

Homework 2

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Exercise 1.

Proof. Since $f \neq 0$, there exists a radius r , on which $\int_{B_r(0)} |f| = c > 0$. For $|x| > r$, we have $B_r(0) \subseteq B_{|x|+r}(x)$ and therefore

$$Mf(x) \geq \frac{1}{\mu(B_{|x|+r}(x))} \int_{B_{|x|+r}(x)} |f| \geq \frac{c}{(|x|+r)^n}$$

So that

$$\int_{\mathbb{R}^n} |Mf(x)| \geq \int_{|x|>r} \frac{c}{(|x|+r)^n} = \infty$$

Thus, $Mf \notin L^1(\mathbb{R}^n)$. □

Exercise 2.

Proof. (a)(i)

$$0 = \|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} \iff \mu\{|f| > \lambda\} = 0 \text{ for all } \lambda > 0$$

$$\iff |f| = 0 \iff f = 0$$

(ii)

$$\|kf\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|kf| > \lambda\} = |k| \cdot \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} = |k| \cdot \|f\|_{L^{1,\infty}}$$

(iii) Since $|f| + |g| \geq |f + g|$, we have

$$\{|f| > \frac{\lambda}{2}\} \cup \{|g| > \frac{\lambda}{2}\} \supseteq \{|f| + |g| > \lambda\} \supseteq \{|f + g| > \lambda\}$$

so that

$$\begin{aligned} 2(\|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}}) &= 2\|f\|_{L^{1,\infty}} + 2\|g\|_{L^{1,\infty}} = \\ &= \sup_{\lambda>0} \lambda \cdot \mu\{2|f| > \lambda\} + \sup_{\lambda>0} \lambda \cdot \mu\{2|g| > \lambda\} = \\ &= \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{2}\} + \sup_{\lambda>0} \lambda \cdot \mu\{|g| > \frac{\lambda}{2}\} \geq \end{aligned}$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f| + |g| > \lambda\} \geq \sup_{\lambda>0} \lambda \cdot \mu\{|f + g| > \lambda\} = \|f + g\|_{L^{1,\infty}}$$

(b) For $x \in [0, 1]$, we have

$$\begin{aligned} |f_\ell(x)| &= \frac{1}{\log \ell} \left| \sum_{j=1}^{\ell} \frac{1}{x\ell - j} \right| = \frac{1}{\log \ell} \left| \sum_{j=1}^{\ell} \frac{1}{j - x\ell} \right| \\ &= \frac{1}{\log \ell} \left| \sum_{j=1}^{\ell} \frac{1}{j} - \frac{x\ell}{j(x\ell - j)} \right| \geq \frac{1}{\log \ell} \left(\sum_{j=1}^{\ell} \left| \frac{1}{j} \right| - \left| \frac{x\ell}{j(x\ell - j)} \right| \right) = \\ &= \frac{1}{\log \ell} \left(\sum_{j=1}^{\ell} \left| \frac{1}{j} \right| - \sum_{j=1}^{\ell} \left| \frac{x\ell}{j(x\ell - j)} \right| \right) \geq \frac{1}{\log \ell} \sum_{j=1}^{\ell} \frac{1}{j} \geq \\ &= \frac{1}{\log \ell} \sum_{j=1}^{\ell-1} \int_j^{j+1} \frac{1}{j} = \frac{1}{\log \ell} \cdot \log \ell = 1 \end{aligned}$$

So that $\|f_\ell\|_{L^{1,\infty}} \geq 1 \cdot \mu\{|f_\ell| > 1\} \geq \mu[0, 1] = 1$.

