

Homework 5

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Exercise 1.

Proof. First, we claim $f_b \in L^1(\mathbb{R}^n)$, and $f_s \in L^r(\mathbb{R}^n)$ -

$$\int |f_b| = \int |f_b|^p |f_b|^{1-p} \leq C(p) \|f\|_p^p < \infty$$

$$\int |f_s|^r = \int |f_s|^{r-p} |f_s|^p \leq C(r, p) \|f\|_p^p < \infty$$

Now, since $|Tf| \leq |Tf_b| + |Tf_s|$, we have $\{|Tf| > t\} \subseteq \{|Tf_b| > \frac{t}{2}\} \cup \{|Tf_s| > \frac{t}{2}\}$, so that -

$$\begin{aligned} \mu\{|Tf| > t\} &\leq \mu\{|Tf_b| > \frac{t}{2}\} + \mu\{|Tf_s| > \frac{t}{2}\} \leq \\ &\frac{2A\|f\|_1}{t} \int |f_b| + \frac{2^r A^r \|f\|_r^r}{t^r} \int |f_s|^r \end{aligned}$$

Now,

$$\int_0^\infty t^{q-1} t^{-1} \int_{|f|>t} |f| = \int_{\mathbb{R}^n} |f| \int_0^{|f|} t^{q-2} = \frac{1}{q-1} \int_{\mathbb{R}^n} |f| |f|^{q-1} = \frac{\|f\|_q^q}{q-1}$$

since $q > p > 1$, and

$$\int_0^\infty t^{q-1} t^{-r} \int_{|f|\leq t} |f|^r = \int_{\mathbb{R}^n} |f|^r \int_{|f|}^\infty t^{q-1-r} = \frac{1}{r-q} \int_{\mathbb{R}^n} |f|^r |f|^{q-r} = \frac{\|f\|_q^q}{r-q}$$

since $q < r$. Altogether,

$$\|Tf\|_q \leq C\|f\|_q$$

□

Exercise 2.

Proof. (a)

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_{2r}(0)} |K(x) - K(x-z)| dx &= \int_{\mathbb{R}^n \setminus B_{2r}(0)} \left| \int_0^1 DK d\gamma \right| dx = \\ \int_{\mathbb{R}^n \setminus B_{2r}(0)} \int_0^1 \left| DK(t)(x - (1-t)z) \cdot z \right| dt dx &\leq \int_{\mathbb{R}^n \setminus B_{2r}(0)} B|x|^{-n-1} |x - (x-z)| dx = \end{aligned}$$

$$B|z| \int_{\mathbb{R}^n \setminus B_{2r}(0)} |x|^{-n-1} dx = C(n)B$$

(b)

$$\begin{aligned} \int_{\mathbb{R}^n} \left(K(x) - K(x - x_\xi) \right) e^{-2\pi i x \xi} dx &= \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx - \int_{\mathbb{R}^n} K(x - x_\xi) e^{-2\pi i x \xi} dx = \\ \hat{K}(\xi) - \int_{\mathbb{R}^n} K(x) e^{-2\pi i (x + x_\xi) \xi} dx &= \hat{K}(\xi) - e^{-i\pi} \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx = \\ \hat{K}(\xi) + \hat{K}(\xi) &= 2\hat{K}(\xi) \end{aligned}$$

(c) By the cancellation condition, we have

$$\begin{aligned} \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) e^{-2\pi i x \xi} \right| &= \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) e^{-2\pi i x \xi} - \int_{B_{\frac{1}{|\xi|}}(0)} K(x) \right| = \\ \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) (e^{-2\pi i x \xi} - 1) \right| &\leq \int_{B_{\frac{1}{|\xi|}}(0)} |K(x)| \cdot |e^{-2\pi i x \xi} - 1| \leq 2\pi |\xi| |x| \int_{B_{\frac{1}{|\xi|}}(0)} |K(x)| \leq \\ 2\pi |\xi| |x| A \int_{B_{\frac{1}{|\xi|}}(0)} |x|^{-n} &= 2\pi |\xi| A \int_{B_{\frac{1}{|\xi|}}(0)} |x|^{-n+1} = C(n)A \end{aligned}$$

