Class 3

Wednesday, October 7, 2020 3:05 PM

We defined the convolution of two $L^1(\mathbb{R}^n)$ functions.

$$f * g(x) = \int f(x-y)\, g(y)\, dy$$

If $f \in L^p$, $g \in L^{p'}$

$$\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

then $f * g(x)$ makes sense for <u>every</u> x

because $y \mapsto f(x-y)$ is L^p

$\Rightarrow$ (Hölder)

$$|f*g|(x) \le \underbrace{\|f(x-\cdot)\|_{L^p}}_{= \|f\|_{L^p}} \|g\|_{L^{p'}}$$

$\Rightarrow f * g \in L^\infty$.

It is always C^0

(if $p < \infty$ you can approx. f by functions f_j in C^0_c

f with function

$f_j * g \xrightarrow[L^\infty]{} f * g$ because

$\|f*g - f_j * g\|_{L^\infty} = \|(f-f_j)*g\|_{L^\infty}$
$\leq \|f-f_j\|_{L^p} \|g\|_{L^{p'}} \to 0$).

$L^p * L^{p'}$ ✓

$L^p * L^1$ can also be done.

<u>Prop</u> If $f \in L^p$, $g \in L^1(\mathbb{R}^n)$ then $\int |f|(x-y)|g|(y)\, dy < \infty$ for a.e. x and $f*g \in L^p$ with $\|f*g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$.

<u>pf</u> We argue by duality. Assume first f, g are bounded & $f, g = 0$ outside some closed ball $\overline{B}_R$.

$L^q \quad \forall q \in [1, \infty]$

$|f| * |g| \in L \quad \forall q$

because it's supported on the bounded set $\overline{B}_{2R}$

and $\| |f| * |g| \|_{L^\infty} \leq \|f\|_{L^1} \|g\|_{L^\infty} < \infty$.

Now if $h \in L^{p'}$, $\|h\|_{L^{p'}} \leq 1$

$$\int (|f| * |g|)\, h(x)\, dx$$

$$= \int \int |f|(x-y)\, |g|(y)\, h(x)\, dy\, dx$$

$$\underset{\text{(Fubini)}}{=} \int \left[\int |f|(x-y)\, h(x)\, dx \right] |g|(y)\, dy$$

$$\underset{\text{(Hölder)}}{\leq} \int \|f\|_{L^p} \|\cancel{h}\|_{L^{p'}} |g|$$

$$\leq \|f\|_{L^p} \|g\|_{L^1}.$$

<u>Fact</u> If $F \in L^p$ then

$$\sup_{\substack{h \in L^{p'} \\ \|h\|_{L^{p'}} \leq 1}} \int F h = \|F\|_{L^p}.$$

Applying this with $F := |f| * |g|$

$\Longrightarrow \| |f| * |g| \|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$.

$\Longrightarrow$ It also holds for any $f \in L^p$, $g \in L^1$ (approximate "from below"), by monotone convergence. $\square$

<u>Rmk</u> We defined $*$ for $L^p * L^1$, $L^p * L^{p'}$

and L^q is an "interpolation" between L^1 and $L^{p'}$.

We expect that if $g \in L^q$ $1 \leq q \leq p'$ and $f \in L^p$ $f * g$ is defined.

<u>True</u> because we can split

$$g = \underset{\cap}{g_1} + \underset{\cap}{g_2}$$

$L^{p'}$ L^1

$$g_1 := g \mathbb{1}_{\{|g| \le 1\}}$$

$$g_2 := g \mathbb{1}_{\{|g| > 1\}}$$

$$\int |g_2| = \int_{\{|g|>1\}} |g|$$

$$\le \int_{\{|g|>1\}} |g|^q < \infty$$

$g_1 \in L^\infty \implies$ if $p' = \infty$ then $g_1 \in L^{p'}$

otherwise $\int |g_1|^{p'} \le \int_{\{|g|\le 1\}} |g|^{q}$

$$\le \int |g|^q < \infty.$$

$$\implies |f| * |g|(x) \le |f| * |g_1|(x) + |f| * |g_2|(x)$$

$\implies$ LHS $< \infty$ for a.e. x.

Then $1 \le q \le p'$

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}$$

with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

(Proof: maybe later)

More stuff on convolution in problem sheet.

Maximal functions

<u>def</u> $f : \mathbb{R}^n \to \mathbb{C}$ measurable.

The Hardy–Littlewood maximal function of f is

$$Mf(x) := \sup_{r>0} ⨍_{B_r(x)} |f|.$$

$$= \sup_{r>0} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} |f|.$$

<u>Rmk</u> $\forall r \quad x \mapsto ⨍_{B_r(x)} |f|$

... if $f \in L^1_{loc}$

is continuous ...

