Homework 5

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Exercise 1.

Proof. First, we claim $f_b \in L^1(\mathbb{R}^n)$, and $f_s \in L^r(\mathbb{R}^n)$ -

$$\int |f_b| = \int |f_b|^p |f_b|^{1-p} \leq C(p)\|f\|_p^p < \infty$$

$$\int |f_s|^r = \int |f_s|^{r-p} |f_s|^p \leq C(r,p)\|f\|_p^p < \infty$$

Now, since $|Tf| \leq |Tf_b| + |Tf_s|$, we have $\{|Tf| > t\} \subseteq \{|Tf_b| > \frac{t}{2}\} \cup \{|Tf_s| > \frac{t}{2}\}$, so that -

$$\mu\{|Tf| > t\} \leq \mu\{|Tf_b| > \frac{t}{2}\} + \mu\{|Tf_s| > \frac{t}{2}\} \leq$$
$$\frac{2A\|f\|_1}{t} \int |f_b| + \frac{2^r A^r \|f\|_r^r}{t^r} \int |f_s|^r$$

Now,

$$\int_0^\infty t^{q-1} t^{-1} \int_{|f|>t} |f| = \int_{\mathbb{R}^n} |f| \int_0^{|f|} t^{q-2} = \frac{1}{q-1} \int_{\mathbb{R}^n} |f||f|^{q-1} = \frac{\|f\|_q^q}{q-1}$$

since $q > p > 1$, and

$$\int_0^\infty t^{q-1} t^{-r} \int_{|f|\leq t} |f|^r = \int_{\mathbb{R}^n} |f|^r \int_{|f|}^\infty t^{q-1-r} = \frac{1}{r-q} \int_{\mathbb{R}^n} |f|^r |f|^{q-r} = \frac{\|f\|_q^q}{r-q}$$

since $q < r$. Altogether,

$$\|Tf\|_q \leq C\|f\|_q$$

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Exercise 2.

Proof. 2

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Exercise 3.

Proof. 3

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