Class 2

Wednesday, September 30, 2020 3:19 PM

<u>Key fact</u> If $f \in \mathcal{S}(\mathbb{R}^n)$ (i.e. $f \in C^\infty$, $\partial^\alpha f$ "have fast decay") then $\hat{f} = \mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$!

This will allow us to define $\hat{T} = \mathcal{F}(T)$ also for a tempered distr. T.

"" Fourier transform trades integrability (or decay) for regularity and vice versa ""

$f \in L^1 \rightsquigarrow \hat{f} \in C^0$

$xf \in L^1 \rightsquigarrow \hat{f} \in C^1$

- - -

(It's difficult to characterize the image of L^p under the Fourier transform though.)

Fourier transform,

<u>Recall</u> $\hat{f}(\xi) := \int f(x) e^{-2\pi i \langle \xi, x \rangle} dx$

for $f \in \mathcal{S}(\mathbb{R}^n)$ or $f \in L^1$,

hence $\hat{f} : \mathbb{R}^n \to \mathbb{C}$.

<u>Recall</u> If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$

(or $\Leftrightarrow$ $f : \mathbb{R} \to \mathbb{C}$ is 1-periodic)

$$\hat{f}(n) := \int f(x) e^{-2\pi i n x} dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Note $\hat{f}$ is defined everywhere,

$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$.

By analogy, we expect

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi$$

= "superposition of waves $e^{2\pi i \langle \xi, x \rangle}$"

proof of key fact

$\hat{f} \in C^1$: fix a direction e_j

$$\frac{\hat{f}(\xi+he_j)-\hat{f}(\xi)}{h}$$

$$= \frac{1}{h}\int f(x)\left[e^{-2\pi i\langle \xi+he_j, x\rangle} - e^{-2\pi i\langle \xi, x\rangle}\right]$$

$$\overset{(f.t.c.)}{=} \frac{1}{h}\int f(x)\int_0^h \partial_t\left(e^{-2\pi i\langle \xi+te_j, x\rangle}\right)$$

$$= \int f(x)\left(\frac{1}{h}\int_0^h -2\pi i x_j e^{-2\pi i\cdots} dt\right) dx$$

$$= \int f(x)\, x_j\, (\text{expr.})$$

expr. is bounded by 2π and converges to $e^{-2\pi i\langle \xi, x\rangle}$ $\forall x$

(dominated conv.) $\Longrightarrow$ $\frac{\hat{f}(\xi+he_j)-\hat{f}(\xi)}{h}$

$$\downarrow$$

$$\int f(x)(-2\pi i x_j)\, e^{-2\pi i\langle \xi, x\rangle}\, dx$$

$\Rightarrow \partial_j \hat{f}$ exists and

$= \mathcal{F}(-2\pi i x_j f(x))$

$\cap$

L^1

$\Rightarrow \hat{f} \in C^1.$

Iterating $\Rightarrow \hat{f} \in C^\infty$

and $\partial^\alpha \hat{f} = \mathcal{F}((-2\pi i)^{|\alpha|} x^\alpha f(x))$

$(x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}).$

Let's show $\xi \hat{f}$ is bounded.

$$\mathcal{F}(\partial_j f) = \int \partial_j f(x) e^{-2\pi i \langle \xi, x \rangle} dx$$

$$= \int \partial_j (f e^{-2\pi i \langle \xi, x \rangle}) + 2\pi i \xi_j \int f e^{\cdots}$$

$$= \lim_{R \to \infty} \int_{B_R} \partial_j (\cdots) + 2^{nd} \text{ term}$$

$$= \underbrace{\lim_{R \to \infty} \int_{\partial B_R} \underbrace{\nu_j f(x) e^{-2\pi i \langle \xi, x \rangle}}_{\leq C R^{-n}}}_{} + 2^{nd} \text{ term}$$

$R^{n-1} \cdot C R^{-n}$

$\leq CR \cdot CR$
(in absolute value) $\leq C^2 R^{-1} \downarrow 0$

$= 2\pi i \xi_j \int f(x) e^{-2\pi i \langle \xi, x \rangle} dx$

$= 2\pi i \xi_j \hat{f}(\xi)$

$\Rightarrow$ last guy $= \mathcal{F}(\partial_j f) \in L^\infty$.

