Homework 6

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Exercise 1.

Proof. (1) (a) (i)

$$\hat{F}(\xi) = \mathcal{F}(f(\xi) - \Delta f(\xi)) = \mathcal{F}(f(\xi) - \sum_{j=1}^{n} \partial_i^2 f(\xi)) = \mathcal{F}(f(\xi)) - \mathcal{F}(\sum_{j=1}^{n} \partial_i^2 f(\xi))) =$$

$$\mathcal{F}(f(\xi)) - \sum_{j=1}^{n} (2\pi i \xi_j)^2 (\mathcal{F} f)(\xi))) = \hat{f}(\xi)\Big(1 + 4\pi^2 |\xi|^2\Big) = C\hat{f}(\xi)$$

So that $f(\xi) = \frac{1}{C} F(\xi) = \frac{1}{1+4\pi^2|\xi|^2} F(\xi)$.
(ii)

$$\mathcal{F}^{-1}(m(\xi)\mathcal{F}((1-\Delta)f)(\xi)) = \mathcal{F}^{-1}(m(\xi)\hat{F}(\xi)) = \mathcal{F}^{-1}(m(\xi) \cdot C\hat{f}(\xi)) = \mathcal{F}^{-1}((2\pi i)^2 \xi_i \xi_j \hat{f}(\xi))) =$$

$$\mathcal{F}^{-1}(-4\pi^2 \xi_i \xi_j \hat{f}(\xi))) = \mathcal{F}^{-1}\Big(\mathcal{F}\Big(\partial_i \partial_j f(\xi)\Big)\Big) = \partial_i \partial_j f(\xi)$$

(b) We have,

$$\|x^\alpha H(x)\|_\infty \le \|\mathcal{F}(x^\alpha H(x))\|_1 = \|\frac{1}{(-2\pi i)^{|\alpha|}} \partial^\alpha \hat{H}(x)\|_1 =$$

$$-\frac{(|\alpha|)! \cdot (8\pi^2)^{|\alpha|}}{(-2\pi i)^{|\alpha|}} \int \frac{|\xi|^\alpha}{|1 + 4\pi^2 |\xi|^2|^{|\alpha|+1}} d\xi < \infty$$

for $|\alpha| \ge n-1$. So that $|H(x)| \le \frac{C}{|x|^\alpha}$ which represents an L^1 function on $\mathbb{R}^n \setminus \{0\}$, hence everywhere on $\mathbb{R}^n$. Now,

$$|f(x)| = |(H * F)(x)| = |\int H(y)F(x-y)dy| \le \|F\|_\infty | \int H(y)| \le \|H\|_1 \cdot \|F\|_\infty$$

So that $\|f\|_\infty \le \|H\|_1 \cdot \|F\|_\infty$.
(c)

$$\Big| \int \partial_i \partial_j f(1-\Delta) g \Big| = \Big| \int \partial_i \partial_j g(1-\Delta) f \Big| \le \int |\partial_i \partial_j g| \cdot |(1-\Delta) f| \le$$

$$\|(1-\Delta)f\|_{L^1} \cdot \|\partial_i \partial_j g\|_{L^\infty} = C\|(1-\Delta)f\|_{L^1} \cdot \|\Delta g\|_{L^\infty}$$

The second inequality Hölder's inequality.
(d)

$$\|\partial_i\partial_j f\|_{L^1} = \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1}\int \partial_i\partial_j f G = \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1}\int \partial_i\partial_j f(1-\Delta)g \le$$

$$\sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} C\|(1-\Delta)f\|_{L^1}\|\Delta g\|_{L^\infty} \le \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} C\|(1-\Delta)f\|_{L^1}\cdot C_2\|G\|_{L^\infty} \le C\|(1-\Delta)f\|_{L^1}$$

