

Homework 6

Dan Sokolsky

November 13, 2020

Exercise 1.

Proof. (1) (a) (i)

$$\hat{F}(\xi) = \mathcal{F}(f(\xi) - \Delta f(\xi)) = \mathcal{F}(f(\xi) - \sum_{j=1}^n \partial_j^2 f(\xi)) = \mathcal{F}(f(\xi)) - \mathcal{F}\left(\sum_{j=1}^n \partial_j^2 f(\xi)\right) =$$

$$\mathcal{F}(f(\xi)) - \sum_{j=1}^n (2\pi i \xi_j)^2 (\mathcal{F}f)(\xi)) = \hat{f}(\xi) \left(1 + 4\pi^2 |\xi|^2\right) = C \hat{f}(\xi)$$

So that $f(\xi) = \frac{1}{C} \hat{F}(\xi) = \frac{1}{1+4\pi^2|\xi|^2} \hat{F}(\xi)$.
(ii)

$$\mathcal{F}^{-1}(m(\xi) \mathcal{F}((1-\Delta)f)(\xi)) = \mathcal{F}^{-1}(m(\xi) \hat{F}(\xi)) = \mathcal{F}^{-1}(m(\xi) \cdot C \hat{f}(\xi)) = \mathcal{F}^{-1}((2\pi i)^2 \xi_i \xi_j \hat{f}(\xi)) =$$

$$\mathcal{F}^{-1}(-4\pi^2 \xi_i \xi_j \hat{f}(\xi)) = \mathcal{F}^{-1}\left(\mathcal{F}\left(\partial_i \partial_j f(\xi)\right)\right) = \partial_i \partial_j f(\xi)$$

(b) We have,

$$\|x^\alpha H(x)\|_\infty \leq \|\mathcal{F}(x^\alpha H(x))\|_1 = \left\| \frac{1}{(-2\pi i)^{|\alpha|}} \partial^\alpha \hat{H}(x) \right\|_1 =$$

$$- \frac{(|\alpha|)! \cdot (8\pi^2)^{|\alpha|}}{(-2\pi i)^{|\alpha|}} \int \frac{|\xi|^\alpha}{|1 + 4\pi^2 |\xi|^2|^{|\alpha|+1}} d\xi < \infty$$

for $|\alpha| \geq n-1$. So that $|H(x)| \leq \frac{C}{|x|^\alpha}$ which represents an L^1 function on $\mathbb{R}^n \setminus \{0\}$, hence everywhere on \mathbb{R}^n . Now,

$$|f(x)| = |(H * F)(x)| = \left| \int H(y) F(x-y) dy \right| \leq \|F\|_\infty \int |H(y)| \leq \|H\|_1 \cdot \|F\|_\infty$$

So that $\|f\|_\infty \leq \|H\|_1 \cdot \|F\|_\infty$.

(c)

Letting $m(\xi) = -\frac{4\pi^2 \xi_i \xi_j}{1+4\pi^2 |\xi|^2}$ as before, we see m is symmetric. I.e., $m(\xi) = m(-\xi)$. By (a),

$$\partial_i \partial_j f(\xi) = \mathcal{F}^{-1}(m(\xi) \mathcal{F}((1-\Delta)f)(\xi)) = (\check{m} * (1-\Delta)f)(\xi)$$

Keeping with the convention $\check{m}^\#(\xi) = \check{m}(-\xi)$, recall that $\check{m}^\#$ has the same properties as \check{m} , and in particular is also a proper multiplier. Now,

$$\begin{aligned}
|\partial_i \partial_j f \cdot (1 - \Delta)g| &= |\check{m} * (1 - \Delta)f \cdot (1 - \Delta)g| = \\
&= \left| \int \int \check{m}(x - y)((1 - \Delta)f)(y) dy ((1 - \Delta)g)(x) dx \right| = \\
&= \left| \int \int \check{m}^\#(y - x)((1 - \Delta)g)(x) dx ((1 - \Delta)f)(y) dy \right| = \\
&= \left| \int \mathcal{F}(m(-y) \cdot \mathcal{F}((1 - \Delta)g)(y)) ((1 - \Delta)f)(y) dy \right| = \\
&= \left| \int \mathcal{F}(m(y) \cdot \mathcal{F}((1 - \Delta)g)(y)) ((1 - \Delta)f)(y) dy \right| = \\
\left| \int \partial_i \partial_j g (1 - \Delta)f \right| &\leq \int |\partial_i \partial_j g| \cdot |(1 - \Delta)f| \leq \|(1 - \Delta)f\|_{L^1} \cdot \|\partial_i \partial_j g\|_{L^\infty} \leq \\
&= C\|(1 - \Delta)f\|_{L^1} \cdot \|\Delta g\|_{L^\infty}
\end{aligned}$$

The second inequality is Hölder's inequality.

