Homework 4

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Exercise 1.

Proof. Let $\{Q_j\}$ be the cubes defined in the proof of the Calderón–Zygmund lemma. Let $S = \{x \in [0,1)^n \mid M_4 f(x) > \lambda\}$, where $x_0 = 0$ in the definition

$$M_4 f(x) = \sup_{Q \in \mathfrak{D} \ : \ Q \in [0,1]^n} \fint_Q |f|$$

($\subseteq$) Let $Q_j = [a,b]^n$. Let $\tilde{Q}_j = [a,b)^n$. Then, $Q_j, \tilde{Q}_j$ only differ on a set of measure 0. By construction,

$$\fint_{\tilde{Q}_j} |f| = \fint_{Q_j} |f| > \lambda$$

so that $M_4 f(x) > \lambda$ for all $x \in Q_j$. Therefore, $Q_j \subseteq S$ (for a.e. $x \in Q_j$). Thus, $\cup Q_j \subseteq S$ up to a set of measure 0.
($\supseteq$) Let $x \in S$. Then there exists a dyadic cube $Q \subseteq [0,1)^n$ such that

$$\fint_Q |f| > \lambda$$

So that $Q = Q_j$ for some j. Thus $S \subseteq \cup Q_j$. Therefore, $S = \cup Q_j$ up to a set of measure 0. $\square$

Exercise 2.

Proof. $(a)(i)$ Since $\lambda > 0$ and $\mu\{|f| > \lambda\} \geq 0$, we have $\|f\|_{L^{p,\infty}} \geq 0$.
(ii)

$$\|kf\|_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda \cdot \mu\{|kf| > \lambda\} = \sup_{\lambda > 0} \lambda \cdot \mu\{|f| > \frac{\lambda}{|k|}\} =$$

$$\sup_{|k|\lambda > 0} |k|\lambda \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} = \sup_{|k|\lambda > 0} |k|\lambda \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} = |k| \cdot \sup_{|k|\lambda > 0} \lambda \cdot \mu\{|f| > \lambda\} =$$

$$|k| \cdot \sup_{\lambda > 0} \lambda \cdot \mu\{|f| > \lambda\} = |k| \cdot \|f\|_{L^{p,\infty}}$$

(iii)

$$\|f + g\|_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda \cdot \mu\{|f + g| > \lambda\} \leq \sup_{\lambda > 0} \lambda \cdot (\mu\{|f| > \frac{\lambda}{2}\} + \mu\{|g| > \frac{\lambda}{2}\}) \leq$$

$$\sup_{\lambda>0}\lambda\cdot(\mu\{|f|>\frac{\lambda}{2}\})+\sup_{\lambda>0}\lambda\cdot(\mu\{|g|>\lambda\})=$$

$$\sup_{2\lambda>0}2\lambda\cdot(\mu\{|f|>\lambda\})+\sup_{2\lambda>0}2\lambda\cdot(\mu\{|g|>\lambda\})=$$

$$2\sup_{\lambda>0}\lambda\cdot(\mu\{|f|>\lambda\})+2\sup_{\lambda>0}\lambda\cdot(\mu\{|g|>\lambda\})=2(\|f\|_{L^{p,\infty}}+\|g\|_{L^{p,\infty}})$$

(b)

$$\int_E|f|\geq\int_E\lambda=\lambda\mu(E)$$

So that,

$$\mu(E)^{\frac{1}{p}-1}\int_E|f|\geq\mu(E)^{\frac{1}{p}-1}\cdot\lambda\mu(E)=\lambda\cdot\mu(E)^{\frac{1}{p}}$$

Thus,

$$\|f\|'=\sup_{E:0<\mu(E)<\infty}\mu(E)^{\frac{1}{p}-1}\int_E|f|\geq\sup_{\lambda>0}\lambda\mu\{|f|>\lambda\}^{\frac{1}{p}}=\|f\|_{L^{p,\infty}}$$

(c)

$$\int_E|f(x)|d\mu=\int_E\Big(\int_0^{|f(x)|}d\lambda\Big)du=\int_E\mu\{|f(x)|>\lambda\}d\lambda=\int_0^\infty\mu\{x\in E\mid|f(x)|>\lambda\}d\lambda\leq$$

$$\int_0^{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}\mu(E)d\lambda+\int_{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}^\infty\mu\{|f|>\lambda\}d\lambda le$$

$$\int_0^{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}\mu(E)d\lambda+\int_{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}^\infty\frac{\|f\|_{L^{p,\infty}}}{\lambda^p}d\lambda=$$

$$\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}\mu(E)+\|f\|_{L^{p,\infty}}^p\cdot\Big[\frac{\lambda^{1-p}}{1-p}\Big]_{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}^\infty=$$

$$\mu(E)^{\frac{p-1}{p}}\|f\|_{L^{p,\infty}}\cdot\frac{p}{p-1}\iff$$

$$\mu(E)^{\frac{1}{p}-1}\int_E|f|d\mu\leq\frac{p}{p-1}\|f\|_{L^{p,\infty}}$$

For all Borel sets E. Therefore,

$$\|f\|'=\sup_{E:0<\mu(E)<\infty}\mu(E)^{\frac{1}{p}-1}\int_E|f|d\mu\leq\frac{p}{p-1}\|f\|_{L^{p,\infty}}$$

(d) We have

$$\|f\|_{L^{p,\infty}}\leq\|f\|'\leq\frac{p}{p-1}\|f\|_{L^{p,\infty}}$$

Thus the two norms are equivalent. □

Exercise 3.

Proof. □