Homework 3

Dan Sokolsky

October 20, 2020

Exercise 1.

Proof. Since $f \neq 0$, there exists a radius r, on which $\int_{B_r(0)} |f| = c > 0$. For $|x| > r$, we have $B_r(0) \subseteq B_{|x|+r}(x)$ and therefore

$$Mf(x) \geq \frac{1}{\mu(B_{|x|+r}(x))} \int_{B_{|x|+r}(x)} |f| \geq \frac{c}{(|x|+r)^n}$$

So that

$$\int_{\mathbb{R}^n} |Mf(x)| \geq \int_{|x|>r} \frac{c}{(|x|+r)^n} = \infty$$

Thus, $Mf \notin L^1(\mathbb{R}^n)$. $\square$

Exercise 2.

Proof. $(a)(i)$

$$0 = \|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} \iff \mu\{|f| > \lambda\} = 0 \text{ for all } \lambda > 0$$

$$\iff |f| = 0 \iff f = 0$$

(ii)

$$\|kf\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|kf| > \lambda\} = \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{|k|}\} =$$

$$\sup_{|k|\lambda>0} |k|\lambda \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} = \sup_{|k|\lambda>0} |k|\lambda \cdot \mu\{|f| > \lambda\} =$$

$$|k| \cdot \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} = |k| \cdot \|f\|_{L^{1,\infty}}$$

(iii) Since $|f| + |g| \geq |f+g|$, we have

$$\{|f| > \frac{\lambda}{2}\} \cup \{|g| > \frac{\lambda}{2}\} \supseteq \{|f| + |g| > \lambda\} \supseteq \{|f+g| > \lambda\}$$

so that

$$2(\|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}}) = 2\|f\|_{L^{1,\infty}} + 2\|g\|_{L^{1,\infty}} =$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{2|f| > \lambda\} + \sup_{\lambda>0} \lambda \cdot \mu\{2|g| > \lambda\} =$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{2}\} + \sup_{\lambda>0} \lambda \cdot \mu\{|g| > \frac{\lambda}{2}\} \geq$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f| + |g| > \lambda\} \geq \sup_{\lambda>0} \lambda \cdot \mu\{|f+g| > \lambda\} =$$

$$\|f+g\|_{L^{1,\infty}}$$

(b) For $x \in [0,1]$, we have

$$|f_\ell(x)| = \frac{1}{\log \ell} |\sum_{j=1}^{\ell} \frac{1}{x\ell - j}| = \frac{1}{\log \ell} |\sum_{j=1}^{\ell} \frac{1}{j - x\ell}|$$

$$= \frac{1}{\log \ell} |\sum_{j=1}^{\ell} \frac{1}{j} - \frac{x\ell}{j(x\ell - j)}| \geq \frac{1}{\log \ell} (\sum_{j=1}^{\ell} |\frac{1}{j}| - |\frac{x\ell}{j(x\ell - j)}|) =$$

$$\frac{1}{\log \ell} (\sum_{j=1}^{\ell} |\frac{1}{j}| - \sum_{j=1}^{\ell} |\frac{x\ell}{j(x\ell - j)}|) \geq \frac{1}{\log \ell} \sum_{j=1}^{\ell} \frac{1}{j} \geq$$

$$\frac{1}{\log \ell} \sum_{j=1}^{\ell-1} \int_j^{j+1} \frac{1}{j} = \frac{1}{\log \ell} \cdot \log \ell = 1$$

So that $\|f_\ell\|_{L^{1,\infty}} \geq 1 \cdot \mu\{|f_\ell| > 1\} \geq \mu[0,1] = 1$.
(c) Since μ is translation invariant, we have

$$\mu\left\{\left|\frac{1}{x - \frac{j}{\ell}}\right| > \lambda\right\} = \mu\left\{\left|\frac{1}{x}\right| > \lambda\right\}$$

So that

$$\left\|\frac{1}{x - \frac{j}{\ell}}\right\|_{L^{1,\infty}} = \left\|\frac{1}{x}\right\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in \mathbb{R} : \left|\frac{1}{x}\right| > \lambda\right\} =$$

$$2 \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in [0,\infty) : \frac{1}{x} > \lambda\right\} = 2 \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in [0,\infty) : x < \frac{1}{\lambda}\right\} =$$

$$2 \sup_{\lambda>0} \lambda \cdot \mu[0, \frac{1}{\lambda}) = 2 \sup_{\lambda>0} \lambda \cdot \frac{1}{\lambda} = 2 \sup_{\lambda>0} 1 = 2 \cdot 1 = 2$$

So that

$$\|f_\ell\|_{L^{1,\infty}} \leq 2 \sum_{j=1}^{\ell} = \frac{1}{\ell \cdot \log \ell} \|\frac{1}{x}\|_{L^{1,\infty}} = \frac{4\ell}{\ell \cdot \log \ell} = \frac{4}{\log \ell}$$

Now, if there exists a norm $\|\cdot\|'$ such that $c\|f_\ell\|' \leq \|f_\ell\|_{L^{1,\infty}} \leq C\|_\ell\|'$, such that $c, C > 0$, then

$$c\|f_\ell\|' \leq \|f_\ell\|_{L^{1,\infty}} \leq \frac{4}{\log \ell} \to 0$$

as $\ell \to \infty$. Since $c > 0$, it follows that $\|f_\ell\|' = 0$, which is a contradiction since $|f_\ell| \geq 1$ on $[0,1]$ for all ℓ. Meaning, $f \neq 0$, so that $\|\cdot\|'$ isn't a proper norm. $\square$

Exercise 3.

