BNS-392 CALCULUS PART II OF II:

FOR LECTURE

▷ REMEMBER
DO PROBLEMS 5.1 → 5.10 IN SCHAUM'S "MATH FOR ECON" TEXTBOOK. (THEY ARE ALREADY SOLVED, JUST READ THEM OVER).

INTEGRATION (AKA ANTI-DERIVATIVES)

THE REVERSE PROCESS OF TAKING A DERIVATIVE.

IT GIVES THE AREA UNDER A CURVE. (NOT THE SLOPE)

IMPORTANT!

▷ BASIC RULES:

(I) $$\int Ax^{B}\,dx = \frac{A}{B+1}x^{B+1} + C$$

A CONSTANT

INCREASE THE POWER BY 1 & DIVIDE BY THE NEW POWER.

FACT: SINCE A IS A CONSTANT, IT CAN BE PULLED OUT FRONT OF THE INTEGRAL SIGN.

ie $$\int Ax^{B}\,dx = A\cdot\int x^{B}\,dx$$
$$= A\left(\frac{1}{B+1}\right)x^{B+1}$$

NOW LETS TAKE THE DERIV OF ↑ WRT X ↓

$$\frac{d}{dx}\ \frac{A}{B+1}X^{B+1}+C=\left(\frac{A}{B+1}\right)(B+1)\ X^{B}\ \underbrace{+\ 0}_{\text{DERIV OF CONST} = \text{ZERO}}$$

$$= A\,x^{B}$$

IF YOU TAKE THE INTEGRAL OF $f(x)$ AND THEN TAKE THE DERIV OF THE INTEGRAL, YOU'LL GET THE ORIG FUNCTION BACK.

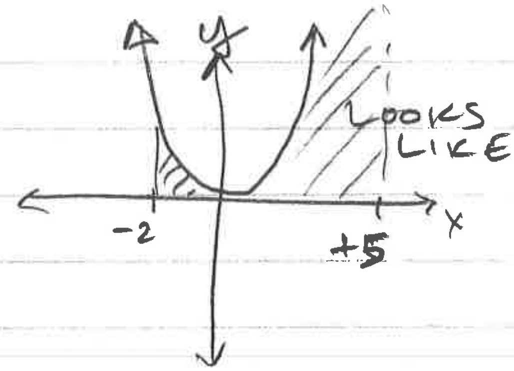
THUS, THE INTEGRAL IS REFERRED TO AS THE ANTI-DERIVATIVE.

TO ACTUALLY FIND THE AREA UNDER A CURVE BETWEEN 2 X VALUES (SAY $X=-1$ AND $X=3.2$) YOU PROCEED AS FOLLOWS

1. COMPUTE THE INTEGRAL PER PREVIOUS PAGES
2. EVALUATE THE INTEGRAL AT THE UPPER VALUE OF X (3.2 IN MY EXAMPLE) AND SUBTRACT FROM THIS THE VALUE OF THE INTEGRAL EVALUATED AT THE LOWER VALUE OF X (-1 IN MY EXAMPLE)
3. THE RESULT IS THE AREA UNDER THE CURVE BETWEEN THE 2 X VALUES.

EX

FIND THE AREA UNDER $f(x) = 3x^2$ FROM $x = -2$ TO $x = +5$



$$\int_{-2}^{+5} 3x^2 \, dx$$

$$= \underbrace{x^3 + C \Big|_{-2}^{+5}}$$

C=CONSTANT

THIS MEANS EVALUATE $x^3 + C$ w/ $x = +5$ AND THE SUBTRACT $x^3 + C$ EVALUATED @ $x = -2$.

(THE CONSTANT CANCELS OUT SO THIS IS THE SAME AS

$x^3 \Big|_{-2}^{+5}$).

$$= x^3 \Big|_{-2}^{+5} = (+5)^3 - (-2)^3$$

$$= 125 - (-8) = 133$$ ANSWER

SOME INTEGRALS ARE TOUGH TO EVALUATE 'CUZ THE FUNCTION IS NOT OF THE FORM Ax^B.

KNOWN AS INTEGRATION BY PARTS.

ONE INTEGRATION TECHNIQUE IS RELATED TO THE PRODUCT RULE. RECALL THE PRODUCT RULE WAS

$$\frac{d}{dx} f(x) \cdot g(x) = f(x) \cdot \frac{dg}{dx} + g(x) \cdot \frac{df}{dx}$$

NOW INTEGRATE BOTH SIDES OF THE PRODUCT RULE. (ie SLAP A $\int \_\ dx$ AROUND EACH TERM). THIS GIVES

$$\underbrace{\int \frac{d}{dx} f(x) \cdot g(x)\, dx}_{= f(x) \cdot g(x)} = \int f(x) \frac{dg}{dx}\, dx + \int g(x) \frac{df}{dx}\, dx$$

(ie THE INTEGRAL OF A DERIVATIVE CANCEL EACH OTHER OUT)

THUS

$$f(x) \cdot g(x) = \int f(x) \frac{dg}{dx}\, dx + \int g(x) \frac{df}{dx}\, dx$$

$\downarrow$

LET'S REARANGE:

$$\int f(x)\cdot\frac{dg}{dx}\,dx = f(x)\cdot g(x) - \int g(x)\cdot\frac{df}{dx}\,dx$$

THIS IS KNOWN AS "INTEGRATION BY PARTS" (IBP)
IT IS USEFUL FOR EVALUATING INTEGRALS THAT ARE EXPRESSED AS A PRODUCT AND INVOLVE e^x, $\ln x$, $\sin x$, OR $\cos x$.

