Class 1

Wednesday, September 23, 2020 3:05 PM

Harmonic analysis

Study of quantitative properties of fct's like integrability, regularity, ... and how particular operators preserve/enhance/worsen these properties.

Question Assume L is a linear differential operator of order k (homogeneous, with const. coefficients).
For a function $f \in C_c^\infty(\mathbb{R}^n)$ can we say that

$$\|\nabla^k f\| \leq C \| Lf \| \; ?$$

$$Lf = c_1 \partial^{\alpha^{(1)}} f + c_2 \partial^{\alpha^{(2)}} f + \dots$$

Recall $\beta = (\beta_1, \dots, \beta_n)$

$$\partial^\beta f = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f$$

If $\beta = (1,1)$ and $n=2$

$$\partial^\beta f = \frac{\partial^2 f}{\partial x_1 \partial x_2}.$$

Homogeneous means

$$|\alpha^{(1)}| = |\alpha^{(2)}| = \dots = k$$

$|\beta| = \beta_1 + \dots + \beta_n$.

<u>Important example</u>

$L = \Delta$, i.e, $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

For Δ answer is yes!

$$\|\nabla^2 f\|_{L^p} \leq C_p \|\Delta f\|_{L^p}$$

for any $1 < p < \infty$.

<u>Distributions</u>

<u>Recall</u> Ω will always denote an open set in $\mathbb{R}^n$.

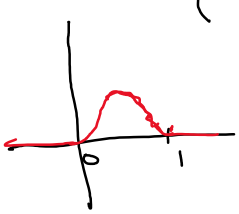
$C^k(\Omega) = \{$ functions $\Omega \to \mathbb{C}$ which are diff.ble k times with cont. derivatives $\}$.

$C^\infty(\Omega) = \{$ f's diff. ∞ many times $\}$

spt$(f) :=$ closure in Ω of $\{f \neq 0\}$.

For instance, $f = 0$ has spt$(f) = \emptyset$.

$$f(x) = \begin{cases} 0 & \text{if } x \notin (0,1) \\ e^{-1/x(1-x)} & \text{if } x \in (0,1) \end{cases}$$



$f : \mathbb{R} \to \mathbb{C}$

spt$(f) = [0,1]$.

$C^\infty(\Omega) = \{ f \in C^\infty(\Omega)$ s.t.,

$C_c(\Omega) = \{ \ldots$ spt(f) is compact $\subseteq \Omega \}$.

$C^\infty(\Omega)$ is canonically a topological space!

A fundamental system of neighborhoods of $\varphi \in C^\infty(\Omega)$ is made of sets of the form

$$\left\{ \psi \in C^\infty(\Omega) \text{ s.t. } \|\psi - \varphi\|_{C^j(K)} < \varepsilon \right\}$$

where $j \in \mathbb{N}$, $K \subset \Omega$ compact.

[Recall: $\max_K |\psi - \varphi| + \max_K |D\psi - D\varphi| + \ldots + \max_K |D^j\psi - D^j\varphi|$ (your favorite norm)]

__def__ A __topol. vector space__ is a vector space X over $\mathbb{C}$ with a topology s.t.

- X is a Hausdorff space
- operations are continuous:

$+ : X \times X \to X$

$\cdot : \mathbb{C} \times X \to X$.

One can check that

$C^\infty(\Omega)$ with the above topology is a t.v.s.

<u>def</u> A Fréchet space is a t.v.s. whose topology is <u>induced</u> by a countable family of seminorms $\| \|_1, \dots \| \|_j, \dots$

(seminorm is like a norm except that $\|x\| = 0 \not\Rightarrow x = 0$)

s.t. for any x

$$x = 0 \Leftrightarrow \forall j \ \|x\|_j = 0.$$

[a fund. system of neighborhoods of x is made of sets like

$\{y : \|y-x\|_1 < \varepsilon, \dots, \|y-x\|_k < \varepsilon\}$

$k \in \mathbb{N}$ arbitrary].

<u>Fact</u> $C^\infty(\Omega)$ is also Fréchet.

pf Take an exhaustion $K_1, K_2, \dots$ of cpt subsets ($K_j \subset \mathring{K}_{j+1}$ & $\cup K_j = \Omega$)

and take $\{\| \|_{C^j(K_\ell)} \mid j, \ell \in \mathbb{N}\}$

as a countable family

of seminorms.