$\Rightarrow$ (Iterating) $\xi^\alpha \hat{f}(\xi) \in L^\infty$.

Note that $\partial^\alpha \hat{f} = \mathcal{F}(c_\alpha x^\alpha f)$

and $c_\alpha x^\alpha f \in \mathcal{S}$

$\Rightarrow$ also $\partial^\alpha \hat{f}$ have rapid decay. $\square$

Actually, recalling the seminorms $\|f\|_{\alpha,\beta} := \max_x |x^\alpha \partial^\beta f|$,

the proof shows that

$\|\hat{f}\|_{\alpha,\beta} \leq C [\|f\|_{\gamma,\delta} + \cdots$

α, β γ, δ

$$+ \| f \|_{\gamma_k, \delta_k} \Big]$$

depending on α, β ~~but~~ but
not on f.

This shows $\mathcal{F} : \mathcal{S} \to \mathcal{S}$
is continuous at $0 \in \mathcal{S}$
$\Longrightarrow$ everywhere (by
translation invariance).

[For instance, let's check
$\| \mathcal{F} f \|_{0,0} \leq$ something

$$\text{LHS} = \| \hat{f} \|_{L^\infty} \leq \| f \|_{L^1}$$

$$|f(x)| \leq C (1 + |x|^2)^{-n}$$

where $C = \max (1 + |x|^2)^n |f(x)|$
$= \max |p(x) f(x)|$
poly. $p(x)$

for some $p > 0$,

$$\Rightarrow \int |f| \leq C \underbrace{\int (1+|x|^2)^{-n} dx}_{" c_n < \infty}$$

$$\leq \left[\left[\text{some seminorms} \right] \cdot c_n \right]$$

We are ready to def. $\hat{T}$ for $T \in \mathcal{S}'$.

If $T = T_f$, $f \in \mathcal{S}$,

we want $\hat{T} = T_{\hat{f}}$.

So $\langle \hat{T}, \varphi \rangle = \langle T_{\hat{f}}, \varphi \rangle$

$$= \int \hat{f} \varphi \, d\cancel{x}y$$

$$= \int \int f(x) e^{-2\pi i \langle y, x \rangle} \varphi(y) \, dx \, dy$$

$$\underset{\text{(Fubini)}}{=} \int \left[\int \varphi(y) e^{-2\pi i \langle x, y \rangle} dy \right] f(x) \, dx$$

$$= \int f(x) \hat{\varphi}(x) \, dx$$

$$= \langle T_f, \hat{\varphi} \rangle$$

$= \langle T, \hat{\varphi} \rangle.$

<u>def</u> If $T \in \mathcal{S}'$,
we let $\langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle$.

<u>Rmk</u> Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$
is continuous, $\hat{T} : \mathcal{S} \to \mathbb{C}$
is a continuous linear functional.

<u>Rmk</u> For $f \in L^1$ (or L^2)
we have other definitions of F.t. but they agree with the above
(see homework).

One could also say that $\mathcal{S} \subset \mathcal{S}'$ is dense
for the weak topo. on $\mathcal{S}'$
(similarly, $\mathcal{D} \subset \mathcal{D}'$ is dense).

Given $f \in L^1$, I can approx. $f = \lim f_j$ both in L^1 & in $\mathcal{S}'$.

$\mathcal{F} f_j \xrightarrow{} \mathcal{F} f$ in $\mathcal{S}'$ (distr.) (distr.)