$\Rightarrow Mf$ is Borel (lower semicontinuous)

because $\{Mf > \lambda\}$

$= \bigcup_{r>0} \underbrace{\left\{x : \fint_{B_r(x)} |f| > \lambda\right\}}_{\text{open}}$

$\Rightarrow \{Mf > \lambda\}$ is open.

<u>Rmk</u> If $f \in L^\infty$

then $\left|\fint_{B_r(x)} f\right| \leq \|f\|_{L^\infty}$

$\forall x \; \forall r > 0$

$\Rightarrow \|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}$.

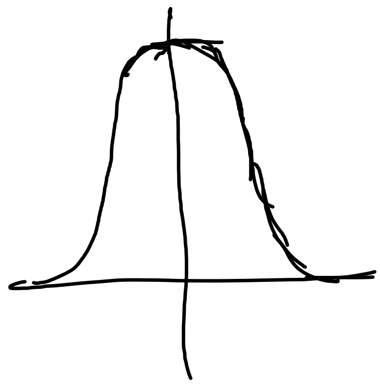
Some motivation:

$\varphi \geq 0$, $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$\int \varphi = 1$, φ radial

(radial = "$\varphi(x) = \Phi(|x|)$" for some Φ)

Φ decreasing.



$$\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right)$$

has $\int \varphi_\varepsilon = 1$ and $\operatorname{spt}(\varphi_\varepsilon) = \varepsilon \operatorname{spt}(\varphi)$.

<u>Prop</u> If $f \in L^1$ then $\varphi_\varepsilon * f \to f$ in L^1.

(φ_ε) is a "family of mollifiers" because $\varphi_\varepsilon * f \in C^\infty$.

<u>pf</u> Exercise
(it's trivial if $f \in C^0_c$
since in this case
$\varphi_\varepsilon * f \to f$ uniformly;
[illegible] $f \in L^1$ by approximation:

follows $\forall$ ... by ...

say $f = \lim_{j\to\infty} f_j$ in L^1,

$f_j \in C_c^0$;

$$
\begin{aligned}
\|\varphi_\varepsilon * f - f\|_{L^1}
&\le \|\varphi_\varepsilon * f - \varphi_\varepsilon * f_j\|_{L^1} \\
&\quad + \|\varphi_\varepsilon * f_j - f_j\|_{L^1} \\
&\quad + \|f_j - f\|_{L^1} \\
&\le \overset{=1}{\|\varphi_\varepsilon\|_{L^1}} \|f - f_j\|_{L^1} \\
&\quad + \|\varphi_\varepsilon * f_j - f_j\|_{L^1} \\
&\quad + \|f_j - f\|_{L^1} \\
&= 2\|f_j - f\|_{L^1} + \|\varphi_\varepsilon * f_j - f_j\|_{L^1}.
\end{aligned}
$$

$$
\Rightarrow \limsup_{\varepsilon\to 0} \|\varphi_\varepsilon * f - f\|_{L^1}
\le 2\|f_j - f\|_{L^1}
+ \limsup \cancel{\|\varphi_\varepsilon * f_j - f_j\|_{L^1}}
$$

$\varepsilon \to 0$

$\Rightarrow \limsup \dots \le 2\|f - f_j\|_{L^1}$

$\Rightarrow$ (let $j \to \infty$) $\limsup = 0$.)

So given a sequence $\varepsilon_k \to 0$

$\varphi_{\varepsilon_k} * f \to f$ in L^1.

$\Rightarrow \exists$ subsequence ε_k'
s.t. $\varphi_{\varepsilon_k'} * f \to f$ a.e.

Question Do we really need a subsequence?

Is $\varphi_\varepsilon * f \to f$ a.e.?

Again argue by approximation.

$f_j \to f$ in L^1 & pointwise a.e.

$\cap$

C_c^0

As before,

$$|\varphi_\varepsilon * f - f|(x)$$
$$\le |\varphi_\varepsilon * (f - f_j)|(x)$$
$$+ |f - f_j|(x)$$
$$+ |\varphi_\varepsilon * f_j - f_j|(x) \to 0$$

(because f_j is C^0)

2^{nd} term goes to 0 when $j \to \infty$.

