Class 7

Wednesday, November 4, 2020 2:58 PM

<u>Question</u> Given $(f_k)_{k\in\mathbb{N}}$ seq. of functions, can we bound $\|\sum c_k f_k\|_{L^p}$ in terms of $|c_k|$'s ?

<u>Weaker question</u>

Assuming $|c_k'| = |c_k|$,

is $\|\sum c_k' f_k\|_{L^p} \lesssim \|\sum c_k f_k\|_{L^p}$?

$\lesssim$ means "LHS $\leq C \cdot$ RHS"

<u>Example</u> If $p=2$ and f_k's are orthogonal (i.e. $\int f_j \overline{f_k} = 0$ $\forall j \neq k$)

then $\|\sum c_k f_k\|_{L^2}^2 = \sum |c_k|^2 \|f_k\|_{L^2}^2$.

This happens e.g. with $f_k = e^{2\pi i k x}$ $(k \in \mathbb{Z})$ on $S^1 = \mathbb{R}/\mathbb{Z}$.

So for $p=2$ answer is <u>yes</u> to both questions! (in this case)

We will see that it's <u>no</u> for $1<p<2$.

Very special case: <u>independent</u> random variables $(X_k)_{k\in\mathbb{N}}$ on a probab. space.

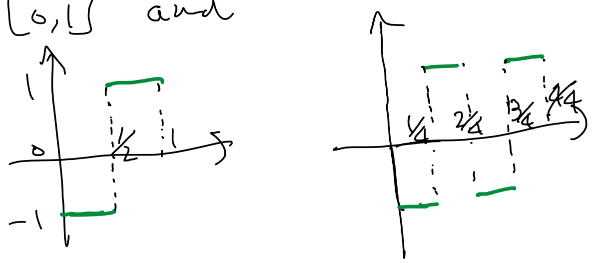
X_k independent & identically distributed

X_k ... distributed

$\mathbb{P}(X_k = 1) = \frac{1}{2}$

$\mathbb{P}(X_k = -1) = \frac{1}{2}$

This is realized e.g. taking $[0,1]$ and



. . .

<u>Thm</u> (Khintchine's inequality)

$$\left(\mathbb{E}\left|\sum c_k X_k\right|^p\right)^{1/p} \approx \left(\sum |c_k|^2\right)^{1/2}$$

$$\forall\, 0 < p < \infty.$$

("$\approx$" means "$\lesssim$" and "$\gtrsim$")

<u>Proof</u> ($\lesssim$) W.l.o.g. $c_k \in \mathbb{R}$ $\forall k$

and we can normalize s.t.

$$\sum |c_k|^2 = 1.$$

Note that

$$\left\{\sum c_k X_k > t\right\} = \left\{e^{\alpha \sum c_k X_k} > e^{\alpha t}\right\}$$

so

$$\mathbb{P}\left\{\sum c_k X_k > t\right\} \le e^{-\alpha t}\, \mathbb{E}\left[e^{\alpha c_1 X_1} \cdot e^{\alpha c_2 X_2} \cdots\right]$$

$$= e^{-\alpha t}\, \mathbb{E}\left[e^{\alpha c_1 X_1}\right] \mathbb{E}\left[e^{\alpha c_2 X_2}\right] \cdots$$

(independence)

$$\mathbb{E}\left[e^{\alpha c_k X_k}\right] = \frac{1}{2} e^{\alpha c_k} + \frac{1}{2} e^{-\alpha c_k}$$

$$= \sum^{\infty} \frac{(\alpha c_k)^{2j}}{\ldots} \le \sum^{\infty} \frac{(\alpha c_k)^{2j}}{\ldots}$$

$$j=0 \quad (2j)! \qquad j=0 \quad 2^j j!$$

$$= e^{\alpha^2 c_k^2/2}$$

$$\Rightarrow \mathbb{P}\{\sum c_k X_k \geq t\} \leq e^{-\alpha t} e^{\alpha^2 c_1^2/2} e^{\alpha^2 c_2^2/2} \cdots$$

$$= e^{-\alpha t + \alpha^2/2}$$

for $\alpha := t$ we get

$$\mathbb{P}\{|\sum c_k X_k| > t\} \leq 2 e^{-t^2/2}$$

(using same procedure for $\{\sum c_k X_k < -t\} = \{\sum (-c_k) X_k > t\}$)

$$\Rightarrow \mathbb{E}|\sum c_k X_k|^p = \int_0^\infty p t^{p-1} \mathbb{P}\{|\sum c_k X_k| > t\} \, dt$$

$$\leq 2p \int_0^\infty t^{p-1} e^{-t^2/2} \, dt$$

$$< \infty.$$

assume $p<4$.

