(1)

FIN. ENGR / BUS 444

LECTURE NOTES

STOCHASTICS AND { WILMOTT CH 3, HULL CH ~~10~~ 11

MODEL OF THE BEHAVIOR OF STOCK PRICES

PHOTOCOPY THIS

WEEK 4

WAS CHAPTER 10 IS NOW CH 11

▷ WE ASSUME:

STOCK RETURNS HAVE 2 COMPONENTS: ONE DUE TO SYSTEMATIC RISK, THE OTHER DUE TO IDIOSYN. RISK. THE SYS. RISK (S.R.) COMPONENT PROVIDES A RETURN AS COMPENSATION FOR S.R., THE REMAINING COMPONENT IS TIED TO IDIOSYN RISK (I.R.) WHICH ON AVERAGE SHOULD NOT BE COMPENSATED. THUS, THE MEAN RETURN ASSOC w/ I.R. SHOULD HAVE MEAN ZERO.

THUS

$$\left(\text{TOTAL RETURN}\right)_t = \underbrace{\left(\text{S.R. RETURN}\right)_t}_{\text{SIZE IS RELATED TO AMT OF SYS. RISK.}} + \underbrace{\left(\text{I.R. RETURN}\right)_t}_{\text{MEAN ZERO}}$$

WE'LL WRITE AS

dt = ELAPSED TIME (A VERY SMALL #)

(*)
$$\text{RETURN} = \frac{dS}{S} = \underbrace{\mu \cdot dt}_{\text{EXPECTED RETURN PART}} + \underbrace{\sigma \cdot dz}_{\text{RANDOM PART OF RETURN}}$$

S IS STOCK PRICE, SO THIS IS LIKE $\frac{\Delta S}{S}$ = RETURN

μ: MEAN RETURN PER PERIOD

σ: STND DEV OF RETURNS

dz: A RANDOM VARIABLE w/ MEAN ZERO

THE LAST TERM, $\sigma \cdot dz$, IS THE RANDOM PORTION OF THE RETURN. SEE FIGURE 10.2 IN HULL, IT SHOWS A PLOT OF dz. A PLOT OF $\sigma \cdot dz$ WOULD JUST BE SCALED IN THE VERTICAL DIRECTION BY A FACTOR σ.

dz IS KNOWN AS A WEINER PROCESS, AKA A BROWNIAN MOTION. IT IS OFTEN USED IN PHYSICS TO MODEL OF MOTION OF ERATIC PARTICLES.

dz IS ALSO EQUAL TO $\varepsilon \cdot \sqrt{dt}$ WHERE $\varepsilon \sim N(0,1)$ (IE ITS DRAWN FROM A STND NORMAL DISTRIBUTION).

MEAN $= 0$
$\sigma' = 1$

FACT IF $\varepsilon \sim N(0,1)$

THEN $k \cdot \varepsilon \sim N(0, k)$ (W/ k = SOME CONSTANT #)

σ'

THUS SINCE $\varepsilon \sim N(0,1)$

THEN

$$dz = \underbrace{\sqrt{dt}}_{\text{A CONST \#}} \cdot \varepsilon \sim N(0, \sqrt{dt})$$

σ'_{dz}

THUS YOU CAN THINK OF dz AS BEING DRAWN FROM A NORMAL DIST$^{\underline{B}}$ W/ MEAN 0 AND $\sigma' = \sqrt{dt}$ (IE $\sigma^2_{dz} = dt$)

LETS COMPUTE THE MEAN AND VARIANCE OF OUR $\frac{dS}{S}$ PROCESS ON PG 1.

* IMPORTANT! *

RECALL, THE VARIANCE OF AN EQUATION W/ 2 TERMS IS FOUND VIA:

IF EQ^n IS: $\tilde{y} = a \cdot \tilde{x} + b \cdot \tilde{y}$ (X & Y ARE RANDOM VAR'S. IE $\sigma > 0$)

a,b = CONSTANTS.

THEN VARIANCE OF $\tilde{y}$ IS:

$$\sigma_y^2 = a^2 \cdot \sigma_x^2 + b^2 \cdot \sigma_y^2 + 2 \cdot a \cdot b \cdot \sigma_x \cdot \sigma_y \cdot \underbrace{\rho_{x,y}}_{\text{CORRELATION}}$$

(BUS-342 VARIANCE OF A 2 ASSET PORTFOLIO)

RECALL EQ^n PG 1

$$\frac{dS}{S} = \underbrace{\mu \cdot dt}_{\substack{\text{THIS IS NOT A} \\ \text{RAND. VAR. IE} \\ \text{IT'S A CONSTANT}}} + \underbrace{\sigma \cdot dz}_{\text{A R.V.}}$$

$$\text{MEAN}\left(\frac{dS}{S}\right) = \mu \cdot dt$$ ← SINCE MEAN OF $\sigma \cdot dz = 0$ PER PG 2.

$$\text{VAR}\left(dS/S\right) = \sigma^2 \cdot \underbrace{dt}_{\text{VAR OF } dz \text{ PER PG 2}}$$ ← ONLY THE SECOND TERM IN dS/S IS RANDOM.