$$g_j := f - f_j$$

$$\varphi_\varepsilon * g_j(x) = \int \varphi_\varepsilon(y)\, g_j(x-y)\, dy$$
$$= \int \left(\int_0^{\varphi_\varepsilon(y)} 1 \, dt \right) g_j(x-y)\, dy$$
$$= \int_0^\infty \left[\int_{\{y : \varphi_\varepsilon(y) > t\}} g_j(x-y)\, dy \right] dt$$

$$\{y : \varphi_\varepsilon(y) > t\} = B_{r(t)}$$

$$\left| \int g_j(x-y)\, dy \right|$$

$$\leq \int_{B_{r(t)}(x)} |g_j| \leq \mathcal{L}^n(B_{r(t)}) Mg_j(x)$$

$$\leq \int_0^\infty \mathcal{L}^n(\{\varphi_\varepsilon > t\})\, Mg_j(x)$$

$$= Mg_j(x) \int \varphi_\varepsilon$$

$$= Mg_j(x).$$

So $|\varphi_\varepsilon * (f - f_j)|(x)$

$$\leq M(f - f_j)(x).$$

<u>def</u> Given $f : \mathbb{R}^n \to \mathbb{C}$

$$\|f\|_{L^{1,\infty}} := \sup_{\lambda > 0} \lambda \mathcal{L}^n\{|f| > \lambda\}.$$

<u>Rmk</u> $\|f\|_{L^{1,\infty}} \leq \|f\|_{L^1}$

because $\lambda \mathcal{L}^n\{|f| > \lambda\}$

$$\int |f| \leq \|f\|_{L^1}.$$

$$\leq \int_{\{|f|>\lambda\}} |f| = \|\cdot\|_{L}$$

$$\Rightarrow L^{1,\infty} := \{ f : \|f\|_{L^{1,\infty}} < \infty \}$$

$$\supseteq L^1.$$

Rmk $\| \ \|_{L^{1,\infty}}$ is not a norm;

$$\|f+g\|_{L^{1,\infty}} \not\leq \|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}}.$$

It is a quasi norm.

Fact $\nexists \ \| \ \|'$ norm

s.t. $\frac{\|f\|'}{C} \leq \|f\|_{L^{1,\infty}} \leq C\|f\|'$

for some C.

"$\| \ \|_{L^{1,\infty}}$ is not comparable with any norm"!

[Because of this fact Calderón – Zygmund theory is not so easy.]

<u>Thm</u> $\|Mf\|_{L^{1,\infty}} \leq 3^n \|f\|_{L^1}$.

<u>pf of $\varphi_\varepsilon * f \to f$</u>

$$\left\{x : \limsup_{\varepsilon \to 0} |\varphi_\varepsilon * f - f|(x) > \lambda\right\} = S_\lambda$$

$$\limsup \cdots \leq M(f - f_j)(x) + |f - f_j|(x)$$

$$\Rightarrow S_\lambda \subseteq \left\{M(f - f_j) > \frac{\lambda}{2}\right\} \cup \left\{|f - f_j| > \frac{\lambda}{2}\right\}$$

$$\Rightarrow \mathcal{L}^n(S_\lambda) \leq \frac{2}{\lambda} \|M(f - f_j)\|_{L^{1,\infty}}$$

def. of $\mathcal{L}$ ...

$$+ \frac{2}{\lambda} \|f - f_j\|_{L^1}$$

(Thm)
$$\leq \frac{2}{\lambda} 3^n \|f - f_j\|_{L^1} + \frac{2}{\lambda} \|f - f_j\|_{L^1}$$

$$\Rightarrow (j \to \infty) \quad \mathcal{L}^n(S_\lambda) = 0.$$

$\Rightarrow$ limsup ... $= 0$ for a.e. x.

$\square$

<u>Lemma</u> (Vitali's covering lemma)

Assume $K \subseteq B_{r_1}(x_1) \cup \dots \cup B_{r_N}(x_N)$

then $\exists J \subseteq \{1, \dots, n\}$

s.t. $\{B_{r_j}(x_j)\}_{j \in J}$ are disjoint

& $K \subseteq \bigcup B_{3r_j}(x_j)$.

a $R = \ j\in J \ \ldots$

pf W.l.o.g $r_1 \geq r_2 \geq \cdots \geq r_N$.

Put $B_{r_1}(x_1)$ in your subcollection.

Choose $k :=$ min. index > 1

s.t. $B_{r_k}(x_k) \cap B_{r_1}(x_1) = \phi$
and put it in your bag.

Choose $l :=$ min. index $> k$

s.t. $B_{r_l}(x_l) \cap \left(B_{r_1}(x_1) \cup B_{r_k}(x_k) \right) = \phi$

and put it in the bag.

- - - .

$J := \{$ selected indices $\}$.

Of course $\{B_{r_j}(x_j)\}$ is a

disjoint collection.

