Homework 2

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Exercise 1.

Proof. (i) Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi = \partial^\alpha \phi$ for some multi-index α. Then $\|\psi\|_{a,\beta} = \|x^\beta \partial^a \psi\| = \|x^\beta \partial^{\alpha+a} \phi\| = \|\phi\|_{\alpha+a,\beta} < \infty$. So that $\psi = \partial^\alpha \phi \in \mathcal{S}(\mathbb{R}^n)$. Now consider the neighborhood $\|\phi\|_{\alpha,\beta} < \epsilon$. We have $\epsilon > \|\phi\|_{\alpha,\beta} = \|x^\beta \partial^\alpha \phi\| = \|\partial^\alpha \phi\|_{0,\beta}$. So that ∂^α is a continuous operator.
(ii)

$$\|\tau_a \phi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x-a)| = \sup_{x \in \mathbb{R}^n} |(x+a)^\beta \partial^\alpha \phi(x)| \leq \sum_{k \leq \beta} \binom{\beta}{k} \sup_{x \in \mathbb{R}^n} |x^k a^{\beta-k} \partial^\alpha \phi(x)|$$

$$= \sum_{k \leq \beta} \binom{\beta}{k} |a|^{\beta-k} \|\phi\|_{\alpha,k} < \infty$$

This shows simultaneously that $\tau_a \phi \in \mathcal{S}(\mathbb{R}^n)$ and that τ_a is a continuous operator.
(iii) We use the equivalent, alternative definition of the Schwartz space here. By the generalized Leibniz rule (for multi-indices), we have –

$$|\partial^\alpha (h\phi)(x)| = |\sum_{k \leq \alpha} \binom{\alpha}{k} \partial^k h(x) \cdot \partial^{\alpha-k} \phi(x)| \leq \sum_{k \leq \alpha} [C_k (1+|x|)^{N_k} + C_{M_k,\alpha-k}(1+|x|)^{-M_k}] \leq$$

$$\sum_{k \leq \alpha} \max\{C_k, C_{M_k,\alpha-k}\}(1+|x|)^{-M_k+N_k} = \sum_{k \leq \alpha} A_k \|\phi\|_{\alpha,(-M_k+N_k)} < \infty$$

where M_k can be chosen sufficiently large. This shows simultaneously that $h\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \mapsto h\phi$ is a continuous operator. □

Exercise 2.

Proof. (i) We want $\partial^\alpha T_f = T_{\partial^\alpha f}$. By integration by parts, we have

$$\langle \partial^\alpha T_f, \phi \rangle = \langle T_{\partial^\alpha f}, \phi \rangle = \int \partial^\alpha f \cdot \phi = -1^{|\alpha|} \int f \cdot \partial^\alpha \phi = -1^{|\alpha|} \langle T_f, \partial^\alpha \phi \rangle$$

We want $\tau_a T_f = T_{\tau_a f}$. Thus,

$$\langle \tau_a T_f, \phi \rangle = \langle T_{\tau_a f}, \phi \rangle = \int_{\mathbb{R}^n} \tau_a f(x) \cdot \phi(x) = \int_{\mathbb{R}^n} f(x-a) \cdot \phi(x) = \int_{\mathbb{R}^n} f(x)\phi(x+a) =$$

$$\langle T_f, \tau_{-a}\phi\rangle$$

We want $hT_f = T_{hf}$. Thus,

$$\langle hT_f, \phi\rangle = \langle T_{hf}, \phi\rangle = \int (hf)\phi = \int f(h\phi) = \langle T_f, h\phi\rangle$$

where the condition on h, $|h(x)| \le C_0(1+|x|)^{N_0}$, is used to guarantee $\int hf\phi$
$\le C\int \dfrac{1}{(1+|x|)^M} < \infty$ for some C, and some integer $M \ge 2$, since $f, \phi \in \mathcal{S}(\mathbb{R}^n)$, and so we can choose constants C_f, C_ϕ so that $|f(x)| \le C_f(1+|x|)^{M_f}$ and $|\phi(x)| \le C_\phi(1+|x|)^{M_\phi}$ for some very large M_f, M_ϕ. This ensures that $hT_f = T_{hf}$ is a well defined distribution.
We want $\mathcal{F}^{-1}(T_f) = T_{\mathcal{F}^{-1}(f)}$. Thus,