FACT USED, IF K = CONST

THEN $\text{VAR}(K\tilde{x}) = K^2 \sigma_x^2$

WE HAD $\underset{\text{CONST}}{\sigma} \cdot \underset{\text{RV}}{dz}$, THUS $\text{VAR}(\sigma \cdot dz) = \sigma^2 \cdot \sigma_{dz}^2$

$$= \sigma^2 \cdot dt$$

THEREFORE WE KNOW:

$$\frac{dS}{S} \sim N\left(\mu dt, \underbrace{\sigma' \cdot \sqrt{dt}}_{\sigma' \text{ OF } \frac{dS}{S}}\right)$$

EQ^N 10.8 OF HULL

$\triangle$

FACT:

Q: SUPPOSE WE HAVE A VARIABLE, $\tilde{x} \sim N(0,1)$ AND WE WANT TO HAVE A VARIABLE w/ MEAN .2 AND $\sigma_y = .3$. HOW DO WE GET IT? (CALL THE NEW VARIABLE $\tilde{y}$).

A: DO THIS

$$\tilde{y} = \tilde{x} \cdot (.3) + .2$$

THAT IS, DRAW A VALUE OF X FROM $N(0,1)$ THEN MULT BY σ_y $(=.3)$ AND ADD THE MEAN OF y $(=.2)$.

YOU NOW HAVE A DRAW FROM $N(.2, .3)$

- eg SIMILAR TO EX 10.3 IN HULL

ASSUME $\mu_{IBM} = .12/\text{YR}$ w/ $\sigma'_{\text{ANNUALIZED}} = .40$

ASSUME THE STOCK RETURNS FOLLOW:

$$\frac{dS}{S} = \mu \cdot dt + \sigma \cdot dz$$

Q: WHAT RANGE OF STOCK PRICES, GIVEN A CURRENT PRICE OF \$100/SH, COULD WE EXPECT TO SEE w/ 95% PROBABILITY IN 1 MONTH? ($dt = 1/12$).

$\downarrow$

$$dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot dz$$ (MOVE S TO RHS)

$$= \mu S dt + \sigma \cdot S \cdot (\varepsilon \cdot \sqrt{dt})$$ ← VIA pg 2

w/ $\varepsilon \sim N(0,1)$

$$= (.12)(100)(1/12) + (.4)(100)\,\varepsilon\sqrt{1/12}$$

$$dS = \underbrace{1.00 + 11.547 \cdot \varepsilon}_{}$$ $\varepsilon \sim N(0,1)$

THIS LOOKS A LOT LIKE MY "FACT" ON pg 4.

THAT IS, THE CHANGE IN THE STOCK PRICE IS DIST^{D} NORMAL w/ MEAN 1.00 AND $\sigma' = 11.547$.

THERE IS A 95% CHANCE dS WILL MOVE ± 1.96 STND DEVIATIONS FROM THE MEAN OF 1.00. THUS WE CAN SAY THAT W/ 95% PROB, dS WILL RANGE FROM −21.63 TO +23.63

$1.00 - 1.96(11.547)$ $\qquad$ $1.00 + 1.96(11.547)$

THUS THERE IS ONLY A 5% CHANCE THAT S IN 1 MONTH WILL BE <u>OUTSIDE</u> THE RANGE OF 78.37 → 123.63

$= 100 - 21.63$ $\qquad$ $= 100 + 23.63$

OUR BEST <u>POINT</u> <u>ESTIMATE</u> GUESS, IS THAT IN 1 MONTH THE PRICE WILL BE \$101.^{00}, BUT THERE IS A LOT OF IDIOSYNCRATIC RISK, SO IT COULD FALL INTO A WIDE RANGE.

§10.4 HULL

$$\frac{ds}{s} = \mu\, dt + \sigma\, dz$$ IS AKA GEOMETRIC BROWNIAN MOTION (GBM)

IF WE DIDN'T DIVIDE ds BY S ON THE LHS THEN IT WOULD BE CALLED BROWNIAN MOTION (THE GEOMETRIC PART IS INCLUDED ONLY IF S IS IN THE LHS DENOM).

