

OF COURSE, THIS INVOLVES TAKING 1ST, 2ND, 3RD & 4TH ORDER DERIVATIVES

NOTE THAT $\left.\frac{d^4f}{dx^4}\right|_{x=c}$ MEANS TO EXALUATE THE 4TH DERIVATIVE OF f AT THE POINT $X=C$.

IN GENERAL, AN N^{TH} ORDER T.S. EXPANSION OF $f(x)$ AROUND $X=C$ CAN BE WRITTEN AS:

$$f(x) \cong \sum_{j=0}^{N} \underbrace{\frac{1}{j!}} \cdot \underbrace{f^{(j)}}\Big|_{c} \cdot (x-c)^{j}$$

j^{TH} DERIVATIVE OF f WITH RESPECT TO X

NOTE: $0! = 1$

$1! = 1$

$2! = 2$

ETC

THAT IS,

$$f(x) \cong \underbrace{\frac{1}{0!}}_{=1.00} \underbrace{f^{(0)}}\Big|_{x=c} \cdot \underbrace{(x-c)^{0}}_{=1.00} + \frac{1}{1!} \cdot \underbrace{f^{(1)}}\Big|_{x=c} \cdot (x-c)^{1} + \frac{1}{2!} \cdot f^{(2)}\Big|_{x=c} \cdot (x-c)^{2} + \ldots$$

$$= f(c) + \frac{df}{dx}\Big|_{c} \cdot (x-c) + \frac{1}{2!} + \frac{d^2f}{dx^2}\Big|_{c} \cdot (x-c)^{2} + \ldots$$

LETS DO AN EXAMPLE

WRITE THE 3RD ORDER T.S. EXPANSION OF

$$f(x) = \frac{1}{1+x}$$

AROUND THE POINT $x=1$.

<u>SOLUTION</u>:

FIRST FIND THE FIRST 4 DERIVATIVES:

$$\left.\begin{aligned} f'(x) &= -(1+x)^{-2} \\ f''(x) &= 2(1+x)^{-3} \\ f'''(x) &= -6(1+x)^{-4} \\ f^{(4)}(x) &= 24(1+x)^{-5} \end{aligned}\right\} \Rightarrow \begin{aligned} f'(1) &= -(1+1)^{-2} = -\frac{1}{4} \\ f''(1) &= 2(2)^{-3} = 1/4 \\ f'''(1) &= -6(2)^{-4} = -3/8 \\ f^{(4)}(1) &= 24(2)^{-5} = 3/4 \end{aligned}$$

ALSO $\left. f(x)\right|_{x=c} = \left. f(x)\right|_{x=1} = f(1) = 1/2$, THUS WE

CAN WRITE

$$f(x) \cong f(1) + f'(1)\cdot(x-1)^1 + \frac{1}{2!} f''(1)\cdot(x-1)^2 +$$

$$+ \frac{1}{3!}\cdot f'''(1)\cdot(x-1)^3 + \frac{1}{4!}\cdot f^{(4)}(1)\cdot(x-1)^4$$

THUS

$$f(x) \cong \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{2}\cdot\frac{1}{4}\cdot(x-1)^2 + \frac{1}{6}\cdot\frac{-3}{8}\cdot(x-1)^3 + \frac{1}{24}\cdot\frac{3}{4}\cdot(x-1)^4$$

WHICH, AFTER A LOT OF SIMPLIFICATION GIVES

$$f(x) \cong \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}\cdot x^4$$

NOTE:

A 1ST ORDER T.S. EXPANSION OF ANY FUNCTION WILL ALWAYS RESULT IN A LINEAR EQUATION (I.E $y = mx + b$ FORM) REPRESENTING THE TANGENT LINE OF THE CURVE $f(x)$ AT THE POINT c.

$\boxed{EX}$

WRITE & GRAPH THE 1ST ORDER T.S. EXPANSION OF $f(x) = 5 + 2x + x^2$ AROUND THE POINT $x = 1/2$.