Proof. (a) (i) Since $\{B_r(x)|x \in \mathbb{R}^n\} \subseteq \{B_r(y)|x \in B_r(y)\}$, we have

$$M_1 f = \sup_{\{B_r(y)|x\in B_r(y)\}} \frac{1}{\mu(B_r(y))} \int_{B_r(y)} |f| \geq \sup_{\{B_r(x)|x\in\mathbb{R}\}} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| = Mf$$

Now, for every $z \in B_r(y)$, $|x-z| \leq |x-y| + |y-z| \leq r + r = 2r$ so that $B_r(y) \subseteq B_{2r}(x)$. Now,

$$M_1 f = \sup_{\{B_r(y)|x\in B_r(y)\}} \frac{1}{\mu(B_r(y))} \int_{B_r(y)} |f| \leq \sup_{\{B_r(x)|x\in\mathbb{R}^n\}} \frac{1}{\mu(B_r(x))} \int_{B_{2r}(x)} |f| =$$

$$\sup_{\{B_r(x)|x\in\mathbb{R}^n\}} \frac{2^n}{\mu(B_r(x))} \int_{B_{2r}(x)} |f| = 2^n Mf$$

So that $Mf \leq M_1 f \leq 2^n Mf$.
(ii) $B_r(x) \subseteq Q_r(x) \subseteq B_{2r}(x)$. Let $c_2 = \frac{\mu(B_r(x))}{\mu(Q_r(x))}$. Observe that c_2 is independent of r. Then,

$$M_2 f = \sup_{\{Q_r(x)|x\in\mathbb{R}^n\}} \frac{1}{\mu(Q_r(x))} \int_{Q_r(x)} |f| \geq \sup_{\{B_r(x)|x\in\mathbb{R}\}} \frac{c_2}{\mu(B_r(x))} \int_{B_r(x)} |f| = c_2 Mf$$

and

$$M_2 f = \sup_{\{Q_r(x)|x\in\mathbb{R}^n\}} \frac{1}{\mu(Q_r(x))} \int_{Q_r(x)} |f| \leq \sup_{\{B_{2r}(x)|x\in\mathbb{R}^n\}} \frac{2^n}{\mu(B_{2r}(x))} \int_{B_{2r}(x)} |f| = 2^n Mf$$

So that $c_2 Mf \leq M_2 f \leq 2^n Mf$.
(iii) Same argument as (i) applied to (ii).
Since $\{Q_r(x)|x \in \mathbb{R}^n\} \subseteq \{Q_r(y)|x \in Q_r(y)\}$, we have

$$M_3 f = \sup_{\{Q_r(y)|x\in Q_r(y)\}} \frac{1}{\mu(Q_r(y))} \int_{Q_r(y)} |f| \geq \sup_{\{Q_r(x)|x\in\mathbb{R}\}} \frac{1}{\mu(Q_r(x))} \int_{Q_r(x)} |f| = M_2 f$$

Now, for every $z \in Q_r(y)$, $|x-z| \leq |x-y| + |y-z| \leq 2r + 2r = 4r$ so that $Q_r(y) \subseteq B_{4r}(x) \subseteq Q_{4r}(x)$. Now,

$$M_3 f = \sup_{\{Q_r(y)|x\in Q_r(y)\}} \frac{1}{\mu(Q_r(y))} \int_{Q_r(y)} |f| \leq \sup_{\{Q_r(x)|x\in\mathbb{R}^n\}} \frac{1}{\mu(Q_r(x))} \int_{Q_{4r}(x)} |f| =$$

$$\sup_{\{Q_r(x)|x\in\mathbb{R}^n\}} \frac{4^n}{\mu(Q_r(x))} \int_{Q_{4r}(x)} |f| = 4^n M_2 f$$

So that $c_2 Mf \leq M_2 f \leq M_3 f \leq 4^n M_2 f \leq 8^n Mf$.

(b) Notice that the fact that the dyadic cubes are defined using half open intervals doesn't change the measure or the value of the integrals. This is practically a particular case of

$(a)(iii)$. Let $O_{x,k}$ be the centerpoint of the dyadic cube $\mathfrak{Q}_{x,k} = 2^k x + [0, 2^k)^n$. Then the radius of . Thus, since

$$\mu(\mathfrak{Q}_{x_0,k}) = \mu(2^k x_0 + [0, 2^k)^n) = \mu(Q_{2^k}(O_{x_0,k}))$$

and

$$\int_{\mathfrak{Q}_{x_0,k}} |f| = \int_{2^k x_0 + [0,2^k)^n} |f| = \int_{Q_{2^k}(O_{x_0,k})} |f|$$

we have

$$M_4 f = \sup_{\mathfrak{Q}_{x,k} \in \mathfrak{D}} \frac{1}{\mu(\mathfrak{Q}_{x,k})} \int_{\mathfrak{Q}_{x,k}} |f| = \sup_{\mathfrak{Q}_{x,k} \in \mathfrak{D}} \frac{1}{\mu(Q_{2^k}(O_{x,k}))} \int_{Q_{2^k}(O_{x,k})} |f| \leq$$

$$\sup_{Q_r(y) : x \in Q_r(y)} \frac{1}{\mu(Q_r(y))} \int_{Q_r(y)} |f| = M_3 f \leq 8^n M f$$

So that $M_4 f \leq 8^n M f$. $\square$