GBM IS WRITTEN 2 WAYS, AS ABOVE, AND AS:

$$ds = \mu \cdot S \cdot dt + \sigma \cdot S \cdot dz$$

MOVING S TO RHS. THIS IS STILL GBM. SINCE YOU COULD WRITE AS $\frac{ds}{s} = \ldots$

IN OUR EXAMPLE S FOLLOWS GBM AND X FOLLOWS BM. (THE GEOMETRIC PART MEANS ds IS ÷ BY S)

NOTE: REGULAR BROWNIAN MOTION (BM) IS $dx = \alpha\, dt + \sigma\, dz$.

MONTE CARLO SIMULATION

THIS MEANS TO DRAW THE INPUT PARAMETERS FROM SOME ASSUMED DISTRIBUTION, AND TO THEN DETERMINE HOW SOMETHING (THE OUTPUT) BEHAVES.

ASSUME $\mu = .14$, $\sigma = .20$ ← ANNUALIZED VALUES, THUS dt MUST BE IN # OF YRS, i.e. $1/12$ ETC (1 MONTH).

LET $dt \cong \Delta t = .01/\text{YR}$ (3.65 DAYS)

Q: HOW IS ds/s THEN DISTRIBUTED?

THAT IS, WE KNOW ds/s WILL BE

DIFFERENT IN DIFFERENT PERIODS (IT'S A $\tilde{RV}$) BUT WHAT DISTRIBUTION DO WE ENVISION IT COMING FROM?

FIND DIST^{B} OF dS/S: ← PER PERIOD RETURN.

WE KNOW:

RETURN PER PERIOD ⟶
$$\frac{dS}{S} = \mu\, dt + \sigma \cdot dz$$
$$= \mu\, dt + \sigma \cdot \varepsilon \cdot \sqrt{dt}$$
$$= (.14)(.01) + (.20)\, \varepsilon \sqrt{.01}$$
$$= .0014 + .02 \cdot \underset{\sim N(0,1)}{\varepsilon}$$

FROM OUR "FACT" pg 4

THIS IMPLIES THAT THE RETURNS ($= dS/S$) ARE DISTRIBUTED $N(\underset{\text{MEAN RETURN}}{.0014}, \underset{\sigma_{RET}}{.02})$

SEE TABLE 10.1 FOR A MONTE CARLO SIMULATION OF THE RETURNS (COLUMN 3) WHICH IMPLIES ΔS (COLUMN 4), WHICH IMPLIES STOCK PRICE (COLUMN 1).

▷ RECALL FROM BUS-342 AND BUS-433 (INT'L) THAT

$$\text{MEAN MONTHLY RETURN} = \frac{\text{MEAN ANNUAL RETURN}}{12}$$

AND

$$\text{STND DEV OF MONTHLY RETURNS} = \frac{\text{STND DEV OF ANNUALIZED RETURNS}}{\sqrt{12}}$$

↓

THESE "ADJUSTMENT" EQUATIONS <u>ARE</u> BUILT INTO GBM. THAT IS, ASSUME $\mu = .12/\text{YR}$

$\sigma = .20/\text{YR}$

WHAT IS μ/mo & σ'/mo ?

FROM OUR "ADJUSTM'T" EQUATIONS, BOTTOM pg 7, IT IS:

$$\mu/\text{mo} = .12/12 = .01$$

$$\sigma'/\text{mo} = .20/\sqrt{12} = .0577$$

THESE MATCH!

USING GBM:

$$\frac{ds}{s} = \mu\, dt + \sigma \cdot dz$$

$$= \mu\, dt + \sigma\, \varepsilon \sqrt{dt}$$

$$= (.12)(1/12) + (.20)(\varepsilon)\sqrt{1/12}$$

$$= \underset{\text{MEAN/mo}}{.01} + \underbrace{.0577}_{\text{STND DEV/mo}} \cdot \varepsilon, \quad \varepsilon \sim N(0,1)$$

§10.6 ITO'S LEMMA

Q: IF AN UNDERLYING VARIABLE FOLLOWS GBM OR BM, THEN WHAT PROCESS DOES THE 'DERIVATIVE' PROCESS FOLLOW?

(BY '<u>DERIVATIVE</u>' WE MEAN THAT ONE VARIABLE IS A FUNCTION OF, ie IT'S <u>DERIVED</u> FROM, ANOTHER 'UNDERLYING' (INPUT) VARIABLE).

TO ANSWER THIS QUESTION YOU <u>MUST</u> USE ITO'S LEMMA.