$$(\gtrsim) \quad \sum |c_k|^2 = \mathbb{E}|\sum c_k X_k|^2$$

$$= \mathbb{E}\left[|\sum c_k X_k|^{p/2} \cdot |\sum c_k X_k|^{2-p/2}\right]$$

$$\underset{(C-S)}{\leq} \mathbb{E}\left[|\sum c_k X_k|^p\right]^{1/2} \underbrace{\mathbb{E}\left[|\sum c_k X_k|^{4-p}\right]^{1/2}}_{\lesssim (\sum |c_k|^2)^{\frac{1}{2} \cdot (4-p) \cdot \frac{1}{2}}}$$

$$\lesssim \mathbb{E}\left[|\sum c_k X_k|^p\right]^{1/2} (\sum |c_k|^2)^{1-p/4}$$

$$\Rightarrow (\sum |c_k|^2)^{p/4} \lesssim \mathbb{E}\left[|\sum c_k X_k|^p\right]^{1/2}$$

$\Rightarrow$ done.

For $p \geq 4$ (or $p > 2$)

$$\left(\sum |c_k|^2\right)^{1/2} = \mathbb{E}\left[\left|\sum c_k X_k\right|^2\right]^{1/2}$$
$$\leq \mathbb{E}\left[\left|\sum c_k X_k\right|^p\right]^{1/p}. \qquad \square$$

<u>Rmk</u> Proof works assuming only finitely ~~many~~ coefficients are $\neq 0$. It also shows that if $\sum |c_k|^2 < \infty$ then ineq. is still true.

For $S^1 = \mathbb{R}/\mathbb{Z}$ and $f_k = e^{2\pi i k x}$ $(k \in \mathbb{Z})$ we can now show that

$$\left\|\sum c_k' f_k\right\|_{L^p} \lesssim \left\|\sum c_k f_k\right\|_{L^p}$$

(whenever $|c_k'| = |c_k|$) for $1 < p < 2$.

We take $c_k' = X_k c_k$ (X_k as above, independent)

$$\mathbb{E}\left[\int_{S^1} \left|\sum c_k X_k f_k\right|^p\right]$$
$$= \int_{S^1} \mathbb{E}\left|\sum c_k X_k f_k\right|^p$$

$\gtrsim \int_{S^1} \left(\sum |c_k f_k|\right)$

(but $|f_k| = 1$, so)

$= \left(\sum |c_k|^2\right)^{p/2}$.

Now take $f \in L^p(S^1) \setminus L^2(S^1)$.

Take $c_k := \hat{f}(k)$ for $-N \leq k \leq N$.

Assume by contradiction

$$\int_{S^1} \left|\sum_{-N}^{N} c_k \kappa_k f_k\right|^p \leq C \int_{S^1} \left|\sum c_k f_k\right|^p$$

RHS $\lesssim \|f\|_{L^p}$ (see one of previous HW's)

$$\Rightarrow \left(\sum_{-N}^{N} |c_k|^2\right)^{p/2} \lesssim \mathbb{E}[\text{LHS}] \lesssim \|f\|_{L^p}.$$

Since $f \notin L^2$, $\sum_{-N}^{N} |c_k|^2 \to \infty$

as $N \to \infty$. $\lightning$

However, if we multiply each block c_k, $2^j \leq k < 2^{j+1}$, by an $\varepsilon_k = \pm 1$

then $\|\sum c_k' f_k\|_p \lesssim \|\sum c_k f_k\|_p$

$(\varepsilon_2 = \varepsilon_3,\ \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7, \ldots)$.

