Calculus Overview

BUS-444

Product Rule, Quotient Rule, Chain Rule
Shenk 3.3, 3.4, 3.7

CALCULUS

Financial Engineering/444

* CHAIN RULE
* PRODUCT RULE
* QUOTIENT RULE

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SHENK CH 3.3, 3.4, 3.7 w/ SOLUTIONS...

3-3 THE DERIVATIVE OF x^n, DERIVATIVES OF LINEAR COMBINATIONS

In Sections 3.1 and 3.2 we saw how the derivatives of functions given by very simple algebraic expressions can be computed by using the definition of the derivative. This procedure would require a great deal of effort if it were applied to a function given by a complicated algebraic expression. The effectiveness of calculus as a mathematical tool rests in part on the fact that derivatives of many functions can be computed without referring back to the definition of the derivative; they can be computed by the application of RULES OF DIFFERENTIATION.

In this section we discuss the rules for differentiating the function x^n and for expressing the derivative of a linear combination of functions in terms of the derivatives of the functions themselves. These rules, along with the product, quotient, and chain rules, which will be discussed in Sections 3.4 and 3.7, will enable us to compute, with relatively little effort, the derivative of any function that is given by an algebraic expression.

The derivative of x^n

Theorem 3.2 For any rational constant n the derivative of the function x^n is

(1) $$\frac{d}{dx}x^n = nx^{n-1}.$$ ◀

Here x must be nonzero if $n - 1$ is negative and must be positive if n is a fraction, $n = p/q$, with p an integer and q an even integer.* In the case of $n = 0$ we interpret nx^{n-1} as the zero function.

Rule (1) is not valid for $x = 0$ if $n - 1$ is negative because then it would involve dividing by zero. It is not valid for $x \leq 0$ if n is a fraction (positive or negative) with an even denominator because then it would involve even roots of the negative number x.

We give here the proof of Theorem 3.2 first for $n = 0$ and 1, then for integers $n \geq 2$, and finally for arbitrary positive rational numbers n. The proof for negative rational n is left as Exercise 27 at the end of this section.

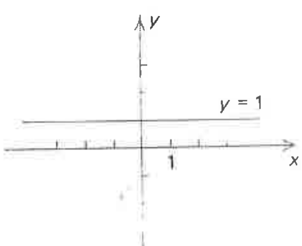

FIGURE 3.11 The graph of the function 1

Proof of Theorem 3.2 for $n = 0$. In the case of $n = 0$ the function x^n denotes the constant function 1, and formula (1) reads

(2) $$\frac{d}{dx}1 = 0.$$ ◀

The graph of the constant function 1 is the horizontal line $y = 1$ (Figure 3.11); its tangent line at any point is the line itself and has zero

*In Chapter 7 we will see that (1) is also valid for positive x and irrational n.

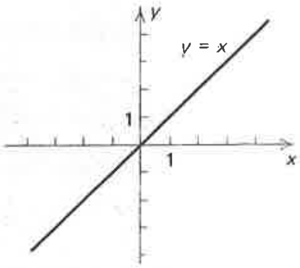

FIGURE 3.12 The graph of the function x

slope. The derivative of the constant function 1, therefore, is zero for all x, as stated in formula (2). Q.E.D.

Proof of Theorem 3.2 for $n = 1$. In this case the function x^n is the function $x^1 = x$, and formula (1) reads

$$(3) \qquad \frac{d}{dx}x = 1.$$

The graph of the function x is the line $y = x$ (Figure 3.12); its tangent line at any point is the line itself and has slope 1. Therefore, the derivative of the function x is 1 as is stated in formula (3). Q.E.D.

Difference quotients

The slope of the secant line $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$, whose limit is the derivative $f'(x)$, is also known as a DIFFERENCE QUOTIENT because the denominator is the difference $\Delta x = (x + \Delta x) - x$ of values of the variable and the numerator is the difference $f(x + \Delta x) - f(x)$ of the corresponding values of the function.

Proof of Theorem 3.2 for integers $n \geq 2$. The graph of x^n for integers $n \geq 2$ is not a line, and we have to compute its derivative as the limit of a difference quotient. We have

$$(4) \qquad \frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.$$

To express the difference quotient in (4) in a form whose limit we can recognize, we use the formula from algebra

$$(5) \qquad (x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \left[\begin{array}{l}\text{Terms involving } (\Delta x)^2 \\ \text{and higher powers of } \Delta x\end{array}\right].$$

Formula (5) is valid for any integer $n \geq 2$. For $n = 2$, for example, we have $(x + \Delta x)^2 = x^2 + 2x(\Delta x) + (\Delta x)^2$, and for $n = 3$, $(x + \Delta x)^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. The proof of (5) for arbitrary integers $n \geq 2$ is given in Example 2 of the Appendix to Chapter 1.

If we subtract x^n from both sides of equation (5) and divide both sides by Δx, we obtain

$$\frac{(x + \Delta x)^n - x^n}{\Delta x} = nx^{n-1} + \left[\begin{array}{l}\text{Terms involving } \Delta x \\ \text{and powers of } \Delta x\end{array}\right].$$

Because the terms involving Δx and powers of Δx tend to zero as Δx tends to zero, this equation yields

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} = nx^{n-1}. \quad \text{Q.E.D.}$$

EXAMPLE 1 Compute the derivative $\left[\dfrac{d}{dx}x^5\right]_{x=2}$.

Solution. Formula (1) with $n = 5$ reads

$$\frac{d}{dx}x^5 = 5x^{5-1} = 5x^4.$$

Therefore,

$$\left[\frac{d}{dx}x^5\right]_{x=2} = [5x^4]_{x=2} = 5(2^4) = 80. \quad \square$$

Proof of Theorem 3.2 for positive rational n. Suppose that $n = p/q$ with p and q positive integers and that the numbers x and a are positive if q is even. Set $y = x^{1/q}$ and $b = a^{1/q}$, so that $x = y^q$ and $a = b^q$. Then for $x \neq a$ we have

$$\frac{x^n - a^n}{x - a} = \frac{(x^{1/q})^p - (a^{1/q})^p}{x - a} = \frac{y^p - b^p}{y^q - b^q} = \frac{\dfrac{y^p - b^p}{y - b}}{\dfrac{y^q - b^q}{y - b}}.$$

As x tends to a, $y = x^{1/q}$ tends to $b = a^{1/q}$, so we have

$$\left[\frac{d}{dx}x^n\right]_{x=a} = \lim_{x\to a}\frac{x^n - a^n}{x - a} = \frac{\displaystyle\lim_{y\to b}\frac{y^p - b^p}{y - b}}{\displaystyle\lim_{y\to b}\frac{y^q - b^q}{y - b}} = \frac{\left[\dfrac{d}{dy}y^p\right]_{y=b}}{\left[\dfrac{d}{dy}y^q\right]_{y=b}}.$$

We have established the differentiation formula for the positive integer powers p and q. Therefore, the last equation gives

$$\left[\frac{d}{dx}x^n\right]_{x=a} = \frac{pb^{p-1}}{qb^{q-1}} = \left(\frac{p}{q}\right)b^{p-q} = \left(\frac{p}{q}\right)(a^{1/q})^{p-q}$$

$$= \left(\frac{p}{q}\right)a^{(p/q)-1} = na^{n-1}.$$

This is differentiation formula (1) at the arbitrary value $x = a$. Q.E.D.

EXAMPLE 2 Compute the derivative $F'(8)$, where $F(x) = x^{4/3}$.

Solution. By formula (1) with $n = \frac{4}{3}$ and $n - 1 = \frac{4}{3} - 1 = \frac{1}{3}$ we have

$$F'(8) = \left[\frac{d}{dx}x^{4/3}\right]_{x=8} = \left[\frac{4}{3}x^{1/3}\right]_{x=8} = \frac{4}{3}(8)^{1/3} = \frac{4}{3}(2) = \frac{8}{3}. \quad \square$$

Derivatives of linear combinations of functions

The next theorem, used in conjunction with rule (1) for differentiating powers, enables us to differentiate any polynomial or other linear combination of powers of x.