(d)

$$\begin{aligned} \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x - x_\xi) e^{-2\pi i x \xi} \right| &= \left| e^{i\pi} \int_{B_{\frac{1}{|\xi|}}(0)} K(x - x_\xi) e^{-2\pi i x \xi} \right| = \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x - x_\xi) e^{-2\pi i (x - x_\xi) \xi} \right| = \\ \left| \int_{B_{\frac{1}{|\xi|}}(x_\xi)} K(x) e^{-2\pi i x \xi} \right| & \end{aligned}$$

Since $|x_\xi| = \frac{1}{2|\xi|} < \frac{1}{|\xi|}$, we have that $B_r(0) \subseteq B_{\frac{1}{|\xi|}}(x_\xi)$ for some $0 < r < \frac{1}{|\xi|}$. Thus, as in (c),

$$\left| \int_{B_r(0)} K(x - x_\xi) e^{-2\pi i x \xi} \right| \leq 2\pi r A \int_{B_r(0)} |x|^{-n+1} \leq 2\pi |\xi| A \int_{B_{\frac{1}{|\xi|}}(0)} |x|^{-n+1} = C(n)A$$

Now,

$$\begin{aligned} \left| \int_{B_{\frac{1}{|\xi|}}(x_\xi)} K(x) e^{-2\pi i x \xi} \right| &= \left| \int_{B_{\frac{1}{|\xi|}}(x_\xi) \setminus B_r(0)} K(x) e^{-2\pi i x \xi} + \int_{B_r(0)} K(x) e^{-2\pi i x \xi} \right| \leq \\ \left| \int_{B_{\frac{1}{|\xi|}}(x_\xi) \setminus B_r(0)} K(x) e^{-2\pi i x \xi} \right| &+ \left| \int_{B_r(0)} K(x) e^{-2\pi i x \xi} \right| \leq \end{aligned}$$

$$A \int_{B_{\frac{1}{|\xi|}}(x_\xi) \setminus B_r(0)} \frac{1}{|x|^n} + C(n)A = C + C(n)A \leq C_1(n)A$$

(e)

$$\begin{aligned} |2\hat{K}(\xi)| &= \left| \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx \right| = \left| \int_{\mathbb{R}^n} (K(x) - K(x - x_\xi)) e^{-2\pi i x \xi} dx \right| = \\ &= \left| \int_{\mathbb{R}^n \setminus B_{\frac{1}{|\xi|}}(0)} (K(x) - K(x - x_\xi)) e^{-2\pi i x \xi} dx + \int_{B_{\frac{1}{|\xi|}}(0)} (K(x) - K(x - x_\xi)) e^{-2\pi i x \xi} dx \right| \leq \\ &\leq \int_{\mathbb{R}^n \setminus B_{\frac{1}{|\xi|}}(0)} |K(x) - K(x - x_\xi)| dx + \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) e^{-2\pi i x \xi} dx \right| + \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x - x_\xi) e^{-2\pi i x \xi} dx \right| \leq \\ &A + 2C(n)A \leq C_1(n)A \iff |\hat{K}(\xi)| \leq \frac{C_1(n)A}{2} \end{aligned}$$

□

Exercise 3.

Proof. (a) (i) $|K \mathbf{1}_{B_r \setminus B_\epsilon}(x)| \leq |K(x)| \leq A|x|^{-n}$

(ii)

If $r < \frac{R}{2}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_{2r}(0)} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x - z)| dx &\leq \int_{\mathbb{R}^n \setminus B_R(0)} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x - z)| dx \leq \\ &\int_{\mathbb{R}^n \setminus B_R(0)} \frac{1}{|x|^n} dx \leq A \int_{B_R(0) \setminus B_{\frac{R}{2}}(0)} \frac{1}{|x - z|^n} dx = \\ &A \int_{B_R(0) \setminus B_{\frac{R}{2}}(0)} \frac{1}{|x|^n} dx = C_1(n)A \end{aligned}$$