(d)

$$\begin{aligned}
\|\partial_i \partial_j f\|_{L^1} &= \sup_{G \in \mathcal{S}: \|G\|_{L^\infty} \leq 1} \int \partial_i \partial_j f G = \sup_{G \in \mathcal{S}: \|G\|_{L^\infty} \leq 1} \int \partial_i \partial_j f (1 - \Delta)g \leq \\
\sup_{G \in \mathcal{S}: \|G\|_{L^\infty} \leq 1} C\|(1 - \Delta)f\|_{L^1} \|\Delta g\|_{L^\infty} &\leq \sup_{G \in \mathcal{S}: \|G\|_{L^\infty} \leq 1} C\|(1 - \Delta)f\|_{L^1} \cdot C_2 \|G\|_{L^\infty} \leq C\|(1 - \Delta)f\|_{L^1}
\end{aligned}$$

Now, by (a), we have $g = \frac{1}{C}G$. So that $\|g\|_{L^\infty} = \frac{1}{C}\|G\|_{L^\infty} < \infty$. It follows that $|\partial_i^2 g| < \infty$, (else, $\partial_i g$, and hence g , explodes at some point). So that $\|\Delta g\|_{L^\infty} = \|\sum_{i=1}^n \partial_i^2 g\|_{L^\infty} < \infty$ and $\|\Delta g\|_{L^\infty} \leq C_1 \|g\|_{L^\infty} \leq C \|G\|_{L^\infty}$.

(e)

$$\begin{aligned}
\|(1 - \Delta)f\|_1 &= \int |f - \Delta f| dx = \int \left| \frac{1}{\lambda} f(\lambda x) - \frac{1}{\lambda} \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx = \\
&= \int \left| \frac{1}{\lambda} f(\lambda x) - \frac{1}{\lambda} \cdot \lambda^2 \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx = \int \left| \frac{1}{\lambda} f(\lambda x) - \lambda \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx \geq \\
&= \left| \frac{1}{\lambda} \int |f(\lambda x)| dx - \int \left| \lambda \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx \right| = \\
\left| \frac{1}{\lambda} \int |f(\lambda x)| dx - \|\Delta f\|_1 \right| &\rightarrow \|0 - \Delta f\|_1 = \|\Delta f\|_1
\end{aligned}$$

Likewise,

$$\int \left| \frac{1}{\lambda} f(\lambda x) - \lambda \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx \leq \int \left| \frac{1}{\lambda} f(\lambda x) \right| + \left| \lambda \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx \leq$$

$$\int \left| \frac{1}{\lambda} f(\lambda x) \right| + \int \left| \lambda \sum_{j=1}^n \partial_j^2 f(\lambda x) \right| dx = \int \left| \frac{1}{\lambda} f(\lambda x) \right| + \|\Delta f\|_1 \rightarrow 0 + \|\Delta f\|_1 = \|\Delta f\|_1$$

as $\lambda \rightarrow \infty$. Now, by (d),

$$\|D^2 f\|_1 \leq C\|(1 - \Delta)f\|_1 = C\|\Delta f\|_1$$

□

Exercise 2.

Proof. (a) By continuity of the differential operator,

$$\lim_{R \rightarrow \infty} (D^k f_R(x)) = D^k \left(\lim_{R \rightarrow \infty} f_R(x) \right) = D^k(\phi(0) \cdot e^{2\pi i x \xi_0}) = \phi(0) D^k e^{2\pi i x \xi_0} =$$

$$\phi(0) \left(\sum \partial_{j_1} \cdots \partial_{j_k} (e^{2\pi i x \xi_0}) \right) = \phi(0) \cdot e^{2\pi i x \xi_0} \cdot \sum (\xi_0)_{j_1} \cdots (\xi_0)_{j_k} \neq 0$$

and

$$\lim_{R \rightarrow \infty} L f_R(x) = L \left(\lim_{R \rightarrow \infty} f_R(x) \right) = L(\phi(0) \cdot e^{2\pi i x \xi_0}) = \sum_{|\alpha| \leq k} c_\alpha \partial^\alpha (\phi(0) \cdot e^{2\pi i x \xi_0}) =$$

$$\phi(0) \sum_{|\alpha| \leq k} c_\alpha (2\pi i x \xi_0)^\alpha e^{2\pi i x \xi_0} = \phi(0) \cdot e^{2\pi i x \xi_0} \sum_{|\alpha| \leq k} c_\alpha (2\pi i x \xi_0)^\alpha = 0$$