(c) Since μ is translation invariant, we have

$$\mu\left\{\left|\frac{1}{x - \frac{j}{\ell}}\right| > \lambda\right\} = \mu\left\{\left|\frac{1}{x}\right| > \lambda\right\}$$

So that

$$\begin{aligned} \left\| \frac{1}{x - \frac{j}{\ell}} \right\|_{L^{1,\infty}} &= \left\| \frac{1}{x} \right\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in \mathbb{R} : \left|\frac{1}{x}\right| > \lambda\right\} = \\ &= 2 \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in [0, \infty) : \frac{1}{x} > \lambda\right\} = 2 \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in [0, \infty) : x < \frac{1}{\lambda}\right\} = \\ &= 2 \sup_{\lambda>0} \lambda \cdot \mu\left[0, \frac{1}{\lambda}\right) = 2 \sup_{\lambda>0} \lambda \cdot \frac{1}{\lambda} = 2 \sup_{\lambda>0} 1 = 2 \cdot 1 = 2 \end{aligned}$$

So that

$$\|f_\ell\|_{L^{1,\infty}} \leq 2 \sum_{j=1}^{\ell} \frac{1}{\ell \cdot \log \ell} \left\| \frac{1}{x} \right\|_{L^{1,\infty}} = \frac{4\ell}{\ell \cdot \log \ell} = \frac{4}{\log \ell}$$

Now, if there exists a norm $\|\cdot\|'$ such that $c\|f_\ell\|' \leq \|f_\ell\|_{L^{1,\infty}} \leq C\|f_\ell\|'$, such that $c, C > 0$, then

$$c\|f_\ell\|' \leq \|f_\ell\|_{L^{1,\infty}} \leq \frac{4}{\log \ell} \rightarrow 0$$

as $\ell \rightarrow \infty$. Since $c > 0$, it follows that $\|f_\ell\|' = 0$, which is a contradiction since $|f_\ell| \geq 1$ on $[0, 1]$ for all ℓ . Meaning, $f \neq 0$, so that $\|\cdot\|'$ isn't a proper norm. \square

Exercise 3.

Proof. (a) (i) Since $\{B_r(x)|x \in \mathbb{R}^n\} \subseteq \{B_r(y)|x \in B_r(y)\}$, we have

$$M_1 f = \sup_{\{B_r(y)|x \in B_r(y)\}} \frac{1}{\mu(B_r(y))} \int_{B_r(y)} |f| \geq \sup_{\{B_r(x)|x \in \mathbb{R}\}} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| = Mf$$

Now, for every $z \in B_r(y)$, $|x - z| \leq |x - y| + |y - z| \leq r + r = 2r$ so that $B_r(y) \subseteq B_{2r}(x)$.
Now,

$$\begin{aligned} M_1 f &= \sup_{\{B_r(y)|x \in B_r(y)\}} \frac{1}{\mu(B_r(y))} \int_{B_r(y)} |f| \leq \sup_{\{B_r(x)|x \in \mathbb{R}^n\}} \frac{1}{\mu(B_r(x))} \int_{B_{2r}(x)} |f| = \\ &\quad \sup_{\{B_r(x)|x \in \mathbb{R}^n\}} \frac{2^n}{\mu(B_r(x))} \int_{B_{2r}(x)} |f| = 2^n Mf \end{aligned}$$

So that $Mf \leq M_1 f \leq 2^n Mf$.

(ii) $B_r(x) \subseteq Q_r(x) \subseteq B_{2r}(x)$. Let $c_2 = \frac{\mu(B_r(x))}{\mu(Q_r(x))}$. Observe that c_2 is independent of r . Then,

$$M_2 f = \sup_{\{Q_r(x)|x \in \mathbb{R}^n\}} \frac{1}{\mu(Q_r(x))} \int_{Q_r(x)} |f| \geq \sup_{\{B_r(x)|x \in \mathbb{R}\}} \frac{c_2}{\mu(B_r(x))} \int_{B_r(x)} |f| = c_2 Mf$$

and

$$M_2 f = \sup_{\{Q_r(x)|x \in \mathbb{R}^n\}} \frac{1}{\mu(Q_r(x))} \int_{Q_r(x)} |f| \leq \sup_{\{B_{2r}(x)|x \in \mathbb{R}^n\}} \frac{2^n}{\mu(B_{2r}(x))} \int_{B_{2r}(x)} |f| = 2^n Mf$$

So that $c_2 Mf \leq M_2 f \leq 2^n Mf$.

(iii)

□