Class 5

Wednesday, October 21, 2020 3:11 PM

Lemma (another version of C–Z decomposition)

If $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$

then $\exists \{Q_j\}$ disjoint cubes

s.t.

- $|f| \le \lambda$ a.e. on $\mathbb{R}^n \setminus \bigcup Q_j$
- $\lambda < ⨍_{Q_j} |f| \le 2^n \lambda$.

Proof Split $\mathbb{R}^n$ into cubes of sidelength ℓ and for each cube Q

$$⨍_Q |f| = \frac{\int_Q |f|}{\ell^n} \le \ell^{-n} \|f\|_{L^1}$$

$\Rightarrow$ for ℓ large enough,

$$⨍_Q |f| \le \lambda \quad \forall Q.$$

Now apply 1^{st} version on every cube $Q = Q_k$

Each produces a collection $\{Q_{k,j}\}$.

The desired collection is

$\bigcup \{Q_{k,j}\}$ $\square$

$\cup \ [\sim_{k}]\cdot$

Recall: we asked:

given $L = \sum_{|\alpha|=k} c_\alpha \partial^\alpha$

when is $\|D^k f\|_{L^p} \le C \|Lf\|_{L^p}$?

$(1 < p < \infty)$

Let's look at $L = \Delta$ (Laplacian)

Is $\|\partial_i \partial_j f\|_{L^p} \le C \|\Delta f\|_{L^p}$?

$\mathcal{F}(\partial_i \partial_j f) = 4\pi^2 \xi_i \xi_j \hat{f}(\xi)$

$\mathcal{F}(\Delta f) = 4\pi^2 |\xi|^2 \hat{f}(\xi)$.

We can recover $\hat{f}(\xi)$ from $\mathcal{F}(\Delta f)$ if $\xi \neq 0$.

Also, $\mathcal{F}(\partial_i \partial_j f) = \underbrace{\frac{\xi_i \xi_j}{|\xi|^2}}_{m(\xi)} \mathcal{F}(\Delta f)$

$m \in L^\infty$.

Rmk If $p = 2$, by Plancherel

$\|\partial_i \partial_j f\|_{L^2} = \|\mathcal{F}(\partial_i \partial_j f)\|_{L^2}$

$$= \| m \, \mathcal{F}(\Delta f) \|_{L^2}$$

$$\leq \| \mathcal{F}(\Delta f) \|_{L^2}$$

$$= \| \Delta f \|_{L^2}.$$

Same holds if

<u>symbol</u> of L, i.e.,

$$\mathbb{L}(\xi) := \sum c_\alpha (2\pi i \xi)^\alpha$$

vanishes only at 0.

Indeed, $\mathbb{L}(\xi) = |\xi|^k \, \mathbb{L}\left(\frac{\xi}{|\xi|}\right)$

and (by compactness of S^{n-1}) $c \leq \left|\mathbb{L}\left(\frac{\xi}{|\xi|}\right)\right| \leq C$

$$\Rightarrow \mathcal{F}(\partial_{i_1} \cdots \partial_{i_k} f) = \underbrace{\frac{(2\pi i)^k \xi_1 \cdots \xi_k}{|\xi|^k \, \mathbb{L}(\xi/|\xi|)}}_{\text{bounded}} \mathcal{F}(Lf)$$

For $p \neq 2$:

$$\mathcal{F}(\partial_{i_1} \cdots \partial_{i_k} f) = m(\xi) \, \mathcal{F}(Lf)$$

m = "multiplier"

$$\partial \cdots = \check{m} * Lf.$$

$\partial_{i_1} \cdots \partial_{i_k} t = m$

Unluckily $\check{m} \notin L^1$

($m \in L^\infty$, so $\check{m} \in \mathcal{S}'$ at least).

One can show:

$\check{m}$ on $\mathbb{R}^n \setminus \{0\}$ is a function

$|\check{m}|(x) \leq C|x|^{-n}$

So $\check{m} \in L^{1,\infty}(\mathbb{R}^n \setminus \{0\})$.