FOR EXAMPLE, A FORWARD PRICE DEPENDS ON THE VALUE OF S. THUS IF S FOLLOWS GBM, THEN WHAT PROCESS DOES F FOLLOW? —MUST USE ITO'S LEMMA!

TO GET (DERIVE) ITO'S LEMMA, WE'LL USE THE IDEA'S BEHIND TAYLOR SERIES EXPANSIONS (SEE 'MATHEMATICAL PRELIMINARIES' HANDOUT FIRST)

SUPPOSE dx FOLLOWS THE PROCESS

$$dx = \alpha\, dt + \sigma\, dz$$

{ X FOLLOWS BM
S " GBM

AND LET $dx = \frac{dS}{S}$ (THINK OF X AS A RETURN)

SUPPOSE G DEPENDS ON X (G IS A DERIVATIVE SECURITY)

Q: WHAT DOES THE PROCESS FOR dG LOOK LIKE? ie HOW DO CHANGES IN G EVOLVE?

A: dG IS LIKELY TO BE A COMPLEX PROCESS SINCE IT DEPENDS ON dx WHICH IS A COMPLEX PROCESS. LETS SIMPLIFY dG VIA TAKING A TAYLOR SERIES EXPANSION OF G (NOT dG) AROUND THE POINT x_0.

RECALL, A TAYLOR SERIES EXPANSION OF $f(x)$ AROUND x_0 IS: (SEE MATH HANDOUT)

$$f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} \cdot (x-x_0)^1 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} \cdot (x-x_0)^2 +$$

$$+ \frac{1}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x_0} \cdot (x-x_0)^3 + \ldots$$

NOW LETS APPLY THIS IDEA IN A 'DIFFERENT' WAY:

ie. LET'S EXPAND $G(x+\Delta x)$ AROUND THE POINT x

(TYPICALLY WE EXPAND $G(x)$ AROUND SOME POINT x_0, BUT THE ABOVE 'DIFFERENT' EXPANSION IS EQUALLY VALID, AND IT WILL LEAD US TO AN EXPRESSION WE WANT)

3RD DEGREE TAYLOR SERIES EXPANSION OF $G(x+dx)$ AROUND x IS:

$$G(x+\Delta x) = G(x) + \left.\frac{\partial G}{\partial x}\right|_x \cdot (x+\Delta x - x) + \frac{1}{2!}\left.\frac{\partial^2 G}{\partial x^2}\right|_x (x+\Delta x - x)^2 + \frac{1}{3!}\left.\frac{\partial^3 G}{\partial x^3}\right|_x (x+\Delta x - x)^3 + \ldots$$

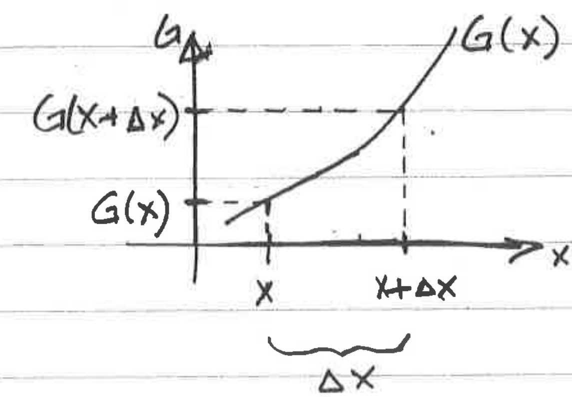
$$= G(x) + \frac{\partial G}{\partial x}(\Delta x) + \frac{1}{2!}\frac{\partial^2 G}{\partial x^2}\cdot(\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 G}{\partial x^3}(\Delta x)^3 + \ldots$$

$\frac{\partial G}{\partial x}$ EVALUATED @ x $\left(\text{ie } \left.\frac{\partial G}{\partial x}\right|_x\right)$ IS JUST $\frac{\partial G}{\partial x}$

NOTICE THAT $G(x+\Delta x) - G(x)$ EQUALS ΔG, ie.

ie

$G(x+\Delta x) - G(x) = \Delta G \rightarrow$



THEREFORE WE CAN WRITE THE ABOVE EXPANSION AS

$$\Delta G = \frac{\partial G}{\partial x}\cdot(\Delta x) + \frac{1}{2!}\frac{\partial^2 G}{\partial x^2}\cdot(\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 G}{\partial x^3}\cdot(\Delta x)^3 + \ldots$$

