Class 6

Wednesday, October 28, 2020 2:59 PM

A "multiplier" version of Calderón-Zygmund estimates:

$$\|\mathcal{F}^{-1}(m\hat{f})\|_{L^p} \leq ?$$

(m = rational function of ξ, e.g.)

<u>Thm</u> (Hörmander-Mikhlin multiplier theorem)

Assume $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is <u>smooth</u> and

$$\forall \alpha \in \mathbb{N}^n \quad |\partial^\alpha m|(\xi) \leq C_\alpha |\xi|^{-|\alpha|}$$

$(|\alpha| = \alpha_1 + \dots + \alpha_n)$

or also for $|\alpha| \leq n+2$.

Then

$$\|\mathcal{F}^{-1}(m\hat{f})\|_{L^p} \le C \|f\|_{L^p}$$

$\forall\, 1<p<\infty$, $f \in \mathcal{S}(\mathbb{R}^n)$.

Rmk $\hat{f} \in \mathcal{S}$ and $m \in L^\infty$

$\Rightarrow m\hat{f}$ makes sense and is L^1 (or L^2).

Rmk It follows that $T(f) := \mathcal{F}^{-1}(m\hat{f})$ extends uniquely to a continuous linear operator $L^p \to L^p$.

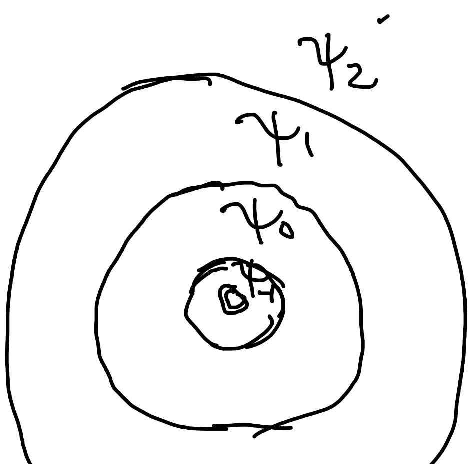
Proof Heuristically, take $K := \check{m}$ and show $f \mapsto K * f$ is bounded from L^p to L^p.

But K is only an element of $\mathcal{S}'$ in general.

We first construct a partition of unity $(\psi_\ell)_{\ell \in \mathbb{Z}}$ s.t.

- $\psi_\ell \in C_c^\infty$
- $\mathrm{spt}(\psi_\ell) \subseteq B_{2^{\ell+1}} \setminus B_{2^{\ell-1}}$
- $\psi_\ell(\xi) = \psi_0(\xi/2^\ell)$
- $\sum_{\ell \in \mathbb{Z}} \psi_\ell(\xi) = 1 \quad \forall \xi \neq 0.$

Morally $\psi_\ell \approx 1_{B_{2^{\ell+1}} \setminus B_{2^\ell}}$.



$\left(\mathrm{spt}(\psi_j) \cap \mathrm{spt}(\psi_\ell) = \emptyset \text{ unless } |j-\ell| \leq 1. \right)$

Then we will take

$$m_M := \sum_{\ell=-M}^{M} m \psi_\ell \in C_c^\infty$$

and $K_M := \mathcal{F}^{-1}(m_M)$.

We will show $K_M \in \mathcal{S}$ satisfies the version of C-Z from last time

$\Rightarrow \|\mathcal{F}^{-1}(m_M \hat{f})\|_{L^p} \le C\|f\|_{L^p}$

$\Rightarrow$ send $M \to \infty$.

Details:

we first construct (ψ_ℓ).

For instance take $\varphi \in C_c^\infty$ radial decreasing (and nonnegative) s.t.

$\varphi = 0$ outside $B_{2-\varepsilon}$,

$\varphi = 1$ inside $B_{1+\varepsilon}$ $\left(\varepsilon = \frac{1}{10}\right)$

Take $\psi_0(\xi) := \varphi(\xi) - \varphi(2\xi)$.

$\mathrm{spt}(\psi_0) \subseteq B_2$ and for ξ near $\overline{B}_{1/2}$, $\psi_0(\xi) = 1 - 1 = 0$

$\Rightarrow \operatorname{spt}(\psi_0) \subseteq B_2 \setminus \bar{B}_{1/2}$.

