Homework 6

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Exercise 1.

Proof. (1) (a) (i)

$$\hat{F}(\xi) = \mathcal{F}(f(\xi) - \Delta f(\xi)) = \mathcal{F}(f(\xi) - \sum_{j=1}^{n} \partial_i^2 f(\xi)) = \mathcal{F}(f(\xi)) - \mathcal{F}(\sum_{j=1}^{n} \partial_i^2 f(\xi))) =$$

$$\mathcal{F}(f(\xi)) - \sum_{j=1}^{n} (2\pi i \xi_j)^2 (\mathcal{F} f)(\xi))) = \hat{f}(\xi)\Big(1 + 4\pi^2 |\xi|^2\Big) = C\hat{f}(\xi)$$

So that $f(\xi) = \frac{1}{C} F(\xi) = \frac{1}{1+4\pi^2|\xi|^2} F(\xi)$.
(ii)

$$\mathcal{F}^{-1}(m(\xi)\mathcal{F}((1-\Delta)f)(\xi)) = \mathcal{F}^{-1}(m(\xi)\hat{F}(\xi)) = \mathcal{F}^{-1}(m(\xi) \cdot C\hat{f}(\xi)) = \mathcal{F}^{-1}((2\pi i)^2 \xi_i \xi_j \hat{f}(\xi))) =$$

$$\mathcal{F}^{-1}(-4\pi^2 \xi_i \xi_j \hat{f}(\xi))) = \mathcal{F}^{-1}\Big(\mathcal{F}\Big(\partial_i \partial_j f(\xi)\Big)\Big) = \partial_i \partial_j f(\xi)$$

(b)
(c)

$$\Big|\int \partial_i \partial_j f (1-\Delta) g\Big| = \Big|\int \partial_i \partial_j g (1-\Delta) f\Big| \le \int |\partial_i \partial_j g| \cdot |(1-\Delta) f| \le$$

$$\|(1-\Delta)f\|_{L^1} \cdot \|\partial_i \partial_j g\|_{L^\infty} = C\|(1-\Delta)f\|_{L^1} \cdot \|\Delta g\|_{L^\infty}$$

The second inequality Hölder's inequality.
(d)

$$\|D^2 f\|_{L^1} = \|\partial_i \partial_j f\|_{L^1} = \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} \int \partial_i \partial_j f G = \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} \int \partial_i \partial_j f (1-\Delta) g \le$$

$$\sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} C\|(1-\Delta)f\|_{L^1}\|\Delta g\|_{L^\infty} \le \sup_{G\in\mathcal{S}:\ \|G\|_{L^\infty}\le 1} C\|(1-\Delta)f\|_{L^1} \cdot C_2\|G\|_{L^\infty} \le C\|(1-\Delta)f\|_{L^1}$$

Now, by (a), we have $g = \frac{1}{C}G$. So that $\|g\|_{L^\infty} = \frac{1}{C}\|G\|_{L^\infty} < \infty$. It follows that $|\partial_i^2 g| < \infty$, (else, $\partial_i g$, and hence g, explodes at some point). So that $\|\Delta g\|_{L^\infty} = \|\sum_{i=1}^n \partial_i^2 g\|_{L^\infty} < \infty$ and $\|\Delta g\|_{L^\infty} \le C_1\|g\|_{L^\infty} \le C\|G\|_{L^\infty}$. □

Exercise 2.

Proof. □

Exercise 3.

Proof. □