

# Homework 4

Dan Sokolsky

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## Exercise 1.

*Proof.* Let  $\{Q_j\}$  be the cubes defined in the proof of the Calderón–Zygmund lemma. Let  $S = \{x \in [0, 1]^n \mid M_4 f(x) > \lambda\}$ , where  $x_0 = 0$  in the definition

$$M_4 f(x) = \sup_{Q \in \mathfrak{D} : x \in Q} \int_Q |f|$$

( $\subseteq$ ) Let  $Q_j = [a, b]^n$ . Let  $\tilde{Q}_j = [a, b]^n$ . Then,  $Q_j, \tilde{Q}_j$  only differ on a set of measure 0. By construction,

$$\int_{\tilde{Q}_j} |f| = \int_{Q_j} |f| > \lambda$$

so that  $M_4 f(x) > \lambda$  for all  $x \in Q_j$ . Therefore,  $Q_j \subseteq S$  (for a.e.  $x \in Q_j$ ). Thus,  $\cup Q_j \subseteq S$  up to a set of measure 0.

( $\supseteq$ ) Let  $x \in S$ . Then there exists a dyadic cube  $Q \subseteq [0, 1]^n$  such that

$$\int_Q |f| > \lambda$$

So that  $Q = Q_j$  for some  $j$ . Thus  $S \subseteq \cup Q_j$ . Therefore,  $S = \cup Q_j$  up to a set of measure 0.  $\square$

## Exercise 2.

*Proof.* (a)(i) Since  $\lambda > 0$  and  $\mu\{|f| > \lambda\} \geq 0$ , we have  $\|f\|_{L^{p,\infty}} \geq 0$ .

(ii)

$$\begin{aligned} \|kf\|_{L^{p,\infty}}^p &= \sup_{\lambda > 0} \lambda^p \cdot \mu\{|kf| > \lambda\} = \sup_{\lambda > 0} \lambda^p \cdot \mu\{|f| > \frac{\lambda}{|k|}\} = \\ \sup_{|k|\lambda > 0} (|k|\lambda)^p \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} &= \sup_{|k|\lambda > 0} (|k|\lambda)^p \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} = |k|^p \cdot \sup_{|k|\lambda > 0} \lambda^p \cdot \mu\{|f| > \lambda\} = \\ |k|^p \cdot \sup_{\lambda > 0} \lambda^p \cdot \mu\{|f| > \lambda\} &= |k|^p \cdot \|f\|_{L^{p,\infty}}^p \iff \|kf\|_{L^{p,\infty}} = |k| \cdot \|f\|_{L^{p,\infty}} \end{aligned}$$

(iii)

$$\|f + g\|_{L^{p,\infty}}^p = \sup_{\lambda > 0} \lambda^p \cdot \mu\{|f + g| > \lambda\} \leq \sup_{\lambda > 0} \lambda^p \cdot (\mu\{|f| > \frac{\lambda}{2}\} + \mu\{|g| > \frac{\lambda}{2}\}) \leq$$

$$\begin{aligned}
& \sup_{\lambda>0} \lambda^p \cdot (\mu\{|f| > \frac{\lambda}{2}\}) + \sup_{\lambda>0} \lambda^p \cdot (\mu\{|g| > \lambda\}) = \\
& \sup_{2\lambda>0} (2\lambda)^p \cdot (\mu\{|f| > \lambda\}) + \sup_{2\lambda>0} (2\lambda)^p \cdot (\mu\{|g| > \lambda\}) = \\
& 2^p \sup_{\lambda>0} \lambda \cdot (\mu\{|f| > \lambda\}) + 2^p \sup_{\lambda>0} \lambda \cdot (\mu\{|g| > \lambda\}) = 2^p (\|f\|_{L^{p,\infty}}^p + \|g\|_{L^{p,\infty}}^p) \\
& \iff \|f+g\|_{L^{p,\infty}} \leq 2(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})
\end{aligned}$$

(b)

$$\int_E |f| \geq \int_E \lambda = \lambda \mu(E)$$

So that,

$$\mu(E)^{\frac{1}{p}-1} \int_E |f| \geq \mu(E)^{\frac{1}{p}-1} \cdot \lambda \mu(E) = \lambda \cdot \mu(E)^{\frac{1}{p}}$$

Thus,

$$\|f\|' = \sup_{E: 0<\mu(E)<\infty} \mu(E)^{\frac{1}{p}-1} \int_E |f| \geq \sup_{\lambda>0} \lambda \mu\{|f| > \lambda\}^{\frac{1}{p}} = \|f\|_{L^{p,\infty}}$$