IN THIS CASE

$f'(x) = 2 + 2x$ & $f'(\frac{1}{2}) = 2 + 2(\frac{1}{2})$
$= 3$

THE SLOPE OF $f(x)$ AT $x = 1/2$

AND $f(\frac{1}{2}) = 5 + 2(\frac{1}{2}) + (\frac{1}{2})^2$
$= 6.25$

THE TS EXPANSION IS

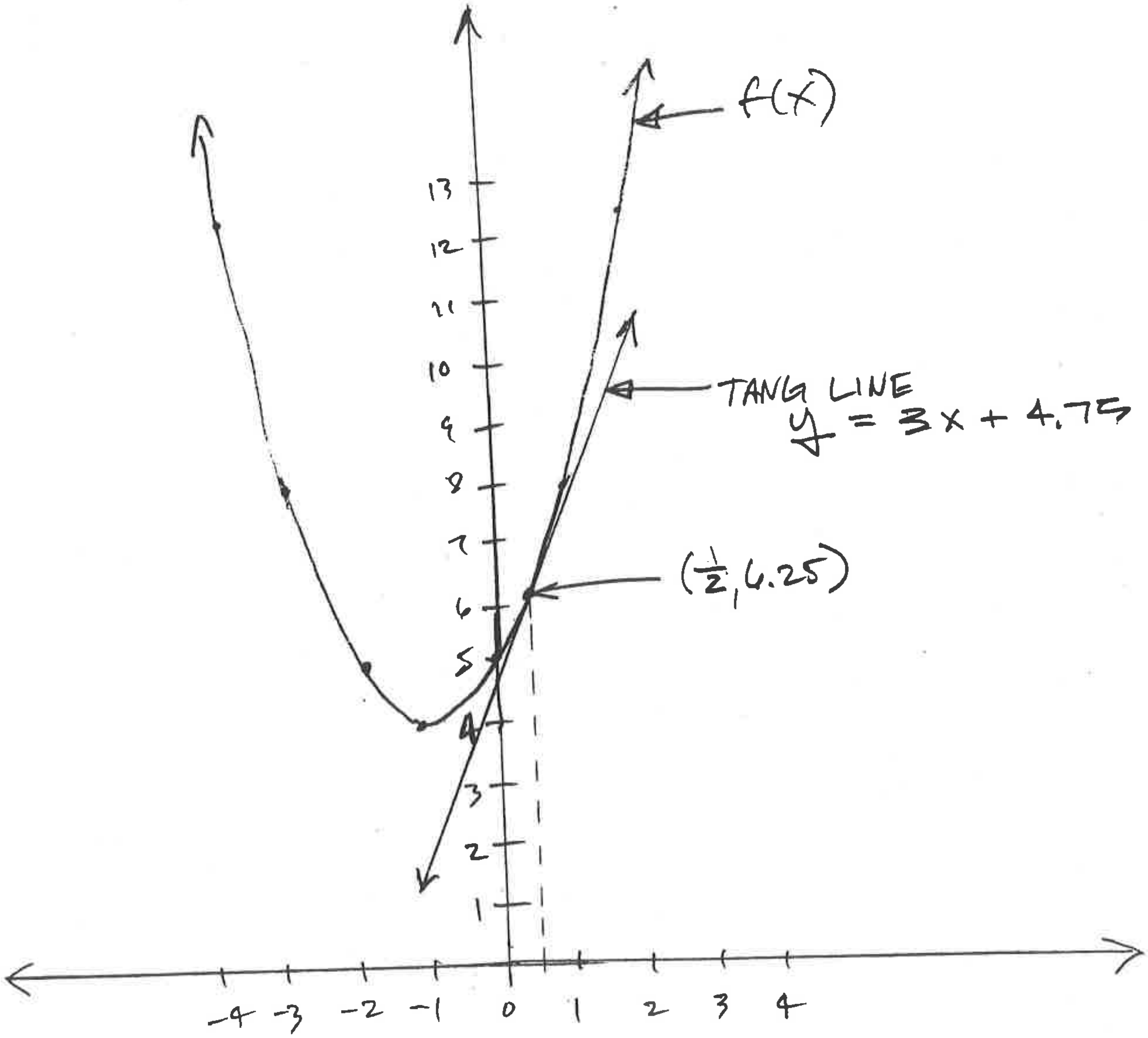
$$f(x) \cong f(c) + f'(x)\Big|_c \cdot (x-c)^1$$

$$\Rightarrow f(x) \cong f(\tfrac{1}{2}) + f'(\tfrac{1}{2}) \cdot (x - 1/2)$$

$$= 6.25 + 3 \cdot (x - \tfrac{1}{2})$$
$$= 6.25 + 3x - 1.5$$
$$= 4.75 + 3 \cdot x$$

EQ^n OF THE TANG LINE @ $x = 1/2$

GRAPH NEXT PAGE



NOTICE HOW "IN THE NEIGHBORHOOD" OF $C = \frac{1}{2}$ THE ERROR BETWEEN THE TRUE FUNCTION AND THE T.S. APPROXIMATION IS "SMALL" AND HOW THE ERROR INCREASES AS YOU MOVE AWAY FROM THE EXPANSION POINT.

TO GET A BETTER FIT, SIMPLY TAKE A HIGHER ORDER T.S. EXPANSION.

