Homework 5

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Exercise 1.

Proof. First, we claim $f_b \in L^1(\mathbb{R}^n)$, and $f_s \in L^r(\mathbb{R}^n)$ -

$$\int |f_b| = \int |f_b|^p |f_b|^{1-p} \leq C(p)\|f\|_p^p < \infty$$

$$\int |f_s|^r = \int |f_s|^{r-p}|f_s|^p \leq C(r,p)\|f\|_p^p < \infty$$

Now, since $|Tf| \leq |Tf_b| + |Tf_s|$, we have $\{|Tf| > t\} \subseteq \{|Tf_b| > \frac{t}{2}\} \cup \{|Tf_s| > \frac{t}{2}\}$, so that -

$$\mu\{|Tf| > t\} \leq \mu\{|Tf_b| > \frac{t}{2}\} + \mu\{|Tf_s| > \frac{t}{2}\} \leq$$
$$\frac{2A\|f\|_1}{t}\int |f_b| + \frac{2^r A^r \|f\|_r^r}{t^r}\int |f_s|^r$$

Now,

$$\int_0^\infty t^{q-1}t^{-1}\int_{|f|>t}|f| = \int_{\mathbb{R}^n}|f|\int_0^{|f|} t^{q-2} = \frac{1}{q-1}\int_{\mathbb{R}^n}|f||f|^{q-1} = \frac{\|f\|_q^q}{q-1}$$

since $q > p > 1$, and

$$\int_0^\infty t^{q-1}t^{-r}\int_{|f|\leq t}|f|^r = \int_{\mathbb{R}^n}|f|^r\int_{|f|}^\infty t^{q-1-r} = \frac{1}{r-q}\int_{\mathbb{R}^n}|f|^r|f|^{q-r} = \frac{\|f\|_q^q}{r-q}$$

since $q < r$. Altogether,

$$\|Tf\|_q \leq C\|f\|_q$$

$\square$

Exercise 2.

Proof. (a)

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)} |K(x) - K(x-z)|dx = \int_{\mathbb{R}^n\setminus B_{2r}(0)} \Big|\int_0^1 DK d\gamma\Big| dx =$$

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)}\int_0^1 \Big|DK(t)(x-(1-t)z)\cdot z\ \Big| dt\ dx \leq \int_{\mathbb{R}^n\setminus B_{2r}(0)} B|x|^{-n-1}|x-(x-z)|dx =$$

$$B|z|\int_{\mathbb{R}^n\setminus B_{2r}(0)}|x|^{-n-1}dx = C(n)B$$

(b)

$$\int_{\mathbb{R}^n}\Big(K(x)-K(x-x_\xi)\Big)e^{-2\pi ix\xi}\ dx = \int_{\mathbb{R}^n}K(x)e^{-2\pi ix\xi}\ dx - \int_{\mathbb{R}^n}K(x-x_\xi)e^{-2\pi ix\xi}\ dx =$$

$$\hat{K}(\xi)-\int_{\mathbb{R}^n}K(x)e^{-2\pi i(x+x_\xi)\xi}\ dx = \hat{K}(\xi)-e^{-i\pi}\int_{\mathbb{R}^n}K(x)e^{-2\pi ix\xi}\ dx =$$

$$\hat{K}(\xi)+\hat{K}(\xi)=2\hat{K}(\xi)$$

(c) By the cancellation condition, we have

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)e^{-2\pi ix\xi}\Big| = \Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)e^{-2\pi ix\xi}-\int_{B_{\frac{1}{|\xi|}}(0)}K(x)\Big| =$$

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)(e^{-2\pi ix\xi}-1)\Big| \le \int_{B_{\frac{1}{|\xi|}}(0)}|K(x)|\cdot|e^{-2\pi ix\xi}-1| \le 2\pi|\xi||x|\int_{B_{\frac{1}{|\xi|}}(0)}|K(x)| \le$$

$$2\pi|\xi||x|A\int_{B_{\frac{1}{|\xi|}}(0)}|x|^{-n} = 2\pi|\xi|A\int_{B_{\frac{1}{|\xi|}}(0)}|x|^{-n+1} = C(n)A$$

