Homework 2

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Exercise 1.

Proof. Since $f \neq 0$, there exists a radius r, on which $\int_{B_r(0)} |f| = c > 0$. For $|x| > r$, we have $B_r(0) \subseteq B_{|x|+r}(x)$ and therefore

$$Mf(x) \geq \frac{1}{\mu(B_{|x|+r}(x))} \int_{B_{|x|+r}(x)} |f| \geq \frac{c}{(|x|+r)^n}$$

So that

$$\int_{\mathbb{R}^n} |Mf(x)| \geq \int_{|x|>r} \frac{c}{(|x|+r)^n} = \infty$$

Thus, $Mf \notin L^1(\mathbb{R}^n)$. $\square$

Exercise 2.

Proof. $(a)(i)$

$$0 = \|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} \iff \mu\{|f| > \lambda\} = 0 \text{ for all } \lambda > 0$$

$$\iff |f| = 0 \iff f = 0$$

(ii)

$$\|kf\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|kf| > \lambda\} = |k| \cdot \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} = |k| \cdot \|f\|_{L^{1,\infty}}$$

(iii) Since $|f| + |g| \geq |f+g|$, we have

$$\{|f| > \frac{\lambda}{2}\} \cup \{|g| > \frac{\lambda}{2}\} \supseteq \{|f| + |g| > \lambda\} \supseteq \{|f+g| > \lambda\}$$

so that

$$2(\|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}}) = 2\|f\|_{L^{1,\infty}} + 2\|g\|_{L^{1,\infty}} =$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{2}\} + \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{2}\} \geq$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f| + |g| > \lambda\} \geq \sup_{\lambda>0} \lambda \cdot \mu\{|f+g| > \lambda\} =$$

$$\|f+g\|_{L^{1,\infty}}$$

(b) $\square$

Exercise 3.

Proof. $\square$