Homework 4

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Exercise 1.

Proof. Let $\{Q_j\}$ be the cubes defined in the proof of the Calderón–Zygmund lemma. Let $S = \{x \in [0,1)^n \mid M_4 f(x) > \lambda\}$, where $x_0 = 0$ in the definition

$$M_4 f(x) = \sup_{Q \in \mathfrak{D} \,:\, Q \in [0,1]^n} \fint_Q |f|$$

($\subseteq$) Let $Q_j = [a,b]^n$. Let $\tilde{Q}_j = [a,b)^n$. Then, $Q_j, \tilde{Q}_j$ only differ on a set of measure 0. By construction,

$$\fint_{\tilde{Q}_j} |f| = \fint_{Q_j} |f| > \lambda$$

so that $M_4 f(x) > \lambda$ for all $x \in Q_j$. Therefore, $Q_j \subseteq S$ (for a.e. $x \in Q_j$). Thus, $\cup Q_j \subseteq S$ up to a set of measure 0.
($\supseteq$) Let $x \in S$. Then there exists a dyadic cube $Q \subseteq [0,1)^n$ such that

$$\fint_Q |f| > \lambda$$

So that $Q = Q_j$ for some j. Thus $S \subseteq \cup Q_j$. Therefore, $S = \cup Q_j$ up to a set of measure 0. □

Exercise 2.

Proof. □

Exercise 3.

Proof. □