Theorem 3.3 If the functions f and g have derivatives at x, then so does the linear combination $Af + Bg$ for any constants A and B. The derivative of the linear combination of the functions is the same linear combination of their derivatives:

$$(6) \qquad \frac{d}{dx}[Af(x) + Bg(x)] = A\frac{df}{dx}(x) + B\frac{dg}{dx}(x). \qquad ◀$$

Proof. To prove this theorem, we notice that a difference quotient for the function $Af + Bg$ is equal to A times the difference quotient for f plus B times the difference quotient for g:

$$(7) \qquad \begin{aligned} &\frac{[Af(x + \Delta x) + Bg(x + \Delta x)] - [Af(x) + Bg(x)]}{\Delta x} \\ &\quad = A\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right] + B\left[\frac{g(x + \Delta x) - g(x)}{\Delta x}\right]. \end{aligned}$$

When Δx tends to zero, equation (7) becomes equation (6). Q.E.D.

EXAMPLE 3 Compute $\frac{d}{dx}(6x^{2/3} - 4x^{-2} + 5)$. For what values of x does this derivative exist?

Solution. By the analog of rule (6) for linear combinations of three functions and by rule (1) with $n = \frac{2}{3}$, $n = -2$, and $n = 0$ we have

$$\begin{aligned} \frac{d}{dx}(6x^{2/3} - 4x^{-2} + 5) &= 6\frac{d}{dx}x^{2/3} - 4\frac{d}{dx}x^{-2} + 5\frac{d}{dx}x^0 \\ &= 6(\tfrac{2}{3})x^{-1/3} - 4(-2)x^{-3} + 5(0) \\ &= 4x^{-1/3} + 8x^{-3}. \end{aligned}$$

This derivative exists for all $x \neq 0$. □

Exercises

†Answer provided.
‡Outline of solution provided

In Exercises 1 through 10 compute the derivatives and describe the values of the variables for which they exist.

1.† $\frac{d}{dx}(5x^3 - 6x^2 + 7x)$

2. $\frac{d}{dx}(x^7 + 3x^2 - 15)$

3.† $\frac{d}{dx}(6x^{-3} - 4x^{-1})$

4. $\frac{d}{dx}(7x^{-8} - 8x^{-7} + 3)$

5.† $\frac{d}{dt}(t^{3/2} - t^{2/3})$

6. $\frac{d}{dt}(4t^{-1/3} - 3t^{1/4})$

7.‡ $\frac{d}{dv}(\sqrt{v} - 4\sqrt[3]{v})$

8.† $\frac{d}{ds}(1 - s + \sqrt{s^5})$

9.† $\frac{d}{dx}\left(x - \frac{3}{\sqrt{x}}\right)$

10. $\frac{d}{dx}\left(\frac{1}{x} - \frac{2}{x^2} + \frac{3}{x^3}\right)$

In Exercises 11 through 16 compute the derivatives.

11.† $F'(2)$, where $F(x) = 6x^3 - x^{-1}$

12. $W'(3)$, where $W(t) = t^{-1} - 3t^2 + 4$

13.† $\left[\dfrac{d}{dx}(4x^4 - 4x^{-4})\right]_{x=1}$

14. $\left[\dfrac{d}{dw}(3w^{4/3} - 6w^{-1/3})\right]_{w=8}$

15.† $f'(10)$, where $f(x) = 3g(x) - 4h(x)$ and $g'(10) = 7, h'(10) = -3$

16. $\left[\dfrac{d}{dt}[5V(t) + 6W(t) - 7Z(t)]\right]_{t=0}$, where $V'(0) = -1$, $W'(0) = 4$, and $Z'(0) = 3$

In Exercises 17 through 20 give equations for the indicated lines.

17.† The tangent line to $y = x^3 - 4x$ at $x = 2$

18. The tangent line to $y = \sqrt[3]{x}$ at $x = 27$

19.† The normal line to $y = 53 + 7x^{10}$ at $x = 1$

20. The normal line to $y = x^{3/2}$ at $x = 9$

21.† Two curves are *tangent* at a point where they intersect if they have the same tangent line at that point. Find a constant k such that the curves $y = 1 + kx - kx^2$ and $y = x^4$ are tangent at the point $(1, 1)$.

22. Two curves are *perpendicular* at a point where they intersect if their tangent lines at that point are perpendicular. Find a constant n such that the curves $y = \frac{1}{2}x^{-2}$ and $y = \frac{1}{2}x^n$ are perpendicular at the point $(1, \frac{1}{2})$, then sketch the two curves in one coordinate plane.

In Exercises 23 through 26 use the definition of the derivative to derive the indicated formula.

23.† Formula (2)

24. Formula (3)

25.† Formula (1) for $n = \frac{1}{2}$

26. Formula (1) for $n = -1$

27.† Prove formula (1) for negative rational numbers n.

Compute the derivatives in Exercises 28 through 37 by expressing the functions as linear combinations of powers.

28.† $\dfrac{d}{dx}\left(\dfrac{\sqrt{x} - 3}{x}\right)$

29.† $\dfrac{d}{dx}(x + 3)^2$ (Compute the square)

30.† $\dfrac{d}{du}[(1 - u^2)/\sqrt{u}]$

31.† $\dfrac{d}{dw}(\sqrt[5]{w^3} + \sqrt[3]{w^5})$

32.† $\dfrac{d}{dy}[y^2(2y^3 + 3y^2 - 4)]$

33.† $\dfrac{d}{dx}[(x + 2)(3x - 4)]$

34.† $\dfrac{d}{dx}\left(\dfrac{2 + 3x^{30} - 5x^{40}}{x^{50}}\right)$

35.† $\dfrac{d}{dx}[x^{1/7}(x^{1/6} - x^{1/5})]$

36.† $\dfrac{d}{dt}\left(\dfrac{t - 1}{t}\right)^2$

37.† $\dfrac{d}{dx}\sqrt{\dfrac{x^{3/4}}{x^{2/3}}}$

38.† Find numbers a and b so that the curves $y = \sqrt{x}$ and $y = ax^2 + b$ are tangent at $(1, 1)$.

39.† Find constants a and n so that the curves $y = x^3 + x^2 + x$ and $y = ax^n$ are perpendicular at $(1, 3)$.

3-4 THE PRODUCT AND QUOTIENT RULES

Many functions may be expressed as products or quotients of simpler functions, and to compute their derivatives, we use the PRODUCT and QUOTIENT RULES, which we discuss in this section.

Theorem 3.4 (The Product Rule) If the functions f and g have derivatives at x, then so does their product. The derivative of the product is

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg}{dx}(x) + g(x)\frac{df}{dx}(x). \tag{1}$$

Formula (1) states that the derivative of the product is equal to the first function multiplied by the derivative of the second, plus the second multiplied by the derivative of the first.

Proof. To prove formula (1), we add and subtract the term $f(x + \Delta x)g(x)$ in the numerator of the difference quotient for the product and rewrite the result. We obtain

$$\begin{aligned} &\frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x} \\ &\quad = \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x)}{\Delta x} \\ &\qquad + \frac{f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x} \\ &\quad = f(x + \Delta x)\left[\frac{g(x + \Delta x) - g(x)}{\Delta x}\right] + g(x)\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]. \end{aligned} \tag{2}$$

Because f has a derivative at x, it is continuous there and $f(x + \Delta x)$ tends to $f(x)$ as Δx tends to zero. The expressions in square brackets on the right side of equation (2) tend to the derivatives of g and f, while the left side of the equation tends to the derivative of their product. Equation (2), therefore, becomes equation (1) as Δx tends to zero. Q.E.D.

EXAMPLE 1 Compute the derivative $\frac{d}{dx}[(x^2 + 3x - 1)(4x^{1/2} - 6)]$.

Solution. According to the product rule (1) with $f(x) = x^2 + 3x - 1$ and $g(x) = 4x^{1/2} - 6$, we have

$$\frac{d}{dx}[(x^2 + 3x - 1)(4x^{1/2} - 6)]$$
$$= (x^2 + 3x - 1)\frac{d}{dx}(4x^{1/2} - 6) + (4x^{1/2} - 6)\frac{d}{dx}(x^2 + 3x - 1)$$
$$= (x^2 + 3x - 1)(2x^{-1/2}) + (4x^{1/2} - 6)(2x + 3). \quad \square$$

EXAMPLE 2 The functions H and K satisfy $H(3) = 4$, $H'(3) = -7$, $K(3) = 10$, and $K'(3) = 2$. Compute the derivative $\left[\frac{d}{dx}(H(x)K(x))\right]_{x=3}$.