So $\psi_\ell(\xi) := \psi_0(\xi/2^\ell)$

has $\operatorname{spt}(\psi_\ell) \subseteq B_{2^{\ell+1}} \setminus B_{2^{\ell-1}}$.

Also, for $\xi \neq 0$,

$$\sum_{\ell=-\infty}^{\infty} \psi_\ell(\xi) = \sum_{\ell=-\infty}^{\infty} \left[\varphi(\xi/2^\ell) - \varphi(\xi/2^{\ell-1})\right]$$

sum is actually finite

$$= \sum_{\ell=a}^{b} \left[\varphi(\xi/2^\ell) - \varphi(\xi/2^{\ell-1})\right]$$

$$= \varphi(\xi/2^b) - \varphi(\xi/2^{a-1})$$

$$= 1 - 0 = 1.$$

<u>Rmk</u> $|\partial^\alpha \psi_\ell(\xi)|$

$$= |\partial^\alpha \psi_0(\xi/2^\ell)| 2^{-|\alpha|\ell}$$

$$\leq C'_\alpha 2^{-|\alpha|\ell}$$

(since $|\xi| \approx 2^\ell$ on $\operatorname{spt}(\psi_\ell)$)

$$\leq C''_\alpha |\xi|^{-|\alpha|}.$$

So by Leibniz

$$|\partial^\alpha (\psi_\ell m)|$$

$$= \left| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta m \, \partial^{\alpha-\beta} \psi_\ell \right|$$

(where $\beta \le \alpha$ means $\beta_i \le \alpha_i$

and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$)

$$\le C \sum |\xi|^{-|\beta|} |\xi|^{-|\alpha-\beta|}$$

$$\le C |\xi|^{-|\alpha|}.$$

So $\psi_\ell m$ has same decay assumptions as m.

Now we check that

$$K_M := \mathcal{F}^{-1}\left(\sum_{-M}^{M} \psi_\ell m\right)$$

satisfies

- $|\hat{K}_M| \le C$
- $|DK_M(x)| \le C / |x|^{n+1}$

with C <u>independent of M</u>.

$$|\hat{K}_M| = \left| \sum_{-M}^{M} \psi_\ell m \right| \le |m|$$

$\Rightarrow$ 1^{st} requirement is OK.

We show $|K_M(x)| \leq C|x|^{-n}$

for simplicity.

(Estimate for DK_M is similar using $\mathcal{F}(DK_M) = 2\pi i \xi m_M$.)

We look at $x^\alpha K_M$

with $|\alpha| = n$ and we want

to show $x^\alpha K_M \in L^\infty$

with $\|x^\alpha K_M\|_{L^\infty} \leq C$

$(\Rightarrow |K_M(x)| \leq C' |x|^{-n})$.

Since $F(x) = \int \hat{F}(\xi) e^{2\pi i \xi x}$,

$\|F\|_{L^\infty} \leq \|\hat{F}\|_{L^1}$.

Now $\mathcal{F}((2\pi i x)^\alpha K_M) =$

$= \partial^\alpha \hat{K}_M$

$= \sum_{l=-M}^{M} \partial^\alpha (\psi_l m)$.

So $\|\mathcal{F}(\cdots)\|_{L^1}$

$$\leq \sum_{\ell=-M}^{M} \|\partial^{\alpha}(\psi_\ell m)\|_{L^1}$$

$$\leq C \sum_{-M}^{M} (2^{\ell})^{n} \, 2^{-|\alpha|\ell}$$

from above

$$\leq C\,(2M+1).$$

So we are stuck!

Idea: generalize this.

Look at general α:

$$\|\mathcal{F}(x^{\alpha}\,\mathcal{F}^{-1}(\psi_\ell m))\|_{L^1}$$

$$= \frac{1}{(2\pi)^{|\alpha|}} \|\partial^{\alpha}(\psi_\ell m)\|_{L^1}$$

$$\leq C\, 2^{\ell n}\, 2^{-\ell|\alpha|}$$

$$= C\, 2^{\ell(n-|\alpha|)}.$$

$$\Rightarrow \|x^{\alpha}\mathcal{F}^{-1}(\psi_\ell m)\|_{L^\infty} \leq C\, 2^{\ell(n-|\alpha|)}$$

... have (combining all $|\alpha|=k$) ... $\ell(n-k)$

So we have ... $\ell (n$,

$$(*) \quad \left| \mathcal{F}^{-1}(\psi_\ell m) \right| \le C \frac{2}{|x|^k} .$$

[We are using:

$$|x^\alpha F(x)| \le C \quad \forall |\alpha| = k$$

$$\Longrightarrow |F(x)| \le C' |x|^{-k}]$$

<u>Idea</u>: Use $k = n+1$ for $\ell >> 0$,

$k = n-1$ for $\ell << 0$.