BUT WHAT IF G DEPENDS ON 2 INPUT VARIABLES, X AND t? HOW TO TAKE THE TAYLOR SERIES EXPANSION? AS SHOWN IN THE "MATH PRELIM" HANDOUT A 2ND DEGREE TAYLOR EXPANSION OF $G(x,t)$ AROUND THE POINT x_0, t_0 IS:

$$G(x,t) = G(x_0,t_0) + \left.\frac{\partial G}{\partial x}\right|_{x_0,t_0} \cdot (x-x_0)^1 + \left.\frac{\partial G}{\partial t}\right|_{x_0,t_0} \cdot (t-t_0)^1 +$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 G}{\partial x^2}\right|_{x_0,t_0} \cdot (x-x_0)^2 + \underbrace{\frac{2}{2!}}_{=1} \cdot \left.\frac{\partial^2 G}{\partial x \partial t}\right|_{x_0,t_0} \cdot (x-x_0)^1 \cdot (t-t_0)^1 + \frac{1}{2!}\left.\frac{\partial^2 G}{\partial t^2}\right|_{x_0,t_0} \cdot (t-t_0)^2 + \ldots$$

NOW, SIMILAR TO WHAT WE DID ON TOP PG 10, LETS EXPAND $G(x+\Delta x, t+\Delta t)$ AROUND X AND t: THE 2ND DEGREE EXPANSION IS:

$$G(x+\Delta x, t+\Delta t) = G(x,t) + \left.\frac{\partial G}{\partial x}\right|_{x,t} \cdot (x+\Delta x - x)^1 + \left.\frac{\partial G}{\partial t}\right|_{x,t} \cdot (t+\Delta t - t)^1 +$$

$$+ \frac{1}{2!}\left.\frac{\partial^2 G}{\partial x^2}\right|_{x,t} \cdot (x+\Delta x - x)^2 + \frac{2}{2!}\left.\frac{\partial^2 G}{\partial x \partial t}\right|_{x,t} \cdot (x+\Delta x - x)^1 \cdot (t+\Delta t - t)^1 + \frac{1}{2!}\left.\frac{\partial^2 G}{\partial t^2}\right|_{x,t} \cdot (t+\Delta t - t)^2 + \ldots$$

WHICH WE CAN REWRITE AS:

$$\Delta G = \frac{\partial G}{\partial x} \cdot (\Delta x)^1 + \frac{\partial G}{\partial t} (\Delta t)^1 +$$

$$(10.A.6) \quad \frac{1}{2} \cdot \frac{\partial^2 G}{\partial x^2} \cdot (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} \cdot (\Delta x) \cdot (\Delta t) + \frac{1}{2} \cdot \frac{\partial^2 G}{\partial t^2} \cdot (\Delta t)^2 + \ldots$$

EQ^N (10.A.6) IN THE APPENDIX OF CH 10.

WE COULD TAKE THE LIMIT OF (10.A.6) AS Δx AND Δt GET SMALLER & SMALLER (ie GO TO ZERO).

WE WRITE:

$$\lim_{\Delta x \to 0} = dx \qquad \lim_{\Delta t \to 0} = dt$$

AFTER WE TAKE THE LIMIT, (10.A.6) WILL HAVE TERMS INVOLVING dx, dt, $(dx)^2$, $dx \cdot dt$, AND $(dt)^2$.

BOTH dx AND dt ARE VERY SMALL, BUT NOT QUITE ZERO. THUS

$$dx \neq 0$$
$$dt \neq 0.$$

IN REGULAR CALCULUS, THE ARGUMENT IS MADE THAT SINCE dx AND dt ARE SO SMALL IT MUST BE TRUE THAT $(dx)^2$, $(dt)^2$ AND $dx \cdot dt$ ARE ZERO.

THEREFORE, IN REGULAR CALCULUS, AFTER TAKING THE LIMIT, (10.A.6) WOULD EQUAL:

$$\text{®} \quad dG = \underbrace{\frac{\partial G}{\partial x} \cdot dx + \frac{\partial G}{\partial t} \cdot dt}_{} + \underset{\substack{\text{SINCE}\\(dx)^2=0}}{0} + \underset{\substack{\text{SINCE}\\dxdt=0}}{0} + \underset{\substack{\text{SINCE}\\(dt)^2=0}}{0} + \underbrace{\ldots}_{\substack{\text{ALSO}\\\text{ALL}\\\text{ZERO'S}}}$$

A KEY RESULT IN REGULAR CALCULUS.
AKA THE TOTAL DIFFERENTIAL.

NOW WE NEED TO ASK, DOES THIS RESULT HOLD TRUE IF X FOLLOWS GBM OR BM?

ASSUME X FOLLOWS THE PROCESS

NOTE: HULL USES $a = \mu$, $b = \sigma$ IN THE APPENDIX

$$dx = \mu\, dt + \sigma\, dz$$ ← B.M. = BROWNIAN MOTION

$$= \mu\, dt + \sigma \cdot \varepsilon \cdot \sqrt{dt}$$ ← BROWNIAN MOTION (B.M.)

