Homework 2

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Exercise 1.

Proof. Since $f \neq 0$, there exists a radius r, on which $\int_{B_r(0)} |f| = c > 0$. For $|x| > r$, we have $B_r(0) \subseteq B_{|x|+r}(x)$ and therefore

$$Mf(x) \geq \frac{1}{\mu(B_{|x|+r}(x))} \int_{B_{|x|+r}(x)} |f| \geq \frac{c}{(|x|+r)^n}$$

So that

$$\int_{\mathbb{R}^n} |Mf(x)| \geq \int_{|x|>r} \frac{c}{(|x|+r)^n} = \infty$$

Thus, $Mf \notin L^1(\mathbb{R}^n)$. $\square$

Exercise 2.

Proof. $(a)(i)$

$$0 = \|f\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} \iff \mu\{|f| > \lambda\} = 0 \text{ for all } \lambda > 0$$

$$\iff |f| = 0 \iff f = 0$$

(ii)

$$\|kf\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\{|kf| > \lambda\} = |k| \cdot \sup_{\lambda>0} \lambda \cdot \mu\{|f| > \lambda\} = |k| \cdot \|f\|_{L^{1,\infty}}$$

(iii) Since $|f| + |g| \geq |f+g|$, we have

$$\{|f| > \frac{\lambda}{2}\} \cup \{|g| > \frac{\lambda}{2}\} \supseteq \{|f| + |g| > \lambda\} \supseteq \{|f+g| > \lambda\}$$

so that

$$2(\|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}}) = 2\|f\|_{L^{1,\infty}} + 2\|g\|_{L^{1,\infty}} =$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{2|f| > \lambda\} + \sup_{\lambda>0} \lambda \cdot \mu\{2|g| > \lambda\} =$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f| > \frac{\lambda}{2}\} + \sup_{\lambda>0} \lambda \cdot \mu\{|g| > \frac{\lambda}{2}\} \geq$$

$$\sup_{\lambda>0} \lambda \cdot \mu\{|f|+|g|>\lambda\} \geq \sup_{\lambda>0} \lambda \cdot \mu\{|f+g|>\lambda\} =$$

$$\|f+g\|_{L^{1,\infty}}$$

(b) For $x \in [0,1]$, we have

$$|f_\ell(x)| = \frac{1}{\log \ell}|\sum_{j=1}^{\ell} \frac{1}{x\ell - j}| = \frac{1}{\log \ell}|\sum_{j=1}^{\ell} \frac{1}{j - x\ell}|$$

$$= \frac{1}{\log \ell}|\sum_{j=1}^{\ell} \frac{1}{j} - \frac{x\ell}{j(x\ell - j)}| \geq \frac{1}{\log \ell}(\sum_{j=1}^{\ell} |\frac{1}{j}| - |\frac{x\ell}{j(x\ell - j)}|) =$$

$$\frac{1}{\log \ell}(\sum_{j=1}^{\ell} |\frac{1}{j}| - \sum_{j=1}^{\ell} |\frac{x\ell}{j(x\ell - j)}|) \geq \frac{1}{\log \ell}\sum_{j=1}^{\ell} \frac{1}{j} \geq$$

$$\frac{1}{\log \ell}\sum_{j=1}^{\ell-1} \int_j^{j+1} \frac{1}{j} = \frac{1}{\log \ell} \cdot \log \ell = 1$$

So that $\|f_\ell\|_{L^{1,\infty}} \geq 1 \cdot \mu\{|f_\ell| > 1\} \geq \mu[0,1] = 1$.
(c) Since μ is translation invariant, we have

$$\mu\left\{\left|\frac{1}{x - \frac{j}{\ell}}\right| > \lambda\right\} = \mu\left\{\left|\frac{1}{x}\right| > \lambda\right\}$$

So that

$$\left\|\frac{1}{x - \frac{j}{\ell}}\right\|_{L^{1,\infty}} = \left\|\frac{1}{x}\right\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda \cdot \mu\left\{x \in \mathbb{R} : \left|\frac{1}{x}\right| > \lambda\right\} =$$

$$2\sup_{\lambda>0} \lambda \cdot \mu\left\{x \in [0,\infty) : \frac{1}{x} > \lambda\right\} = 2\sup_{\lambda>0} \lambda \cdot \mu\left\{x \in [0,\infty) : x < \frac{1}{\lambda}\right\} =$$

$$2\sup_{\lambda>0} \lambda \cdot \mu[0, \frac{1}{\lambda}) = 2\sup_{\lambda>0} \lambda \cdot \frac{1}{\lambda} = 2\sup_{\lambda>0} 1 = 2 \cdot 1 = 2$$

So that

$$\|f_\ell\|_{L^{1,\infty}} \leq 2\sum_{j=1}^{\ell} = \frac{1}{\ell \cdot \log \ell}\|\frac{1}{x}\|_{L^{1,\infty}} = \frac{4\ell}{\ell \cdot \log \ell} = \frac{4}{\log \ell}$$

Now, if there exists a norm $\|\cdot\|'$ such that $c\|f_\ell\|' \leq \|f_\ell\|_{L^{1,\infty}} \leq C\|_\ell\|'$, such that $c, C > 0$, then

$$c\|f_\ell\|' \leq \|f_\ell\|_{L^{1,\infty}} \leq \frac{4}{\log \ell} \to 0$$

as $\ell \to \infty$. Since $c > 0$, it follows that $\|f_\ell\|' = 0$, which is a contradiction since $|f_\ell| \geq 1$ on $[0,1]$ for all ℓ. Meaning, $f \neq 0$, so that $\|\cdot\|'$ isn't a proper norm. □

Exercise 3.

Proof. □