Q: HOW TO DO IT?
A: WHAT YOU DO, IS WHEN YOU HAVE SOMETHING LIKE

$$\int x\cdot\sin(6x)\,dx$$

YOU MUST PUT THIS IN THE FORM OF

$$\int f(x)\cdot\frac{dg}{dx}\cdot dx$$

SO, YOU MUST DETERMINE WHAT IS $f(x)$ AND WHAT IS $\frac{dg}{dx}$

YOU ASSIGN f & g IN A "PRUDENT" MANNER (TO MAKE THE INTEGRAL EASIER)

WE ASSIGN:

LETS SET $f(x) = x$

AND $\frac{dg}{dx} = \sin(6x)$

THIS IMPLIES THAT $\frac{df}{dx} = 1$

ONCE f & $\frac{dg}{dx}$ ARE FOUND, THEN FIND $\frac{df}{dx}$ & g

AND $g = -\frac{1}{6}\cos(6x)$

SIDE NOTE:

Q: HOW DID WE GET $g = \frac{-1}{6}\cos(6x)$ FROM KNOWING $\frac{dg}{dx} = \sin(6x)$?

$\left(\frac{df}{dx} = 6\right)$

LET $f(x) = 6 \cdot x \implies df = 6 \cdot dx$

AND $g(f) = \sin(f) \implies dx = \frac{1}{6} \cdot df$

THEN:

$$g = \int \frac{dg}{dx}\,dx = \int \sin(6x)\,dx = \int \sin(f)\,dx = \int \sin(f)\underbrace{\left(\frac{1}{6} \cdot df\right)}_{=dx}$$

$$= \frac{1}{6}\int \sin(f)\,df = \frac{1}{6}\left[-\cos f\right]$$

NOTE: THIS TECHNIQUE IS KNOWN AS INT-BY-SUBSTITUTION.

$$= \frac{-1}{6}\cos(6x) \quad \leftarrow \text{ANSWER.}$$

✓ IT. (TAKE THE DERIV, YOU GET $\sin(6x)$)

CHECK ANSWER:

$$\frac{d}{dx}\, \frac{-1}{6}\cos(6x)$$

LET $f = 6x$

$g(f) = \frac{-1}{6}\cos(f)$

$$= \frac{dg}{df} \cdot \frac{df}{dx} \quad \leftarrow \text{CHAIN RULE}$$

$$= \underbrace{\frac{-1}{6} \cdot (-\sin(f))}_{\frac{dg}{df}} \cdot \underbrace{6}_{\frac{df}{dx}} = \sin(f) = \sin(6x) \checkmark$$

END OF "SIDENOTE"

GOTO PAGE 19.10 FOR MORE ON INT-BY-SUBSTITUTION

BVS 343

PAGES TO GO BETWEEN pg 19 & 20...

$\longrightarrow$ 19.10

FOLLOWS PAGE 19.0

SOME MORE COMMENT ON INT-BY-SUBSTITUTION:

w/ THE INT-BY-SUB ~~METHOD~~, NEW VARIABLES ARE DEFINED, AND THESE VARIABLES ARE SUBBED INTO THE INTEGRAL TO BE EVALUATED (IN THE HOPE THAT THE NEW INTEGRAL, i.e. WITH THE SUBSTITUTIONS, WILL BE EASY TO EVALUATE)

SOME EXAMPLES:

$$\int_0^{\pi/2} 3x^2 \cdot \cos(x^3)\, dx$$

DEFINE $f(x) = x^3$

THUS $\frac{df}{dx} = 3x^2$

$df = 3x^2\, dx$

REWRITE AS:

$$= \int_0^{\frac{\pi}{2}} \frac{df}{dx} \cdot \cos(f)\, dx$$

$$= \int_0^{\pi/2} \cos(f) \cdot \underbrace{3x^2 \cdot dx}_{= df}$$

$$= \int_{f(0)}^{f(\pi/2)} \cos(f)\, df$$

COMPARE

NOTICE WE HAD TO FIND NEW LIMITS OF INTEGRATION (IN TERMS OF f) AND WE HAD TO REPLACE dx w/ df

$$= \sin(f)\Big|_{0^3}^{(\pi/2)^3} = \sin\left(\frac{\pi^3}{8}\right) - \sin(0)$$

$$= \sin(\pi^3/8) - 0 = \sin(\pi^3/8)$$

INT BY SUBSTITUTION CAN BE FORMALLY WRITTEN AS

$$\int g(f) \cdot \frac{df}{dx} \, dx = \int g(f) \cdot df$$

IN OUR LAST EXAMPLE $f(x) = x^3$ $\quad \frac{\partial f}{dx} = 3x^2$
$g(f) = \cos(f)$

THUS

ORIGINAL PROBLEM

THE "NEW" PROBLEM (i.e. w/ SUBSTITUTION)

$$\int \cos(f) \cdot 3x^2 \cdot dx = \underbrace{\int \cos(f) \, df}_{\text{EASY TO EVALUATE!}}$$

▷ ANOTHER EXAMPLE:

FIND $\int_1^2 t \cdot e^{t^2+1} \, dt$ $\qquad$ SET $f(t) = t^2 + 1$

$$\Rightarrow \frac{\partial f}{\partial t} = 2 \cdot t$$

SO REWRITE AS

$$\Rightarrow df = 2 \cdot t \cdot dt$$

$$\int_1^2 e^f \cdot \underbrace{t \cdot dt}$$

ALMOST EQUAL TO df, BUT NOT QUITE (WE'RE OFF BY A FACTOR OF 2)