$\varepsilon \sim N(0,1)$

Q: WHAT DOES $(dx)^2$ LOOK LIKE?

DOES IT GO TO ZERO IN THE LIMIT? — LETS FIND OUT...

$$dx^2 = (\mu\, dt + \sigma \varepsilon \sqrt{dt}) \cdot (\mu\, dt + \sigma \cdot \varepsilon \cdot \sqrt{dt})$$

$$= \mu^2 dt^2 + 2\mu\sigma\varepsilon\, dt^{3/2} + \sigma^2 \varepsilon^2 (dt)^1$$

- WE KNOW $(dt)^2 = 0$ (SINCE dt IS VERY SMALL)
- IT'S ALSO TRUE THAT dt RAISED TO ANY POWER > 1.00 EQUALS ZERO TOO, THUS $(dt)^{3/2} = 0$

SO WE HAVE

$$(dx)^2 = \sigma^2 \varepsilon^2 (dt) + \text{TERMS INVOLVING } dt \text{ THAT EQUAL ZERO IN THE LIMIT.}$$

$$(dx)^2 = \sigma^2 \varepsilon^2 (dt) \neq 0$$

BUT LOOK!, IF X FOLLOWS BM, THEN $(dx)^2 \neq 0$!

IN REGULAR CALC $(dx)^2 = 0$

IN $\underbrace{\text{STOCHASTIC}}_{}$ CALC $(dx)^2 \neq 0$!

MEANS THAT SOME VARIABLE FOLLOWS BM OR GBM.

LETS LOOK CLOSER @ THE EXPRESSION

$$(dx)^2 = \sigma^2 \varepsilon^2 \cdot dt$$

LETS THINK ABOUT $\varepsilon \sim N(0,1)$.

ONE WAY TO WRITE THE VARIANCE OF A RV, $\tilde{X}$ IS

$$E[X^2] - (E[X])^2 = \text{VARIANCE OF } X$$

VERY IMPORTANT TO KNOW.

ie $E[X^2] - (\text{MEAN OF } X)^2 = VAR(X)$

SQUARE ALL VALUES, THEN FIND THEIR MEAN

ie $$\underbrace{\frac{\sum_{i=1}^{N} x_i^2}{N} - \left(\frac{\sum_{i=1}^{N} x_i}{N}\right)^2}_{} = VAR(X)$$

JUST ANOTHER WAY TO WRITE:

$$\frac{\sum_{i=1}^{N} (x_i - \bar{X})^2}{N} = VAR(X)$$

SINCE $\varepsilon \sim N(0,1)$

WE KNOW $E[\varepsilon] = 0$, $VAR(\varepsilon) = 1$

THUS WE KNOW

$$E[\varepsilon^2] - (E[\varepsilon])^2 = VAR(\varepsilon)$$

$$E[\varepsilon^2] - (0)^2 = 1$$

$$\Rightarrow E[\varepsilon^2] = 1$$

THUS WHEN WE TAKE THE EXPECTATION OF $(dx)^2$ (IN ORDER TO FIGURE OUT WHAT WE EXPECT IT TO BE ON AVERAGE) AND THEREFORE TAKE THE EXPECTATION OF $\sigma^2 \tilde{\varepsilon}^2 dt$ WE GET

$$E[\sigma^2 \tilde{\varepsilon}^2 dt] = \underbrace{\sigma^2 \cdot dt}_{\text{PULL THE CONSTANTS OUT FRONT.}} \cdot E[\tilde{\varepsilon}^2] = \sigma^2 dt \cdot 1$$

$$= \sigma^2 \cdot dt$$

THUS $E[dx^2] = \sigma^2 \cdot dt$ ← MEAN OF $(dx)^2$
(THINK OF EXPECTATION AS BEING THE MEAN)

BUT WHAT IS THE VARIANCE OF $(dx)^2$?

$$VAR(dx^2) = VAR(\underbrace{\sigma^2}_{\text{A CONSTANT}} \tilde{\varepsilon}^2 \underbrace{dt}_{\text{A CONSTANT}})$$

FACT: $VAR(\text{CONST} \cdot \tilde{x}) = (\text{CONST})^2 \cdot \sigma_x^2$

THUS

$$VAR(dx^2) = \sigma^4 (dt)^2 \underbrace{VAR(\tilde{\varepsilon}^2)}_{\text{WHAT IS THIS?}}$$

MEAN = 0
VAR = 1

WE KNOW $\varepsilon \sim N(0,1)$, ALSO FROM STAT'S, IF YOU TAKE N, STANDARD NORMALLY DIST^B VARIABLES, SQUARE EACH OF THEM, AND THEN ADD THEM UP, THE RESULTING SUM IS DIST^B CHI-SQUARED W/ DEGREE OF FREEDOM = N, ie χ^2_N

THE MEAN OF A χ^2_N VARIABLE IS N

THE VARIANCE OF A χ^2_N VARIABLE IS ALSO N.