$$\langle \mathcal{F}^{-1}(T_f), \phi\rangle = \langle T_{\mathcal{F}^{-1}(f)}, \phi\rangle = \int \mathcal{F}^{-1}(f)\cdot\phi = \int \mathcal{F}^{-1}(f)(x)\phi(x) = \int\left(\int f(\xi)e^{2\pi i x\xi}d\xi\right)\phi(x)dx =$$

$$\int\left(\int \phi(x)e^{2\pi i x\xi}dx\right)f(\xi)d\xi = \int f\cdot\mathcal{F}^{-1}(\phi) = \langle T_f, \mathcal{F}^{-1}(\phi)\rangle$$

(ii)

$$\langle \mathcal{F}^{-1}(\mathcal{F}(T)), \phi\rangle = \langle \mathcal{F}(T), \mathcal{F}^{-1}(\phi)\rangle = \langle T_f, \mathcal{F}(\mathcal{F}^{-1}(\phi))\rangle = \langle T_f, \phi\rangle$$

The last equality we proved in class $\mathcal{F}(\mathcal{F}^{-1}(\phi)) = \phi = \mathcal{F}^{-1}(\mathcal{F}(\phi))$. Since this holds for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{F}^{-1}(\mathcal{F}(T)) = T$. Likewise, $\mathcal{F}(\mathcal{F}^{-1}(T)) = T$.

$$\langle \partial_j\mathcal{F}(T), \phi\rangle = -\langle \mathcal{F}(T), \partial_j\phi\rangle = -\langle T, \mathcal{F}(\partial_j\phi)\rangle = -\langle T, 2\pi i x_j\mathcal{F}(\phi)\rangle =$$
$$\langle -2\pi i x_j T, \mathcal{F}(\phi)\rangle = \langle \mathcal{F}(-2\pi i x_j T), \phi\rangle$$

So that $\partial_j\mathcal{F}(T) = \mathcal{F}(-2\pi i x_j T)$.

$$\langle \mathcal{F}(\partial_j T), \phi\rangle = \langle \partial_j T, \mathcal{F}(\phi)\rangle = -\langle T, \partial_j\mathcal{F}(\phi)\rangle = -\langle T, \mathcal{F}(-2\pi i\xi_j\phi)\rangle = -\langle \mathcal{F}(T), -2\pi i\xi_j\phi\rangle = \langle 2\pi i\xi_j\mathcal{F}(T), \phi\rangle$$

So that $\mathcal{F}(\partial_j T) = 2\pi i\xi_j\mathcal{F}(T)$.

$$\langle \mathcal{F}(\tau_a T), \phi\rangle = \langle \tau_a T, \mathcal{F}(\phi)\rangle = \langle T, \tau_{-a}\mathcal{F}(\phi)\rangle = \int f(x)\int \phi(\xi)e^{-2\pi i\xi(x+a)}d\xi dx =$$

$$e^{-2\pi i\xi a}\int f\hat{\phi} = e^{-2\pi i\xi a}\int \hat{f}\phi = \langle e^{-2\pi i\xi a}\mathcal{F}(T), \phi\rangle$$

So that $\mathcal{F}(\tau_a T) = e^{-2\pi i\xi a}\mathcal{F}(T)$.

$$\langle \mathcal{F}(e^{2\pi i\xi a}T), \phi\rangle = \langle e^{2\pi i\xi a}T, \mathcal{F}(\phi)\rangle = \langle T, e^{2\pi i\xi a}\mathcal{F}(\phi)\rangle = \int f(x)\int \phi(\xi)e^{-2\pi i\xi(x-a)}d\xi =$$

$$\langle T, \mathcal{F}(\tau_a\phi))\rangle = \langle \tau_{-a}\mathcal{F}(T), \phi\rangle$$

So that $\mathcal{F}(e^{2\pi i\xi a}T) = \tau_{-a}\mathcal{F}(T)$. □

Exercise 3.

Proof. (i)

$$|\langle T_f, \phi\rangle| = |\int f\phi| \le \int |f\phi| = \|f\phi\|_{L^1} \le \|f\|_{L^p}\cdot\|\phi\|_{L^q} < \infty$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n) \subseteq L^q$ and where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Hence T_f defines a tempered distribution, indeed.