We prove

<u>Thm</u> For $f \in \mathcal{S}(\mathbb{R}^n)$

$\|f\|_{L^p} \approx L^p$-norm of

$$x \mapsto \Big(\sum_{k \in \mathbb{Z}} |P_k f|^2(x) \Big)^{1/2}.$$

$(1 < p < \infty)$

Here $P_k f := \mathcal{F}^{-1}(\psi_k \hat{f})$

(recall: $\sum \psi_k = 1$ on $\mathbb{R}^n \setminus \{0\}$

and $\mathrm{spt}(\psi_k) \subseteq B_{2^{k+1}} \setminus \overline{B}_{2^{k-1}}$).

We will write

$\|(P_k f)\|_{L^p(\ell^2)}$ to mean the above RHS.

(1^{st}: take ℓ^2-norm of functions at x,

2^{nd}: take L^p-norm of the result).

In 1D one can really prove that $\|f\|_{L^p} \approx \| \mathcal{F}^{-1} 1_{(2^k, 2^{k+1})} \hat{f} \|_{L^p(\ldots}$

$+ \| \mathcal{F}^{-1} 1_{(-2^{k+1}, -2^k)} \hat{f} \|_{L\ldots}$

First proof of thm

Basically Hörmander-Mikhlin holds also for vector valued functions:

assume $f: \mathbb{R}^n \to X$

(X finite dim. vector space over $\mathbb{C}$ with a norm $\| \|_X$),

$m: \mathbb{R}^n \setminus \{0\} \to L(X, Y)$

(Y finite dim. ......)

$L(X,Y) = \{\text{linear maps } X \to Y\}$.

Assume

- $T: f \to \mathcal{F}^{-1}(m\hat{f})$ is bounded from L^2 to L^2.

- $\|\partial^\alpha m(\xi)\|_{L(X,Y)} \leq C_\alpha |\xi|^{-|\alpha|}$

then $\|Tf\|_{L^p} \leq C \|f\|_{L^p}$ $\forall\, 1<p<\infty$

Rmk $\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle \xi, x \rangle} dx \in X$

and "$m(\xi)\hat{f}(\xi)$" is $m(\xi) \in L(X$ applied to the vector $\hat{f}(\xi)$.

So $Tf(x) \in Y$.

$\|f\|_{L^p} := \left(\int \|f(x)\|_X^p \, dx\right)^{1/p}$.

The proof for the scalar version carries over (

Let's go back to the theorem!

$(\gtrsim)$ $X := \mathbb{C}$, $Y := \mathbb{C}^{2N+1}$

(with Eucl. norm)

$m(\xi)$ is multiplication by

$$\begin{pmatrix} \psi_{-N}(\xi) \\ \vdots \\ \psi_N(\xi) \end{pmatrix}.$$

L^2 bound holds by Plancherel.

$m(\xi)$ satisfies the required decay since $\forall \xi \neq 0$ at most two components matter.

$$\Rightarrow \|\mathcal{F}^{-1}(m\hat{f})\|_{L^p} \lesssim \|f\|_{L^p}$$

$$\|$$

L^p-norm of

$$x \mapsto \left(\sum_{-N}^{N} |P_k f|^2\right)^{1/2}$$

and claim follows sending $N \to \infty$.

$(\lesssim)$ follows from this lemma:

Lemma Assume $f_{-N}, \dots, f_N \in \mathcal{S}$ s.t. $\hat{f}_k$ has support $\subseteq B_{2^{k+1}} \setminus B_{2^{k-1}}$.

Then $\| \sum_{-N}^{N} f_k \|_{L^p} \lesssim \| (f_k) \|_{L^p(\ell^2)}$.

We can apply lemma with $f_k := P_k f$ and note that

$\| f \|_{L^p} \leq \liminf_{N \to \infty} \| \sum_{-N}^{N} P_k f \|_{L^p} \lesssim \| (P_k f) \|_{L}$

(as $\sum_{-N}^{N} P_k f \to f$ in L^2 ...)

