Class 4

Wednesday, October 14, 2020 3:21 PM

Recall: $Mf(x) := \sup_{r>0} ⨍_{B_r(x)} |f|$

(for $f \in L^1_{loc}(\mathbb{R}^n)$)

is the Hardy–Littlewood maximal function.

<u>Thm</u> $\|Mf\|_{L^{1,\infty}} \leq C(n) \|f\|_{L^1}$

& $\forall 1 < p \leq \infty$ $\|Mf\|_{L^p} \leq C(p,n)\|f\|_{L^p}$

$L^{1,\infty}$ is a special case of $L^{p,q}$ $p,q \in [1,\infty]$

Lorentz spaces

($L^p = L^{p,p}$).

Recall: an application is

$\varphi_\varepsilon * f \to f$ a.e. as $\varepsilon \to 0$.

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"mollifiers" / "approximations of identity"

Another (essentially the same):

<u>Def</u> A <u>Lebesgue point</u> for $f \in L^1(\mathbb{R}^n)$ is $x_0 \in \mathbb{R}^n$ s.t.

$$\fint_{B_r(x_0)} |f - f(x_0)| \to 0 \quad \text{as } r \to 0.$$

(Also called <u>approximate continuity point</u>).

<u>Def'</u> Ask instead that $\exists \lambda \in \mathbb{C}$ s.t.

$$\fint_{B_r(x_0)} |f - \lambda| \to 0 \quad \text{as } r \to 0.$$

In the second case, you can def. $\tilde{f}(x_0) := \lambda$

$\tilde{f}$ well-defined

and leave $\tilde{f}$ undef.

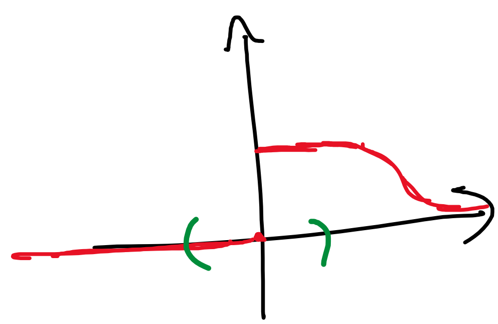
if no such λ exists.

<u>Thm</u> A.e. x_0 is a Lebesgue point.

<u>Coroll</u> $\tilde{f}$ is def. almost everywhere and $\tilde{f}=f$ a.e.

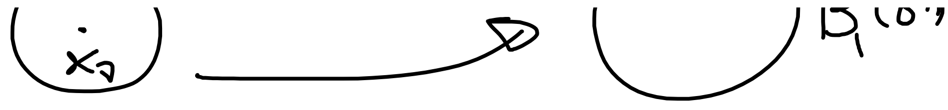
$\tilde{f}$ is the <u>robust representative</u> of $[f] \in L^1$.

<u>Example</u>



$x_0=0$ makes Def and Def' fail.

<u>Rmk</u> Def' is saying that $f(x_0+r\,\cdot) \xrightarrow[L^1(B_1)]{} \lambda$



Proof of Thm

Take $\varphi \in C_c^0$.

$$\underbrace{\left\{ x_0 : \limsup_{r \to 0} ⨍_{B_r(x_0)} |f - f(x_0)| > \alpha \right\}}$$

$$\leq \limsup ⨍_{B_r(x_0)} |f - \varphi| + \cancel{\limsup ⨍ |\varphi - \varphi(x_0)|} + |f(x_0) - \varphi(x_0)|$$

$$\leq M(f-\varphi)(x_0) + |f-\varphi|(x_0)$$

$$\left\{ \cdots > \alpha \right\} \subseteq \left\{ x_0 : M(f-\varphi)(x_0) > \frac{\alpha}{2} \right\} \cup \left\{ x_0 : |f-\varphi|(x_0) > \frac{\alpha}{2} \right\}$$

$$\Rightarrow \mathcal{L}^n \left\{ \cdots > \alpha \right\}$$

by $L^{1,\alpha}$ bound

$$\leq \frac{2}{\alpha} 3^n \|f-\varphi\|_{L^1}$$

$$+ \frac{2}{\alpha} \|f-\varphi\|_{L^1}.$$

Now take $\varphi \to f$ in L^1

$$\Rightarrow \mathcal{L}^n \left\{ x_0 : \limsup_{r \to 0} \fint_{B_r(x_0)} |f - f(x_0)| > \alpha \right\} = 0$$

$$\Rightarrow \mathcal{L}^n \{ \limsup > 0 \} = 0.$$

$\square$

Another application: Sobolev's inequality

Thm For $f \in \mathcal{S}(\mathbb{R}^n)$ and $p \in [1, n)$

$$\|f\|_{L^{p^*}} \leq C(p,n) \|Df\|_{L^p}.$$

$$\boxed{\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}}$$

Rmk $p^* > p$, so

"f is more integrable than Df".

