Homework 6

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Exercise 1.

Proof. (1) (a) (i)

$$\hat{F}(\xi) = \mathcal{F}(f(\xi) - \Delta f(\xi)) = \mathcal{F}(f(\xi) - \sum_{j=1}^{n} \partial_i^2 f(\xi)) = \mathcal{F}(f(\xi)) - \mathcal{F}(\sum_{j=1}^{n} \partial_i^2 f(\xi))) =$$

$$\mathcal{F}(f(\xi)) - \sum_{j=1}^{n}(2\pi i\xi_j)^2(\mathcal{F}f)(\xi))) = \hat{f}(\xi)\Big(1 + 4\pi^2|\xi|^2\Big) = C\hat{f}(\xi)$$

So that $f(\xi) = \frac{1}{C}F(\xi) = \frac{1}{1+4\pi^2|\xi|^2}F(\xi)$.
(ii)

$$\mathcal{F}^{-1}(m(\xi)\mathcal{F}((1-\Delta)f)(\xi)) = \mathcal{F}^{-1}(m(\xi)\hat{F}(\xi)) = \mathcal{F}^{-1}(m(\xi) \cdot C\hat{f}(\xi)) = \mathcal{F}^{-1}((2\pi i)^2\xi_i\xi_j\hat{f}(\xi))) =$$

$$\mathcal{F}^{-1}(-4\pi^2\xi_i\xi_j\hat{f}(\xi))) = \mathcal{F}^{-1}\Big(\mathcal{F}\Big(\partial_i\partial_j f(\xi)\Big)\Big) = \partial_i\partial_j f(\xi)$$

(b) We have,

$$\|x^\alpha H(x)\|_\infty \leq \|\mathcal{F}(x^\alpha H(x))\|_1 = \|\frac{1}{(-2\pi i)^{|\alpha|}}\partial^\alpha \hat{H}(x)\|_1 =$$

$$-\frac{(|\alpha|)! \cdot (8\pi^2)^{|\alpha|}}{(-2\pi i)^{|\alpha|}}\int \frac{|\xi|^\alpha}{|1+4\pi^2|\xi|^2|^{|\alpha|+1}}d\xi < \infty$$

for $|\alpha| \geq n-1$. So that $|H(x)| \leq \frac{C}{|x|^\alpha}$ which represents an L^1 function on $\mathbb{R}^n \setminus \{0\}$, hence everywhere on $\mathbb{R}^n$. Now,

$$|f(x)| = |(H * F)(x)| = |\int H(y)F(x-y)dy| \leq \|F\|_\infty|\int H(y)| \leq \|H\|_1 \cdot \|F\|_\infty$$

So that $\|f\|_\infty \leq \|H\|_1 \cdot \|F\|_\infty$.
(c)
Letting $m(\xi) = -\frac{4\pi^2\xi_i\xi_j}{1+4\pi^2|\xi|^2}$ as before, we see m is symmetric. I.e., $m(\xi) = m(-\xi)$. By (a),

$$\partial_i\partial_j f(\xi) = \mathcal{F}^{-1}(m(\xi)\mathcal{F}((1-\Delta)f)(\xi)) = (\check{m} * (1-\Delta)f)(\xi)$$

Keeping with the convention $\check{m}^{\#}(\xi) = \check{m}(-\xi)$, recall that $\check{m}^{\#}$ has the same properties as $\check{m}$, and in particular is also a proper multiplier. Now,

$$|\partial_i\partial_j f \cdot (1-\Delta)g| = |\check{m} * (1-\Delta)f \cdot (1-\Delta)g| =$$

$$\Big|\int\int \check{m}(x-y)((1-\Delta)f)(y)\ dy\ ((1-\Delta)g)(x)\ dx\Big| =$$

$$\Big|\int\int \check{m}^{\#}(y-x)((1-\Delta)g)(x)\ dx\ ((1-\Delta)f)(y)\ dy\Big| =$$

$$\Big|\int \mathcal{F}(m(-y)\cdot\mathcal{F}((1-\Delta)g)(y))\ ((1-\Delta)f)(y)\ dy\Big| =$$

$$\Big|\int \mathcal{F}(m(y)\cdot\mathcal{F}((1-\Delta)g)(y))\ ((1-\Delta)f)(y)\ dy\Big| =$$

$$\Big|\int \partial_i\partial_j g(1-\Delta)f\Big| \le \int |\partial_i\partial_j g|\cdot|(1-\Delta)f| \le \|(1-\Delta)f\|_{L^1}\cdot\|\partial_i\partial_j g\|_{L^\infty} \le$$

$$C\|(1-\Delta)f\|_{L^1}\cdot\|\Delta g\|_{L^\infty}$$

The second inequality is Hölder's inequality.