Proof of Lemma

Choose $\tilde{\psi}_k$ s.t. $\tilde{\psi}_k = 1$ on $\overline{B}_{2^{k+1}} \setminus B_{2^{k-1}}$,

$\operatorname{spt} \tilde{\psi}_k \subseteq B_{2^{k+2}} \setminus \overline{B}_{2^{k+1}}$,

$| \partial^\alpha \psi_k (\xi) | \lesssim C_\alpha | \xi |^{-|\alpha|}$.

$($\tilde{\psi}_k := \psi_{k-1} + \psi_k + \psi_{k+1}$ works$)$

Now $X = \mathbb{C}^{2N+1}$, $Y = \mathbb{C}$,

$$m(\xi)\begin{bmatrix} a_{-N} \\ \vdots \\ a_N \end{bmatrix} := \sum_k \tilde{\psi}_k(\xi)\, a_k .$$

We get $\|Tf\|_{L^p} \lesssim \|(f_k)\|_{L^p(\ell^2)}$

but $Tf = \sum f_k$ because

$$Tf = \mathcal{F}^{-1}\left(\sum \tilde{\psi}_k \hat{f}_k\right)$$
$$= \mathcal{F}^{-1}\left(\sum \hat{f}_k\right) = \sum f_k . \quad \square$$

P_k is called the k-th <u>Littlewood–Paley projection</u> of f.

The last theorem is called the <u>Littlewood–Paley characterization</u> of L^p.

Application

We want to show

$$\|Df\|_{L^q}^2 \lesssim \|f\|_{L^p} \|D^2 f\|_{L^r}$$

where $\boxed{\frac{2}{q} = \frac{1}{p} + \frac{1}{r}}$

$(f \in \mathcal{S},\ 1 < p, q, r < \infty)$.

By L-P characterization,

$$\|Df\|_{L^q}^2 \approx \|(D P_k f)\|_{L^q(\ell^2)}^2$$

$$= \Big\| \sum_k |D P_k f|^2 \Big\|_{L^{q/2}}.$$

(we used that D and P_k commute since in $\mathcal{F}$ they are multiplications)

Call $f_k := P_k f$.

Take $h \in (L^{q/2})'$

$$\int \sum_k |Df_k|^2 h$$

$$= \sum_k \int |Df_k|^2 h$$

$$= \sum_k \int |Df_k|^2 h^k$$

$$\left(h^k := \sum_{j \le k+2} h_j \right)$$

last equality holds because

$$\int |Df_k|^2 h = \int \widehat{|Df_k|^2}\, \hat{h}$$

$$= \int \left[(\widehat{Df_k}) * (\widehat{Df_k}) \right] \hat{h}$$

$$\operatorname{spt}(\widehat{Df_k}) \subseteq B_{2^{k+1}}$$

$$\Rightarrow \operatorname{spt}(\widehat{Df_k} * \widehat{Df_k}) \subseteq B_{2^{k+2}}.$$

$h = \sum_{j \in \mathbb{Z}} h_j$ and h_j will vanish

$|Df_k|^2 \quad \forall j > k+3.$

against ...

Now $\left| |Df_k|^2 h^k \right| \leq |2^{-k} Df_k| \; |2^k \mathrm{I}$

$P_k f \qquad P_k \mathrm{I}$

$$\overset{(HW)}{\lesssim} |2^{-k} Df_k| \; |2^k Df_k| \; Mh$$

$$\left| \sum_k \cdots \right| \lesssim \|(2^{-k} Df_k)\|_{\ell^2} \|(2^k Df_k)\|$$

$$\underset{(\text{Hölder})}{\Longrightarrow} \left| \int \sum_k \cdots \right| \lesssim \|(2^{-k} Df_k)\|_{L^p(\ell^2)} \|(2$$

$$\left(\frac{1}{p} + \frac{1}{r} + \frac{1}{(2/q)'} = 1 \right)$$

Now 1^{st} factor $\lesssim \|f\|_{L^p}$,

$2^{nd} \lesssim \|D^2 f\|_{L^r}$,

$3^{rd} \lesssim \|h\|_{L^{(q/2)'}} \leq 1.$

(H