<u>Rmk</u> If $L^{1,\infty}$ had an equiv. norm, we would easily get

$\|\check{m} * Lf\|_{L^{1,\infty}} \leq \|\check{m}\|_{L^{1,\infty}} \|Lf\|_{L^1}$

(interpolation)

$\Longrightarrow$ T is bounded from L^p to L^p

$(1 \leq p \leq 2)$.

$T\varphi := \check{m} * \varphi$.

$(Lf = \sum c_\alpha \partial^\alpha f)$

<u>Thm</u> (1st version of C-Z estimate)

Take $K \in \mathcal{S}(\mathbb{R}^n)$ s.t.

- $|\hat{K}| \le B$
- $|DK|(\xi) \le B/|\xi|^{n+1}$.

Then $\|K*f\|_{L^p} \le C(n,p)B\|f\|_{L^p}$ for $f \in L^p$.

Rmk We already know $\|K*f\|_{L^p} \le \|K\|_{L^1}\|f\|_{L^p}$ but we want to use ~~K~~ approximation of $\check{m}$.

Proof W.l.o.g. $B=1$. Step 1 It is enough to show $\|K*f\|_{L^{1,\infty}} \le C(n)\|f\|_{L^1}$.

Indeed, we know

$$\|K*f\|_{L^2} = \|\hat{K}\hat{f}\|_{L^2} \le \|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

(Marcinkiewicz interpolation)

$\ldots C(\ldots) \|f\|_p$

$$\|K * f\|_{L^p} \le C(n,p) \;\;\text{"}\;\; L^p$$

$$\forall\, 1 < p \le 2.$$

We obtain claim for $2 < p < \infty$ by duality:

if $f \in L^p$,

$$\|K * f\|_{L^p} = \sup_{\|h\|_{L^{p'}} \le 1} \int (K * f) h$$

$$= \int \int K(x-y) f(y) h(x) \, dy \, dx$$

$$= \int \left(\int K(x-y) h(x) \, dx \right) f(y) \, dy$$

$K^{\#}(z) := K(-z)$

$$= \int (K^{\#} * h) f$$

Now $1 < p' < 2$, so

$$\int (K^{\#} * h) f \underset{\text{(Hölder)}}{\le} \|K^{\#} * h\|_{L^{p'}} \|f\|_{L^p}$$

and the "kernel" $K^{\#}$ satisfies same hypotheses as K

S (previous case: $p' \in (1,2)$) /

$\Longrightarrow$ (previous

$$\| K^{\#} * h \|_{L^{p'}} \le C(n,p) \cancel{\| h \|_{L^{p'}}}$$

$$\Longrightarrow \int (K * f) h \le C(n,p) \| f \|_{L^p}.$$

Step 2 We want to show

$$\mathcal{L}^n \{ |K * f| > \lambda \} \le \frac{C(n) \| f \|_{L^1}}{\lambda}$$

for all $\lambda > 0$.

Fix $\lambda > 0$, and use C-Z dec.:

$$f = g + b$$

$$g(x) := \begin{cases} f(x) & \text{if } x \notin \cup Q_j \\ \fint_{Q_j} f & \text{if } x \in Q_j \end{cases}$$

$$b = f - g$$

Recall: $|g| \le 2^n \lambda$, $\int_{Q_j} b = 0$.

$$b = \sum_j b_j, \quad b_j := b \mathbb{1}_{Q_j}.$$

Also, $\| g \|_{L^1} = \int_{\mathbb{R}^n \setminus \cup Q_j} |f| + \sum_j \mathcal{L}^n(Q_j) \left| \fint_{Q_j} f \right|$

$$\le \int |f| + \sum_j \mathcal{L}^n(Q_j) \fint_{Q_j} |f|$$

$\mathbb{R}^n \setminus \cup Q_j$

$$= \int_{\mathbb{R}^n \setminus \cup Q_j} |f| + \sum_j \int_{Q_j} |f|$$

$$= \|f\|_{L^1}$$

$$\Rightarrow \|b\|_{L^1} = \sum \|b_j\|$$

$$\|b\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1} \leq 2\|f\|_{L^1}.$$