$(*)$ gives

$$\left| \mathcal{F}^{-1}(\psi_\ell m) \right| \le C \min \left\{ \frac{2^{-\ell}}{|x|^{n+1}}, \frac{2^{\ell}}{|x|^{n-1}} \right\}.$$

Look for $\ell_0 \in \mathbb{Z}$ s.t.

$$\frac{2^{-\ell_0}}{|x|^{n+1}} \approx \frac{2^{\ell_0}}{|x|^{n-1}} \Longleftrightarrow 2^{\ell_0} \approx |x|^{-1}$$

$$\Longrightarrow \sum_{\ell=-M}^{M} \left| \mathcal{F}^{-1}(\psi_\ell m)(x) \right|$$

$2^{-\ell}$

$$\leq C \sum_{l \leq l_0} \frac{2^{l}}{|x|^{n-1}} + C \sum_{l > l_0} \frac{2^{-l}}{|x|^{n+1}}$$

$$= C \frac{2 \cdot 2^{l_0}}{|x|^{n-1}} + C \frac{2^{-l_0}}{|x|^{n+1}}$$

$$\left(2^{l_0} \approx |x|^{-1}\right)$$

$$\leq C |x|^{-n}.$$

So we know

$$\| \mathcal{F}^{-1}(m_M \hat{f}) \|_{L^p}$$

$$= \| K_M * f \|_{L^p}$$

$$\leq C \| f \|_{L^p}.$$

Now $m_M \hat{f} \to m \hat{f}$ in L^2

(by dominated convergence)

$\Longrightarrow$ (Plancherel) $\mathcal{F}^{-1}(m_M \hat{f})$

$$\downarrow$$

$\mathcal{F}^{-1}(m \hat{f})$

in L^2

a.e.

So for fixed f we have

$\mathcal{F}^{-1}(m_{M_j}\hat{f}) \to \mathcal{F}^{-1}(m\hat{f})$ a.e.

$\Longrightarrow$ by Fatou

$\|\mathcal{F}^{-1}(m\hat{f})\|_{L^p}$

$\leq \liminf\limits_{j\to\infty} \|\mathcal{F}^{-1}(m_{M_j}\hat{f})\|_{L^p}$

$\leq C\|f\|_{L^p}$. $\square$

<u>Corollary</u> If L differential operator, $L = \sum\limits_{|\alpha|=k} c_\alpha \partial^\alpha$,

s.t. $\sum c_\alpha \xi^\alpha$ vanishes only

at 0, then

$\|D^k f\|_{L^p} \leq C\|Lf\|_{L^p}$

$\forall f \in \mathcal{S}$ $\forall 1<p<\infty$.

Proof $\partial_{j_1}\cdots\partial_{j_k} f = \mathcal{F}^{-1}\left(\frac{\xi_{j_1}\cdots\xi_{j_k}}{\sum \xi^\alpha}\widehat{Lf}\right)$

$$\ldots \partial_{j_1} \ldots \partial_{j_k} \ldots \underbrace{\le C_\alpha}_{m(\xi)}$$

Since denominator is $\neq 0$ for $\xi \neq 0$, $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

Also, $m(\lambda \xi) = m(\xi)$

("m is 0-homogeneous").

So by chain rule

$$\partial^\alpha m(\xi) = \partial^\alpha (m(\lambda \xi))$$
$$= \partial^\alpha m(\lambda \xi) \cdot \lambda^{|\alpha|}$$

("$\partial^\alpha m$ is $(-|\alpha|)$-homog.")