(d)

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi ix\xi}\Big| = \Big|e^{i\pi}\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi ix\xi}\Big| = \Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi i(x-x_\xi)\xi}\Big| =$$

$$\Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)}K(x)e^{-2\pi ix\xi}\Big|$$

Since $|x_\xi|=\frac{1}{2|\xi|}<\frac{1}{|\xi|}$, we have that $B_r(0)\subseteq B_{\frac{1}{|\xi|}}(x_\xi)$ for some $0<r<\frac{1}{|\xi|}$. Thus, as in (c),

$$|\int_{B_r(0)}K(x-x_\xi)e^{-2\pi ix\xi}| \le 2\pi rA\int_{B_r(0)}|x|^{-n+1} \le 2\pi|\xi|A\int_{B_{\frac{1}{|\xi|}}(0)}|x|^{-n+1} = C(n)A$$

Now,

$$\Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)}K(x)e^{-2\pi ix\xi}\Big| = \Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)\setminus B_r(0)}K(x)e^{-2\pi ix\xi}+\int_{B_r(0)}K(x)e^{-2\pi ix\xi}\Big| \le$$

$$\Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)\setminus B_r(0)}K(x)e^{-2\pi ix\xi}\Big|+\Big|\int_{B_r(0)}K(x)e^{-2\pi ix\xi}\Big| \le$$

$$A\int_{B_{\frac{1}{|\xi|}}(x_\xi)\setminus B_r(0)}\frac{1}{|x|^n}+C(n)A=C+C(n)A\leq C_1(n)A$$

(e)

$$\left|2\hat{K}(\xi)\right|=\Big|\int_{\mathbb{R}^n}K(x)e^{-2\pi ix\xi}\ dx\Big|=\Big|\int_{\mathbb{R}^n}\Big(K(x)-K(x-x_\xi)\Big)e^{-2\pi ix\xi}\ dx\Big|=$$

$$\Big|\int_{\mathbb{R}^n\setminus B_{\frac{1}{|\xi|}}(0)}\Big(K(x)-K(x-x_\xi)\Big)e^{-2\pi ix\xi}\ dx+\int_{B_{\frac{1}{|\xi|}}(0)}\Big(K(x)-K(x-x_\xi)\Big)e^{-2\pi ix\xi}\ dx\Big|\leq$$

$$\int_{\mathbb{R}^n\setminus B_{\frac{1}{|\xi|}}(0)}\Big|K(x)-K(x-x_\xi)\Big|\ dx+\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)e^{-2\pi ix\xi}\Big|+\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi ix\xi}\Big|\leq$$

$$A+2C(n)A\leq C_1(n)A\iff\left|\hat{K}(\xi)\right|\leq\frac{C_1(n)A}{2}$$

□

Exercise 3.

Proof. (a) (i) $|K\mathbf{1}_{B_r\setminus B_\epsilon}(x)|\leq|K(x)|\leq A|x|^{-n}$
(ii)
If $r<\frac{R}{2}$, we have

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)}|K_{\epsilon,R}(x)-K_{\epsilon,R}(x-z)|dx\leq\int_{\mathbb{R}^n\setminus B_R(0)}|K_{\epsilon,R}(x)-K_{\epsilon,R}(x-z)|dx\leq$$

$$\int_{\mathbb{R}^n\setminus B_R(0)}\frac{1}{|x|^n}dx\leq A\int_{B_R(0)\setminus B_{\frac{R}{2}}(0)}\frac{1}{|x-z|^n}dx=$$