Solution. By the product rule

$$\frac{d}{dx}(H(x)K(x)) = H(x)\frac{d}{dx}K(x) + K(x)\frac{d}{dx}H(x)$$
$$= H(x)K'(x) + K(x)H'(x).$$

At $x = 3$ this expression reads

$$\left[\frac{d}{dx}(H(x)K(x))\right]_{x=3} = H(3)K'(3) + K(3)H'(3)$$
$$= 4(2) + 10(-7) = -62. \quad \square$$

Theorem 3.5 (The Quotient Rule) If the functions f and g have derivatives at x and if $g(x)$ is not zero, then the quotient f/g also has a derivative at x. The derivative is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{df}{dx}(x) - f(x)\frac{dg}{dx}(x)}{g(x)^2}. \tag{3}$$

Formula (3) states that the derivative of the quotient is equal to the denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

Proof. We will deal here with the case where $f(x)$ is the constant function 1 and formula (3) gives the derivative of the reciprocal function $g(x)^{-1}$. The derivation of the general case then follows from the product rule and is left as Exercise 16.

Because g has a derivative at x, it is continuous there and $g(x + \Delta x)$ is not zero for sufficiently small $\Delta x \neq 0$. For such values of Δx we have

$$\frac{\dfrac{1}{g(x + \Delta x)} - \dfrac{1}{g(x)}}{\Delta x} = \frac{g(x) - g(x + \Delta x)}{\Delta x g(x + \Delta x)g(x)}$$
$$= \frac{-1}{g(x + \Delta x)g(x)}\left[\frac{g(x + \Delta x) - g(x)}{\Delta x}\right].$$

As Δx tends to 0, $g(x + \Delta x)$ tends to $g(x)$ and the expression in square brackets on the right side of the last equation tends to $\frac{dg}{dx}(x)$. Hence

$$\frac{d}{dx}\left[\frac{1}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{\dfrac{1}{g(x + \Delta x)} - \dfrac{1}{g(x)}}{\Delta x} = \frac{-\dfrac{dg}{dx}(x)}{g(x)^2}.$$

This is formula (3) if f is the constant function 1 because then df/dx is the zero function. Q.E.D.

EXAMPLE 3 Compute the derivative $\dfrac{d}{dx}\left[\dfrac{x^2}{2x - 1}\right]$.

Solution. According to the quotient rule (3) with $f(x) = x^2$ and $g(x) = 2x - 1$ we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^2}{2x - 1}\right] &= \frac{(2x - 1)\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(2x - 1)}{(2x - 1)^2}\\ &= \frac{(2x - 1)(2x) - x^2(2)}{(2x - 1)^2} = \frac{4x^2 - 2x - 2x^2}{(2x - 1)^2}\\ &= \frac{2x^2 - 2x}{(2x - 1)^2}. \quad \square\end{aligned}$$

Exercises

†Answer provided.
‡Outline of solution provided.

In Exercises 1 through 6 compute the derivatives.

1.‡ $\dfrac{d}{dx}[(1 + 3x - x^2)(x^2 - 5)]$

2.† $\dfrac{d}{dx}[(x - x^2)(1 + x^{-1} + x^{-2})]$

3. $\dfrac{d}{dx}[(4x^2 + 5)(6x^{-2} - 3)]$

4.‡ $\dfrac{d}{dt}[(\sqrt{t} + \sqrt[3]{t})(\sqrt[4]{t} + \sqrt[5]{t})]$

5.† $\dfrac{d}{dw}\left[\left(w^2 + \dfrac{1}{w^2}\right)(2 + 3w)\right]$

6. $\dfrac{d}{dz}[(1 - \sqrt{z})(1 + \sqrt{z} - z)]$

7.‡ Compute $F'(-2)$ where $F(x) = x^2G(x)$, $G(-2) = 3$, and $G'(-2) = 5$.

8.† Compute $\left[\dfrac{d}{dx}\left(\dfrac{P(x)}{x}\right)\right]_{x=5}$, where $P(5) = 10$ and $P'(5) = 2$

9. Compute $\left[\dfrac{d}{dt}R(t)\right]_{t=6}$, where $R(t) = P(t)Q(t)$, $P(6) = -1$, $Q(6) = 5$, $\dfrac{dP}{dt}(6) = 4$, and $\dfrac{dQ}{dt}(6) = 7$

10.‡ Compute $\left[\dfrac{d}{dw}\left(\dfrac{f(w) + g(w)}{h(w)}\right)\right]_{w=0}$, where $f(0) = 1$, $g(0) = 2$, $h(0) = 3$, $f'(0) = 4$, $g'(0) = 5$, and $h'(0) = 6$

11.† Compute $\frac{dV}{dy}(7)$, where $V(y) = \frac{y^2}{y - T(y)}$ and $T(7) = -3, \frac{dT}{dy}(7) = -5$.

12. Compute $K'(1)$, where $K(z) = \frac{z + f(z)}{z - f(z)}$ and $f(1) = 4, f'(1) = 2$.

In Exercises 13 through 15 give equations for the tangent lines.

13.‡ The tangent line to $y = \frac{2x + 3}{4x + 5}$ at $x = 0$

14.† The tangent line to $y = \frac{2}{2 - x}$ at $x = -2$

15. The tangent line to $y = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$ at $x = 9$

16. Use the product rule and the quotient rule for the special case of $f = 1$ to derive the quotient rule for general f.

Compute the derivatives of the functions in Exercises 17 through 25.

17.† $\frac{x^2 + 1}{x^3 - 1}$

18.† $(\sqrt{x} - 1)(\sqrt{x} + 1)$

19.† $(x^{-1} + 1 + x)(x^{-2} + 2 + x^2)$

20.† $\frac{\sqrt{x}}{3x - 4}$

21.† $\frac{1 + x + x^2}{1 + x^4}$

22.† $\frac{x + x^{-1}}{x - x^{-1}}$

23.† $(x^{1/3} + 1)(x^{2/3} - 1)$

24.† $\frac{x + 3}{x - 4}$

25.† $x^3 - \frac{3x}{x^2 + 1}$

26.† Derive the formula

$$\frac{d}{dx}[f(x)g(x)h(x)] = \frac{df}{dx}(x)g(x)h(x) + f(x)\frac{dg}{dx}(x)h(x) + f(x)g(x)\frac{dh}{dx}(x)$$

for the derivative of the product of three functions. Use the formula in Exercise 26 to compute the derivatives of the functions in Exercises 27 and 28.

27.† $(x^2 + 1)(x^3 + 2)(x^4 + 3)$

28.† $(x + 1)(x^{1/2} + x)(x^{-1} + x - 3)$

3-5 AVERAGE VELOCITY AND VELOCITY

We can say that a car has the constant velocity of 50 miles per hour if in every time interval of length T it travels (50 miles/hour)(T hours) $= 50T$ miles. This calculation requires only arithmetic. If, however, the car is accelerating or decelerating, then its velocity *varies*, and our calculations require the use of a derivative.

In each of Exercises 13 through 16 compute the acceleration of the object whose coordinate on an s-axis is given as a function of time.

13.† $s = t^4 + 2t - 1$ (meters) at time t (seconds)

14.† $s = \dfrac{t+1}{t+4}$ (centimeters) at time t (seconds)

15.† $s = t^{10} - 9\sqrt{t}$ (kilometers) at time t (hours)

16.† $s = t^{11/3}$ (kilometers) at time t (hours)

3-7 COMPOSITE FUNCTIONS AND THE CHAIN RULE

An important procedure for constructing functions is the *composition* of other functions. Suppose that we have a function $G = G(s)$ of the variable s and that we consider s to be a function $s = s(t)$ of the variable t. Then the function $G(s(t))$ of the variable t is a COMPOSITE FUNCTION; it is the composition of the functions G and s. The domain of $G(s(t))$ consists of all numbers t in the domain of the function s such that $s(t)$ is in the domain of the function G.

EXAMPLE 1 What is the value of the composite function $G(s(t))$ at $t = 5$ if $s(5) = 20$ and $G(20) = 8$?

Solution. At $t = 5$ we have $G(s(5)) = G(20) = 8$. □

EXAMPLE 2 The function s is defined by the formula $s(t) = 2t - 1$ and the function G by $G(s) = 4\sqrt{s}$. Give a formula for the composite function $G(s(t))$ in terms of its variable t. What is its domain?

Solution. Replacing s by $s(t) = 2t - 1$ in the formula for $G(s)$ gives $G(s(t)) = 4\sqrt{s(t)} = 4\sqrt{2t-1}$. The domain of $G(s) = 4\sqrt{s}$ is the closed half line $s \geq 0$, so the domain of the composite function consists of all t such that $s(t) = 2t - 1$ is nonnegative. The domain is the closed half line $t \geq \frac{1}{2}$. We could also determine this by examining the formula $4\sqrt{2t-1}$ for the composite function. □

The chain rule

The CHAIN RULE tells us how to compute the derivative of a composite function from the derivatives of the functions used to form it.

