Homework 5

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Exercise 1.

Proof. First, we claim $f_b \in L^1(\mathbb{R}^n)$, and $f_s \in L^r(\mathbb{R}^n)$ -

$$\int |f_b| = \int |f_b|^p |f_b|^{1-p} \leq C(p)\|f\|_p^p < \infty$$

$$\int |f_s|^r = \int |f_s|^{r-p} |f_s|^p \leq C(r,p)\|f\|_p^p < \infty$$

Now, since $|Tf| \leq |Tf_b| + |Tf_s|$, we have $\{|Tf| > t\} \subseteq \{|Tf_b| > \frac{t}{2}\} \cup \{|Tf_s| > \frac{t}{2}\}$, so that -

$$\mu\{|Tf| > t\} \leq \mu\{|Tf_b| > \frac{t}{2}\} + \mu\{|Tf_s| > \frac{t}{2}\} \leq$$
$$\frac{2A\|f\|_1}{t} \int |f_b| + \frac{2^r A^r \|f\|_r^r}{t^r} \int |f_s|^r$$

Now,

$$\int_0^\infty t^{q-1} t^{-1} \int_{|f|>t} |f| = \int_{\mathbb{R}^n} |f| \int_0^{|f|} t^{q-2} = \frac{1}{q-1} \int_{\mathbb{R}^n} |f||f|^{q-1} = \frac{\|f\|_q^q}{q-1}$$

since $q > p > 1$, and

$$\int_0^\infty t^{q-1} t^{-r} \int_{|f|\leq t} |f|^r = \int_{\mathbb{R}^n} |f|^r \int_{|f|}^\infty t^{q-1-r} = \frac{1}{r-q} \int_{\mathbb{R}^n} |f|^r |f|^{q-r} = \frac{\|f\|_q^q}{r-q}$$

since $q < r$. Altogether,

$$\|Tf\|_q \leq C\|f\|_q$$

$\square$

Exercise 2.

Proof. (a)

$$\int_{\mathbb{R}^n \setminus B_{2r}(0)} |K(x) - K(x-z)| dx = \int_{\mathbb{R}^n \setminus B_{2r}(0)} \Big| \int_{x-z}^x DK(t) dt \Big| dx \leq$$

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)}\int_{x-z}^{x}\Big|DK(t)\Big|dt\ dx \leq \int_{\mathbb{R}^n\setminus B_{2r}(0)} B|x|^{-n-1}|x-(x-z)|dx =$$

$$B|z|\int_{\mathbb{R}^n\setminus B_{2r}(0)}|x|^{-n-1}dx = C(n)B$$

(b)

$$\int_{\mathbb{R}^n}\Big(K(x)-K(x-x_\xi)\Big)e^{-2\pi i x\xi}\ dx = \int_{\mathbb{R}^n}K(x)e^{-2\pi i x\xi}\ dx - \int_{\mathbb{R}^n}K(x-x_\xi)e^{-2\pi i x\xi}\ dx =$$

$$\hat{K}(\xi)-\int_{\mathbb{R}^n}K(x)e^{-2\pi i(x+x_\xi)\xi}\ dx = \hat{K}(\xi)-e^{-i\pi}\int_{\mathbb{R}^n}K(x)e^{-2\pi i x\xi}\ dx =$$

$$\hat{K}(\xi)+\hat{K}(\xi)=2\hat{K}(\xi)$$

(c) Since K vanishes outside the annulus $B_R(0)\setminus B_\epsilon(0)$, we have

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x)e^{-2\pi i x\xi}\Big|dx \leq \int_{B_{\frac{1}{|\xi|}}(0)\cap(B_R(0)\setminus B_\epsilon(0))}|K(x)e^{-2\pi i x\xi}|dx \leq$$

$$A\int_{B_{\frac{1}{|\xi|}}(0)\cap(B_R(0)\setminus B_\epsilon(0))}\frac{1}{|x|^n} = C(n)A$$

(d)

$$\Big|\int_{B_{\frac{1}{|\xi|}}(0)}K(x-x_\xi)e^{-2\pi i x\xi}\Big|dx \leq \int_{B_{\frac{1}{|\xi|}+x_\xi}(x_\xi)\cap(B_{R+x_\xi}(x_\xi)\setminus B_{\epsilon+x_\xi}(x_\xi))}|K(x)e^{-2\pi i(x+x_\xi)\xi}|dx =$$

$$\int_{B_{\frac{1}{|\xi|}+x_\xi}(x_\xi)\cap(B_{R+x_\xi}(x_\xi)\setminus B_{\epsilon+x_\xi}(x_\xi))}|e^{-i\pi}|\cdot|K(x)e^{-2\pi i(x)\xi}|dx \leq C(n)A$$