FACT:

IF THE ORIGINAL FUNCTION IS AN N^{TH} ORDER POLYNOMIAL, THEN AN N^{TH} ORDER T.S. EXPANSION WILL GIVE A PERFECT FIT. THIS IS BECAUSE ANY N^{TH} ORDER TS EXPANSION <u>IS</u> AN N^{TH} ORDER POLYNOMIAL.

JUST WHAT IS A POLYNOMIAL? (IN CASE YOU WERE ASLEEP IN HIGH SCHOOL)

WELL, A 4^{TH} ORDER POLYNOMIAL IS OF THE FORM

$$y = \underbrace{\underbrace{\overbrace{A + Bx}^{\text{1ST ORDER (LINEAR)}} + Cx^2}_{\text{2ND ORDER POLYNOMIAL (QUADRATIC)}} + Dx^3}_{\text{3RD ORDER POLYNOMIAL (CUBIC)}} + Ex^4$$

SO BASICALLY A T.S. EXPANSION SEEKS TO MIMIC A FUNCTION BY FINDING THE BEST FIT N^{TH} ORDER POLYNOMIAL.

IF YOU MIMIC A 3^{RD} ORDER FN W/ A 3^{RD} ORDER TS EXPANSION - YOU'LL GET AN EXACT FIT. IF YOU USE A 1^{ST} ORDER T.S. EXPANSION YOU GET A TANGENT LINE (@ $x=c$). IF YOU USE A 2^{ND} ORDER T.S. EXPANSION →

YOU'LL GET A QUADRATIC ~~THAT~~ IS THE ~~BEST~~ POSSIBLE FIT TO THE CUBIC (IN THE NEIGHBORHOOD OF $X=C$).

SO WHAT IS <u>NOT</u> A POLYNOMIAL?
I.E. WHAT CANNOT BE WRITTEN AS

$$A + Bx^1 + Cx^2 + Dx^3 \text{ ETC}$$

<u>NON-</u> POLYNOMIALS INCLUDE:

$$y = \frac{1}{1-x}$$

$$y = \sqrt{x}$$

$$y = \sin(x) \quad \& \quad \cos(x)$$

$$y = e^x$$

BUT BY USING AN N^{TH} ORDER T.S. EXPANSION OF THESE FUNCTIONS, WE CAN APPROXIMATE THEM AS N^{TH} ORDER POLYNOMIALS.

LETS LOOK AT A "HIGH ORDER" T.S. EXPANSION OF e^x AROUND $X=0$

ALL DERIVATIVES OF e^x ARE e^x
AND e^x EVALUATED AT $X=0$ EQUALS 1.00,
THUS,

$$f(x) = e^x \cong e^0 + e^0(x-0)^1 + \frac{1}{2!}e^0(x-0)^2 + \frac{1}{3!}e^0(x-0)^3 + \ldots$$

$$f(x) = e^x \cong 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots$$

AND IF WE CONTINUE THIS FOREVER YOU'LL GET THE APPROXIMATION TO BE EXACT.

Q: WHAT HAPPENS IF $X = 1$

A: FROM ABOVE

$$e = 1 + 1 + \frac{1}{2}(1)^2 + \frac{1}{3!}(1)^3 + \frac{1}{4!}(1)^4 + \frac{1}{5!}(1)^5 + \ldots$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = 2.71828\ldots$$

THIS IS ONE OF THE POSSIBLE EXPRESSIONS FOR e.

LET'S LOOK AT ANOTHER EXPRESSION FOR e:

CONSIDER $f(x) = \left(1 + \frac{1}{x}\right)^x$

LET'S EVALUATE THIS AT $X = 1, 2, 3$ ETC...

$$f(1) = \left(1 + \frac{1}{1}\right)^1 = 2.00$$

$$f(2) = \left(1 + \frac{1}{2}\right)^2 = 2.250$$

$$f(3) = \left(1 + \frac{1}{3}\right)^3 = 2.37037$$

$$f(4) = \left(1 + \frac{1}{4}\right)^4 = 2.44141$$

AS X INCREASES TOWARD ∞ WE WRITE

$$\lim_{x \to \infty} f(x) = \underbrace{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e}_{\text{THIS IS THE ACTUAL DEFINITION OF } e}$$

▷

HOW DOES e RELATE TO FINANCE?

RECALL FROM BUS-342

$$\left(1 + \frac{r^{APR}}{f}\right)^f = 1 + r^{\substack{\text{ANNUAL} \\ \text{EFFECTIVE}}}$$

w/ f = FREQUENCY OF COMPOUNDING

EX 10%^{APR} w/ MONTHLY COMPOUNDING GIVES AN EFFECTIVE (ANNUALIZED) RETURN OF 10.47%

ie $\left(1 + \frac{.10}{12}\right)^{12} = 1.1047$