(d)

$$\|\partial_i\partial_j f\|_{L^1} = \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1}\int \partial_i\partial_j f G = \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1}\int \partial_i\partial_j f(1-\Delta)g \le$$

$$\sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} C\|(1-\Delta)f\|_{L^1}\|\Delta g\|_{L^\infty} \le \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} C\|(1-\Delta)f\|_{L^1}\cdot C_2\|G\|_{L^\infty} \le C\|(1-\Delta)f\|_{L^1}$$

Now, by (a), we have $g = \frac{1}{C}G$. So that $\|g\|_{L^\infty} = \frac{1}{C}\|G\|_{L^\infty} < \infty$. It follows that $|\partial_i^2 g| < \infty$, (else, $\partial_i g$, and hence g, explodes at some point). So that $\|\Delta g\|_{L^\infty} = \|\sum_{i=1}^n \partial_i^2 g\|_{L^\infty} < \infty$ and $\|\Delta g\|_{L^\infty} \le C_1\|g\|_{L^\infty} \le C\|G\|_{L^\infty}$.

(e)

$$\|(1-\Delta)f\|_1 = \int |f-\Delta f|dx = \int |\frac{1}{\lambda}f(\lambda x) - \frac{1}{\lambda}\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx =$$

$$\int |\frac{1}{\lambda}f(\lambda x) - \frac{1}{\lambda}\cdot\lambda^2\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx = \int |\frac{1}{\lambda}f(\lambda x) - \lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx \ge$$

$$\Big|\frac{1}{\lambda}\int |f(\lambda x)|dx - \int |\lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx\ \Big| =$$

$$\Big|\frac{1}{\lambda}\int |f(\lambda x)|dx - \|\Delta f\|_1\ \Big| \to \|0-\Delta f\|_1 = \|\Delta f\|_1$$

Likewise,

$$\int |\frac{1}{\lambda}f(\lambda x) - \lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|\ dx \le \int |\frac{1}{\lambda}f(\lambda x)| + |\lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|\ dx \le$$

$$\int |\frac{1}{\lambda}f(\lambda x)| + \int |\lambda \sum_{j=1}^{n} \partial_j^2 f(\lambda x)|\ dx = \int |\frac{1}{\lambda}f(\lambda x)| + \|\Delta f\|_1 \to 0 + \|\Delta f\|_1 = \|\Delta f\|_1$$

as $\lambda \to \infty$. Now, by (d),

$$\|D^2 f\|_1 \leq C\|(1-\Delta)f\|_1 = C\|\Delta\|_1$$

$\square$

Exercise 2.

Proof. (a) By continuity of the differential operator,

$$\lim_{R\to\infty}(D^k f_R(x)) = D^k(\lim_{R\to\infty} f_R(x)) = D^k(\phi(0)\cdot e^{2\pi i x\xi_0}) = \phi(0)D^k e^{2\pi i x\xi_0} =$$

$$\phi(0)\Big(\sum \partial_{j_1}\cdots\partial_{j_k}(e^{2\pi i x\xi_0})\Big) = \phi(0)\cdot e^{2\pi i x\xi_0}\cdot \sum(\xi_0)_{j_1}\cdots(\xi_0)_{j_k} \neq 0$$

and

$$\lim_{R\to\infty} Lf_R(x) = L(\lim_{R\to\infty} f_R(x)) = L(\phi(0)\cdot e^{2\pi i x\xi_0}) = \sum_{|\alpha|\leq k} c_\alpha \partial^\alpha(\phi(0)\cdot e^{2\pi i x\xi_0}) =$$

$$\phi(0)\sum_{|\alpha|\leq k} c_\alpha(2\pi i x\xi_0)^\alpha e^{2\pi i x\xi_0} = \phi(0)\cdot e^{2\pi i x\xi_0}\sum_{|\alpha|\leq k} c_\alpha(2\pi i x\xi_0)^\alpha = 0$$