Theorem 3.6 (The Chain Rule) The derivative of the composition $G(s(t))$ of the functions $G(s)$ and $s(t)$ is

$$\frac{d}{dt}G(s(t)) = \frac{dG}{ds}(s(t))\frac{ds}{dt}(t). \tag{1}$$

Here we assume that the function $s(t)$ has a derivative at t and that the function $G(s)$ has a derivative at $s = s(t)$.

Proof. We will derive (1) at $t = t_0$. We set $s_0 = s(t_0)$. Because we assume that $G(s)$ has a derivative at s_0, we have

$$(2) \qquad \lim_{s \to s_0} \frac{G(s) - G(s_0)}{s - s_0} = \frac{dG}{ds}(s_0).$$

We cannot set $s - s_0 = s(t) - s(t_0)$ in the denominator of the difference quotient in (2) because even for $t \neq t_0$, $s(t)$ might be equal to $s(t_0)$ and $s(t) - s(t_0)$ might be zero. Therefore, we need to reformulate definition (2) of the derivative so that it does not involve $s - s_0$ in the denominator.

To do this, we let $E(s)$ $(s \neq s_0)$ denote the difference between the slope of the tangent line at $(s_0, G(s_0))$ and the slope of the secant line through the points $(s_0, G(s_0))$ and $(s, G(s))$. Then

$$(3) \qquad \frac{G(s) - G(s_0)}{s - s_0} = \frac{dG}{ds}(s_0) + E(s)$$

and statement (2) implies that $E(s)$ tends to zero as s tends to s_0.

Multiplying equation (3) by $s - s_0$ gives

$$(4) \qquad G(s) - G(s_0) = \left[\frac{dG}{ds}(s_0) + E(s)\right](s - s_0).$$

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We set $E(s_0) = 0$ so that equation (4) is also valid at $s = s_0$. Equation (4) with the statement that $E(s)$ tends to zero is the reformulation of definition (2) that we need.

Making the substitutions $s = s(t)$ and $s_0 = s(t_0)$ in equation (4) and dividing by $t - t_0$ $(t \neq t_0)$ yields

$$(5) \qquad \frac{G(s(t)) - G(s(t_0))}{t - t_0} = \left[\frac{dG}{ds}(s_0) + E(s(t))\right]\left[\frac{s(t) - s(t_0)}{t - t_0}\right].$$

Because the derivative

$$(6) \qquad \frac{ds}{dt}(t_0) = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

exists, $s = s(t)$ tends to $s_0 = s(t_0)$ and $E(s(t)) = E(s)$ tends to zero as t tends to t_0. Hence equations (2), (5), and (6) give

$$\begin{aligned}\left[\frac{d}{dt}G(s(t))\right]_{t=t_0} &= \lim_{t \to t_0} \frac{G(s(t)) - G(s(t_0))}{t - t_0} \\ &= \frac{dG}{ds}(s(t_0))\,\frac{ds}{dt}(t_0)\end{aligned}$$

which is formula (1) at $t = t_0$. Q.E.D

EXAMPLE 3 Suppose that $s(5) = 20$, $\frac{ds}{dt}(5) = 6$, and $\frac{dG}{ds}(20) = -4$. What is $\frac{d}{dt}G(s(t))$ at $t = 5$?

Solution. The chain rule (1) at $t = 5$ reads

$$\left[\frac{d}{dt}G(s(t))\right]_{t=5} = \frac{dG}{ds}(s(5))\frac{ds}{dt}(5).$$

Making the substitution $s(5) = 20$, we obtain

$$\left[\frac{d}{dt}G(s(t))\right]_{t=5} = \frac{dG}{ds}(20)\frac{ds}{dt}(5) = (-4)(6) = -24. \quad \square$$

The chain rule in abbreviated Leibniz notation

The form of the chain rule (1) can be emphasized by writing it in abbreviated notation. We use the letter "G" to denote both the function $G(s)$ of s and the composite function $G(s(t))$ of t, and we drop the references to where the derivatives are evaluated. Then equation (1) reads

$$\frac{dG}{dt} = \frac{dG}{ds}\frac{ds}{dt}. \quad \blacktriangleleft$$

This abbreviated formulation of the chain rule is useful because it is easy to remember. Even though the symbols representing the derivatives are *not* actual fractions, the *ds* terms appear to cancel giving the equation:

$$\frac{dG}{dt} = \frac{dG}{\cancel{ds}}\frac{\cancel{ds}}{dt}.$$

Interpreting the chain rule

Although the notation can make the general statement of the chain rule somewhat difficult to understand, in situations where the functions and their derivatives have everyday meaning, it follows from a common sense understanding of rates of change.

Let us examine the meaning of the chain rule (1) in the case where $s(t)$ is the distance (miles) a car has traveled along a route after t hours, and $G(s)$ is the amount (gallons) of gas it takes the car to travel s miles along the route. Then $G(s(t))$ is the amount of gas it takes the car up to time t, when it is at $s = s(t)$. The chain rule (1) reads

$$(7) \qquad \frac{d}{dt}G(s(t))\,\frac{\text{gallons}}{\text{hour}} = \left[\frac{dG}{ds}(s(t))\,\frac{\text{gallons}}{\text{mile}}\right]\left[\frac{ds}{dt}(t)\,\frac{\text{miles}}{\text{hour}}\right]$$

and states that the car's rate of consumption in gallons per hour is equal to its rate of consumption in gallons per mile, multiplied by its velocity in miles per hour.

Notice that in equation (7) the words "mile" and "miles," which give the units in the symbolic fractions, seem to cancel:

$$\frac{\text{gallons}}{\cancel{\text{mile}}}\,\frac{\cancel{\text{miles}}}{\text{hour}} = \frac{\text{gallons}}{\text{hour}}.$$

Symbolic calculations such as this are called DIMENSION ANALYSIS. They make it easier to remember equations such as (7).

The geometric meaning of the chain rule

Suppose that the functions $s(t)$ and $G(s)$ are those whose graphs are shown in Figures 3.28 and 3.29. The graph of the composite function $G(s(t))$ is then shown in Figure 3.30. The chain rule

$$(1)\qquad \frac{d}{dt}G(s(t)) = \frac{dG}{ds}(s(t))\,\frac{ds}{dt}(t)$$

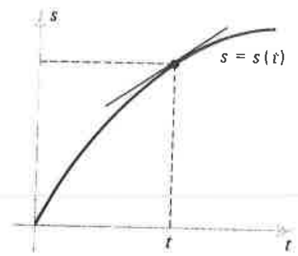


FIGURE 3.28 Graph of the function $s(t)$ and the tangent line of slope $\dfrac{ds}{dt}(t)$

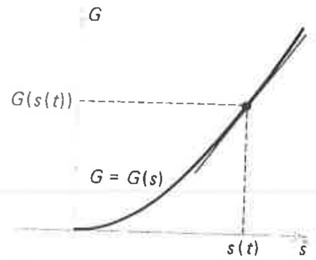


FIGURE 3.29 Graph of the function $G(s)$ and the tangent line of slope $\dfrac{dG}{ds}(s(t))$

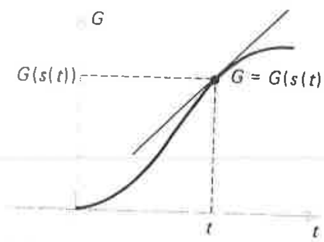


FIGURE 3.30 Graph of the composite function $G(s(t))$ and the tangent line of slope $\dfrac{d}{dt}G(s(t))$ $= \dfrac{dG}{dt}(s(t))\,\dfrac{ds}{dt}(t)$

states that the slope $\frac{d}{dt}G(s(t))$ of the tangent line to the graph of the composite function is equal to the *product* of the slopes $\frac{dG}{ds}(s(t))$ and $\frac{ds}{dt}(t)$ of the tangent lines to the graphs of $G(s)$ and $s(t)$ at the corresponding points.

The use of prime notation in the chain rule

It is sometimes preferable to use the chain rule with some or all of the derivatives expressed in prime notation. To do this, we follow the convention that *the prime always denotes the function's derivative with respect to its own variable.*

The variable of the function G in the chain rule (1) is s. Therefore, G' stands for $\frac{dG}{ds}$, and we can write (1) in the form

$$(8)\qquad \frac{d}{dt}G(s(t)) = G'(s(t))\,\frac{ds}{dt}(t).$$ ◀

The variable of the function s is t, so s' stands for $\frac{ds}{dt}$ and we can also express the chain rule in the form

$$(9)\qquad \frac{d}{dt}G(s(t)) = G'(s(t))\,s'(t).$$ ◀

EXAMPLE 4 Suppose that $G'(8) = -7$. What is $\frac{d}{dt}G(t^3)$ at $t = 2$?