SO MULT BY $2 \cdot \frac{1}{2}$ GIVES

$$\frac{1}{2} \int e^f \cdot 2 \cdot t \, dt = \frac{1}{2} \int e^f \, df$$

NEXT PAGE

THUS $\frac{1}{2}\int e^f\,df = \frac{1}{2}e^f + \text{CONST}\Big|_{f(1)}^{f(2)}$

THIS CAN BE EVALUATED 2 DIFFERENT WAYS:

I. COMPLETELY IN TERMS OF f

NOTE: $f(1) = 2$ w/ $f(t) = t^2+1$

$f(2) = 5$

THUS $\frac{1}{2}e^f\Big|_2^5 = \frac{1}{2}\{e^5 - e^2\}$ ✱

OR

II. COMPLETELY IN TERMS OF t (WHICH REQUIRES SUBBING $f(t)$ BACK IN FOR f) AND USING t VALUES AS THE LIMITS OF INTEGRATION.

THUS,

$$\frac{1}{2}e^f\Big|_{f(1)}^{f(2)} = \frac{1}{2}e^{t^2+1}\Big|_{x=1}^{x=2} \leftarrow ✱$$

$$= \frac{1}{2}\{e^5 - e^2\}$$

SAME ANSWER

OFTEN W/ INT-BY-SUB WE'LL BE FORTUNATE, IN THAT AFTER WE PICK $f(x)$, AND THUS FIND $\frac{df}{dx}$ IT JUST HAPPENS THAT $\frac{df}{dx}$ IS INCLUDED IN THE ORIGINAL PROBLEM.

IE

$$\int x e^{x^2} dx \qquad f(x) = x^2$$

$$\frac{df}{dx} = 2x$$

INCLUDED IN THE ORIGINAL PROBLEM

THE DERIV OF f JUST "HAPPENS" TO BE PART OF THE PROBLEM.

ALL EXAMPLES ON PAGES 19.10 → 19.12 WERE OF THIS "CONVIENIENT" TYPE.

BUT WHAT IF THIS IS NOT TRUE?

IN THIS CASE, INT-BY-SUB CAN STILL HELP:

$\boxed{EX}$

$$\int_2^5 (x+7)\sqrt[3]{3-2x}\, dx$$

LET $f(x) = 3-2x$

$$\Rightarrow \frac{df}{dx} = -2 \Rightarrow df = -2 \cdot dx$$

WRITE AS:

$$\int_2^5 (x+7) \cdot f^{1/3} \cdot dx$$

NEED A -2

$$= -\frac{1}{2}\int_2^5 (x+7) \cdot f^{1/3} \cdot (-2)\, dx = -\frac{1}{2}\int_{f(2)}^{f(5)} (x+7) f^{1/3} \cdot df$$

WE STILL HAVE X'S IN THE INTEGRAL & WE WANT ONLY f'S. THUS GIVEN $f = 3-2x$, WE HAVE $x = \frac{3-f}{2}$

$$x = \frac{3}{2} - \frac{1}{2} \cdot f$$

THUS, REWRITE

$$-\frac{1}{2} \cdot \int (x+7) f^{1/3} \cdot df$$

AS

$$= -\frac{1}{2} \int \left(\frac{3}{2} - \frac{1}{2}f + 7 \right) f^{1/3} \cdot df$$

$$= -\frac{1}{2} \int \left(8.5 - \frac{1}{2}f \right) f^{1/3} \, df$$

$$= -\frac{1}{2} \int 8.5 f^{1/3} - \frac{1}{2} f^{4/3} \, df$$

$$= -\frac{1}{2} \left[8.5 \left(\frac{3}{4}\right) f^{4/3} - \frac{1}{2} \left(\frac{3}{7}\right) f^{7/3} + \text{CONST} \right] \Bigg|_{f(2)}^{f(5)}$$

BOTH GIVE THE SAME ANSWER.

COULD EVALUATE USING ALL f'S (USING LIMITS OF INTEGRATION $f(5)$ & $f(2)$)

OR

COULD PLUG IN $f(x) = 3-2x$ AND EVALUATE @ LIMITS OF $X_{LOWER} = 2$, $X_{UPPER} = 5$

I.E.

$$= -\frac{1}{2} \left[8.5 \left(\frac{3}{4}\right) (3-2x)^{4/3} - \frac{1}{2} \left(\frac{3}{7}\right) (3-2x)^{7/3} + \text{CONST} \right] \Bigg|_{x=2}^{x=5}$$

GO TO PAGE 20

...NOW BACK TO THE PROBLEM: (SEE PAGE 18)
...WE NOW KNOW, $f, g, \dot{f}, \& \dot{g}$ SO CAN
...USE INT-BY-PARTS:

INT.-BY-PARTS IS: $\int f \cdot \dot{g}\, dx = f \cdot g - \int g \cdot \dot{f}\, dx$

THAT IS: (w/ $f = x$, $g = \frac{-1}{6}\cos(6x)$, $\dot{f} = 1$, $\dot{g} = \sin(6x)$)
WE CAN WRITE:

$$\int x \cdot \sin(6x)\, dx = x \cdot \left(\frac{-1}{6}\cos 6x\right) - \underbrace{\int \frac{-1}{6} \cdot \cos(6x) \cdot 1\, dx}$$

SO WE STILL NEED TO EVALUATE THIS INTEGRAL
BUT ITS "EASY". (CAN DO FORMALLY VIA INT-BY-SUBSTITUTION AS WELL)

$$+\frac{1}{6}\int \cos(6x)\, dx$$

$$= +\frac{1}{6}\left[\sin(6x)\right] \cdot \frac{1}{6} = \frac{1}{36} \cdot \sin(6x)$$

✓ IT: $\frac{d}{dx} \frac{+1}{6} \sin(6x) =$
$= \frac{1}{6} \cdot \cos(6x) \cdot 6$
$= \cos(6x)$ ✓

THUS VIA INT-BY-PARTS WE HAVE

$$\int x \cdot \sin(6x)\, dx = -\frac{x}{6} \cdot \cos(6x) + \frac{1}{36} \cdot \sin(6x) + C$$

THE ANSWER.