(e)

$$\Big|\hat{K}(\xi)\Big| = \Big|\int_{\mathbb{R}^n}K(x)e^{-2\pi i x\xi}\ dx\Big| = \Big|\int_{B_R(0)\setminus B_\epsilon(0)}K(x)e^{-2\pi i x\xi}\ dx\Big| \leq$$

$$\int_{B_R(0)\setminus B_\epsilon(0)}|K(x)e^{-2\pi i x\xi}|dx \leq A\int_{B_R(0)\setminus B_\epsilon(0)}\frac{1}{|x|^n}\ dx = C(n)A$$

□

Exercise 3.

Proof. (a) (i) $|K\mathbf{1}_{B_r\setminus B_\epsilon}(x)| \leq |K(x)| \leq A|x|^{-n}$
(ii)

$$\int_{\mathbb{R}^n\setminus B_{2r}(0)}|K_{\epsilon,R}(x)-K_{\epsilon,R}(x-z)|dx \leq \int_{\mathbb{R}^n\setminus B_{2r}(0)}|K_{\epsilon,R}(x)|+|K_{\epsilon,R}(x-z)|dx \leq$$

$$\frac{2A}{|\epsilon|^n} \cdot \mu\{B_R(0)\} = C(n)A$$

(iii)

$$\int_{B_s(0)\setminus B_r(0)} K = \int_{B_s(0)} K - \int_{B_r(0)} K = 0 - 0 = 0$$

(b)

$$\|K_{\epsilon,R} * f\|_p^p = \int |K_{\epsilon,R}(x-y)f(y)|^p dy \leq \frac{A}{|\epsilon|^{np}} \int_{B_R(0)\setminus B_\epsilon(0)} |f|^p \leq C(n,p)\|f\|_p^p$$

(c) Since f is smooth and has compact support, by the Extreme Value Theorem, f achieves a minimum and a maximum on it's domain. Let m, M be the minimal, and maximal values of f, respectively. WLOG, suppose $R \geq a$. Then,

$$|(K_{\epsilon,R}*f)(x)| = \Big|\int K_{\epsilon,R}(y)f(x-y)dy\Big| = \Big|\int_{|x|\geq a} K_{\epsilon,R}(y)f(x-y)dy + \int_{B_a(0)} K_{\epsilon,R}(y)f(x-y)dy\Big| \leq$$

$$\Big|\int_{|x|\geq a} K_{\epsilon,R}(y)f(x-y)dy\Big| + \Big|\int_{B_a(0)} K_{\epsilon,R}(y)f(x-y)dy\Big| \leq$$

$$\int_{|x|\geq a} |K_{\epsilon,R}(y)f(x-y)|dy + \Big|\int_{B_a(0)\setminus B_\epsilon(0)} K(y)f(x-y)dy\Big| \leq$$

$$|M|\Big(A\int_{\{|x|\geq a\}\cap B_R(0)} \frac{1}{|a|^n} + |\int_{B_a(0)\setminus B_\epsilon(0)} K(y)dy|\Big) \leq |M|\Big(\mu\{B_R(0)\}\cdot\frac{A}{|a|^n} + 0\Big) = C(n)A\cdot|M|$$

for every $\epsilon > 0$, and for every $R \geq a$. Thus the integral is absolutetely convergent and the limit exists. Now, since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we can approximate $g \in L^p(\mathbb{R}^n)$ with a sequence $C_c^\infty(\mathbb{R}^n) \ni g_j \to g$, with convergence in L^p. Thus, by the dominated convergence theorem, we have

$$\|K_{\epsilon,R} * g_j\|_{L^p}^p = \int_{\mathbb{R}^n} |(K_{\epsilon,R} * g_j)(x)|^p = \int_{\mathbb{R}^n} \Big|\int K_{\epsilon,R}(y)g_j(x-y)dy\Big|^p \nearrow$$

$$\int_{\mathbb{R}^n} \Big|\int K_{\epsilon,R}(y)g(x-y)dy\Big|^p \nearrow \int_{\mathbb{R}^n} \Big|\int K(y)g(x-y)dy\Big|^p =$$

$$\int_{\mathbb{R}^n} |(K * g)(x)|^p = \|K * g\|_{L^p}^p$$

as $\epsilon \to 0,\ R, j \to \infty$. $\square$