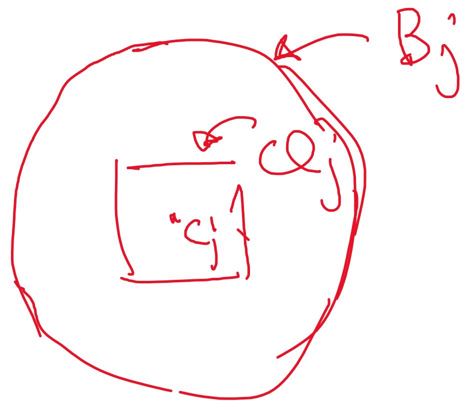
$$\mathcal{L}^n\{|K*f| > \lambda\}$$
$$\leq \mathcal{L}^n\{|K*g| > \tfrac{\lambda}{2}\}$$
$$+ \mathcal{L}^n\{|K*b| > \tfrac{\lambda}{2}\}$$

Strategy:

- g bounded in L^2
  $\Rightarrow$ also $K*g$ bded in L^2
  $\Rightarrow$ can use Chebyshev to bound $\mathcal{L}^n\{|K*g| > \tfrac{\lambda}{2}\}$

- call $c_j :=$ center of Q_j

$B_j := B_{\sqrt{n}\,\mathrm{side}(Q_j)}(c_j)$



For $x \notin B_j$, $K * b_j(x)$ has good pointwise bound thanks to $\int b_j = 0$ & regularity of K.

We will estimate $\|K * b_j\|_{L^1(\mathbb{R}^n \setminus B_j)} \leq \ldots$

$\Rightarrow$ can use Chebyshev.

- We then just note that $\mathcal{L}^n(\bigcup B_j)$

$$\leq \sum_j \mathcal{L}^n(B_j)$$

$$\leq C(n) \sum \mathcal{L}^n(Q_j)$$

$$= C(n) \sum_j \lambda^{-1} \ldots)$$

$$\left(\text{since } ⨍_{Q_j} |f| > \lambda, \text{ then } \int_{Q_j} |f| > \lambda \mathcal{L}^n(Q_j) \right)$$

$$\leq C(n) \sum_j \lambda^{-1} \int_{Q_j} |f|$$

$$\leq C(n) \lambda^{-1} \|f\|_{L^1}.$$

Step 3 Contribution of g:

$$\|g\|_{L^2}^2 = \int |g|^2 = \int |g| \cdot |g|$$

$$\leq 2^n \lambda \int |g| \leq 2^n \lambda \|f\|_{L^1}.$$

By Chebyshev,

$$\mathcal{L}^n \{ |K * g| > \tfrac{\lambda}{2} \} \leq \frac{\|K * g\|_{L^2}^2}{(\lambda/2)^2}$$

(L^2 bound)

$$\leq \frac{4}{\lambda^2} \|g\|_{L^2}^2 \leq \frac{2^{n+2}}{\lambda} \|f\|_{L^1}.$$

Step 4 We know

$$\mathcal{L}^n(\bigcup B_j) \le \frac{C(n)}{\lambda} \|f\|_{L^1}.$$

So it's enough to show

$$\mathcal{L}^n\{x \notin \bigcup B_j : |K * b| > \tfrac{\lambda}{2}\} \le \frac{C(n)}{\lambda} \|f\|_{L^1}.$$