The topo. is the same:

- a neigh. of φ in the new topo. is $\{\psi : \|\psi-\varphi\|_{C^{j_1}(K_{l_1})} < \varepsilon, \ldots, \|\psi-\varphi\|_{C^{j_m}(K_{l_m})} < \varepsilon\}$

$$\supseteq \{\psi : \|\psi-\varphi\|_{C^{\max(j_1,\ldots,j_m)}(K_{l_1}\cup\ldots\cup K_{l_m})} < \varepsilon\}$$

neighborhood in old topo.

- a neighborhood in the old topo. is $\{\psi : \|\psi-\varphi\|_{C^j(K)} < \varepsilon\}$

There exists l s.t. $K \subseteq K_l$

$$\Rightarrow \{\psi : \|\psi-\varphi\|_{C^j(K)} < \varepsilon\}$$
$$\supseteq \{\psi : \|\psi-\varphi\|_{C^j(K_l)} < \varepsilon\}.$$

$\square$

More importantly, also $C^\infty_c(\Omega)$ is canonically a t.v.s.

(the topo. is designed to

be locally convex)

~~It's a direct limit of $C^\infty(K)$ as K ranges among subsets of Ω in the category of locally convex t.v.s.'s.~~

<u>Rmk</u> $C_c^\infty(\Omega) \subseteq C^\infty(\Omega)$.

topo. here is <u>not</u> the subspace topo.

<u>Rmk</u> It's simple to characterize converging sequences $\varphi_j \to \varphi_\infty$ in $C_c^\infty(\Omega)$:

this happens when there exists K compact s.t. $\operatorname{spt}(\varphi_j) \subseteq K$ (hence also $\operatorname{spt}(\varphi_\infty)$) and $\varphi_j \to \varphi_\infty$ unif.,

$D^\ell \varphi_j \to D^\ell \varphi_\infty$ unif.

For example, take $\varphi_1 =$

and $\varphi_j(x) := \frac{1}{j}\varphi_1\left(\frac{x}{j}\right)$

$\varphi_j \to 0$ unif. but $\varphi_j \not\to 0$ in $C_c^\infty(\mathbb{R})$

<u>def</u> A classical distribution is a linear functional $T : C_c^\infty(\Omega) \to \mathbb{C}$ which is continuous.

$C_c^\infty(\Omega) = \mathcal{D}(\Omega)$

$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is the space of "test functions".

<u>Rmk</u> A linear functional T is continuous iff

is continuous

$T(\varphi_j) \to 0$ whenever $\varphi_j \to 0$

Key idea of distributions

f function defines a distr.

T_f: $T_f(\varphi) = \int f\varphi \, d\mathcal{L}^n$.

Let's work on $\mathbb{R}$.

$T_{f'}(\varphi) = \int f'\varphi \, dx$

$= -\int f\varphi' \, dx$

$= -\langle T_f, \varphi' \rangle$

$T \in D'(\Omega)$

def Given T, its derivative (in distributional sense) is $\partial_1 T = \frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x}$,

given by $\langle \partial_1 T, \varphi \rangle := -\langle T, \partial_1 \varphi \rangle$

Rmk This makes sense since $\varphi \in C^\infty$ and $\partial_1 \varphi \in$

def Given $T \in \mathcal{D}'(\Omega)$, Ω open in $\mathbb{R}^n$, α multi-

$$\langle \partial^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$$

$(|\alpha| = \alpha_1 + \dots + \alpha_n)$.

Rmk Note that $\partial^\alpha T$ is still continuous: indeed,

$$\mathcal{D}(\Omega) \xrightarrow{\partial^\alpha} \mathcal{D}(\Omega) \xrightarrow{T}$$

φ

is a composition of continu maps.

- We can sum $S, T \in \mathcal{D}'($
just by saying $\langle S+T, \varphi\rangle$
$:= \langle S, \varphi\rangle + \langle T,$

$[\langle T, \varphi\rangle = T(\varphi)]$,

- we can multiply by $\lambda \in$
$\langle \lambda T, \varphi\rangle := \langle T, \lambda\varphi\rangle$

There is no reasonable product on distributions!