LOOK @ $VAR(\tilde{\varepsilon}^2)$. THIS IS THE SUM OF 1 $(N=1)$ STND NORMAL $\tilde{RV}$ (SINCE $\varepsilon \sim N(0,1)$) THUS ε^2 IS DISTRIBUTED χ^2_1 AND THUS HAS MEAN = 1 AND VARIANCE = 1.

THUS $VAR(dx^2) = \sigma^4 (dt)^2 \cdot 1$

BUT $(dt)^2 = 0$ SO $VAR(dx^2) = 0$

SO WE KNOW

$$E[dx^2] = \sigma^2 dt$$ TOP pg 13.3

AND

$$VAR(dx^2) = 0$$ BOTTOM pg 13.3

THEREFORE dx^2 IS A CONSTANT w/ VALUE OF $\sigma^2 dt$. (AS Δt GOES TO ZERO IN THE LIMIT)

WE SAY THAT $(dx)^2$ IS NON-STOCHASTIC (FANCY TALK FOR "WE KNOW ITS ACTUAL VALUE NO MATTER WHAT THE SITUATION" — A STOCHASTIC VALUE HAS SOME "NOISE" ASSOCIATED w/ IT SO THAT WE CANT PREDICT IT FOR SURE. A NON-STOCHASTIC VARIABLE CAN ALWAYS BE PREDICTED w/ 100% ACCURACY. IT'S AKA A DETERMINISTIC VARIABLE, ie IT'S EASILY DETERMINED).

THUS $(dx)^2$ IS NON-STOCHASTIC (ie DETERMINISTIC) AND EQUAL TO $\sigma^2 \cdot dt$ AS $\Delta t \to 0$.

NOW THAT WE FULLY UNDERSTAND HOW $(dx)^2$ BEHAVES LETS SEE WHAT OUR TAYLOR SERIES EXPANSION (10.A.6) pg 11 REDUCES TO, WE WONT GET ⊕ pg 12, INSTEAD WE GET

$$(10.A.9) \quad dG = \frac{\partial G}{\partial x}\cdot dx + \frac{\partial G}{\partial t}\cdot dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\cdot (dx)^2 + 0 + 0 + \ldots$$

EQ^N 10.A.9 IN APPENDIX CH 10.

CUZ $dx \cdot dt = 0$

CUZ $dt^2 = 0$

(10.A.9) IS WHAT YOU GET w/ STOCHASTIC CALC.
(*) pg 12, " " " " " REGULAR ".

THEY DIFFER BY THE TERM $\frac{1}{2}\cdot\frac{\partial^2 G}{\partial x^2}(dx)^2$

▷ SINCE ALL THIS WAS DRIVEN BY OUR ASSUMPTION THAT X FOLLOWS BROWNIAN MOTION, LETS PLUG $dx = \mu dt + \sigma dz$ INTO (10.A.9) AND SEE WHAT WE GET.

RECALL (10.A.9) IS:

(10.A.9)
$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2$$

pg 13.4

THUS,

$$dG = \frac{\partial G}{\partial x}(\mu dt + \sigma dz) + \frac{\partial G}{\partial t}\cdot dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\sigma^2 dt)$$

NOW COLLECT dt AND dz TERMS:

BM*
$$dG = \left(\frac{\partial G}{\partial x}\cdot\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2\right)dt + \frac{\partial G}{\partial x}\sigma\, dz$$

EQ^n (10.A.9) IS ITO'S LEMMA. IN FINANCE WE OFTEN WORK DIRECTLY w/ BM* ABOVE. BM* TELLS YOU HOW A VARIABLE (G IN THIS CASE) MOVES WHEN ONE OF ITS INPUTS FOLLOWS BM.

IF INSTEAD X FOLLOWS GBM THEN WE GET

(10.A.6) $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (dx)^2$$ ITO'S LEMMA

w/ $dx = \mu \underline{\underline{X}}\, dt + \sigma \underline{\underline{X}}\, dz$ ← $\underline{\underline{GBM}}$

COMBINING WE GET:

$dx^2 = \sigma^2 X^2 dt$ (GBM)

$$dG = \frac{\partial G}{\partial x}(\mu X dt + \sigma X dz) + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \underbrace{(\sigma^2 X^2 dt)}$$

THIS IS WHAT $(dx)^2$ CONVERGES TO FOR $\underline{GBM}$, PER PG 13.4 IT CONVERGES TO $\sigma^2 dt$ FOR $\underline{\underline{BM}}$.