C=CONSTANT (DON'T FORGET C)

FACT: OFTEN IN ORDER TO SOLVE THE "RESIDUAL RHS INTEGRAL" ie

IBP: $$\underbrace{\int f \dot{g}\, dx}_{\text{ORIGINAL INTEGRAL TO FIND}} = f \cdot g - \underbrace{\int g \dot{f}\, dx}_{\text{"THE RESIDUAL RHS INTEGRAL"}} \leftarrow \text{INT-BY-PARTS (IBP)}$$

INTEG: BY PARTS

(SOMETIMES WE MUST USE INT-BY-PARTS AGAIN — IN ORDER TO EVALUATE THIS INTEGRAL)

YOU MUST SOMETIMES DO INT-BY-PARTS AGAIN (AND AGAIN?)

FACT:

INTEGRALS OF THE FORM:

(i) $x^n e^{ax}$

(ii) $x^n \sin(ax)$

(iii) $x^n \cos(ax)$

w/ n = POSITIVE INTEGER ie. $\underbrace{\in \{1, 2, 3, \ldots\}}$

AND a = CONSTANT

"A MEMBER OF ($\in$) THE SET $\{\cdot\}$."

MUST BE SOLVED VIA APPLYING INTEG-BY-PARTS n-TIMES, EACH TIME w/ $f = x^n$

$\dot{g} = e^{ax}$, $\sin(ax)$, OR $\cos(ax)$.

LETS DO AN EXAMPLE (IBP'S TWICE)

INT-BY-PARTS

ID

EVALUATE: $\int x^2 \cdot e^{-x(30)}\,dx$

(THINK OF X AS A RATE OF RETURN, IRR, DISCOUNT RATE ETC...), THIS TYPE OF INTEGRAL IS COMMON WHEN COMPUTING THE VARIANCE OF RETURNS W/ CALCULUS:

LET: $f = x^2 \implies \dot{f} = 2x$

$\dot{g} = e^{-x(30)} \implies g = \frac{-1}{30} \cdot e^{-x(30)}$

MAKE SURE THE DERIV OF g IS EQUAL TO $\dot{g}$ IE CHECK YOUR WORK!

THEN VIA INT-BY-PARTS:

$$\int f \cdot \dot{g}\,dx = x^2 \cdot \left(\frac{-1}{30} e^{-x(30)}\right) - \int \frac{-1}{30} e^{-x(30)} \cdot 2x\,dx$$

(**)
$$= \frac{-x^2}{30} e^{-x(30)} + \frac{1}{15} \underbrace{\int x \cdot e^{-x(30)}\,dx}$$

BUT, THIS IS STILL OF THE FORM $x^n e^{ax}$ — SO DO INTEG-BY-PARTS AGAIN (UNTIL THE "RESIDUAL RHS INTEGRAL" IS NOT OF THIS FORM.)

SO INT-BY-PARTS AGAIN W/ $f = x$, $\dot{g} = e^{-x(30)} \implies \dot{f} = 1$, $g = \frac{-1}{30} e^{-x(30)}$

THUS

THUS,

$$\frac{+1}{15}\int \underbrace{x}_{f=x}\,\underbrace{e^{-30\cdot x}}_{\dot{g}}\,dx = \underbrace{x}_{f=x}\Big(\underbrace{\tfrac{-1}{30}e^{-30x}}_{g}\Big) - \underbrace{\int \underbrace{\tfrac{-1}{30}e^{-30x}}_{g=} \cdot \underbrace{1}_{\dot{f}=1} \cdot dx}$$

NOW THE "RESIDUAL RHS INTEGRAL" IS NOT OF THE FORM $x^n \cdot x^{ax}$ (W/ n A POSITIVE INTEGER)

i.e. ZERO = n IS NOT ALLOWED. — IT'S NOT A POSITIVE INTEGER.

SO NOW JUST EVALUATE THE LAST RESIDUAL RHS INTEGRAL (AND THEN SUBSTITUTE OUR RESULTS FROM THE TWO STAGES OF INT-BY-PARTS).

THE LAST INTEGRAL IS:

(**) $\frac{+1}{30}\int e^{-30\cdot x}\,dx$ IT EQUALS

$$\underbrace{\frac{-1}{900}e^{-30\cdot x}}_{\text{CHECK IT.}\ \checkmark} + \text{CONST}$$

THUS OUR OVERALL ANSWER IS:

USING

THE ORIGINAL PROBLEM WAS TO FIND $\int x^2 e^{-30 \cdot x} dx$

THE SOLUTION IS:

(VIA (**) PG 22, AND (**) PG 23):

$$= \underbrace{\frac{-x^2}{30} e^{-30 \cdot x}}_{\text{FROM (**) PG 22}} + \underbrace{\left\{ \frac{-x}{30} e^{-30x} - \frac{1}{900} e^{-30x} \right\}}_{\text{FROM 2ND STAGE INT-BY-PARTS}} \cdot \left(\frac{1}{15} \right)$$

NOW LETS TALK ABOUT THE LIMITS (UPPER & LOWER) OF INTEGRATION:

EX

$$\int_2^7 3x^2 dx = \left(x^3 + \text{CONST} \right) \Big|_2^7$$

$$= \left(7^3 + \cancel{\text{CONST}}\right) - \left(2^3 + \cancel{\text{CONST}}\right) = 7^3 - 2^3$$
$$= 335$$

IN THE ABOVE EXAMPLE, THE LOWER LIMIT WAS 2, THE UPPER LIMIT = 7.

