Homework 5

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Exercise 1.

Proof. First, we claim $f_b \in L^1(\mathbb{R}^n)$, and $f_s \in L^r(\mathbb{R}^n)$ -

$$\int |f_b| = \int |f_b|^p |f_b|^{1-p} \leq C(p)\|f\|_p^p < \infty$$

$$\int |f_s|^r = \int |f_s|^{r-p} |f_s|^p \leq C(r,p)\|f\|_p^p < \infty$$

Now, since $|Tf| \leq |Tf_b| + |Tf_s|$, we have $\{|Tf| > t\} \subseteq \{|Tf_b| > \frac{t}{2}\} \cup \{|Tf_s| > \frac{t}{2}\}$, so that -

$$\mu\{|Tf| > t\} \leq \mu\{|Tf_b| > \frac{t}{2}\} + \mu\{|Tf_s| > \frac{t}{2}\} \leq$$
$$\frac{2A\|f\|_1}{t} \int |f_b| + \frac{2^r A^r \|f\|_r^r}{t^r} \int |f_s|^r$$

Now,

$$\int_0^\infty t^{q-1} t^{-1} \int_{|f|>t} |f| = \int_{\mathbb{R}^n} |f| \int_0^{|f|} t^{q-2} = \frac{1}{q-1} \int_{\mathbb{R}^n} |f||f|^{q-1} = \frac{\|f\|_q^q}{q-1}$$

since $q > p > 1$, and

$$\int_0^\infty t^{q-1} t^{-r} \int_{|f|\leq t} |f|^r = \int_{\mathbb{R}^n} |f|^r \int_{|f|}^\infty t^{q-1-r} = \frac{1}{r-q} \int_{\mathbb{R}^n} |f|^r |f|^{q-r} = \frac{\|f\|_q^q}{r-q}$$

since $q < r$. Altogether,

$$\|Tf\|_q \leq C\|f\|_q$$

□

Exercise 2.

Proof. (a)

$$\int_{\mathbb{R}^n \setminus B_{2r}(0)} |K(x) - K(x-z)| dx = \int_{\mathbb{R}^n \setminus B_{2r}(0)} \Big| \int_0^1 DK d\gamma \Big| dx =$$

$$\int_{\mathbb{R}^n \setminus B_{2r}(0)} \int_0^1 \Big| DK(t)(x-(1-t)z) \cdot z \Big| dt\ dx \leq \int_{\mathbb{R}^n \setminus B_{2r}(0)} B|x|^{-n-1} |x-(x-z)| dx =$$

$$B|z|\int_{\mathbb{R}^n\setminus B_{2r}(0)}|x|^{-n-1}dx = C(n)B$$

(b)

$$\int_{\mathbb{R}^n}\Big(K(x)-K(x-x_\xi)\Big)e^{-2\pi ix\xi}\ dx = \int_{\mathbb{R}^n}K(x)e^{-2\pi ix\xi}\ dx - \int_{\mathbb{R}^n}K(x-x_\xi)e^{-2\pi ix\xi}\ dx =$$

$$\hat{K}(\xi)-\int_{\mathbb{R}^n}K(x)e^{-2\pi i(x+x_\xi)\xi}\ dx = \hat{K}(\xi)-e^{-i\pi}\int_{\mathbb{R}^n}K(x)e^{-2\pi ix\xi}\ dx =$$

$$\hat{K}(\xi)+\hat{K}(\xi)=2\hat{K}(\xi)$$

(c) By the cancellation condition, we have

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)e^{-2\pi ix\xi}\Big| = \Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)e^{-2\pi ix\xi}-\int_{B_{\frac{1}{|\xi|}}(0)}K(x)\Big| =$$

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)(e^{-2\pi ix\xi}-1)\Big| \le \int_{B_{\frac{1}{|\xi|}}(0)}|K(x)|\cdot|e^{-2\pi ix\xi}-1| \le 2\pi|\xi||x|\int_{B_{\frac{1}{|\xi|}}(0)}|K(x)| \le$$

$$2\pi|\xi||x|A\int_{B_{\frac{1}{|\xi|}}(0)}|x|^{-n} = 2\pi|\xi|A\int_{B_{\frac{1}{|\xi|}}(0)}|x|^{-n+1} = C(n)A$$

(d)

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi ix\xi}\Big| = \Big|e^{i\pi}\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi ix\xi}\Big| = \Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi i(x-x_\xi)\xi}\Big| =$$

$$\Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)}K(x)e^{-2\pi ix\xi}\Big|$$

Since $|x_\xi| = \frac{1}{2|\xi|} < \frac{1}{|\xi|}$, we have that $B_r(0)\subseteq B_{\frac{1}{|\xi|}}(x_\xi)$ for some $0<r<\frac{1}{|\xi|}$. Thus, as in (c),

$$|\int_{B_r(0)}K(x-x_\xi)e^{-2\pi ix\xi}| \le 2\pi rA\int_{B_r(0)}|x|^{-n+1} \le 2\pi|\xi|A\int_{B_{\frac{1}{|\xi|}}(0)}|x|^{-n+1} = C(n)A$$