Rmk $f(x) = \int_{-\infty}^{x} f'(t)\,dt$

$\Rightarrow \|f\|_{L^\infty} \le \|Df\|_{L^1}$ when $n=1$.

$(p=1,\ n=1$ gives $p^* = \infty)$

But in general $(n>2)$

$\|f\|_{L^\infty} \le C(n)\,\|Df\|_{L^n}$.

We will see the proof for $1<p<n$.

It's true also for $p=1$

It's true

(it is basically equivalent to the isoperimetric ineq.).

Proof of Thm

We will prove

$$\boxed{\left| f(x) - \fint_{B_r(x)} f \right| \leq C(n) r \cdot M(Df)(x).} \quad (*)$$

Let's see how Thm follows:

$$|f(x)| \leq C r M(Df)(x) + \fint_{B_r(x)} |f|$$

$$\leq C r M(Df)(x) + C r^{-n} \left(\int_{B_r} |f|^{p^*} \right)^{1/p^*} \left(\int_{B_r} 1 \right)^{1 - 1/p^*}$$

2^{nd} term is $\le C r \, r^{n(\cancel{1}-1/p^*)} \|f\|_{L^{p^*}}$

$$= C r^{-n/p^*} \|f\|_{L^{p^*}}.$$

(I could assume $\|f\|_{L^{p^*}} = 1$.)

$$\Rightarrow |f(x)| \le C r M(Df)(x) + C r^{-n/p^*} \|f\|_{L^{p^*}}$$

r arbitrary.

We then minimize R.H.S. in r (i.e., choose r making the ineq. strongest).

$$0 = \frac{d}{dr} RHS = C M(Df)(x) + C\left(-\frac{n}{p^*}\right) r^{-1-n/p^*} \|f\|_{L^{p^*}}$$

$$-1 - \frac{n}{p^*} = -1 - n\left(\frac{1}{p} - \frac{1}{n}\right)$$

$$= -\frac{n}{p}$$

$$\Longrightarrow \text{ choose } r := \left(\frac{M(Df)}{\|f\|_{L^{p^*}}}\right)^{-p/n}$$

$$RHS = C\, M(Df)(x)^{1-p/n} \|f\|_{L^{p^*}}^{p/n}$$

$$+ \text{ same}$$

$$= C\, M(Df)(x)^{1-p/n} \|f\|_{L^{p^*}}^{p/n}$$

$$\int_{\mathbb{R}^n} |f(x)|^{p^*} \le C \int M(Df)^{p^*(1-p/n)} \cdot \|f\|_{L^{p^*}}^{p^* \cdot p/n}$$

$$= C \|f\|_{L^{p^*}}^{p^* \cdot p/n} \int M(Df)^p$$

$$\text{because } p^*(1-p/n) = \frac{pn}{n-p}(1-p/n) = \frac{p}{n-p}(n-p) = p$$

$$\Rightarrow \|f\|_{L^{p^*}}^{p^*} \le C \|f\|_{L^{p^*}}^{p^* \cdot p/n} \|MDf\|^p$$

$$\Rightarrow \|f\|_{L^{p^*}} - C\|\ \|_{L^p}^{p^*} \qquad L^p$$

divide by this

$$\Rightarrow \|f\|_{L^{p^*}}^{p^*(1-p/n)} \le C\|M(Df)\|_{L^p}^{p}$$

$$\|f\|_{L^{p^*}}^{p}$$

By L^p bound on maximal function:

$$\|f\|_{L^{p^*}} \le C\,\|M(Df)\|_{L^p}$$
$$\le C\,\|Df\|_{L^p}. \quad \square$$