(c)

$$\begin{aligned}
\int_E |f(x)| d\mu &= \int_E \left( \int_0^{|f(x)|} d\lambda \right) du = \int_E \mu\{|f(x)| > \lambda\} d\lambda = \int_0^\infty \mu\{x \in E \mid |f(x)| > \lambda\} d\lambda \leq \\
& \int_0^{\mu(E)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}}} \mu(E) d\lambda + \int_{\mu(E)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}}}^\infty \mu\{|f| > \lambda\} d\lambda \leq \\
& \int_0^{\mu(E)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}}} \mu(E) d\lambda + \int_{\mu(E)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}}}^\infty \frac{\|f\|_{L^{p,\infty}}^p}{\lambda^p} d\lambda = \\
& \mu(E)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}} \mu(E) + \|f\|_{L^{p,\infty}}^p \cdot \left[ \frac{\lambda^{1-p}}{1-p} \right]_{\mu(E)^{-\frac{1}{p}} \|f\|_{L^{p,\infty}}}^\infty = \\
& \mu(E)^{\frac{p-1}{p}} \|f\|_{L^{p,\infty}} \cdot \frac{p}{p-1} \iff \\
& \mu(E)^{\frac{1}{p}-1} \int_E |f| d\mu \leq \frac{p}{p-1} \|f\|_{L^{p,\infty}}
\end{aligned}$$

For all Borel sets  $E$ . Therefore,

$$\|f\|' = \sup_{E: 0<\mu(E)<\infty} \mu(E)^{\frac{1}{p}-1} \int_E |f| d\mu \leq \frac{p}{p-1} \|f\|_{L^{p,\infty}}$$

(d) We have

$$\|f\|_{L^{p,\infty}} \leq \|f\|' \leq \frac{p}{p-1} \|f\|_{L^{p,\infty}}$$

Thus the two norms are equivalent. □

**Exercise 3.**

*Proof.* (a) Let  $R = |x - y|$ . Since  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $Df$  is bounded. Let  $Df \leq K$ . Let us first estimate

$$\begin{aligned} \left| \int_{B_R(x)} f - \int_{B_R(y)} f \right| &= \left| \int_{B_R(x)} f(z + (y - x)) - f(z) dz \right| = \left| \int_{B_R(x)} \left( \int_z^{z+(y-x)} Df(t) dt \right) dz \right| \leq \\ &\int_{B_R(x)} K \cdot |z + (y - x) - z| dz = \int_{B_R(x)} K \cdot |x - y| dz = K \cdot |x - y| \cdot \frac{1}{\mu(B_R(x))} \int_{B_R(x)} dz = \\ &K \cdot |x - y| \cdot 1 = K \cdot |x - y| \end{aligned}$$

Thus, by the lemma we proved in class, we have -

$$\begin{aligned} |f(x) - f(y)| &\leq \left| f(x) - \int_{B_R(x)} f \right| + \left| \int_{B_R(x)} f - \int_{B_R(y)} f \right| + \left| \int_{B_R(y)} f - f(y) \right| \leq \\ &C_1 \cdot |x - y| \cdot Mf(x) + K \cdot |x - y| + C_2 \cdot |x - y| \cdot Mf(y) \leq \\ &C(Mf(x) + Mf(y)) \cdot |x - y| \end{aligned}$$

where  $C = \max\{C_1, C_2, K\}$ .

(b) By Hölder's inequality, we have, for all  $s$ ,

$$s \int_{B_s(x)} |Df| \leq C(p, n) s^{1-n+\frac{n}{p'}} \|Df\|_{L^p} = C(p, n) s^\alpha \|Df\|_{L^p}$$

Thus

$$s \cdot M(Df)(x) \leq C(p, n) s^\alpha \|Df\|_{L^p}$$

Substituting  $s = \frac{r}{2}, \frac{r}{4}, \frac{r}{8}, \dots$ , we have

$$\begin{aligned} r \cdot M(Df)(x) &= \sum_{k=1}^{\infty} \frac{r}{2^k} \cdot M(Df)(x) \leq \sum_{k=1}^{\infty} C(p, n) \left(\frac{r}{2^k}\right)^\alpha \|Df\|_{L^p} \leq \\ &\sum_{k=1}^{\infty} C(p, n) r^\alpha \frac{1}{2^k} \|Df\|_{L^p} = C(p, n) r^\alpha \|Df\|_{L^p} \end{aligned}$$

Now, by the lemma we proved preceeding Sobolev's inequality, we have

$$\left| f(x) - \int_{B_R(x)} f \right| \leq C \cdot r \cdot M(Df)(x) \leq C(p, n) r^\alpha \|Df\|_{L^p}$$

(c) Let  $R = |x - y|$ , as before. We had shown in (b) that

$$R \cdot M(Df)(x) \leq C(p, n) R^\alpha \|Df\|_{L^p}$$

It follows that

$$M(Df)(x) \leq C(p, n) R^{\alpha-1} \|Df\|_{L^p}$$

Now from (a),

$$\begin{aligned} |f(x) - f(y)| &\leq C(Mf(x) + Mf(y)) \cdot |x - y| \leq \\ &2 \cdot C(p, n) \cdot R^{\alpha-1} \cdot \|Df\|_{L^p} \cdot |x - y| = C(p, n) |x - y|^\alpha \|Df\|_{L^p} \end{aligned}$$

□