Solution. Formulation (9) of the chain rule gives

$$\frac{d}{dt}G(t^3) = G'(t^3)\frac{d}{dt}t^3 = 3t^2\, G'(t^3).$$

At $t = 2$, this equation reads

$$\left[\frac{d}{dt}G(t^3)\right]_{t=2} = [3t^2\, G'(t^3)]_{t=2} = 12G'(8) = 12(-7) = -84$$

The derivative of $s(t)^n$

If we take G in the chain rule (1) to be the function $G(s) = s^n$, where n is a rational constant, we obtain

$$\frac{dG}{ds}(s) = \frac{d}{ds}s^n = ns^{n-1}.$$

Since in this case $G(s(t)) = s(t)^n$, the chain rule gives

$$\frac{d}{dt}s(t)^n = ns(t)^{n-1}\frac{ds}{dt}(t). \tag{10}$$

Note that in expressions such as on the left side of equation (10), as in expressions of the form $\frac{d}{dt}t^n$, we adopt the convention that the power is to be computed before the derivative. Thus, $\frac{d}{dt}s(t)^n$ stands for $\frac{d}{dt}[s(t)^n]$ (see the discussion of notation at the end of Section 3.2).

EXAMPLE 5 Suppose that $s(5) = 4$ and $s'(5) = -1$. What is $\frac{d}{dt}s(t)^3$ at $t = 5$?

Solution. Applying rule (10) with $n = 3$ yields

$$\frac{d}{dt}s(t)^3 = 3s(t)^{3-1}\frac{d}{dt}s(t) = 3s(t)^2s'(t).$$

Then, setting $t = 5$ and using the conditions $s(5) = 4$ and $s'(5) = -1$, we obtain

$$\begin{aligned}\left[\frac{d}{dt}s(t)^3\right]_{t=5} &= [3s(t)^2s'(t)]_{t=5} = 3[s(5)]^2\, s'(5)\\ &= 3[4]^2(-1) = -48. \quad \square\end{aligned}$$

EXAMPLE 6 Compute the derivative with respect to t of $\sqrt{t^2+4}$.

Solution. We apply rule (10) with $s(t) = t^2 + 4$ and $n = \frac{1}{2}$ to obtain

$$\frac{d}{dt}\sqrt{t^2+4} = \frac{d}{dt}(t^2+4)^{1/2} = \frac{1}{2}(t^2+4)^{-1/2}\frac{d}{dt}(t^2+4)$$
$$= \frac{1}{2}(t^2+4)^{-1/2}(2t) = t(t^2+4)^{-1/2}. \quad \square$$

The differentiation of a complex expression may require the product rule or the quotient rule and one or more applications of the chain rule. We can determine the order in which to apply these operations by noting the order of the steps used in calculating values of the function. The differentiation proceeds in the reverse order. This is illustrated in the following three examples.

EXAMPLE 7 Find the derivative of $\left(\frac{t^2+1}{t-3}\right)^5$.

Solution. To compute $\left(\frac{t^2+1}{t-3}\right)^5$ for a particular value of t, we first compute $t^2 + 1$ and $t - 3$, then we divide, and then we take the 5th power. Accordingly, to find the derivative we first apply the chain rule (10) with $n = 5$ and $s(t) = (t^2 + 1)/(t - 3)$; then we use the quotient rule; and finally we differentiate $t^2 + 1$ and $t - 3$:

$$\frac{d}{dt}\left(\frac{t^2+1}{t-3}\right)^5 = 5\left(\frac{t^2+1}{t-3}\right)^4 \frac{d}{dt}\left(\frac{t^2+1}{t-3}\right)$$
$$= 5\left(\frac{t^2+1}{t-3}\right)^4 \left[\frac{(t-3)\frac{d}{dt}(t^2+1) - (t^2+1)\frac{d}{dt}(t-3)}{(t-3)^2}\right]$$
$$= 5\left(\frac{t^2+1}{t-3}\right)^4 \left[\frac{(t-3)(2t) - (t^2+1)(1)}{(t-3)^2}\right]$$
$$= 5\left(\frac{t^2+1}{t-3}\right)^4 \left[\frac{t^2-6t-1}{(t-3)^2}\right]. \quad \square$$

EXAMPLE 8 What is the derivative of $x^2(x^3 + 2x)^{10}$?

Solution. To compute a value of $x^2(x^3 + 2x)^{10}$, we first compute $x^3 + 2x$, then the 10th power, then x^2, and finally the product. Therefore, to find the derivative, we first use the product rule; then we differentiate x^2 and use the chain rule (10) with x replacing t and $s(x) = x^3 + 2x$; and finally we differentiate $x^3 + 2x$:

$$\frac{d}{dx}[x^2(x^3+2x)^{10}] = \left(\frac{d}{dx}x^2\right)(x^3+2x)^{10} + x^2\left[\frac{d}{dx}(x^3+2x)^{10}\right]$$

$$= 2x(x^3+2x)^{10} + x^2\left[10(x^3+2x)^9\frac{d}{dx}(x^3+2x)\right]$$

$$= 2x(x^3+2x)^{10} + 10x^2(x^3+2x)^9(3x^2+2)$$

$$= 2x(x^3+2x)^{10} + (30x^4+20x^2)(x^3+2x)^9. \quad \square$$

EXAMPLE 9 Find the derivative of $[x^2 + (5x+1)^3]^7$.

Solution. To compute $[x^2 + (5x+1)^3]^7$ from x, we first compute $5x+1$, then its third power and x^2, then the sum, and finally the seventh power. Accordingly, to find the derivative, we first apply the chain rule (10) with $s(x) = x^2 + (5x+1)^3$; then we differentiate x^2 and use (10) with $s(x) = 5x+1$; and finally, we differentiate $5x+1$:

$$\frac{d}{dx}[x^2 + (5x+1)^3]^7 = 7[x^2+(5x+1)^3]^6\frac{d}{dx}[x^2+(5x+1)^3]$$

$$= 7[x^2+(5x+1)^3]^6\left[2x + 3(5x+1)^2\frac{d}{dx}(5x+1)\right]$$

$$= 7[x^2+(5x+1)^3]^6[2x+3(5x+1)^2(5)]$$

$$= 7[x^2+(5x+1)^3]^6[2x+15(5x+1)^2]. \quad \square$$

Exercises

†*Answer provided.*
‡*Outline of solution provided.*

In Exercises 1 through 17 compute the derivatives.

1.† $\dfrac{d}{dx}(1-x^3)^5$

2.† $f'(t)$, where $f(t) = (t^3+2)^{1/2}$

3. $\dfrac{d}{dx}(5x^4 - 4x^3 + 2)^{11}$

4.† $\dfrac{d}{dx}\dfrac{1}{\sqrt{3x-4}}$

5. $\dfrac{d}{du}\sqrt{u^3+u^5}$

6.‡ $\left[\dfrac{d}{dx}\sqrt{f(x)}\right]_{x=1}$, where $f(1) = 4$ and $f'(1) = -5$

7.† $\dfrac{dW}{dt}(0)$, where $W(t) = v(t)^5$ and $v(0) = 2$, $\dfrac{dv}{dt}(0) = -4$

8. $p'(5)$, where $p(x) = g(x)^{-1}$ and $g(5) = 10$, $g'(5) = 4$

9.‡ $\left[\dfrac{d}{dx}G(x^3)\right]_{x=2}$, where $G'(8) = 5$

10.† $W'(9)$, where $W(t) = S(\sqrt{t})$ and $S'(3) = -4$

11. $\left[\dfrac{d}{dx}F(x^3+2x)\right]_{x=-1}$, where $F'(-3) = 10$

12.‡ $\left[\dfrac{dy}{dt}\right]_{t=1}$, where $y = y(x(t))$, and $x(1) = 3$, $\dfrac{dx}{dt}(1) = 5$, $\dfrac{dy}{dx}(3) = 6$