Choose $\lambda = |\xi|^{-1}$

$$\Rightarrow \partial^\alpha m(\xi) = \underbrace{\partial^\alpha m\left(\frac{\xi}{|\xi|}\right)}_{|\,| \le C_\alpha} |\xi|^{-|\alpha|}. \quad \square$$

<u>Examples</u> • $L = \Delta = \partial_1^2 + \cdots + \partial_n^2$.

• $L = \partial_{\bar z} = \partial_x + i \partial_y$

is also good.

(One has local versions such as

$$\|Df\|_{L^p(B)} \leq C\|\partial_{\bar z} f\|_{L^p} + C\|f - f_B f\|_{L^p \cdot})$$

Application: convergence of "partial Fourier representations"

(In HW you will deduce from this the analogue for Fourier series...)

In 1D $(n=1)$ we have

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

$$= \lim_{a \to \infty} \int_{-a}^{a} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

For $f \in L^p$, $1 \leq p \leq 2$,

then $f \in L^1 + L^2 \Rightarrow \hat{f} = L^\infty + L^2$

$\Rightarrow$ so $f_a(x) := \int_{-a}^{a} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$

and we wonder if $f_a \to f$ in L^p.

<u>Thm</u> If $1 < p \leq 2$, $f_a \to f$ in L^p as $a \to \infty$.

<u>Proof</u> As usual, we need to show $\|f_a\|_{L^p} \leq C \|f\|_{L^p}$ with C <u>independent of f, a</u>.

[<u>Rmk</u> If T_k is a family of operators such that $T_k f \xrightarrow{L^p} T_\infty f \ \forall f \in \mathcal{S}$ then $\|T_k f\|_{L^p} \leq C \|f\|_{L^p}$ (uniform) implies $T_k f \xrightarrow{L^p} T_\infty f \ \forall f \in L^p$.

lication

And the implication can be somehow reversed, by functional analysis nonsense.

Pointwise a.e. convergence $T_k f \to T_\infty f$ turns out to be equivalent to bounds for $\tilde{T}f(x) := \sup_k |T_k f|(x)$

$\tilde{T}$ "maximal operator".
(pointwise $\Rightarrow$ maximal op. bound was observed by Stein).

In our case we are looking at $\mathcal{F}^{-1}(\mathbb{1}_{(-a,a)}\hat{f}) = f_a$

$$= \mathcal{F}^{-1}(\mathbb{1}_{(-\infty,a)}\hat{f}) - \mathcal{F}^{-1}(\mathbb{1}_{(-\infty,-a)}\hat{f})$$

so it is enough to show

$$\|\mathcal{F}^{-1}(\mathbb{1}_{(-\infty,t)}\hat{f})\|_{L^p} \leq C\|f\|_{L^p}$$

with C indep. of t and f.

For $t=0$ it is "just" an application of Hörmander-Mikhlin $\left(m := \begin{cases} 1 & \text{if } \xi<0 \\ 0 & \text{if } \xi>0 \end{cases}\right)$.

In general,

$$\mathcal{F}^{-1}(\mathbb{1}_{(-\infty,t)}\hat{f})$$

$$= \mathcal{F}^{-1}\left[\tau_t(\mathbb{1}_{(-\infty,0)}\tau_{-t}\hat{f})\right]$$

$$= e^{2\pi i t x}\mathcal{F}^{-1}\left[\mathbb{1}_{(-\infty,0)}\mathcal{F}(e^{2\pi i t x} f)\right]$$

up to wrong signs.

$$\Rightarrow \|\mathcal{F}^{-1}(\mathbb{1}_{(-\infty,t)}\hat{f})\|_{L^p}$$

$$= \|\mathcal{F}^{-1}(\mathbb{1}_{(-\infty,0)}\mathcal{F}(e^{2\pi i t x} f))\|_{L^p}$$

$$\leq C\|e^{2\pi i t x} f\|_{L^p}$$

$$= C \|f\|_{L^p}. \qquad \square$$

<u>Rmk</u> Hilbert transform is what you get in 1D using $m(\xi) = -i \operatorname{sgn}(\xi)$

$$= \begin{cases} i & \text{if } \xi < 0 \\ -i & \text{if } \xi > 0. \end{cases}$$

$$\check{m} = \text{"}\frac{1}{\pi x}\text{"}$$

$$= \frac{1}{\pi}\, \text{p.v.}$$