Homework 1

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Exercise 1. Let X be a Fréchet space, and let us index the countable family of seminorms with positive integers, i.e., $\|\cdot\|_1, \|\cdot\|_2$, and so on. Check that

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-y\|_j, 1\}$$

defines a distance on X and that it induces the same topology as the Fréchet structure.

Proof. Let us verify the d is a distance function. First observe that all terms are nonnegative, so

$$(i)\ 0 \leq d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-y\|_j, 1\} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} = 1 < \infty$$

$$(ii)\ d(x,x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-x\|_j, 1\} = \sum_{j=1}^{\infty} 0 = 0$$

$$(iii)\ d(x,y) = 0 \implies \|x-y\|_j = 0\ \forall j \in \mathbb{N} \iff x-y=0 \iff x=y$$

$$(iv)\ d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-y\|_j, 1\} = \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|y-x\|_j, 1\} = d(y,x)$$

$$(v)\ d(x,z) = \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-z\|_j, 1\} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-y\|_j + \|y-z\|_j, 1\}$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-y\|_j, 1\} + \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|y-z\|_j, 1\} = d(x,y) + d(y,z)$$

Now, define $B_r^j(x) = \{y \in X : \|x-y\|_j < r\}$. Let $B_r^d(x) = \{y \in X : d(x,y) < r\}$. Define $B_r^{\leq k} = \cap_{j \leq k} B_r^j(x) = \{y \in X : \|x-y\|_1 < r, \ldots, \|x-y\|_k < r\}$. Let the bases of the Fréchet topology and the new topology defined by the metric d, be

$$\mathscr{B}_1 = \{B_\epsilon^{\leq k}(x) | x \in X, \epsilon \in \mathbb{R}, k \in \mathbb{N}\}$$

$$\mathscr{B}_2 = \{B_\epsilon^d(x) | x \in X, \epsilon \in \mathbb{R}\}$$

respectively. We prove $\mathscr{B}_1 = \mathscr{B}_2$.

$(\supseteq)$ $\sum_{j=1}^{\infty} \frac{1}{2^j} \min\{\|x-z\|_j, 1\} = d(x,y) < \epsilon < 1 \implies \|x-y\|_1 < \frac{\epsilon}{2} \implies B^d_\epsilon(x) \subseteq B^1_{\frac{\epsilon}{2}}(x) \subseteq \mathscr{B}_1$

Now an arbitrary open set $B^1_r(x) \subseteq \mathscr{B}_1$ can be written as

$$B^1_r(x) = \cup_{y \in B^1_r(x)} B_{\epsilon_y}(y)$$

where $\epsilon_y < 1$. So that $\mathscr{B}_1 \supseteq \mathscr{B}_2$. $(\subseteq)$ Now recall that $\sum_{j=k+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^k}$. So that

$$\sum_{j=1}^{k} \frac{\epsilon}{2^j} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} < \epsilon + \frac{1}{2^k}$$

It follows that $B^{\leq k}_\epsilon(x) \subseteq B^d_{\epsilon + \frac{1}{2^k}}(x) \subseteq \mathscr{B}_2$; so that $\mathscr{B}_1 \subseteq \mathscr{B}_2$. Thus $\mathscr{B}_1 = \mathscr{B}_2$ and $\mathscr{B}_2$ generates the Fréchet topology, indeed. $\square$

Lemma 1. $\int_\Omega f\phi = 0$ for all $\phi \in \mathcal{D}(\Omega, \mathbb{C}) \implies f = 0$ a.e.

Proof. Recall that for an arbitrary ball $B_r(x_0) \subseteq \Omega$, $\mathcal{D}(B_r(x_0), \mathbb{C})$ is dense in $L^1(B_r(x_0))$. Thus for $\operatorname{sgn}(f) \in L^1(B_r(x_0))$, there exists a sequence $\phi_n \to \operatorname{sgn}(f)$ in L^1, where $\phi_n \in \mathcal{D}(B_r(x_0), \mathbb{C})$. Then some subsequence $\phi_{n_j} \to f$ pointwise a.e. Now let $g(x) = x$ if $|x| \leq 1$, and $g(x) = \frac{x}{|x|}$, otherwise. Define $\psi_j = g \circ \phi_{n_j}$. Then $\psi_j \in \mathcal{D}(B_r(x_0), \mathbb{C})$, $|\psi_j| \leq 1$, and $\psi_j \nearrow \operatorname{sgn}(f)$. Now, by the Dominated Convergence Theorem, we have,