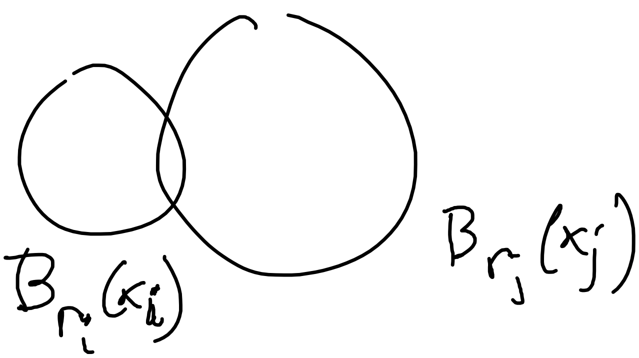
If $x \in K$ then $\exists B_{r_i}(x_i) \ni x$.

either $i \in J \Rightarrow x \in \bigcup_{j \in J} B_{3r_j}(x_j)$

or $i \notin J$

$\Rightarrow B_{r_i}(x_i) \cap B_{r_j}(x_j) \neq \emptyset$

for some $j < i$.



$\Rightarrow$ since $r_i \leq r_j$

$B_{r_i}(x_i) \subseteq B_{3r_j}(x_j)$. $\square$

<u>pf of Thm</u> Fix K compact,

$K \subseteq \{ Mf > \lambda \}$.

For all $x \in K$ $\sup_r \fint_{B_r(x)} |f| > \lambda$

$\Rightarrow \exists B_{r(x)}(x)$ s.t.

$$\int_{B_{r(x)}(x)} |f| > \lambda \mathcal{L}^n(B_{r(x)}(x))$$

$\Rightarrow$ (K compact)

$$K \subseteq \bigcup_{i=1}^{N} B_{r_i}(x_i).$$

Use Lemma:

$$\mathcal{L}^n(K) \leq \sum_{j \in J} \mathcal{L}^n(B_{3r_j}(x_j))$$

$$= 3^n \sum_{j \in J} \mathcal{L}^n(B_{r_j}(x_j))$$

$$\underset{\text{(hypothesis)}}{\leq} 3^n \sum_{j \in J} \frac{1}{\lambda} \int_{B_{r_j}(x_j)} |f|$$

$$\leq 3^n \frac{1}{\lambda} \int_{\mathbb{R}^n} |f|$$

(balls are disjoint)

$$\Rightarrow \lambda \mathcal{L}^n(K) \leq 3^n \|f\|_{L^1}$$

$$\Rightarrow \lambda \mathcal{L}^n(\{Mf > \lambda\}) \leq 3^n \|f\|_{L^1}. \quad \square$$

Actually,

$$\|Mf\|_{L^p} \leq C(p,n) \|f\|_{L^p}$$

$1 < p \leq \infty$.

pf "Marcinkiewicz interpolation"

$$\int |Mf|^p(x)\, dx = \int \int_0^{Mf(x)} p t^{p-1}\, dt\, dx$$

$$\int^\infty p t^{p-1} \mathcal{L}^n\{Mf > t\}\, dt$$

$= \int_0$

Split $f = \underset{\underset{L^\infty}{\Uparrow}}{f_{low}} + \underset{\underset{L^1}{\Uparrow}}{f_{high}}$

$$f_{low} := f \, \mathbb{1}_{\{|f| \leq t/4\}}$$

$$f_{high} := f \, \mathbb{1}_{\{|f| > t/4\}}$$

Observe that

$$Mf \leq Mf_{low} + Mf_{high}$$

$$\Rightarrow \{Mf > t\}$$

$$\subseteq \{Mf_{low} > t/2\} \cup \{Mf_{high} > t/2\}$$

Since $\|Mf_{low}\|_{L^\infty} \leq \frac{t}{4}$

"trivial" case $p = \infty$

$$\{Mf_{low} > t/2\} = \phi$$

$$\Rightarrow \{Mf > t\} \subseteq \{Mf_{high} > t/2\}$$

$$\Rightarrow \mathcal{L}^n(1^{st} \ldots) \leq \mathcal{L}^n(2^{nd} \ldots)$$

$$\leq \frac{2}{t} 3^n \|f_{high}\|_{L^1}$$

$$\int |Mf|^p$$

$$\leq \int p t^{p-1} \frac{2}{t} 3^n \left(\int_{\{|f| > t/4\}} |f| \, dx \right) dt$$

$$= \int 2p \cdot 3^n |f| \int_0^{4|f|(x)} t^{p-2} \, dt \, dx$$

$$= \underline{2p \cdot 3^n \cdot 4^{p-1}} \int |f|^p \, dx$$

$p-1$

$$\Rightarrow \|Mf\|_{L^p} \le \underbrace{\left(\frac{2p \cdot 3^n \cdot 4^{p-1}}{p-1}\right)^{1/p}}_{C(p,n)} \|f\|_{L^p}.$$

$\square$