Suppose by contradiction that $\|D^k f\|_p \leq C\|Lf\|_p$ for some nontrivial $f \in C_c^\infty$, with support$(f) = K$. Then,

$$\|D^k f_R(x)\|_p = |R|\cdot\|D^k f(x)\|_p \leq |R|\cdot C\|Lf(x)\|_p = C\|Lf_R(x)\|_p$$

for all R, so

$$\lim_{R\to\infty}\|D^k f_R\|_p = \mu(K)\cdot\phi(0)\sum(\xi_0)_{j_1}\cdots(\xi_0)_{j_k} > 0 = \lim_{R\to\infty}\|Lf_R\|_p$$

is a contradiction.

(b)
For $i \leq k$,

$$\partial_{j_1}\cdots\partial_{j_i} f = \mathcal{F}^{-1}\Big(\frac{\xi j_1\cdots\xi j_i}{\sum_{|\alpha|\leq k} c_\alpha \xi^{|\alpha|}}\cdot\mathcal{F}(Lf)\Big)$$

where $m(\xi) = \frac{\xi j_1\cdots\xi j_i}{\sum_{|\alpha|\leq k} c_\alpha\xi^\alpha}$. First, observe that $m \in L^\infty$. Let $p(\xi) = \frac{1}{m(\xi)}$. Then,

$$p(\xi) = \frac{\sum_{|\alpha|\leq k} c_\alpha\xi^\alpha}{\xi j_1\cdots\xi j_i} = \sum_{\ell\leq k}\sum_{|\alpha|=\ell}\frac{c_\alpha\xi^\alpha}{\xi j_1\cdots\xi j_i} = \sum_{\ell\leq k}\sum_{|\alpha|=l} p_\ell(\xi)$$

where $p_\ell(\xi) = \frac{c_\alpha\xi^\alpha}{\xi j_1\cdots\xi j_i}$. Now,

$$p_\ell(\lambda\cdot\xi) = \frac{\lambda^\ell c_\alpha\xi^\alpha}{\lambda^i\cdot\xi j_1\cdots\xi j_i} = \lambda^{\ell-i}\cdot p_\ell(\xi)$$

Therefore, take $\lambda = \dfrac{1}{|\xi|^{\frac{|\beta|+\ell-i}{|\beta|}}}$. We have,

$$\partial^\beta p(\xi) = \partial^\beta(\lambda^{\ell-i} \cdot p_\ell(\lambda\xi)) = \lambda^{|\beta|+\ell-i} \cdot \partial^\beta p_\ell(\lambda\xi) =$$

$$|\xi|^{-|\beta|} \cdot \partial^\beta p\left(\frac{\xi}{|\xi|^{\frac{|\beta|+\ell-i}{|\beta|}}}\right) \leq c_\beta |\xi|^{-|\beta|}$$

since $\mathbb{S}^{n-1}$ is compact $\left(\frac{\xi}{|\xi|^{\frac{|\beta|+\ell-i}{|\beta|}}} \text{ takes values on a sphere, albeit not of radius } 1\right)$.
This means

$$|\partial^\alpha p(\xi)| = |\sum_{\ell\leq k}\sum_{|\alpha|=l} \partial^\alpha p_\ell(\xi)| \leq \sum_{\ell\leq k}\sum_{|\alpha|=l} |\partial^\alpha p_\ell(\xi)| \leq c_\alpha |\xi|^{-|\alpha|}$$

Finally,

$$\partial^\alpha p(\xi) = \partial^\alpha\Big(\frac{1}{m(\xi)}\Big) = -\frac{\partial^\alpha m(\xi)}{m(\xi)^2} \iff$$

$$|\partial^\alpha m(\xi)| = |\partial^\alpha p(\xi)| \cdot |m(\xi)|^2 \leq c_\alpha |\xi|^{-|\alpha|} \cdot C \leq c_\alpha |\xi|^{-|\alpha|}$$

since m is bounded. Now, by Hörmander-Mikhlin, it follows that

$$\|\partial_{j_1} \cdots \partial_{j_i} f\|_{L^p} \leq C\|Lf\|_{L^p}$$

for every $0 \leq i \leq k$, as required. $\square$

Exercise 3.