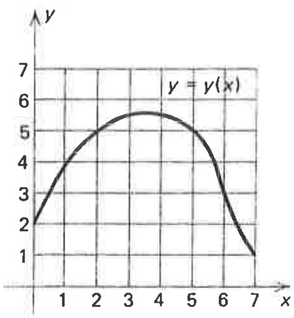


FIGURE 3.31

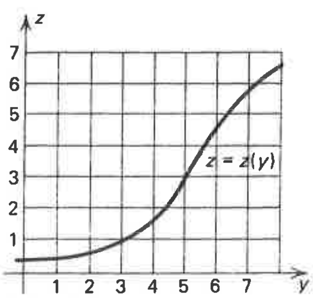


FIGURE 3.32

7 **13.**† $\left[\dfrac{dw}{dx}\right]_{x=0}$, where $w = w(v(x))$, and $v(0) = -1, \dfrac{dv}{dx}(0) = 2,$
$\dfrac{dw}{dv}(-1) = -3$

14. $\left[\dfrac{dF}{dt}\right]_{t=5}$, where $F = F(v(t))$, and $v(5) = 300, \dfrac{dv}{dt}(5) = 40, \dfrac{dF}{dv}(300) = 25$

15.‡ $\left[\dfrac{d}{dx}g(f(x))\right]_{x=3}$, where $f(3) = 5, f(6) = 4, f'(3) = 7, f'(6) = 10,$
$g(3) = 6, g(5) = 11, g'(3) = 10,$ and $g'(5) = -1$

7 **16.**† $C'(-1)$, where $C(x) = B(A(x))$, and $A(-1) = -3, A'(-1) = -4,$
$B(-3) = -6, B'(-3) = 9$

17. $\dfrac{dF}{dx}(3)$, where $F(x) = G(H(x))$, and $H(3) = 4, H'(3) = 7, G(4) = 5,$
$G'(4) = 6, G'(7) = 10, G'(3) = -2$

18.† The graphs of differentiable functions $y = y(x)$ and $z = z(y)$ are sketched in Figures 3.31 and 3.32. Give approximate values for $z(y(x))$ at $x = 2$ and $x = 5$ and for the derivative dz/dx at those two values of x.

7 **19.** The functions whose graphs are sketched in Figures 3.33 and 3.34 give an ice skater's velocity $v(t)$ (feet/minute) t minutes after he has begun skating and his oxygen consumption $A(v)$ (liters/minute) when he is skating v feet/minute. At what approximate rate is his oxygen consumption increasing 2 minutes after he starts skating?

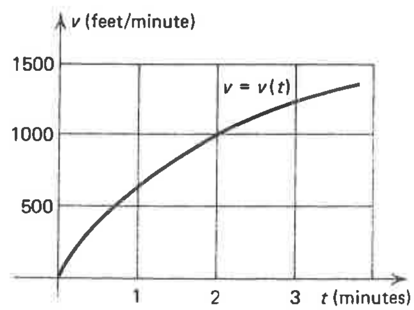


FIGURE 3.33

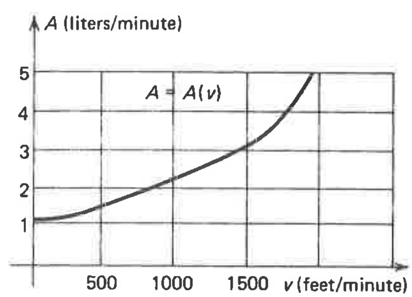


FIGURE 3.34 (Adapted from R. Shephard, [1] *The Physiology of Physical Activity*, p. 537.)

20.† The force of air resistance (drag) on a certain car is $D = \frac{1}{30}v^2$ pounds when the velocity is v miles per hour. The car is accelerating at the constant rate of 2 miles per hour every second. At what rate is the drag increasing when the car is going 50 miles per hour? (Adapted from S. Hoerner, [1] *Fluid Dynamic Drag*, p. 12.7.)

21. The function $D(v)$ of Figure 3.35 gives the air resistance on a certain blunt object as a function of its velocity. (Notice how sharply the curve rises near $v = 740$ miles per hour, the speed of sound. This represents the "sound barrier.") When the object is traveling 725 miles per hour, it is accelerating at the constant rate of 10 miles per hour every second. At approximately what rate is the drag increasing at that moment?

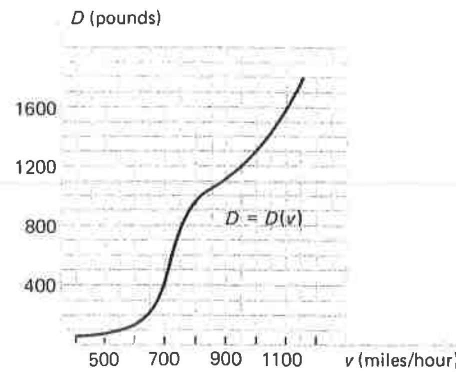


FIGURE 3.35 (Adapted from S. Hoerner, [1] *Fluid Dynamic Drag*, p. 16.26.)

Compute the derivatives of the functions in Exercises 22 through 36.

22.† $\sqrt{6x^2 + 3}$ **23.**† $(6\sqrt{x} + 3)^3$

24.† $\left(\dfrac{x+1}{x-1}\right)^5$ **25.**† $(8x)^{1/3} + (8x + 2)^{1/3}$

26.† $\dfrac{\sqrt{3x+2}}{\sqrt{5x-2}}$ **27.**† $\sqrt{\dfrac{3x+2}{5x-2}}$

28.† $x(5x + 4)^{1/4}$ **29.**† $[1 - \sqrt{2x+1}]^{1/2}$

30.† $\dfrac{x + \sqrt{5x-2}}{x^2 + 9}$ **31.**† $(x^2 + 3)^{1/3}(x^3 + 2)^{1/2}$

32. $\dfrac{x^2 - \sqrt{1 + x^2}}{x}$ **33.** $\dfrac{(2x+1)^{10} - (3x+1)^9}{(4x+1)^9}$

34. $(3x^4 + 1)(3x + 1)^4$ **35.** $(x^{1/2} + 1)^2(x^{1/3} + 1)^3$

36. $(x^2 + 1)^2(x^3 + 1)^3(x^4 + 1)^4$

In each of Exercises 37 through 40 find **a.** $\dfrac{dG}{du}(u)$ and **b.** $\dfrac{du}{dx}(x)$. **c.** Use the results of parts (a) and (b) to find $\dfrac{d}{dx}G(u(x))$. **d.** Then give a formula for $G(u(x))$ in terms of x. **e.** Use that formula to compute $\dfrac{d}{dx}G(u(x))$ directly.

37.† $G(u) = \sqrt{u}$; $u(x) = x^3 + 1$ **38.**† $G(u) = u^3 + 1$; $u(x) = \sqrt{x}$

Section 3.2

1. 11 **3.** $\frac{1}{6}$ **5.** $-\frac{1}{2}x$; Fig. 3.2.5a, b **7.** 5; Fig. 3.2.7a, b **9.** $\frac{3}{8}x^2$; Fig. 3.2.9a, b
11. $4x + 3$ **13.** $1/\sqrt{2x+3}$ **15.** 6 **16.** $1 - x^{-2}$ **17.** $-3(3x+1)^{-2}$
18. $-\frac{1}{2}x^{-3/2}$ **19.** $\frac{1}{2}$ **20.** 13 **21.** $\frac{1}{4}$ **22.** $(x+1)^{-2}$