$$\int_{B_r(x_0)} |f| = \int_{B_r(x_0)} f \cdot \operatorname{sgn}(f) = \int_{B_r(x_0)} f \cdot \lim_{j\to\infty} \psi_j = \lim_{j\to\infty} \int_{B_r(x_0)} f \cdot \psi_j = \lim_{j\to\infty} 0 = 0$$

Thus $|f| = 0$ a.e. on $B_r(x_0)$, iff $f = 0$ a.e. on $B_r(x_0)$. Since $B_r(x_0)$ was arbitrary, it follows that $f = 0$ a.e. on all of Ω. $\square$

Exercise 2. (i) Prove $R \in \mathcal{D}'(\Omega, \mathbb{C})$ can be written as $R = S + iT$ with $S, T \in \mathcal{D}'(\Omega, \mathbb{R})$. (ii) Write the most reasonable defintion for $\overline{R}$, for $R \in \mathcal{D}'(\Omega, \mathbb{C})$

Proof. (i) For $\phi \in \mathcal{D}(\Omega, \mathbb{C})$, let $R \in \mathcal{D}'(\Omega, \mathbb{C})$. Then, R is a linear functional, and

$R(\phi) = R(\mathfrak{R}\phi + i\mathfrak{I}\phi) = R(\mathfrak{R}\phi) + iR(\mathfrak{I}\phi) = (R \circ \mathfrak{R})(\phi) + i(R \circ \mathfrak{I})(\phi) = ((R \circ \mathfrak{R}) + i(R \circ \mathfrak{I}))(\phi)$

For all $\phi \in \mathcal{D}(\Omega, \mathbb{C})$. Now $R \circ \mathfrak{R}, R \circ \mathfrak{I} \in \mathcal{D}(\Omega, \mathbb{R})$. Let $S = R \circ \mathfrak{R}$, $T = R \circ \mathfrak{I}$. We have $R = S + iT$, as desired.
(ii) Let $R = R_f$. Then,

$$\langle R_{\overline{f}}, \phi \rangle = \int \overline{f}\phi = \int \overline{\overline{\overline{f}\phi}} = \overline{\int f\overline{\phi}} = \overline{\langle R_f, \overline{\phi} \rangle}$$

Now,

$$\int f\phi = \langle R_f, \phi \rangle = \langle R_{\overline{f}}, \phi \rangle = \int \overline{f}\phi \iff \int (f - \overline{f})\phi = 0$$

for all $\phi \in \mathcal{D}(\Omega, \mathbb{C})$. By Lemma 1, it follows that $f - \overline{f} = 0 \iff f = \overline{f} \iff$
$f \in \mathcal{D}(\Omega, \mathbb{R})$. $\square$

Exercise 3. Let $f \in L^p(\Omega), g \in L^q(\Omega),\ 1 \leq p, q \leq \infty$. Prove $T_f = T_g \iff f = g$ a.e.

Proof. We have $f \in L^p(\Omega) \subseteq L^1(\Omega),\ g \in L^q(\Omega) \subseteq L^1(\Omega)$.

($\Leftarrow$) $f = g$ a.e. $\implies f\phi = g\phi$ a.e. for all $\phi \in \mathcal{D}(\Omega, \mathbb{R}) \implies \langle T_f, \phi \rangle = \int_\Omega f\phi = \int_\Omega g\phi = \langle T_g, \phi \rangle$ for all $\phi \in \mathcal{D}(\Omega, \mathbb{R}) \iff T_f = T_g$. ($\Rightarrow$) Now suppose $T_f = T_g$. Then,

$$\int f\phi = \langle T_f, \phi \rangle = \langle T_g, \phi \rangle = \int g\phi \iff \int (f - g)\phi = 0$$

for all $\phi \in \mathcal{D}(\Omega, \mathbb{R})$. By Lemma 1, it follows that $f - g = 0$ a.e. on $\Omega \iff f = g$ a.e. on Ω. $\square$