THIS GIVES THE AREA UNDER $y = 3x^2$ FROM $x = 2$ TO $x = 7$. THE AREA IS EXACTLY 335.

SOME INTEGRALS CAN'T BE SOLVED VIA INT-BY-PARTS. SOME OF THESE INTEGRALS REQUIRE A "SUBSTITUTION OF VARIABLES" (SECTION 8.2 OF SHENK CALC TEXT)

FOR EXAMPLE, INTEGRALS INVOLVING $\ln x$ OFTEN REQUIRE SUBSTITUTION.

THE TRICKY THING ABOUT INTEGRATION-BY-SUBSTITUTION IS THAT YOU MUST ALSO ADJUST THE LIMITS OF INTEGRATION AND ADJUST "dx"

EXAMPLE EVALUATE $\displaystyle\int_0^2 \ln(9-3x)\,dx$

INT-BY-PARTS WON'T HELP US HERE — HERE WE MAKE THE SUBSTITUTION

$$f(x) = 9-3x$$

W/ THIS SUBSTITUTION WE NEED TO CHANGE THE FORM OF THE INTEGRAL FROM:

$$\int_{x_{LOWER}}^{x_{UPPER}} g(f(x))\,dx$$

IN OUR CASE:

$g(x) = \ln x$

$f(x) = 9-3x$

TO:

$$\int_{f(x_{LOWER})}^{f(x_{UPPER})} g(f)\cdot df$$

ALSO NEED TO WRITE df IN TERMS OF dx (AND VICE VERSA)

NEED TO DETERMINE THE NEW LIMITS OF INTEGRATION IN TERMS OF f

I. LOWER LIMIT OF INTEGRATION IS $0 = X$ $\leftarrow = 0$

$$f(0) = 9 - 3(0) = 9 = f_{\text{LOWER}} = f(X_{\text{LOWER}})$$

THUS, THE NEW LOWER LIMIT OF INTEGRATION IS $f = 9$.

II THE UPPER LIMIT OF INT WILL BE

$$f(2) = 9 - 3(2) = 3 = f_{\text{UPPER}} = f(\underbrace{X_{\text{UPPER}}}_{=2})$$

III WRITE dx IN TERMS OF df:
(WE DO THIS BASED UPON THE RELATION $f = 9 - 3X$, AND THEN TAKING THE DERIV).

w/ $f = 9 - 3x$

THE FIRST DERIV IS: $\dfrac{df}{dx} = -3$

WHICH CAN BE WRITTEN AS A DIFFERENTIAL EQUATION:

$$df = -3 \cdot dx$$

OR AS

$$dx = -\frac{1}{3} \cdot df$$

WE NEED TO MAKE THIS CHANGE IN THE INTEGRAL TOO. (MUST NOT FORGET!)

THUS...

$$\int_0^2 \ln(9-3x)\,dx = \int_9^3 \ln(f) \cdot \underbrace{\left(-\frac{1}{3}\right)df}_{=dx}$$

$$= -\frac{1}{3}\int_9^3 \ln(f)\,df$$

KEY RESULT

TRICK:

IF YOU WANT TO SWAP THE UPPER & LOWER VALUES OF INTEGRATION, YOU CAN DO SO IF YOU MULT. THE INTEGRAL BY -1.

THAT IS, SIMPLY SWAPING LIMITS OF INTEGRATION ONLY CHANGES THE SIGN OF THE INTEGRAL

USING THE TRICK WE HAVE

UPPER & LOWER LIMITS WERE SWAPPED.

$$-\frac{1}{3}\cdot\int_9^3 \ln f \cdot df = +\frac{1}{3}\cdot\underbrace{\int_3^9 \ln f \, df}$$

DOING THIS GETS THE LOWER INTEGRAND TO BE LESS THAN THE UPPER INTEGRAND. - WHICH "LOOKS NICE" BUT IT'S NOT NECESSARY.

NOTE: WE DON'T HAVE TO EMPLOY THE TRICK. WE'LL GET THE SAME ANSWER EITHER WAY.

NOW EVALUATE.

UNFORTUNATELY WE'RE NOT READY TO FIND THE INTEGRAL OF THE NATURAL LOG.

Δ

AT THIS POINT WE HAVE A PRETTY DECENT EXPOSURE TO THE BASICS OF CALCULUS.

LETS LOOK @ WHY WE CARE IN FINANCE...

IT TURNS OUT THAT THE PV OF ANNUITIES (ie CASH FLOW STREAMS) CAN BE COMPUTED VIA INTEGRALS. (IF THE FLOW RATE IS CONTINUOUS — AS INTEREST IS OFTEN PAID).