THUS, COLLECTING TERMS WE GET

GBM*

$$dG = \underbrace{\left(\frac{\partial G}{\partial X} \cdot \mu X + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} \sigma^2 X^2\right) dt}_{\text{TREND}} + \underbrace{\left(\frac{\partial G}{\partial X} \sigma X\right) dz}_{\text{NOISE.}}$$

(DUE TO NOISE IN x)

SIMILAR TO, BUT DIFFERENT THAN (BM*) PG 14.

DESCRIBES HOW G EVOLVES IF IT DEPENDS ON X AND X FOLLOWS $\underline{\underline{GBM}}$.

IN FINANCE (GBM*) IS USED A LOT, MORE THAN (BM*) SINCE WE USUALLY ASSUME S FOLLOWS GBM.

HULL — SEE TEXT

$\boxed{\text{EXAMPLE 1}}$

IF A STOCK PRICE FOLLOWS GBM, THEN WHAT PROCESS DOES A FORWARD PRICE FOLLOW? SINCE F DEPENDS ON S VIA $F = Se^{r(T-t)}$ WE MUST USE ITO'S LEMMA.

WE NEED TO TAKE A FEW DERIVATIVES.

$$\frac{\partial F}{\partial S} = e^{r(T-t)} \qquad \frac{\partial^2 F}{\partial S^2} = 0$$

$$\frac{\partial F}{\partial t} = -r \cdot S e^{r(T-t)}$$

$dS = \mu \cdot S dt + \sigma \cdot S \cdot dz$

WE ASSUME S FOLLOWS GBM, THUS GBM* ON pg 15 APPLIES:

$$dF = \left(\frac{\partial F}{\partial S}\mu \cdot S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)dt + \left(\frac{\partial F}{\partial S}\sigma S\right)dz$$

$$dF = \left(e^{r(T-t)}\mu \cdot S + -rSe^{r(T-t)} + 0\right)dt + \left(e^{r(T-t)}\sigma \cdot S\right)dz$$

NOW SUB IN $F = Se^{rt}$

$$dF = (\mu \cdot F - rF)\,dt + (F \cdot \sigma)dz$$

$$dF = \underbrace{(\mu - r)}_{\mu_{\text{OF } F}} F\,dt + \underbrace{F\sigma}_{\sigma_F} dz$$

← GBM w/ $\mu_F = \mu_S - r$ AND $\sigma_F = F \cdot \sigma_S$

F TENDS TO GROW AT RATE $\mu - r$ AS HULL SAID §3.7, pg 60. ($\mu - r$ = EXCESS RETURN ON S)

— SEE TEXT

$\boxed{\text{EXAMPLE 2}}$

LET $G = \ln S$

WHAT PROCESS DOES G FOLLOW IF S FOLLOWS GBM ie

$$dS = \mu S\,dt + \sigma S\,dz$$

① FIND DERIVATIVES

$$\frac{\partial G}{\partial S} = \frac{1}{S} \qquad \frac{\partial^2 G}{\partial S^2} = \frac{-1}{S^2}$$

$$\frac{\partial G}{\partial t} = 0$$

USING GBM* pg 15 WE GET

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \left(\frac{\partial G}{\partial S}\sigma S\right)dz$$

$$= \left(\frac{1}{S}\mu S + 0 + \frac{1}{2}\left(\frac{-1}{S^2}\right)\sigma^2 S^2\right)dt + \left(\frac{1}{S}\sigma S\right)dz$$

$$dG = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dz$$

THEREFORE

$$dG = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma\,\varepsilon\sqrt{dt}$$

$\varepsilon \leftarrow \sim N(0,1)$

THUS dG IS DISTRIBUTED NORMAL w/ MEAN $\left(\mu - \frac{1}{2}\sigma^2\right)$ AND STND DEVIATION OF $\sigma\sqrt{dt}$ ($\Rightarrow$ VARIANCE OF $\sigma^2 dt$)

↓

THEREFORE

$$\underbrace{\ln S_T - \ln S}_{\Delta G} \sim N\left(\mu - \tfrac{1}{2}\sigma^2,\ \sigma\sqrt{T-t}\right)$$

THIS WILL TURN OUT TO BE A VERY IMPORTANT RESULT IN CH. 11.