<u>Proof of (*)</u>

$$\left| \fint_{B_r(x)} f - \fint_{B_{r/2}} f \right|$$

$$\left| \fint [f(x+ry) - f(x+\tfrac{r}{2}y)]\,dy \right|$$

$= \left| \int_{B_1(0)} \right.$

$$= \left| \fint_{B_1(0)} \int_{r/2}^{r} Df(x+sy)[y] \, ds \, dy \right|$$

$$\underset{(|y|\le 1)}{\le} \int_{r/2}^{r} \fint_{B_1(0)} |Df|(x+sy) \, dy \, ds$$

$$= \int_{r/2}^{r} \underbrace{\fint_{B_s(x)} |Df|}_{\le M(Df)(x)} \, ds$$

$$\le \frac{r}{2} M(Df)(x).$$

Apply above with r replaced by $r/2, r/4, r/8, \dots$

$$\left| \fint_{B_r(x)} f - \fint_{B_{r/2}(x)} f \right| \le \frac{1}{2} r \, MDf(x)$$

$$\left| \fint_{B_{r/2}(x)} f - \fint_{B_{r/4}(x)} f \right| \le \frac{1}{2} \frac{r}{2} \, MDf(x)$$

$\dots$

$\left. f \right|$

$$\Rightarrow \left| \fint_{B_r(x)} f - \fint_{B_{r/2^k}(x)} f \right|$$

$$\leq \frac{1}{2}\left(r + \frac{r}{2} + \dots + \frac{r}{2^{k-1}}\right) MDf(x)$$

$$\leq r\, M(Df)(x).$$

let $k \to \infty$ so that

$$\fint_{B_{r/2^k}(x)} f \to f(x)$$

$$\Rightarrow \left| \fint_{B_r(x)} f - f(x) \right|$$

$$\leq r\, M(Df)(x). \quad \square$$

<u>Rmk</u> The rule $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$

is <u>forced</u> by dilations.

($\|f\|_{L^q} \leq C \|f\|_{W^{1,p}}$

$p \leq q \leq p^*$ works because

R.H.S. contains $\|f\|_{L^p}$

and $\|Df\|_{L^p}$).

Assume $\|f\|_{L^{p^*}} \le C \|Df\|_{L^p}$

$\forall f \in \mathcal{S}(\mathbb{R}^n)$.

You can fix f and apply it for $f_\lambda(x) := f(\lambda x)$ in place of f

$$\Longrightarrow \|f_\lambda\|_{L^{p^*}} \le C \|Df_\lambda\|_{L^p}$$

$$\|f_\lambda\|_{L^{p^*}} = \left(\int |f|(\lambda x)^{p^*} dx\right)^{1/p^*} = \lambda^{-n/p^*} \|f\|_{L^{p^*}}$$

$$\|Df_\lambda\|_{L^p} = \left(\int \lambda^p |Df|(\lambda x)^p \, dx\right)^{1/p} = \lambda^{1-n/p} \|Df\|_{L^p}.$$

$$\Rightarrow \lambda^{-\frac{n}{p^*} - 1 + \frac{n}{p}} \leq C \frac{\|Df\|_{L^p}}{\|f\|_{L^{p^*}}}$$

$$\forall \lambda \in (0, \infty).$$

If $\lambda \to \infty$ then we get contradiction unless

$$-\frac{n}{p^*} - 1 + \frac{n}{p} \leq 0.$$

Sending $\lambda \to 0$ we similarly find $-\frac{n}{p^*} - 1 + \frac{n}{p} \geq 0$

$$\Rightarrow -\frac{n}{p^*} - 1 + \frac{n}{p} = 0$$

$$\Rightarrow \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

HW

$$\sup \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq C \|Df\|_{L^p}$$

$x \neq y$ $|x-y|$

$p > n$

(Like saying f is C^α in quantitative way)

Check: using dilations,

$$\boxed{\alpha = 1 - \frac{n}{p}}.$$

Lemma (Calderón–Zygmund decomposition)

$f \in L^1([0,1]^n)$. Take $\lambda > 0$ and assume $\int_{[0,1]^n} |f| \leq \lambda$.

Then there exist (dyadic) cubes $\{Q_j\}$ s.t.

- $Q_j \cap Q_{j'} = \phi$ (up to negligible sets...)