FACT: IF \$ IS FLOWING INTO AN ACCT AT ANY TIME t, AT THE RATE $R(t)$, THEN THE PRESENT VALUE OF THIS FLOW FROM TIME t_1 TO TIME t_2 IS COMPUTED AS

$$\underset{(@\, t=0)}{\text{PRESENT VALUE}} = \int_{t_1}^{t_2} R(t) \cdot \underbrace{e^{-r \cdot t}}_{\text{FROM: } P = Fe^{-rt}} \, dt$$

AND

$$\underset{(@\, t=T)}{FV} = \int_{t_1}^{t_2} R(t) \cdot e^{+r(T-t)} \, dt$$

FROM: $P = Fe^{-rt}$

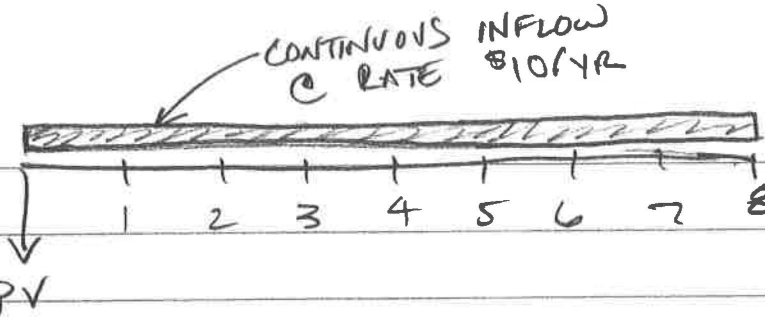
FROM BUS-342

PRESENT VALUE w/ CONTINUOUS COMPOUNDING

FOR EXAMPLE

ASSUME MONEY FLOWS INTO AN ACCOUNT AT THE RATE $R(t) = \$10/\text{YR}$ FOR ALL TIMES BETWEEN $t=0$ & $t=8$. IF THE DISCOUNT RATE IS CONSTANT @ $r = .12$ THEN THE PV OF THE 8 YEAR STREAM IS:

TIMELINE LOOKS LIKE:



$$PV = \int_0^8 10 \cdot e^{-.12t}\, dt$$

$$= 10 \int_0^8 e^{-.12t} \cdot dt = 10 \left[\frac{-1}{.12} e^{-.12t} \right] \Bigg|_0^8$$

$$= -83\tfrac{1}{3}\, e^{-.12t} \Big|_0^8$$

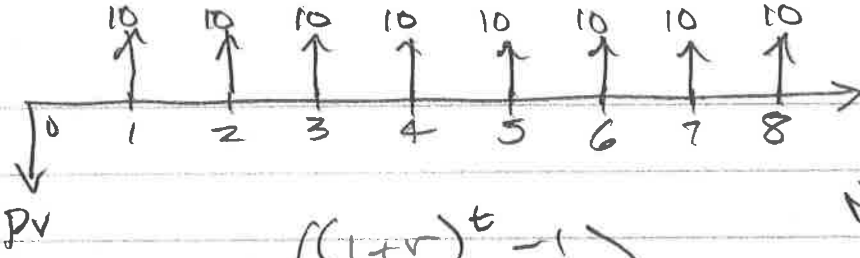
$$= -83\tfrac{1}{3} \left[e^{-.12(8)} - e^{-.12(0)} \right]$$

$$= -83\tfrac{1}{3} \left[.38289 - 1 \right]$$

$= \$51.426$ ← PV OF \$80 TOTAL, FLOWING IN AT THE RATE OF \$10/YR FOR EACH OF 8 YEARS

SIDENOTE:

USING $r^{APR} = .12/\text{PERIOD}$ AND A CONVENTIONAL 8 PERIOD ANNUITY OF \$10/PERIOD, THE PV (AS IT WOULD BE COMPUTED IN BUS-342) WOULD BE



$$P = A \cdot \left(\frac{(1+r)^t - 1}{r(1+r)^t} \right)$$

$$= 10 \left(\frac{(1.12)^8 - 1}{.12(1.12)^8} \right) = \$49.676$$

NOTE* WE REALLY SHOULD USE r_{EFF} ASSOC w/ $.12 = r_{APR}$ CONT. COMPOUNDED i.e. BETTER TO USE $r = .1274$ ($\Rightarrow PV = 48.448$)

THIS IS LESS THAN \$51.426 BECAUSE YOU HAVE TO WAIT A FULL YEAR TO GET THE \$10. w/ A CONTINUOUS FLOW RATE OF R = \$10 - YOU STILL GET A TOTAL OF \$10/YR - BUT YOU GET PAID THE \$10 EARLIER - AND THIS FEATURE IS VALUABLE (IN FACT ITS WORTH ABOUT \$1.750)

$= 51.426 - 49.676$

ANOTHER COMMON USE OF CALCULUS IS IN ESTIMATING THE MEAN AND VARIANCE OF A FUNCTION.

OFTEN WE'LL WANT TO ESTIMATE THE MEAN RETURN AND VARIANCE OF RETURNS OF A TRADED ASSET (eg STOCK, BOND, OPTION, SWAP ETC), ONE SLICK WAY TO DO THIS IS WITH INTERALS.

FOR OPTION PRICING THERE ARE SEVERAL PARTIAL DERIVATIVES THAT DESCRIBE THE RELATION BETWEEN THE OPTION'S VALUE

AND THE VALUE OF THE UNDERLYING ASSET. (SAY THE STOCK OF IBM, OR THE PRICE OF PORK BELLIES).

THESE DERIVATIVES, DELTA, GAMMA, THETA, RHO KAPPA, AND VEGA ARE COLLECTIVELY REFERED TO AS "THE GREEKS". IN ORDER TO REALLY UNDERSTAND WHAT THEY MEAN, YOU MUST UNDERSTAND PARTIAL DERIVATIVES AT AN INTUITIVE LEVEL.

— THERE ARE <u>SEVERAL</u> OTHER WAYS CALCULUS IS APPLIED TO FINANCE.