$\mathcal{F} f_j \to \mathcal{F} f$ in L^∞. (integral) (integral)

Parseval identity

$\varphi, \psi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$.

$$\int \varphi \overline{\psi} = \int \hat{\varphi} \overline{\hat{\psi}}.$$

Before proving it, Let's prove the "inversion formula"

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi.$$

$$\check{g}(x) = \mathcal{F}^{-1}(g)(x) := \int g(\xi) e^{2\pi i \langle \xi, x \rangle} d\xi$$

we are asserting $\mathcal{F}^{-1} \mathcal{F} f = f$.

proof of inversion

We will use that it holds for translations and dilations of Gaussian $e^{-\pi|x|^2}$ (homework).

$$\int \check{\hat{f}}\, \varphi \underset{\uparrow}{=} \int f \varphi$$

want to show this for appropriate choices of φ

If $\varphi \in$ above class,

$$\int \check{\hat{f}}\, \varphi = \int \hat{f}\, \check{\varphi} = \int f\, \hat{\check{\varphi}}$$

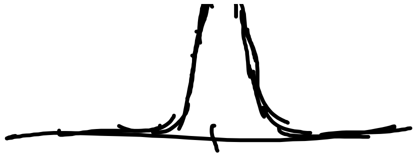
(Fubini) (Fubini)

$$\underset{\uparrow}{=} \int f \varphi.$$

(since inversion holds for φ)

This implies $\check{\hat{f}} = f$:
you can fix $x_0 \in \mathbb{R}^n$ and take $\varphi(x) := e^{-|x-x_0|^2/\varepsilon^2}$

$\sim \varepsilon$



x_0

$$\Rightarrow \quad \int f \varphi \approx f(x_0) \int \varphi + o\left(\int \varphi\right)$$

$$\|$$

$$\int \check{\hat{f}} \varphi \approx \check{\hat{f}}(x_0) \int \varphi + o\left(\int \varphi\right)$$

$$\Rightarrow (\text{as } \varepsilon \to 0) \quad f(x_0) = \check{\hat{f}}(x_0).$$

$\square$

proof of Parseval

$$\int \varphi \overline{\psi} = \int \check{\hat{\varphi}} \overline{\psi} \underset{(\text{Fubini})}{=} \int \hat{\varphi} \check{\overline{\psi}} = \int \hat{\varphi} \overline{\hat{\psi}}$$

(inversion formula)

because $\check{\overline{\psi}} = \overline{\hat{\psi}}$:

$$\check{\overline{\psi}}(y) = \int \overline{\psi}(x) e^{2\pi i \langle y, x \rangle} dx = \overline{\int \psi(x) e^{-2\pi i \langle y, x \rangle} dx}$$

$$\hat{\psi}(y) = \int \psi(x) e$$

$$= \int \hat{\hat{\psi}}(x) e^{2\pi i \langle y, x \rangle} dx.$$

$\square$

Corollary (If $\psi = \varphi$)

$$\|\varphi\|_{L^2} = \|\hat{\varphi}\|_{L^2}.$$

$(\varphi \in \mathcal{S})$

So $\mathcal{F}$ extends (uniquely) to an isometry $L^2 \to L^2$.

(If $f \in L^2$, $\widehat{T_f} = T_{\hat{f}}$
$\hat{f}$ is given by this extension.)

Convolution

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y) \, dy$$

$$\underset{y \leftrightarrow x-y}{\overset{\text{change of var.}}{=}} \int f(y) g(x-y) \, dy$$

In part., $f * g = g * f$.

Rmk If $f, g \in L^1(\mathbb{R}^n)$,
then $f * g$ is not defined
everywhere, but it is def.
at a.e. x:

$$\int_{\mathbb{R}^n} |f(x-y)\, g(y)|\, dy \in [0, +\infty]$$

is defined $\forall x$

and
$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)\, g(y)|\, dy\, dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \cdots\ dx\, dy$$

$$= \int |g(y)| \int |f(x-y)|\, dx\, dy$$

$$= \int |g(y)|\ \|f\|_{L^1}\, dy$$

$$= \|f\|_{L^1} \|g\|_{L^1} < \infty.$$

$\Rightarrow$ for a.e. x
$$\int |f(x-y)\, g(y)|\, dy < \infty.$$

So $f * g$ is def. a.e.

and $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$.