(ii) Let $f \in L^1$, and let $\hat{f}(x) = \int f(\xi)e^{-2\pi i x\xi}d\xi$. We want to verify $\hat{T}_f = T_{\hat{f}}$. I.e., we want to show

$$\int_{\mathbb{R}^n} f\hat{\phi} = \langle T_f, \hat{\phi}\rangle = \langle \hat{T}_f, \phi\rangle = \int_{\mathbb{R}^n} \hat{f}\phi$$

for all $\phi \in L^1$. Observe, $\hat{f} \in L^\infty$, so,

$$\int_{\mathbb{R}^n} |\hat{f}(x)\phi(x)| \le \|\hat{f}\|_{L^\infty}\int_{\mathbb{R}^n} |\phi(x)| < \infty$$

Therefore, Fubini's theorem applies, and

$$\int_{\mathbb{R}^n} \hat{f}(x)\phi(x) = \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} f(\xi)e^{-2\pi i x\xi}d\xi\Big)\phi(x)dx = \int_{\mathbb{R}^n} f(\xi)\Big(\int_{\mathbb{R}^n}\phi(x)e^{-2\pi i x\xi}dx\Big)d\xi = \int_{\mathbb{R}^n} f(x)\hat{\phi}(x)$$

So that $\hat{T}_f = T_{\hat{f}}$, indeed.

(iii) Let $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ for all $f \in L^2(\mathbb{R}^n)$, let $f, g \in L^2(\mathbb{R}^n)$ and let $f_i \to f, g_j \to g$ in L^2 where $f_i, g_j \in \mathcal{S}(\mathbb{R}^n)$. Then, since the Fourier transform is a unique operator in $\mathcal{S}(\mathbb{R}^n)$, the Plancherel identity holds in $\mathcal{S}(\mathbb{R}^n)$, and the inner product is a continuous operator in any vector space, we have -

$$\langle f, \hat{g}\rangle = \lim_{i,j\to\infty}\langle f_i, \hat{g_j}\rangle = \lim_{i,j\to\infty}\langle \hat{f_i}, g_j\rangle = \langle \hat{f}, g\rangle$$

So that $\hat{T}_f = T_{\hat{f}}$, indeed. □

Exercise 4.

Proof. (i)

$$\|\delta_\lambda\phi(x)\|_{\alpha,\beta} = \|\phi(\frac{x}{\lambda})\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}|x^\beta\partial^\alpha\phi(\frac{x}{\lambda})| = \frac{1}{\lambda^{|\alpha|}}\sup_{x\in\mathbb{R}^n}|x^\beta\partial^\alpha\phi(\frac{x}{\lambda})| =$$

$$\frac{1}{\lambda^{|\alpha|}}\sup_{x\in\mathbb{R}^n}|(\lambda x)^{\beta}\partial^{\alpha}\phi(x)| = \lambda^{|\beta|-|\alpha|}\|\phi\|_{\alpha,\beta}$$

This proves that $\delta_\lambda \in \mathcal{S}$ and that it is a continuous operator.
(ii) Now, we want $\delta_\lambda T_f = T_{\delta_\lambda f}$. Thus,

$$\langle \delta_\lambda T_f, \phi\rangle = \langle T_{\delta_\lambda f}, \phi\rangle = \int \delta_\lambda f(x)\phi(x)dx = \int f(\frac{x}{\lambda})\phi(x)dx = \lambda\int f(x)\phi(\lambda x)dx = \lambda\langle T_f, \delta_{\frac{1}{\lambda}}\phi\rangle$$

(iii)

$$\mathcal{F}(\delta_\lambda \phi) = \mathcal{F}(\phi(\frac{x}{\lambda})) = \int \phi(\xi)e^{-2\pi i\frac{x}{\lambda}\xi}d\xi = e^{\frac{-2\pi i}{\lambda}}\int \phi(\xi)e^{-2\pi i x\xi}d\xi = e^{\frac{-2\pi i}{\lambda}}\mathcal{F}(\phi)$$

For a tempered distribution T_f,

$$\langle \mathcal{F}(\delta_\lambda T_f), \phi\rangle = \langle \delta_\lambda T_f, F(\phi)\rangle = \lambda\langle T_f, \delta_{\frac{1}{\lambda}}F(\phi)\rangle$$

$\square$