- $|f| \le \lambda$ a.e. on $[0,1]^n \setminus \bigcup Q_j$
- $\lambda < \fint_{Q_j} |f| \le 2^n \lambda \quad \forall j.$

<u>Corollary</u> $f = g + b$

$g =$ "good part"
$b =$ "bad part"

$$g(x) := \begin{cases} f(x) & \text{if } x \notin \bigcup_j Q_j \\ \fint_{Q_j} f & \text{if } x \in Q_j \end{cases}$$

$b = f - g.$

This decomposition has

- $|g| \le 2^n \lambda$ a.e.
- $\fint_{Q_j} b = 0$ because

$= \fint_{Q_j} f$

$$\fint_{Q_j} b = \fint_{Q_j} f - f_{Q_j}(y) = 0.$$

Proof of Thm

We use the following algorithm.

$Q^{(0)} := [0,1]^n$

(Note $\fint_{Q^{(0)}} |f| = \int_{Q^{(0)}} |f| \leq \lambda$.)

Split $Q^{(0)}$ into 2^n cubes

- $Q_1^{(1)}, \ldots, Q_{2^n}^{(1)}$

if $Q_j^{(1)}$ has

- $\fint_{Q_j^{(1)}} |f| > \lambda$ we put it in the final collection,

otherwise repeat

the $Q^{(1)}$ in place of $Q^{(0)}$

with Q_j in p

. . .

We get a collection of cubes $\{Q_j\}$.

$Q^{(0)}$

. . . . . .

Clear:

- Q_j's are disjoint
- $\int_{Q_j} |f| > \lambda$.

Note that each Q_j had a "parent" $\hat{Q}_j$. This means that Q_j was one of the 2^n pieces in which I split $\hat{Q}_j$.

$$\Rightarrow \quad \fint_{\hat{Q}_j} |f| \le \lambda$$

$$\fint_{\hat{Q}_j} |f| = \frac{1}{|\hat{Q}_j|} \int_{\hat{Q}_j} |f| \ge \frac{1}{2^n |Q_j|} \int_{Q_j} |f|$$

$$\Rightarrow \quad \fint_{Q_j} |f| \le 2^n \lambda.$$

To conclude, we show

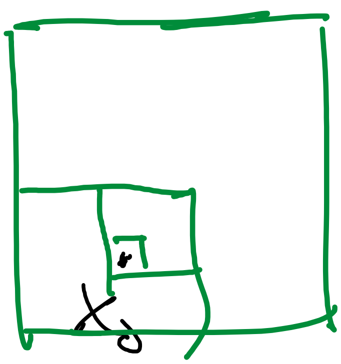
$|g| \le \lambda$ a.e. on $(\bigcup Q_j)^c$.

$|f| \leq \lambda$ a.e. ...

Take x_0 Lebesgue point for f, $x_0 \notin \bigcup Q_j$, $x_0 \in [0,1]$. ($f = 0$ outside $[0,1]^n$.)

Since x_0 is never in one of the final cubes,

$\forall k \quad x_0 \in Q^{(k)}_{i_k}$ ~~=~~ "cube in k-th generation"



and each cube is being split $\Rightarrow$ $\fint_{Q^{(k)}_{i_k}} |f| \leq \lambda$

Morally, this average → to $|f(x_0)|$.

converges to ...

Notice that $Q_{i_k}^{(k)} \subseteq \overline{B}_{2^{-k}\sqrt{n}}(x)$

because diameter of a cube with side 2^{-k} is $2^{-k}\sqrt{n}$.

$$⨍_{Q^{(k)}} |f| = 2^{kn} \int_{Q^{(k)}} |f|$$

$$\Rightarrow |f(x_0)|$$

$$\leq ⨍_{Q^{(k)}} |f - f(x_0)|$$

$$+ ⨍_{Q^{(k)}} |f|$$

$$\leq \lambda + 2^{kn} \int_{Q^{(k)}} |f - f(x_0)|$$

$$\leq \lambda + 2^{kn} \int_{B_{2^{-k}\sqrt{n}}(x_0)} |f - f(x_0)|$$

$$= \lambda + 2^{kn} \mathcal{L}^n(B_{2^{-k}\sqrt{n}}) ⨍_{B_{2^{-k}\sqrt{n}}(x_0)} \cdots$$

$$2^{-k}\sqrt{n}$$

$$\downarrow$$

$$0$$

as x_0 is Lebesgue

and $2^{kn}\mathcal{L}^n(B_{2^{-k}\sqrt{n}})$

$$= \omega_n (\cancel{2}^{-k}\sqrt{n})^n \, \cancel{2}^{kn}$$

$$= \omega_n \sqrt{n}^n$$

$\Longrightarrow$ (let $k \to \infty$)

$$|f(x_0)| \leq \lambda. \qquad \square$$