$$A\int_{B_R(0)\setminus B_{\frac{R}{2}}(0)}\frac{1}{|x|^n}dx=C_1(n)A$$

If $\frac{R}{2}\leq r\leq R$, we have

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)}|K_{\epsilon,R}(x)-K_{\epsilon,R}(x-z)|dx=\int_{\mathbb{R}^n\setminus B_{2r}(0)\cap\{K(x)\neq0\}\cap\{K(x-z)\neq0\}}|K_{\epsilon,R}(x)-K_{\epsilon,R}(x-z)|dx\leq$$

$$A\int_{B_R(0)\setminus B_{\frac{R}{2}}(0)}\frac{1}{|x|^n}dx+A\Big(\mu\{B_R(0)-B_{\frac{R}{2}}(0)\}\Big)\leq C_2(n)A$$

If $r>R$, we have

$$|x-z|\geq|x|-|z|\geq2R-R$$

so that the region of integration falls outside the support of $K(x)$, and $K(x-z)$. Therefore,

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)}|K_{\epsilon,R}(x)-K_{\epsilon,R}(x-z)|dx=0$$

(iii)

$$\int_{B_s(0)\setminus B_r(0)} K = \int_{B_s(0)} K - \int_{B_r(0)} K = 0 - 0 = 0$$

(b) We will prove $\|K_{\epsilon,R} * f\|_{L^2} \leq \|\hat{K}_{\epsilon,R}\|_{L^\infty}\|f\|_{L^2}$. The general inequality $\|K_{\epsilon,R} * f\|_{L^2} \leq C(n,p)A\|f\|_{L^p}$ then follows by the Marcinkiewicz Interpolation Theorem, the same way we proved the Calderon-Zygmund estimate. To that end, by Plancheral's identity, we have -

$$\|K_{\epsilon,R} * f\|_{L^2} = \|\mathcal{F}(K_{\epsilon,R} * f)\|_{L^2} = \|\hat{K}_{\epsilon,R} \cdot \hat{f}\|_{L^2} \leq \|\hat{K}_{\epsilon,R}\|_{L^\infty} \cdot \|\hat{f}\|_{L^2} = C(n)A\|\hat{f}\|_{L^2}$$

(c) Since f is smooth and has compact support, by the Extreme Value Theorem, f achieves a minimum and a maximum on it's domain. Let m, M be the minimal, and maximal values of f, respectively. WLOG, suppose $R \geq a$. Then,

$$|(K_{\epsilon,R}*f)(x)| = \Big|\int K_{\epsilon,R}(y)f(x-y)dy\Big| = \Big|\int_{|x|\geq a} K_{\epsilon,R}(y)f(x-y)dy + \int_{B_a(0)} K_{\epsilon,R}(y)f(x-y)dy\Big| \leq$$

$$\Big|\int_{|x|\geq a} K_{\epsilon,R}(y)f(x-y)dy\Big| + \Big|\int_{B_a(0)} K_{\epsilon,R}(y)f(x-y)dy\Big| \leq$$

$$\int_{|x|\geq a} |K_{\epsilon,R}(y)f(x-y)|dy + \Big|\int_{B_a(0)\setminus B_\epsilon(0)} K(y)f(x-y)dy\Big| \leq$$

$$|M|\Big(A\int_{\{|x|\geq a\}\cap B_R(0)} \frac{1}{|a|^n} + |\int_{B_a(0)\setminus B_\epsilon(0)} K(y)dy|\Big) \leq |M|\Big(\mu\{B_R(0)\} \cdot \frac{A}{|a|^n} + 0\Big) = C(n)A \cdot |M|$$

for every $\epsilon > 0$, and for every $R \geq a$. Thus the integral is absolutetely convergent and the limit exists. Now, since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we can approximate $g \in L^p(\mathbb{R}^n)$ with a sequence $C_c^\infty(\mathbb{R}^n) \ni g_j \to g$, with convergence in L^p. Thus, by the dominated convergence theorem, we have