If $\frac{R}{2} \leq r \leq R$, we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_{2r}(0)} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x - z)| dx &= \int_{\mathbb{R}^n \setminus B_{2r}(0) \cap \{K(x) \neq 0\} \cap \{K(x - z) \neq 0\}} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x - z)| dx \leq \\ &A \int_{B_R(0) \setminus B_{\frac{R}{2}}(0)} \frac{1}{|x|^n} dx + A \left(\mu\{B_R(0) - B_{\frac{R}{2}}(0)\} \right) \leq C_2(n)A \end{aligned}$$

If $r > R$, we have

$$|x - z| \geq |x| - |z| \geq 2R - R$$

so that the region of integration falls outside the support of $K(x)$, and $K(x - z)$. Therefore,

$$\int_{\mathbb{R}^n \setminus B_{2r}(0)} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x - z)| dx = 0$$

(iii)

$$\int_{B_s(0) \setminus B_r(0)} K = \int_{B_s(0)} K - \int_{B_r(0)} K = 0 - 0 = 0$$

(b) We will prove $\|K_{\epsilon,R} * f\|_{L^2} \leq \|\hat{K}_{\epsilon,R}\|_{L^\infty} \|f\|_{L^2}$. The general inequality $\|K_{\epsilon,R} * f\|_{L^2} \leq C(n,p)A\|f\|_{L^p}$ then follows by the Marcinkiewicz Interpolation Theorem, the same way we proved the Calderon-Zygmund estimate. To that end, by Plancherel's identity, we have -

$$\|K_{\epsilon,R} * f\|_{L^2} = \|\mathcal{F}(K_{\epsilon,R} * f)\|_{L^2} = \|\hat{K}_{\epsilon,R} \cdot \hat{f}\|_{L^2} \leq \|\hat{K}_{\epsilon,R}\|_{L^\infty} \cdot \|\hat{f}\|_{L^2} = C(n)A\|\hat{f}\|_{L^2}$$

(c) Since f is smooth and has compact support, by the Extreme Value Theorem, f achieves a minimum and a maximum on it's domain. Let m, M be the minimal, and maximal values of f , respectively. WLOG, suppose $R \geq a$. Then,

$$\begin{aligned} |(K_{\epsilon,R} * f)(x)| &= \left| \int K_{\epsilon,R}(y) f(x-y) dy \right| = \left| \int_{|x| \geq a} K_{\epsilon,R}(y) f(x-y) dy + \int_{B_a(0)} K_{\epsilon,R}(y) f(x-y) dy \right| \leq \\ &\quad \left| \int_{|x| \geq a} K_{\epsilon,R}(y) f(x-y) dy \right| + \left| \int_{B_a(0)} K_{\epsilon,R}(y) f(x-y) dy \right| \leq \\ &\quad \int_{|x| \geq a} |K_{\epsilon,R}(y) f(x-y)| dy + \left| \int_{B_a(0) \setminus B_\epsilon(0)} K(y) f(x-y) dy \right| \leq \\ |M| &\left(A \int_{\{|x| \geq a\} \cap B_R(0)} \frac{1}{|a|^n} + \left| \int_{B_a(0) \setminus B_\epsilon(0)} K(y) dy \right| \right) \leq |M| \left(\mu\{B_R(0)\} \cdot \frac{A}{|a|^n} + 0 \right) = C(n)A \cdot |M| \end{aligned}$$

for every $\epsilon > 0$, and for every $R \geq a$. Thus the integral is absolutely convergent and the limit exists. Now, since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we can approximate $g \in L^p(\mathbb{R}^n)$ with a sequence $C_c^\infty(\mathbb{R}^n) \ni g_j \rightarrow g$, with convergence in L^p . Thus, by the dominated convergence theorem, we have

$$\begin{aligned} \|K_{\epsilon,R} * g_j\|_{L^p}^p &= \int_{\mathbb{R}^n} |(K_{\epsilon,R} * g_j)(x)|^p = \int_{\mathbb{R}^n} \left| \int K_{\epsilon,R}(y) g_j(x-y) dy \right|^p \nearrow \\ &\int_{\mathbb{R}^n} \left| \int K_{\epsilon,R}(y) g(x-y) dy \right|^p \nearrow \int_{\mathbb{R}^n} \left| \int K(y) g(x-y) dy \right|^p = \\ &\int_{\mathbb{R}^n} |(K * g)(x)|^p = \|K * g\|_{L^p}^p \end{aligned}$$

as $\epsilon \rightarrow 0, R, j \rightarrow \infty$. □

Exercise 4.

