Homework 4

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Exercise 1.

Proof. Let $\{Q_j\}$ be the cubes defined in the proof of the Calderón–Zygmund lemma. Let $S = \{x \in [0,1)^n \mid M_4 f(x) > \lambda\}$, where $x_0 = 0$ in the definition

$$M_4 f(x) = \sup_{Q \in \mathfrak{D} \,:\, Q \in [0,1]^n} \fint_Q |f|$$

($\subseteq$) Let $Q_j = [a,b]^n$. Let $\tilde{Q}_j = [a,b)^n$. Then, $Q_j, \tilde{Q}_j$ only differ on a set of measure 0. By construction,

$$\fint_{\tilde{Q}_j} |f| = \fint_{Q_j} |f| > \lambda$$

so that $M_4 f(x) > \lambda$ for all $x \in Q_j$. Therefore, $Q_j \subseteq S$ (for a.e. $x \in Q_j$). Thus, $\cup Q_j \subseteq S$ up to a set of measure 0.
($\supseteq$) Let $x \in S$. Then there exists a dyadic cube $Q \subseteq [0,1)^n$ such that

$$\fint_Q |f| > \lambda$$

So that $Q = Q_j$ for some j. Thus $S \subseteq \cup Q_j$. Therefore, $S = \cup Q_j$ up to a set of measure 0. $\square$

Exercise 2.

Proof. $(a)(i)$ Since $\lambda > 0$ and $\mu\{|f| > \lambda\} \geq 0$, we have $\|f\|_{L^{p,\infty}} \geq 0$.
(ii)

$$\|kf\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|kf| > \lambda\} = \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{|k|}\} =$$
$$\sup_{|k|\lambda>0} |k|\lambda \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} = \sup_{|k|\lambda>0} |k|\lambda \cdot \mu\{|f| > \frac{|k|\lambda}{|k|}\} = |k| \cdot \sup_{|k|\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} =$$
$$|k| \cdot \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} = |k| \cdot \|f\|_{L^{p,\infty}}$$

(iii)

$$\|f+g\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|f+g| > \lambda\} \leq \sup_{\lambda>0} \lambda \cdot (\mu\{|f| > \frac{\lambda}{2}\} + \mu\{|g| > \frac{\lambda}{2}\}) \leq$$

$$\sup_{\lambda>0} \lambda \cdot (\mu\{|f| > \frac{\lambda}{2}\}) + \sup_{\lambda>0} \lambda \cdot (\mu\{|g| > \lambda\}) =$$

$$\sup_{2\lambda>0} 2\lambda \cdot (\mu\{|f| > \lambda\}) + \sup_{2\lambda>0} 2\lambda \cdot (\mu\{|g| > \lambda\}) =$$

$$2\sup_{\lambda>0} \lambda \cdot (\mu\{|f| > \lambda\}) + 2\sup_{\lambda>0} \lambda \cdot (\mu\{|g| > \lambda\}) = 2(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$$

(b)

$$\int_E |f| \geq \int_E \lambda = \lambda\mu(E)$$

So that,

$$\mu(E)^{\frac{1}{p}-1} \int_E |f| \geq \mu(E)^{\frac{1}{p}-1} \cdot \lambda\mu(E) = \lambda \cdot \mu(E)^{\frac{1}{p}}$$

Thus,

$$\|f\|' = \sup_{E:0<\mu(E)<\infty} \mu(E)^{\frac{1}{p}-1} \int_E |f| \geq \sup_{\lambda>0} \lambda\mu\{|f| > \lambda\}^{\frac{1}{p}} = \|f\|_{L^{p,\infty}}$$