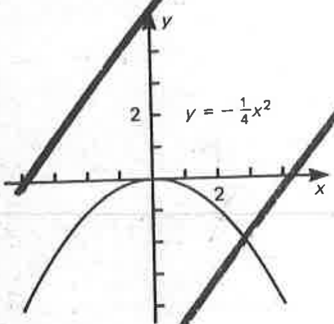


Fig. 3.2.5a

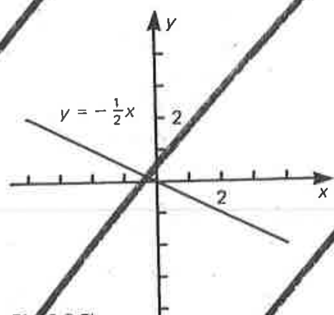


Fig. 3.2.5b

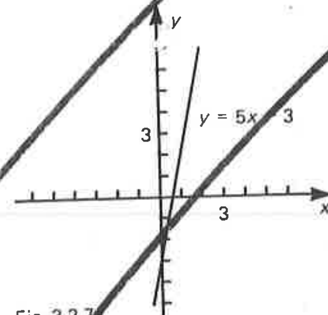


Fig. 3.2.7a

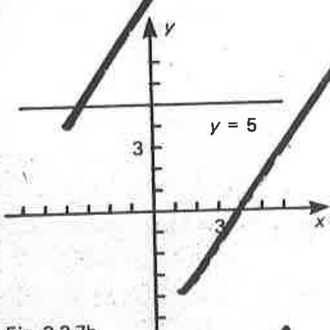


Fig. 3.2.7b

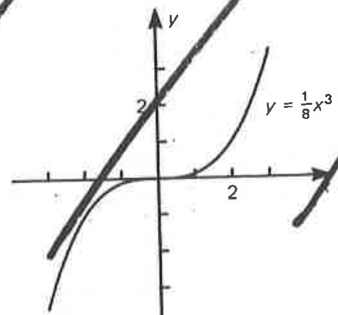


Fig. 3.2.9a

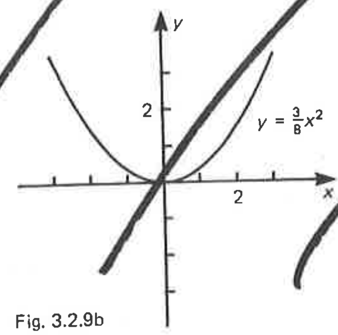


Fig. 3.2.9b

SOLUTIONS

$\boxed{\text{Section 3.3}}$

1. $15x^2 - 12x + 7$; all x **3.** $-18x^{-4} + 4x^{-2}$; $x \neq 0$ **5.** $\frac{3}{2}t^{1/2} - \frac{2}{3}t^{-1/3}$; $t > 0$

7.‡ $\dfrac{d}{dv}(\sqrt{v} - 4\sqrt[3]{v}) = \dfrac{d}{dv}(v^{1/2} - 4v^{1/3}) = \dfrac{1}{2}v^{-1/2} - \dfrac{4}{3}v^{-2/3}$; $v > 0$

8. $-1 + \frac{5}{2}s^{3/2}$; $s > 0$ **9.** $1 + \frac{3}{2}x^{-3/2}$; $x > 0$ **11.** $\frac{289}{4}$ **13.** 32 **15.** 33
17. $y = 8x - 16$ **19.** $y = -\frac{1}{70}x + \frac{4201}{70}$ **21.** $k = -4$

23. $\dfrac{d}{dx}(1) = \lim_{\Delta x \to 0}\left[\dfrac{1-1}{\Delta x}\right] = 0$

25. Rationalize by multiplying and dividing by $\sqrt{x+\Delta x} + \sqrt{x}$:

$$\frac{d}{dx}(x^{1/2}) = \lim_{\Delta x \to 0}\left[\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}\right] = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$$

27. Set $m = -n$ with n a positive rational number. As $\Delta x \to 0$,

$$\frac{1}{\Delta x}[(x+\Delta x)^m - x^m]$$

$$= \frac{1}{\Delta x}\left[\frac{1}{(x+\Delta x)^n} - \frac{1}{x^n}\right] = \left[\frac{x^n - (x+\Delta x)^n}{\Delta x}\right]\frac{1}{(x+\Delta x)^n x^n}$$

$$\to -\left(\frac{d}{dx}x^n\right)\frac{1}{x^{2n}} = -nx^{n-1}x^{-2n} = -nx^{-n-1} = mx^{m-1}$$

28. $-\frac{1}{2}x^{-3/2} + 3x^{-2}$ **29.** $2x+6$ **30.** $-\frac{1}{2}u^{-3/2} - \frac{3}{2}u^{1/2}$ **31.** $\frac{3}{5}w^{-2/5} + \frac{5}{3}w^{2/3}$
32. $10y^4 + 12y^3 - 8y$ **33.** $6x+2$ **34.** $-100x^{-51} - 60x^{-21} + 50x^{-11}$
35. $\frac{13}{42}x^{-29/42} - \frac{12}{35}x^{-23/35}$ **36.** $2t^{-2} - 2t^{-3}$ **37.** $\frac{1}{24}x^{-23/24}$ **38.** $a = \frac{1}{4}, b = \frac{3}{4}$
39. $a = 3, n = -\frac{1}{18}$

Section 3.4

1.‡ $\frac{d}{dx}[(1+3x-x^2)(x^2-5)] = (1+3x-x^2)\frac{d}{dx}(x^2-5)$

$+\left[\frac{d}{dx}(1+3x-x^2)\right](x^2-5)$

$= (1+3x-x^2)(2x) + (3-2x)(x^2-5)$

2. $(x-x^2)(-x^{-2}-2x^{-3}) + (1-2x)(1+x^{-1}+x^{-2})$

4.‡ $\frac{d}{dt}[(\sqrt{t}+\sqrt[3]{t})(\sqrt[4]{t}+\sqrt[5]{t})]$

$= (t^{1/2}+t^{1/3})\frac{d}{dt}(t^{1/4}+t^{1/5}) + \left[\frac{d}{dt}(t^{1/2}+t^{1/3})\right](t^{1/4}+t^{1/5})$

$= (t^{1/2}+t^{1/3})\left(\frac{1}{4}t^{-3/4}+\frac{1}{5}t^{-4/5}\right) + \left(\frac{1}{2}t^{-1/2}+\frac{1}{3}t^{-2/3}\right)(t^{1/4}+t^{1/5})$

5. $3(w^2+w^{-2}) + (2w-2w^{-3})(2+3w)$ **7.‡** $F'(x) = x^2G'(x) + 2xG(x)$;
$F'(-2) = (-2)^2G'(-2) + 2(-2)G(-2) = 4(5) - 4(3) = 8$ **8.** 0

10.‡ $\dfrac{[f'(0)+g'(0)]h(0) - [f(0)+g(0)]h'(0)}{h(0)^2}$

$= \dfrac{(4+5)(3) - (1+2)(6)}{3^2} = 1$

11. $-\frac{77}{50}$

13.‡ $\dfrac{dy}{dx} = \dfrac{2(4x+5) - (2x+3)(4)}{(4x+5)^2}$; For $x = 0$, $y = \dfrac{3}{5}$ and $dy/dx = -\dfrac{2}{25}$;

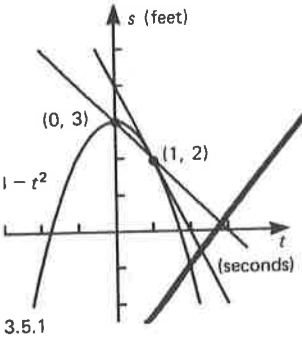
Tangent line: $y - \dfrac{3}{5} = -\dfrac{2}{25}x$

14. $y - \frac{1}{2} = \frac{1}{8}(x+2)$ **17.** $(-x^4 - 3x^2 - 2x)(x^3-1)^{-2}$ **18.** 1
19. $(-x^{-2}+1)(x^{-2}+2+x^2) + (x^{-1}+1+x)(-2x^{-3}+2x)$
20. $-\frac{1}{2}[\sqrt{x}(3x-4)]^{-1}$ **21.** $(-2x^5 - 3x^4 - 4x^3 + 2x + 1)(1+x^4)^{-2}$
22. $-4x(x^2-1)^{-2}$ **23.** $1 + \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-2/3}$ **24.** $-7(x-4)^{-2}$
25. $3x^2 - (3-3x^2)(x^2+1)^{-2}$

26. $\dfrac{d}{dx}\{f(x)[g(x)h(x)]\} = \dfrac{df}{dx}(x)g(x)h(x) + f(x)\dfrac{d}{dx}[g(x)h(x)]$
$= \dfrac{df}{dx}(x)g(x)h(x) + f(x)\left[\dfrac{dg}{dx}(x)h(x) + g(x)\dfrac{dh}{dx}(x)\right]$
$= \dfrac{df}{dx}(x)g(x)h(x) + f(x)\dfrac{dg}{dx}(x)h(x) + f(x)g(x)\dfrac{dh}{dx}(x)$
27. $2x(x^3 + 2)(x^4 + 3) + 3x^2(x^2 + 1)(x^4 + 3) + 4x^3(x^2 + 1)(x^3 + 2)$
28. $(x^{1/2} + x)(x^{-1} + x - 3) + (x + 1)(\frac{1}{2}x^{-1/2} + 1)(x^{-1} + x - 3)$
$+ (x + 1)(x^{1/2} + x)(-x^{-2} + 1)$