Proof. (a)

$$\int_{B_2(0)} \frac{\epsilon^2}{(|x|^2 + \epsilon^2)^{\frac{n}{2}+1}} dx = \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi \int_0^2 \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} \sin \phi_1 \cdots \sin \phi_{n-1} \cdot r^{n-1} dr d\phi_1 \cdots d\phi_{n-1}$$

We will bound $\int_0^2 \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} r^{n-1} dr$. The rest of the integrals are obviously bounded (In particular, $|\sin \phi_i| \leq 1$). To this end,

$$\int_0^2 \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} r^{n-1} dr = \int_0^\epsilon \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} r^{n-1} dr + \int_\epsilon^2 \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} r^{n-1} dr = I_1 + I_2$$

We will bound I_1, I_2 separately. On $(0, \epsilon)$,

when $r \leq \frac{\epsilon}{2}$, we have -

$$\frac{\epsilon^n}{r^{n-1}} \geq \frac{\epsilon^n \cdot 2^{n-1}}{\epsilon^{n-1}} = 2^{n-1} \cdot \epsilon$$

when $r \geq \frac{\epsilon}{2}$, we have -

$$\frac{r^3}{\epsilon^2} \geq \frac{\epsilon^3}{\epsilon^2 \cdot 2^3} = \frac{\epsilon}{8}$$

So that

$$\frac{\epsilon^n}{r^{n-1}} + \frac{r^3}{\epsilon^2} \geq \min\{2^{n-1} \cdot \epsilon, \frac{\epsilon}{8}\}$$

Now, $(r^2 + \epsilon^2)^{\frac{n}{2}+1} \geq r^{n+2} + \epsilon^{n+2}$ by the binomial theorem, so -

$$I_1 = \int_0^\epsilon \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} r^{n-1} dr = \int_0^\epsilon \frac{1}{\frac{\epsilon^n}{r^{n-1}} + \frac{r^3}{\epsilon^2}} dr \leq \epsilon \cdot \max\left\{\frac{2^{1-n}}{\epsilon}, \frac{8}{\epsilon}\right\} = \max\{2^{1-n}, 8\} = 8$$

On $(\epsilon, 2)$,

when $\epsilon \leq r \leq \frac{\epsilon+2}{2}$, we have -

$$\frac{r^3}{\epsilon^2} \geq \frac{(\frac{\epsilon+2}{2})^3}{\epsilon^2} \geq \frac{(\epsilon+2)^3}{\epsilon^2 \cdot 2^3} \geq \frac{\epsilon}{8}$$

when $\frac{\epsilon+2}{2} \leq r \leq 2$, we have -

$$\frac{\epsilon^n}{r^{n-1}} \geq \frac{\epsilon^n \cdot 2^{n-1}}{\epsilon^{n-1}} = 2^{n-1} \cdot \epsilon$$

So that

$$\frac{\epsilon^n}{r^{n-1}} + \frac{r^3}{\epsilon^2} \geq \min\{2^{n-1} \cdot \epsilon, \frac{\epsilon}{8}\}$$

Whence,

$$I_2 = \int_\epsilon^2 \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{n}{2}+1}} r^{n-1} dr = \int_\epsilon^2 \frac{1}{\frac{\epsilon^n}{r^{n-1}} + \frac{r^3}{\epsilon^2}} dr \leq (2 - \epsilon) \cdot \max\left\{\frac{2^{1-n}}{\epsilon}, \frac{8}{\epsilon}\right\} \leq 2 \cdot 8 = 16$$

□