FACT: INTEGRALS ARE JUST FANCY SUMMATION SIGNS (ie Σ)

THAT IS, FOR ANY INTEGRAL THERE IS AN ANALAGOUS <u>APPROXIMATION</u> USING SUMMATION NOTATION. (SEE RIEMANN SUMS IN ANY CALC BOOK. e.g. SHENK pg 223)

IDEA

FOR $\displaystyle\int_{x_1}^{x_2} f(x)\,dx$

THE ANALOGY IS $\displaystyle\sum_{x=x_1}^{x_2} f(x)(\Delta x)$

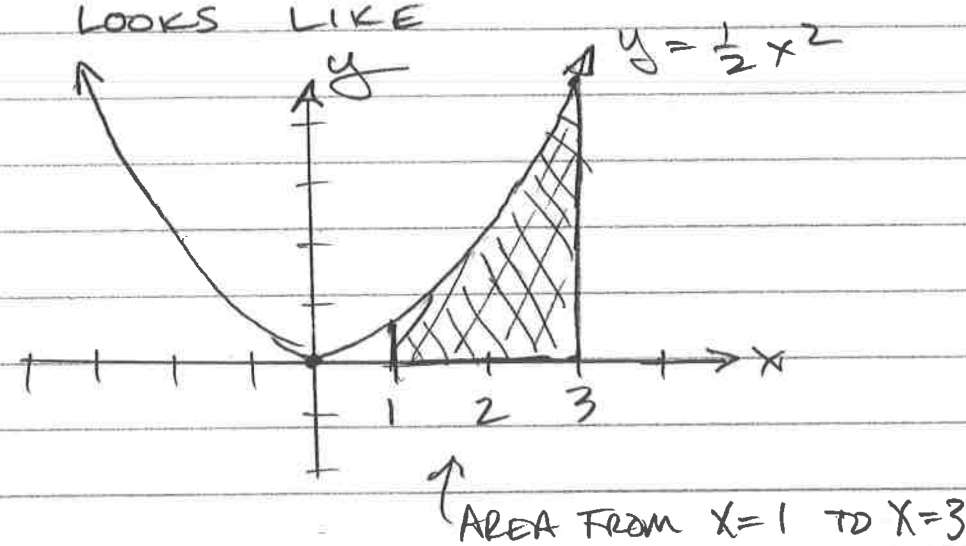
W/ Δx BEING A SMALL PART OF THE TOTAL SPAN FROM x_1 TO x_2 (THE LIMITS OF INTEGRATION).

FOR EXAMPLE, WE KNOW THE INTEGRAL GIVES THE AREA UNDER THE CURVE, BUT WHAT IF WE DONT KNOW HOW TO EVALUATE THE INTEGRAL (SAY $\int \ln(9-3x)\,dx$) WHAT SHOULD WE DO?

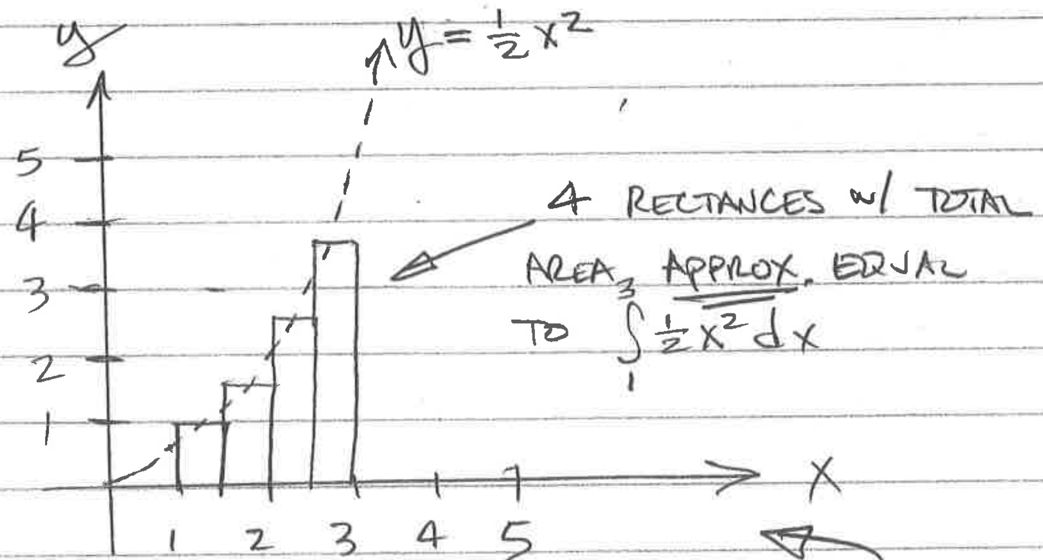
WE CAN APPROXIMATE THE INTEGRAL AS FOLLOWS.

(LETS ASSUME WE DIDN'T KNOW HOW TO EVALUATE $\int_1^3 \frac{1}{2}x^2\,dx$)

$y = \frac{1}{2}x^2$ LOOKS LIKE



WE COULD APPROXIMATE THE AREA AS A FEW RECTANGLES THAT LOOK SOMETHING LIKE:



LETS COMPUTE THE AREA OF

NEXT PAGE

I'VE CHOSEN TO HAVE THE HEIGHT OF EACH RECTANGLE EQUAL TO THE MIDPOINT OF THE VALUE OF $f(x)$ AT THE 2 SIDES OF THE RECTANGLE.