Idea of convolution:

$f * g$ is a linear combination of translations $\tau_y f$ with "coeff." $g(y)$.

$\tau_y f(x) := f(x-y)$

$$f * g = \int \tau_y f \; g(y)$$

$$= \int \tau_y g \; f(y).$$

From this one expects that $f * g$ inherits the best properties of each factor (if $f, g \in L^1$).

[illegible] if $f \in C^1$ with

e.g. if $f \in C^1$, we
$f, Df \in L^\infty$,

$$f * g \approx \sum_i g(y_i)\, \tau_{y_i} f$$

$$\|f * g\|_{C^1} \le \sum |g(y_i)| \, \|\tau_{y_i} f\|_{C^1}$$

$y_j \in$ grid of size 1

$$\sum |g(y_i)| \approx \int |g|.$$

This heuristic argument tells you that $f * g \in C^1$.

<u>proof of $f * g \in C^1$</u>

We also expect $\boxed{D(f*g) = (Df) * g}$.

$$\frac{f*g(x+he_j) - f*g(x)}{h}$$

$$= \int \frac{f(x+he_j-y) - f(x-y)}{h} g(y)\, dy$$

$$= \int \frac{1}{h} \int_0 \underbrace{\partial_j f(x+te_j-y)}_{\to\ \partial_j f(x-y) \text{ pointwise and is bounded by } \|Df\|_{L^\infty}} g(y)\, dy$$

$$\xrightarrow{\text{(dom. conv.)}} \int \partial_j f(x-y)\, g(y)\, dy$$

$$= \partial_j f * g.$$

Also, $\partial_j f \in L^\infty$ and so $\partial_j f * g$ is cont.

(if $f \in L^\infty$, $g \in L^1 \Rightarrow f * g \in C$;
if $g \in C_c^0$ $\int f(y)\, g(x-y)\, dy$ is cont. in x;

$g = \lim g_j$ in L^1

$f * g = f * g_j$ in L^∞

$\Rightarrow f * g$ is also cont.)

$\|f * g - f * g_j\|_{L^\infty}$

$$= \|f * (g - g_j)\|_{L^\infty}$$
$$\le \|f\|_{L^\infty} \|g - g_j\|_{L^1} \to 0.$$

Rmk If μ, ν are measures on $\mathbb{R}^n$ with finite mass (= total variation) we can def. $\mu * \nu$ as

$$+_\#(\mu \times \nu) =: \mu * \nu$$

measure on $\mathbb{R}^n \times \mathbb{R}^n$

($F : X \to Y$ pushes a measure on X to one on Y
$F_\# \alpha(E) := \alpha(F^{-1}(E))$)

Example • If $\mu = f\mathcal{L}^n$,
$\nu = g\mathcal{L}^n$

$\mu * \nu$ (as measures) $= (f * g)\mathcal{L}^n$.

• If $\mu = \delta_{x_0}$ then $\mu * \nu$ is "ν translated by x_0":

$\mu * \nu (E) = \nu(E - x_0)$.

• $spt(\mu * \nu) \subseteq spt(\mu) + spt(\nu)$

$spt(\mu)$ = complement of the biggest open set where $\mu \equiv 0$.
(or $\Leftrightarrow$ $|\mu| \equiv 0$)

Assume $x_0 \notin spt(f) + spt(g)$

$\Rightarrow \int f(x_0 - y)\, g(y) = 0$

because either
- $y \notin spt(g) \Rightarrow g(y) = 0$
- $x_0 - y \notin spt(f) \Rightarrow f(x_0 - y) = 0$

$\Rightarrow f * g = 0$ near x_0
since spt f + spt g closed.

(... of the two

(Assume one o[illegible]

is also compact.)