Proof. Let $\mathbf{1}_{R^+} = \mathbf{1}_{(-\infty,a_1)\times\ldots\times(-\infty,a_n)}$; $\mathbf{1}_{R^-} = \mathbf{1}_{(-\infty,-a_1)\times\ldots\times(-\infty,-a_n)}$. We have,

$$f_R = \mathcal{F}^{-1}(\mathbf{1}_R \hat{f}) = \mathcal{F}^{-1}(\mathbf{1}_{R^+}\hat{f}) - \mathcal{F}^{-1}(\mathbf{1}_{R^-}\hat{f})$$

As before, suffices to prove the bound on $\mathbf{1}_{\mathbb{R}^n_{<0}} = \mathbf{1}_{(-\infty,a_1)\times\ldots\times(-\infty,a_n)}$ since translation leaves the L^p norm invariant. First, observe that $\mathbf{1}_{\mathbb{R}^n_{<0}}$ is a proper multiplier - it is bounded, and any derivative vanishes a.e. So that by Hörmander-Mikhlin,

$$\|\mathcal{F}^{-1}(\mathbf{1}_{\mathbb{R}^n_{<0}} \cdot \hat{f})\|_{L^p} \leq C\|f\|_{L^p}$$

$$\mathcal{F}^{-1}(\mathbf{1}_{R^+} \cdot \hat{f}) = \mathcal{F}^{-1}(\tau_t(\mathbf{1}_{\mathbb{R}^n_{<0}} \cdot \tau_{-t}\hat{f})) = e^{-2\pi i t x} \cdot \mathcal{F}^{-1}((\mathbf{1}_{\mathbb{R}^n_{<0}} \cdot \tau_{-t}\hat{f}) =$$
$$e^{-2\pi i t x} \cdot \mathcal{F}^{-1}((\mathbf{1}_{\mathbb{R}^n_{<0}} \cdot \mathcal{F}(e^{2\pi i t x} f)))$$

where $t = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. So that,

$$\|\mathcal{F}^{-1}(\mathbf{1}_{R^+} \cdot \hat{f})\|_{L^p} = \|e^{-2\pi i t x} \cdot \mathcal{F}^{-1}((\mathbf{1}_{\mathbb{R}^n_{<0}} \cdot \mathcal{F}(e^{2\pi i t x} f)))\|_{L^p} =$$

$$\|\mathcal{F}^{-1}((\mathbf{1}_{\mathbb{R}^n_{<0}} \cdot \mathcal{F}(e^{2\pi i t x} f)))\|_{L^p} \leq C\|\mathcal{F}(e^{2\pi i t x} f)\|_{L^p} = C\|e^{2\pi i t x} \cdot \hat{f}\|_{L^p} = C\|\hat{f}\|_{L^p}$$

Likewise,

$$\|\mathcal{F}^{-1}(\mathbf{1}_{R^-} \cdot \hat{f})\|_{L^p} \leq C\|\hat{f}\|_{L^p}$$

and therefore,

$$\|f_R\|_{L^p} = \|\mathcal{F}^{-1}(\mathbf{1}_R \hat{f})\|_{L^p} \leq \|\mathcal{F}^{-1}(\mathbf{1}_{R^+} \hat{f})\|_{L^p} + \|\mathcal{F}^{-1}(\mathbf{1}_{R^-} \hat{f})\|_{L^p} \leq C\|f\|_{L^p}$$

This holds for every $f \in \mathcal{S}$, with $\hat{f} \in C_c^\infty$, a dense set in L^p. By the addendum note, it follows that $f_R = \mathcal{F}^{-1}(\mathbf{1}_R \hat{f}) \to f$ in L^p, extends continuously to all $f \in L^p$. $\square$