- We can multiply T by a smooth $h : \Omega \to \mathbb{C}$:
if $T = T_f$ $\langle T_{hf}, \varphi\rangle$

$$= \int hf\varphi$$
$$= \int f\,(h\varphi)$$
$$= \langle T_f, h\varphi \rangle$$

In general, $\langle h\cdot T, \varphi \rangle := \langle$

(again, $\varphi \mapsto h\varphi$ is con
$\mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$)

<u>Example</u> Take $p \in [1, \infty]$,
$f \in L^p(\Omega)$.
$$\langle T_f, \varphi \rangle := \int f\varphi$$
makes sense because
$\varphi \in L^q(\Omega)$ $\forall q$; in particula
$\varphi \in L^{p'}(\Omega)$

$f \in L^p$, $g \in L^s$ give rise to the same distr. $\Longleftrightarrow$ $f = $ a.e.

So $L^p(\Omega) = \{ \cdots \} / \text{a.e. equ}$

is identified as a <u>subset</u> of $D'(\Omega)$.

<u>Example</u> Given $x_0 \in \Omega$

$\langle \delta_{x_0}, \varphi \rangle := \varphi(x_0)$.

If you take a positive (or real) measure μ on

it defines a distribution

$$\langle T_\mu, \varphi \rangle := \int \varphi \, d\mu.$$

One can see $\delta_{x_0} = T_\mu$:

$$\mu(S) = \begin{cases} 1 & \text{if } x_0 \in S \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example</u> $\delta_0' = \partial_1 \delta_0 = \frac{\partial \delta_0}{\partial x}$

on $\Omega = \mathbb{R}$.

$$\langle \delta_0', \varphi \rangle = - \langle \delta_0, \varphi' \rangle$$
$$= - \varphi'(0).$$

<u>Tempered distributions</u>

$\Omega = \mathbb{R}^n$

def Rather than $\mathcal{D}(\Omega) =$

we take $\mathcal{S}(\mathbb{R}^n)$, the sp

of Schwartz functions.

Tempered distributions wil

be $\mathcal{S}'(\mathbb{R}^n)$, the dual of

A function φ is Schwar

if $\varphi \in C^\infty$ and "decays

at ∞": this means

$$x^\alpha \partial^\beta \varphi \quad \text{is bounded},$$

$\forall \alpha, \beta \in \mathbb{N}^n$ mu

indi

$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

<u>Rmk</u> Hence, $x^\alpha \partial^\beta \varphi(x) \to 0$ as $|x| \to \infty$

(because $|x|^2 x^\alpha \partial^\beta \varphi(x)$
$= (x_1^2 x^\alpha + \dots + x_n^2 x^\alpha) \partial^\beta \varphi$
is bded).

On $\mathcal{S}(\mathbb{R}^n)$ we put the Fréchet structure coming from seminorms
$\|x^\alpha \partial^\beta \varphi\|_{C^0(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |x$

Rmk $\mathcal{D}'(\Omega)$, $\mathcal{S}'(\mathbb{R}^n)$

are t.v.s., whose topo.
the coarsest making
evaluations $T \mapsto \langle T,$
continuous. (φ fixed)

It follows that $T_j \to T_\infty$
$\iff \forall \varphi \ \langle T_j, \varphi \rangle \to \langle T_\infty,$

$\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$
with continuous inclusio
and it's dense in $\mathcal{S}(\mathbb{R}^n$
If T is a tempered dis
then we can view it a

Then we ...
a classical one ($\in \mathcal{D}'$).

Also, $T=0$ as classical distri. $\Rightarrow T=0$

($\operatorname{Ker} T := \{\varphi : \langle T, \varphi \rangle = 0\} \subseteq \mathcal{S}($
is closed and inc
$\mathcal{D}(\mathbb{R}^n) \Rightarrow \{\ldots\} = \mathcal{S}($

Will see: $\mathcal{F}(\mathcal{S}(\mathbb{R}^n)) = \mathcal{S}($

[Ref.: e.g. Rudin
Functional
Analysis]