$$\|K_{\epsilon,R} * g_j\|_{L^p}^p = \int_{\mathbb{R}^n} |(K_{\epsilon,R} * g_j)(x)|^p = \int_{\mathbb{R}^n} \Big|\int K_{\epsilon,R}(y)g_j(x-y)dy\Big|^p \nearrow$$

$$\int_{\mathbb{R}^n} \Big|\int K_{\epsilon,R}(y)g(x-y)dy\Big|^p \nearrow \int_{\mathbb{R}^n} \Big|\int K(y)g(x-y)dy\Big|^p =$$

$$\int_{\mathbb{R}^n} |(K * g)(x)|^p = \|K * g\|_{L^p}^p$$

as $\epsilon \to 0,\ R, j \to \infty$. $\square$

Exercise 4.

Proof. (a)

$$\int_{B_2(0)} \frac{\epsilon^2}{(|x|^2+\epsilon^2)^{\frac{n}{2}+1}}dx = \int_0^{2\pi}\int_0^\pi \cdots \int_0^\pi \int_0^2 \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} \sin\phi_1 \cdots \sin\phi_{n-1} \cdot r^{n-1}\, dr\, d\phi_1 \cdots d\phi_{n-1}$$

We will bound $\int_0^2 \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr$. The rest of the integrals are obviously bounded (In particular, $|\sin \phi_i| \leq 1$). To this end,

$$\int_0^2 \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr = \int_0^\epsilon \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr + \int_\epsilon^2 \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr = I_1 + I_2$$

We will bound I_1, I_2 separately. On $(0, \epsilon)$,

when $r \leq \frac{\epsilon}{2}$, we have -

$$\frac{\epsilon^n}{r^{n-1}} \geq \frac{\epsilon^n \cdot 2^{n-1}}{\epsilon^{n-1}} = 2^{n-1} \cdot \epsilon$$

when $r \geq \frac{\epsilon}{2}$, we have -

$$\frac{r^3}{\epsilon^2} \geq \frac{\epsilon^3}{\epsilon^2 \cdot 2^3} = \frac{\epsilon}{8}$$

So that

$$\frac{\epsilon^n}{r^{n-1}} + \frac{r^3}{\epsilon^2} \geq \min\{2^{n-1} \cdot \epsilon, \frac{\epsilon}{8}\}$$

Now, $(r^2+\epsilon^2)^{\frac{n}{2}+1} \geq r^{n+2} + \epsilon^{n+2}$ by the binomial theorem, so -

$$I_1 = \int_0^\epsilon \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr = \int_0^\epsilon \frac{1}{\frac{\epsilon^n}{r^{n-1}} + \frac{r^3}{\epsilon^2}}\ dr \leq$$

$$\epsilon \cdot \max\{\frac{2^{1-n}}{\epsilon}, \frac{8}{\epsilon}\} = \max\{2^{1-n}, 8\} = 8$$

On $(\epsilon, 2)$,

$$r^2 + \epsilon^2 \geq r^2$$

So that,

$$I_2 = \int_\epsilon^2 \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr \leq \int_\epsilon^2 \frac{\epsilon^2}{(r^2)^{\frac{n}{2}+1}} r^{n-1}\ dr =$$

$$\int_\epsilon^2 \frac{\epsilon^2}{r^{n+2}} \cdot r^{n-1}\ dr \leq \int_\epsilon^2 \frac{r^2}{r^{n+2}} \cdot r^{n-1}\ dr = \int_\epsilon^2 \frac{1}{r}\ dr =$$

$$\ln(2) - \ln(\epsilon) \leq \ln(2)$$

So that,

$$\int_0^2 \frac{\epsilon^2}{(r^2+\epsilon^2)^{\frac{n}{2}+1}} r^{n-1}\ dr \leq I_1 + I_2 \leq 8 + \ln(2)$$

It follows that

$$\int_{B_2(0)} \frac{\epsilon^2}{(|x|^2+\epsilon^2)^{\frac{n}{2}+1}} dx \leq C$$

□