Section 3.5

1a. -1 ft./sec. **1b.** -2 ft./sec.; Fig. 3.5.1 **3a.** 37 mi./hr. **3b.** 37 mi./hr.; Fig. 3.5.3 **5a.** 5 furlongs per fortnight **5b.** 0 furlongs per fortnight; Fig. 3.5.5 **7a.** 32 ft./sec. **7b.** $t = 1$; $-1 \le t < 1$; $1 < t \le 3$; **7c.** 64 ft. **8a.** $-\frac{15}{2}$ mi./hr.; traveling west **8b.** 5 mi. east of the rest stop



3.5.1

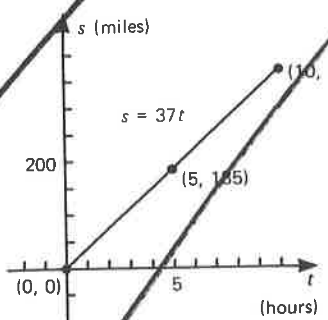


Fig. 3.5.3

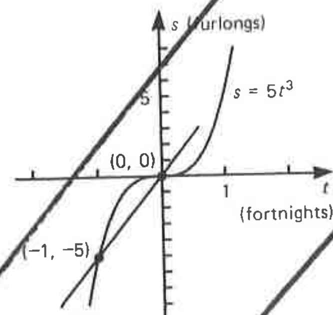


Fig. 3.5.5

Section 3.6

1a. $w = V^{1/3}$ **1b.** $\frac{1}{13}$ meters^{-2} **1c.** $\frac{1}{12}$ meters^{-2} **3a.** $-399/950{,}000$ $= -0.00042$ dynes/centimeter **3b.** -0.002 dynes/centimeter **5a.** $3t^2 - 24t + 45$ ft./sec. **5b.** -15 ft./sec.^{2} **5c.** -12 ft./sec.^{2} **5d.** 0 ft./sec.^{2} **7a.** 0.1% **7b.** 0.13% **7c.** 0.2% per hour **7d.** -0.025% per hour **10a.** $R(x) = 0.75x$ dollars; $P(x) = \frac{1}{2}x + \frac{1}{30}x^2 - 50$ dollars **10b.** Average profit $= -\frac{1}{6}$ dollar/dozen; Marginal profit $= \frac{5}{2}$ dollars/dozen **11a.** $R(p) = 1000p - 300p^2 + 30p^3 - p^4$ cents **11b.** -250 **13.** $12t^2$ meters/second^2 **14.** $-6(t + 4)^{-3}$ centimeters/second^2 **15.** $90t^8 + \frac{9}{4}t^{-3/2}$ kilometers/hour^2 **16.** $\frac{88}{9}t^{5/3}$ kilometers/hour^2

Section 3.7

1.‡ $\dfrac{d}{dx}(1 - x^3)^5 = 5(1 - x^3)^4\dfrac{d}{dx}(1 - x^3) = -15x^2(1 - x^3)^4$

2. $\frac{3}{2}t^2(t^3 + 2)^{-1/2}$

4.‡ $\dfrac{d}{dx}(3x - 4)^{-1/2} = -\dfrac{1}{2}(3x - 4)^{-3/2}\dfrac{d}{dx}(3x - 4) = -\dfrac{3}{2}(3x - 4)^{-3/2}$

6.‡ $\left[\dfrac{d}{dx}(f(x)^{1/2})\right]_{x=1} = \left[\dfrac{1}{2}f(x)^{-1/2}f'(x)\right]_{x=1} = \dfrac{1}{2}f(1)^{-1/2}f'(1)$

$= \dfrac{1}{2}(4)^{-1/2}(-5) = -\dfrac{5}{4}$

7. -320

9.‡ $\left[\dfrac{d}{dx}G(x^3)\right]_{x=2} = \left[G'(x^3)\dfrac{d}{dx}x^3\right]_{x=2} = [3x^2G'(x^3)]_{x=2} = 3(2)^2G'(8$

$= 12(5) = 60$

10. $-\frac{2}{3}$

12.‡ $\left[\dfrac{d}{dt}y(x(t))\right]_{t=1} = [y'(x(t))x'(t)]_{t=1} = y'(x(1))x'(1)$

$= y'(3)x'(1) = 6(5) = 30$

13. -6

15.‡ $\left[\dfrac{d}{dx}g(f(x))\right]_{x=3} = [g'(f(x))f'(x)]_{x=3} = g'(f(3))f'(3)$

$= g'(5)f'(3) = (-1)(7) = -7$

16. -36 **18.** At $x = 2$: $y \approx 5, z \approx 3$; $dy/dx \approx 1$ and $dz/dy \approx 2$, so $dz/dx \approx 2$; At $x = 5$: $y \approx 5, z \approx 3$; $dy/dx \approx -1$ and $dz/dy \approx 2$, so $dz/dx \approx -2$ **20.** $\frac{20}{3}$ pounds/second **22.** $6x(6x^2 + 3)^{-1/2}$
23. $9x^{-1/2}(6\sqrt{x} + 3)^2$ **24.** $-10(x + 1)^4(x - 1)^{-6}$
25. $\frac{8}{3}[8x^{-2/3} + (8x + 2)^{-2/3}]$ **26.** $-8\sqrt{(5x - 2)/(3x + 2)}(5x - 2)^{-2}$
27. $-8\sqrt{(5x - 2)/(3x + 2)}(5x - 2)^{-2}$ **28.** $(5x + 4)^{1/4} + \frac{5}{4}x(5x + 4)^{-3/}$
29. $\frac{1}{2}[1 - (2x + 1)^{1/2}]^{-1/2}[-(2x + 1)^{-1/2}]$
30. $[(x^2 + 9)(1 + \frac{5}{2}(5x - 2)^{-1/2}) - 2x(x + \sqrt{5x - 2})]/(x^2 + 9)^2$
31. $\frac{2}{3}x(x^2 + 3)^{-2/3}(x^3 + 2)^{1/2} + \frac{3}{2}x^2(x^2 + 3)^{1/3}(x^3 + 2)^{-1/2}$
37a. $\frac{1}{2}u^{-1/2}$ **37b.** $3x^2$ **37c.** $\frac{3}{2}x^2(x^3 + 1)^{-1/2}$ **37d.** $\sqrt{x^3 + 1}$
37e. $\frac{3}{2}x^2(x^3 + 1)^{-1/2}$ **38a.** $3u^2$ **38b.** $\frac{1}{2}x^{-1/2}$ **38c.** $\frac{3}{2}x^{1/2}$ **38d.** $x^{3/2} + 1$
38e. $\frac{3}{2}x^{1/2}$ **41.** $1 \le x < 2$ **42.** $|x - \frac{3}{2}| \ge \frac{5}{2}$

Section 3.8

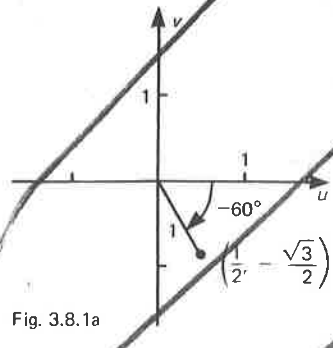


Fig. 3.8.1a

1a.‡ $-60°$ is $-60(\pi/180) = -\pi/3$ radians; $\sin(-\pi/3) = -\sqrt{3}/2$; $\cos(-\pi/3) = \frac{1}{2}$; Fig. 3.8.1a. **1b.**‡ 135° is $135(\pi/180) = 3\pi/4$ radians; $\sin(3\pi/4) = 1/\sqrt{2}$; $\cos(3\pi/4) = -1/\sqrt{2}$; Fig. 3.8.1b **1c.** $5\pi/2$ radians; $\sin(5\pi/2) = 1$; $\cos(5\pi/2) = 0$; Fig. 3.8.1c **2a.** $\sin(7\pi) = 0$; $\cos(7\pi) = -1$; 1260°; Fig. 3.8.2a **2b.** $\sin(-10\pi/3) = \frac{1}{2}\sqrt{3}$; $\cos(-10\pi/3) = -\frac{1}{2}$; $-600°$; Fig. 3.8.2b **2c.** $\sin(-3\pi/2) = 1$; $\cos(-3\pi/2) = 0$; $-270°$; Fig. 3.8.2c

3. $\pm\dfrac{\pi}{12} + 2k\pi$ radians, k any integer

4a. $\sin(1) \approx 0.841$; $\cos(1) \approx 0.540$ **4b.** $\sin(-\frac{1}{5}) \approx -0.199$; $\cos(-\frac{1}{5}) \approx 0.980$ **4c.** $\sin(10) \approx -0.544$; $\cos(10) \approx -0.839$
5a.‡ Table 3.8.5a; Fig. 3.8.5a **5b.** Fig. 3.8.5b