We estimate

$$\int_{\mathbb{R}^n \setminus \bigcup B_j} |K * b|$$

$$\le \sum_j \int_{\mathbb{R}^n \setminus \bigcup_k B_k} |K * b_j|$$

$$\le \sum_j \int_{\mathbb{R}^n \setminus B_j} |K * b_j|.$$

Now: take $x \notin B_j$

$$K * b_j(x) = \int K(x-y)\, b_j(y)\, dy$$

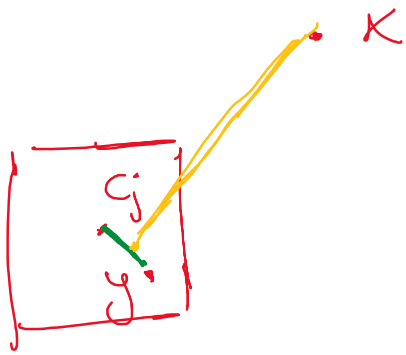
$$= \int_{Q_j} \big(K(x-y) - K(x-c_j)\big)\, b_j(y)\, dy$$

$\int b_j = 0$

Also, $|K(x-y) - K(x-c_j)|$

$$\le |y-c_j| \max_{\substack{z \in \text{segment} \\ \text{from } y \text{ to } c_j}} |DK|(x-z)$$

$$\le \operatorname{diam}(Q_j) \max_{\dots} |x-z|^{-n-1}$$



$|x-z| = \text{length}(\ )$ is bounded below

Indeed, $|x-z| \ge |x-c_j| - |z-c_j|$

$$\ge |x-c_j| - \sqrt{n}\,\frac{\text{side}(Q_j)}{2}$$

$$\geq \frac{1}{2}|x-c_j|$$

because $|x-c_j| \geq \sqrt{n}\, \text{side}(Q_j)$.

$$\Longrightarrow |K(x-y)-K(x-c_j)|$$
$$\leq \text{diam}(Q_j)\, 2^{n+1} |x-c_j|^{-n-1}.$$

We integrate over x:

$$\int_{\mathbb{R}^n \setminus B_j} |K * b_j|$$
$$= \int_{\mathbb{R}^n \setminus B_j} \left| \int_{Q_j} K(x-y)\, b_j(y)\, dy \right| dx$$
$$\leq \int_{\mathbb{R}^n \setminus B_j} \int_{Q_j} |K(-)-K(-)|\, |b_j|(y)\, dy\, dx$$
$$\leq \left(\int_{Q_j} |b_j| \right) \left(\int_{\mathbb{R}^n \setminus B_j} \text{diam}(Q_j)\, 2^{n+1} |x-c_j|^{-n-1} \right)$$

2nd factor $= \int^{\infty} 2^{n+1} C(n) \frac{\rho^{n-1}}{\rho^{n+1}}\, d\omega$

$$\le \quad \int \quad \frac{\quad}{\sqrt{n}\,\text{side}(Q_j)}$$

polar coordinates
with center c_j

$$= \int_{\sqrt{n}\,\text{side}}^{\infty} C(n)\, \rho^{-2}\, d\rho \cdot \text{diam}(Q_j)$$

$$= C(n) \frac{\text{diam}(Q_j)}{\text{side}(Q_j)} = C(n).$$

$$\sum_j \int_{\mathbb{R}^n \setminus B_j} |K * b_j|$$

$$\le \sum_j C(n) \int_{Q_j} |b_j|$$

$$= C(n) \|b\|_{L^1}$$

$$\le 2C(n) \|f\|_{L^1}. \qquad \square$$

<u>Rmk</u> If we try to est.

$$\int_{\mathbb{R}^n \setminus B_j} |K * b_j|$$

$$\le \int \int |K(x-y)|\,|b_j(y)|\,dy\,dx$$

$$\leq \int_{\mathbb{R}^n \setminus B_j} \int_{Q_j}$$

$$\leq C \int \int |x-y|^{-n} |b_j(y)|$$

$$\approx \int_{\mathbb{R}^n \setminus B_j} \int_{Q_j} |x-c_j|^{-n} |b_j(y)| \, dy\, dx$$

but $\int_{\mathbb{R}^n \setminus B_j} |x-c_j|^{-n} = \infty$!