(c)

$$\int_E |f(x)|d\mu = \int_E \Big(\int_0^{|f(x)|} d\lambda\Big)du = \int_E \mu\{|f(x)| > \lambda\}d\lambda = \int_0^\infty \mu\{x \in E \mid |f(x)| > \lambda\}d\lambda \leq$$

$$\int_0^{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}} \mu(E)d\lambda + \int_{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}^\infty \mu\{|f| > \lambda\}d\lambda le$$

$$\int_0^{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}} \mu(E)d\lambda + \int_{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}^\infty \frac{\|f\|_{L^{p,\infty}}}{\lambda^p}d\lambda =$$

$$\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}\mu(E) + \|f\|_{L^{p,\infty}}^p \cdot \Big[\frac{\lambda^{1-p}}{1-p}\Big]_{\mu(E)^{-\frac{1}{p}}\|f\|_{L^{p,\infty}}}^\infty =$$

$$\mu(E)^{\frac{p-1}{p}}\|f\|_{L^{p,\infty}} \cdot \frac{p}{p-1} \iff$$

$$\mu(E)^{\frac{1}{p}-1} \int_E |f|d\mu \leq \frac{p}{p-1}\|f\|_{L^{p,\infty}}$$

For all Borel sets E. Therefore,

$$\|f\|' = \sup_{E:0<\mu(E)<\infty} \mu(E)^{\frac{1}{p}-1} \int_E |f|d\mu \leq \frac{p}{p-1}\|f\|_{L^{p,\infty}}$$

(d) We have

$$\|f\|_{L^{p,\infty}} \leq \|f\|' \leq \frac{p}{p-1}\|f\|_{L^{p,\infty}}$$

Thus the two norms are equivalent. □

Exercise 3.

Proof. (a) Let $R = |x-y|$. Since $f \in \mathcal{S}(\mathbb{R}^n)$, Df is bounded. Let $Df \leq K$. Let us first estimate

$$\fint_{B_R(x)} f - \fint_{B_R(y)} f = \fint_{B_R(x)} f(z+(y-x)) - f(z)dz = \fint_{B_R(x)} \Big(\int_z^{z+(y-x)} Df(t)dt \Big) dz \leq$$

$$\fint_{B_R(x)} K \cdot |z+(y-x)-z| dz = \fint_{B_R(x)} K \cdot |x-y| dz = K \cdot |x-y| \cdot \frac{1}{\mu(B_r(x))} \int_{B_r(x)} dz =$$

$$K \cdot |x-y| \cdot 1 = K \cdot |x-y|$$

Thus, by the lemma we proved in class, we have -

$$|f(x)-f(x)| \leq \Big| f(x) - \fint_{B_R(x)} f \Big| + \Big| \fint_{B_R(x)} f - \fint_{B_R(y)} f \Big| + \Big| \fint_{B_R(y)} f - f(y) \Big| \leq$$

$$C_1 \cdot |x-y| \cdot Mf(x) + K \cdot |x-y| + C_2 \cdot |x-y| \cdot Mf(y) \leq$$

$$C(Mf(x)+Mf(y)) \cdot |x-y|$$

where $C = \max\{C_1, C_2, K\}$.

(b) By Hölder's inequality, we have, for all s,

$$s \fint_{B_s(x)} |Df| \leq C(p,n) s^{1-n+\frac{n}{p'}} \|Df\|_{L^p} = C(p,n) s^{\alpha} \|Df\|_{L^p}$$

Thus

$$s \cdot M(Df)(x) \leq C(p,n) s^{\alpha} \|Df\|_{L^p}$$

Substituting $s = \frac{r}{2}, \frac{r}{4}, \frac{r}{8}, \ldots$, we have

$$r \cdot M(Df)(x) = \sum_{k=1}^{\infty} \frac{r}{2^k} \cdot M(Df)(x) \leq \sum_{k=1}^{\infty} C(p,n) (\frac{r}{2^k})^{\alpha} \|Df\|_{L^p} \leq$$

$$\sum_{k=1}^{\infty} C(p,n) r^{\alpha} \frac{1}{2^k} \|Df\|_{L^p} = C(p,n) r^{\alpha} \|Df\|_{L^p}$$

Now, by the lemma we proved preceeding Sobolev's inequality, we have

$$\Big| f(x) - \fint_{B_R(x)} f \Big| \leq C \cdot r \cdot M(Df)(x) \leq C(p,n) r^{\alpha} \|Df\|_{L^p}$$

□