Now,

$$\Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)}K(x)e^{-2\pi ix\xi}\Big| = \Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)\setminus B_r(0)}K(x)e^{-2\pi ix\xi}+\int_{B_r(0)}K(x)e^{-2\pi ix\xi}\Big| \le$$

$$\Big|\int_{B_{\frac{1}{|\xi|}}(x_\xi)\setminus B_r(0)}K(x)e^{-2\pi ix\xi}\Big| + \Big|\int_{B_r(0)}K(x)e^{-2\pi ix\xi}\Big| \le C + C_1(n)A \le C(n)A$$

(*e*)

$$\left|\hat{K}(\xi)\right| = \left|\int_{\mathbb{R}^n} K(x)e^{-2\pi i x\xi}\ dx\right| = \left|\int_{B_R(0)\setminus B_\epsilon(0)} K(x)e^{-2\pi i x\xi}\ dx\right| \leq$$

$$\int_{B_R(0)\setminus B_\epsilon(0)} |K(x)e^{-2\pi i x\xi}|dx \leq A\int_{B_R(0)\setminus B_\epsilon(0)} \frac{1}{|x|^n}\ dx = C(n)A$$

□

Exercise 3.

Proof. (*a*) (*i*) $|K\mathbf{1}_{B_r\setminus B_\epsilon}(x)| \leq |K(x)| \leq A|x|^{-n}$
(*ii*)

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x-z)|dx \leq \int_{\mathbb{R}^n\setminus B_{2r}(0)} |K_{\epsilon,R}(x)| + |K_{\epsilon,R}(x-z)|dx \leq$$

$$\frac{2A}{|\epsilon|^n}\cdot\mu\{B_R(0)\} = C(n)A$$

(*iii*)

$$\int_{B_s(0)\setminus B_r(0)} K = \int_{B_s(0)} K - \int_{B_r(0)} K = 0 - 0 = 0$$

(*b*) We will prove $\|K_{\epsilon,R} * f\|_{L^2} \leq \|\hat{K}_{\epsilon,R}\|_{L^\infty}\|f\|_{L^2}$. The general inequality $\|K_{\epsilon,R} * f\|_{L^2} \leq C(n,p)A\|f\|_{L^p}$ then follows by the Marcinkiewicz Interpolation Theorem, the same way we proved the Calderon-Zygmund estimate. To that end, by Plancheral's identity, we have -

$$\|K_{\epsilon,R} * f\|_{L^2} = \|\mathcal{F}(K_{\epsilon,R} * f)\|_{L^2} = \|\hat{K}_{\epsilon,R}\cdot\hat{f}\|_{L^2} \leq \|\hat{K}_{\epsilon,R}\|_{L^\infty}\cdot\|\hat{f}\|_{L^2} = C(n)A\|\hat{f}\|_{L^2}$$

(*c*) Since f is smooth and has compact support, by the Extreme Value Theorem, f achieves a minimum and a maximum on it's domain. Let m, M be the minimal, and maximal values of f, respectively. WLOG, suppose $R \geq a$. Then,

$$|(K_{\epsilon,R}*f)(x)| = \left|\int K_{\epsilon,R}(y)f(x-y)dy\right| = \left|\int_{|x|\geq a} K_{\epsilon,R}(y)f(x-y)dy + \int_{B_a(0)} K_{\epsilon,R}(y)f(x-y)dy\right| \leq$$

$$\left|\int_{|x|\geq a} K_{\epsilon,R}(y)f(x-y)dy\right| + \left|\int_{B_a(0)} K_{\epsilon,R}(y)f(x-y)dy\right| \leq$$

$$\int_{|x|\geq a} |K_{\epsilon,R}(y)f(x-y)|dy + \left|\int_{B_a(0)\setminus B_\epsilon(0)} K(y)f(x-y)dy\right| \leq$$

$$|M|\Big(A\int_{\{|x|\geq a\}\cap B_R(0)} \frac{1}{|a|^n} + |\int_{B_a(0)\setminus B_\epsilon(0)} K(y)dy|\Big) \leq |M|\Big(\mu\{B_R(0)\}\cdot\frac{A}{|a|^n} + 0\Big) = C(n)A\cdot|M|$$

for every $\epsilon > 0$, and for every $R \geq a$. Thus the integral is absolutetely convergent and the limit exists. Now, since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we can approximate $g \in L^p(\mathbb{R}^n)$ with a sequence $C_c^\infty(\mathbb{R}^n) \ni g_j \to g$, with convergence in L^p. Thus, by the dominated convergence theorem, we have

$$\|K_{\epsilon,R} * g_j\|_{L^p}^p = \int_{\mathbb{R}^n} |(K_{\epsilon,R} * g_j)(x)|^p = \int_{\mathbb{R}^n} \Big| \int K_{\epsilon,R}(y) g_j(x-y)dy \Big|^p \nearrow$$
$$\int_{\mathbb{R}^n} \Big| \int K_{\epsilon,R}(y) g(x-y)dy \Big|^p \nearrow \int_{\mathbb{R}^n} \Big| \int K(y) g(x-y)dy \Big|^p =$$
$$\int_{\mathbb{R}^n} |(K * g)(x)|^p = \|K * g\|_{L^p}^p$$

as $\epsilon \to 0,\ R, j \to \infty$. $\square$