Now, by (a), we have $g = \frac{1}{C}G$. So that $\|g\|_{L^\infty} = \frac{1}{C}\|G\|_{L^\infty} < \infty$. It follows that $|\partial_i^2 g| < \infty$, (else, $\partial_i g$, and hence g, explodes at some point). So that $\|\Delta g\|_{L^\infty} = \|\sum_{i=1}^n \partial_i^2 g\|_{L^\infty} < \infty$ and $\|\Delta g\|_{L^\infty} \le C_1\|g\|_{L^\infty} \le C\|G\|_{L^\infty}$.
(e)

$$\|(1-\Delta)f\|_1 = \int |f - \Delta f|dx = \int |\frac{1}{\lambda}f(\lambda x) - \frac{1}{\lambda}\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx =$$

$$\int |\frac{1}{\lambda}f(\lambda x) - \frac{1}{\lambda}\cdot\lambda^2\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx = \int |\frac{1}{\lambda}f(\lambda x) - \lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx \ge$$

$$\Big|\frac{1}{\lambda}\int |f(\lambda x)|dx - \int |\lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|dx\ \Big| =$$

$$\Big|\frac{1}{\lambda}\int |f(\lambda x)|dx - \|\Delta f\|_1\ \Big| \to \|0 - \Delta f\|_1 = \|\Delta f\|_1$$

Likewise,

$$\int |\frac{1}{\lambda}f(\lambda x) - \lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|\ dx \le \int |\frac{1}{\lambda}f(\lambda x)| + |\lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|\ dx \le$$

$$\int |\frac{1}{\lambda}f(\lambda x)| + \int |\lambda\sum_{j=1}^n \partial_j^2 f(\lambda x)|\ dx = \int |\frac{1}{\lambda}f(\lambda x)| + \|\Delta f\|_1 \to 0 + \|\Delta f\|_1 = \|\Delta f\|_1$$

as $\lambda\to\infty$. Now, by (d),

$$\|D^2 f\|_1 \le C\|(1-\Delta)f\|_1 = C\|\Delta\|_1$$

□

Exercise 2.

Proof. (a) By continuity of the differential operator,

$$\lim_{R\to\infty}(D^k f_R(x)) = D^k(\lim_{R\to\infty} f_R(x)) = D^k(\phi(0)\cdot e^{2\pi i x\xi_0}) = \phi(0)D^k e^{2\pi i x\xi_0} =$$

$$\phi(0)\Big(\sum \partial_{j_1}\cdots\partial_{j_k}(e^{2\pi i x\xi_0})\Big) = \phi(0)\cdot e^{2\pi i x\xi_0}\cdot\sum(\xi_0)_{j_1}\cdots(\xi_0)_{j_k} \ne 0$$

and

$$\lim_{R\to\infty} Lf_R(x) = L(\lim_{R\to\infty} f_R(x)) = L(\phi(0)\cdot e^{2\pi i x\xi_0}) = \sum_{|\alpha|\le k} c_\alpha\partial^\alpha(\phi(0)\cdot e^{2\pi i x\xi_0}) =$$

$$\phi(0) \sum_{|\alpha| \le k} c_\alpha (2\pi i x \xi_0)^\alpha e^{2\pi i x \xi_0} = \phi(0) \cdot e^{2\pi i x \xi_0} \sum_{|\alpha| \le k} c_\alpha (2\pi i x \xi_0)^\alpha = 0$$

Suppose by contradiction that $\|D^k f\|_p \le C\|Lf\|_p$ for some nontrivial $f \in C_c^\infty$, with support$(f) = K$. Then,

$$\|D^k f_R(x)\|_p = |R| \cdot \|D^k f(x)\|_p \le |R| \cdot C\|Lf(x)\|_p = C\|Lf_R(x)\|_p$$

for all R, so

$$\lim_{R\to\infty} \|D^k f_R\|_p = \mu(K) \cdot \phi(0) \sum (\xi_0)_{j_1} \cdots (\xi_0)_{j_k} > 0 = \lim_{R\to\infty} \|Lf_R\|_p$$

is a contradiction.

(b) □

Exercise 3.

Proof. □