THAT IS,

W/ 4 RECTANGLES SPANNING $X=1$ TO $X=3$ WE HAVE THE FOLLOWING X VALUES TO WORRY ABOUT: $X = \{1, 1.5, 2, 2.5, 3\}$

GIVING 4 RECTANGLES W/ BASES OF

RECTANGLE #	X VALUE AT LEFT SIDE	X VALUE AT RIGHT SIDE	RECTANGLE WIDTH
1	$X=1.0$	$X=1.5$	0.5
2	$X=1.5$	$X=2.0$	0.5
3	$X=2.0$	$X=2.5$	0.5
4	$X=2.5$	$X=3.0$	0.5

LETS LOOK @ RECTANGLE #1

WE HAVE $f(x) = \frac{1}{2}x^2$ THUS,

$$f(1.0) = \tfrac{1}{2}(1)^2 = \tfrac{1}{2} = .500$$

$$f(1.5) = \tfrac{1}{2}(1.5)^2 = 1.125$$

SO I TAKE THE HEIGHT OF RECTANGLE #1 TO BE THE MIDPOINT BETWEEN 0.50 AND 1.125 WHICH IS 0.8125

$$= \left(\frac{f(1.0) + f(1.5)}{2}\right)$$

FOR RECTANGLE #2 THE HEIGHT IS

$$\frac{f(1.5)+f(2.0)}{2} = \frac{\frac{1}{2}(1.5)^2 + \frac{1}{2}(2.0)^2}{2}$$

$$= \frac{1.125 + 2.00}{2}$$

$$= 1.5625$$

IN A SIMILAR MANNER WE CAN GET THE HEIGHTS OF RECTANGLES 3 AND 4 TO BE

RECT #3: $\text{height} = \frac{f(2.0)+f(2.5)}{2} = \frac{\frac{1}{2}(2)^2 + \frac{1}{2}(2.5)^2}{2}$

$$= 2.5625$$

RECT #4: $\text{height} = \frac{f(2.5)+f(3.0)}{2} = \frac{\frac{1}{2}(2.5)^2 + \frac{1}{2}(3.0)^2}{2}$

$$= 3.8125$$

NOTE: BY CHOOSING THE HEIGHTS IN THIS (MIDPOINT) MANNER WE ARE USING "THE MIDPOINT RULE" THERE ARE MORE ACCURATE WAYS AS WELL (BUT REQUIRE A BIT MORE WORK)
SEE SHENK CALCULUS TEXT SECTION 8-9 pg 423.

THE TOTAL AREA OF THE 4 RECTANGLES (EACH W/ WIDTH = 0.5) IS:

$$\sum \text{AREA'S} = \sum (\text{height}) \cdot (\Delta x) \qquad w/\ \Delta x = 0.5$$
$$= \sum f(\text{MIDPOINT}) \cdot (\Delta x)$$
$$= (.8125)(0.5) + (1.5625)(0.5) + (2.5625)(0.5) + (3.8125)(0.5)$$
$$= 4.375$$

COMPARE THIS TO THE TRUE VALUE OF $\int_1^3 \frac{1}{2}x^2\,dx$

$$\frac{1}{6}x^3 \Big|_1^3 = \frac{1}{6}(3)^3 - \frac{1}{6}(1)^3$$
$$= \frac{27}{6} - \frac{1}{6} = \frac{26}{6} = 4\tfrac{1}{3} = 4.333$$

THE APPROXIMATION HAS ABOUT 1% ERROR

$$\left(\text{ERROR\%} = \frac{\text{ESTIMATE} - \text{TRUTH}}{\text{TRUTH}} = \frac{4.375 - 4.\overline{333}}{4.\overline{333}} \cong .96\%\right)$$

THUS, THE APPROXIMATION IS NOT TOO BAD.
IF WE USED > 4 RECTANGLES, THE APPROXIMATION WOULD IMPROVE. SEE SHENK CALC BOOK, SEC 8.9, PAGE 424.

BUT NOTICE THE ANALOGY BETWEEN

$$\sum f(\text{MIDPOINT}) \cdot (\Delta x)$$

AND

$$\int f(x)\,dx$$

IMPORTANT!

SEE RIEMANN SUMS pg. 223

D NOW THINK OF $y = \frac{1}{2}x^2$ AS BEING A STANDARD PHYSICS PROBLEM RESTATED AS:

$$V = \frac{1}{2}t^2$$ ← VELOCITY $\left(\frac{\text{METERS}}{\text{SECOND}}\right)$

SUPPOSE WE WANTED TO KNOW THE TOTAL DISTANCE (IN METERS) TRAVELED FROM $t=1$ TO $t=3$ SECONDS.

WELL.... WHAT IF WE BROKE THE PROBLEM INTO 4 SEPERATE PROBLEMS, AND FOR EACH, ESTIMATE DISTANCE TRAVELED BY

$$\text{DISTANCE} = \text{VELOCITY} \cdot \text{TIME}$$

$$(\text{METERS}) = \left(\frac{\text{METERS}}{\cancel{\text{SEC}}}\right) \cdot (\cancel{\text{SEC}})$$ ← MAKE SURE ANYTHING YOU WRITE HAS CORRECT UNITS.

$$X = V \cdot t$$

SO WE NEED ONLY ESTIMATE THE MEAN VELOCITY IN EACH OF THE 4 PERIODS.

PER PREVIOUS PROBLEM, WE ESTIMATE THE MIDPOINT OF VELOCITY IN EACH PERIOD TO BE

PERIOD #	EST OF MEAN VELOCITY		Δt IN WINDOW		EST. OF DISTANCE TRAVELED
1	0.8125		0.5 SEC		.40625
2	1.5625	×	0.5	=	.78125
3	2.5625		0.5		1.28125
4	3.8125		0.5		1.90625

OUR ESTIMATE ⟶ 4.375 METERS