Suppose by contradiction that $\|D^k f\|_p \leq C\|L f\|_p$ for some nontrivial $f \in C_c^\infty$, with $\text{support}(f) = K$. Then,

$$\|D^k f_R(x)\|_p = |R| \cdot \|D^k f(x)\|_p \leq |R| \cdot C\|L f(x)\|_p = C\|L f_R(x)\|_p$$

for all R , so

$$\lim_{R \rightarrow \infty} \|D^k f_R\|_p = \mu(K) \cdot \phi(0) \sum (\xi_0)_{j_1} \cdots (\xi_0)_{j_k} > 0 = \lim_{R \rightarrow \infty} \|L f_R\|_p$$

is a contradiction.

(b)

For $i \leq k$,

$$\partial_{j_1} \cdots \partial_{j_i} f = \mathcal{F}^{-1} \left(\frac{\xi_{j_1} \cdots \xi_{j_i}}{\sum_{|\alpha| \leq k} c_\alpha \xi^{|\alpha|}} \cdot \mathcal{F}(L f) \right)$$

where $m(\xi) = \frac{\xi_{j_1} \cdots \xi_{j_i}}{\sum_{|\alpha| \leq k} c_\alpha \xi^\alpha}$. First, observe that $m \in L^\infty$. Let $p(\xi) = \frac{1}{m(\xi)}$. Then,

$$p(\xi) = \frac{\sum_{|\alpha| \leq k} c_\alpha \xi^\alpha}{\xi_{j_1} \cdots \xi_{j_i}} = \sum_{\ell \leq k} \sum_{|\alpha| = \ell} \frac{c_\alpha \xi^\alpha}{\xi_{j_1} \cdots \xi_{j_i}} = \sum_{\ell \leq k} \sum_{|\alpha| = \ell} p_\ell(\xi)$$

where $p_\ell(\xi) = \frac{c_\alpha \xi^\alpha}{\xi_{j_1} \cdots \xi_{j_i}}$. Now,

$$p_\ell(\lambda \cdot \xi) = \frac{\lambda^\ell c_\alpha \xi^\alpha}{\lambda^i \cdot \xi_{j_1} \cdots \xi_{j_i}} = \lambda^{\ell-i} \cdot p_\ell(\xi)$$

Therefore, take $\lambda = \frac{1}{|\xi|^{\frac{|\beta|+\ell-i}{|\beta|}}}$. We have,

$$\partial^\beta p(\xi) = \partial^\beta (\lambda^{\ell-i} \cdot p_\ell(\lambda\xi)) = \lambda^{|\beta|+\ell-i} \cdot \partial^\beta p_\ell(\lambda\xi) =$$

$$|\xi|^{-|\beta|} \cdot \partial^\beta p\left(\frac{\xi}{|\xi|^{\frac{|\beta|+\ell-i}{|\beta|}}}\right) \leq c_\beta |\xi|^{-|\beta|}$$

since \mathbb{S}^{n-1} is compact $\left(\frac{\xi}{|\xi|^{\frac{|\beta|+\ell-i}{|\beta|}}}\right)$ takes values on a sphere, albeit not of radius 1).

This means

$$|\partial^\alpha p(\xi)| = \left| \sum_{\ell \leq k} \sum_{|\alpha|=\ell} \partial^\alpha p_\ell(\xi) \right| \leq \sum_{\ell \leq k} \sum_{|\alpha|=\ell} |\partial^\alpha p_\ell(\xi)| \leq c_\alpha |\xi|^{-|\alpha|}$$

Finally,

$$\partial^\alpha p(\xi) = \partial^\alpha \left(\frac{1}{m(\xi)} \right) = -\frac{\partial^\alpha m(\xi)}{m(\xi)^2} \iff$$

$$|\partial^\alpha m(\xi)| = |\partial^\alpha p(\xi)| \cdot |m(\xi)|^2 \leq c_\alpha |\xi|^{-|\alpha|} \cdot C \leq c_\alpha |\xi|^{-|\alpha|}$$

since m is bounded. Now, by Hörmander-Mikhlin, it follows that

$$\|\partial_{j_1} \cdots \partial_{j_i} f\|_{L^p} \leq C \|Lf\|_{L^p}$$

for every $0 \leq i \leq k$, as required. □

Exercise 3.

Proof. □