

# Homework 5

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## Exercise 1.

*Proof.* First, we claim  $f_b \in L^1(\mathbb{R}^n)$ , and  $f_s \in L^r(\mathbb{R}^n)$  -

$$\int |f_b| = \int |f_b|^p |f_b|^{1-p} \leq C(p) \|f\|_p^p < \infty$$

$$\int |f_s|^r = \int |f_s|^{r-p} |f_s|^p \leq C(r, p) \|f\|_p^p < \infty$$

Now, since  $|Tf| \leq |Tf_b| + |Tf_s|$ , we have  $\{|Tf| > t\} \subseteq \{|Tf_b| > \frac{t}{2}\} \cup \{|Tf_s| > \frac{t}{2}\}$ , so that -

$$\begin{aligned} \mu\{|Tf| > t\} &\leq \mu\{|Tf_b| > \frac{t}{2}\} + \mu\{|Tf_s| > \frac{t}{2}\} \leq \\ &\frac{2A\|f\|_1}{t} \int |f_b| + \frac{2^r A^r \|f\|_r^r}{t^r} \int |f_s|^r \end{aligned}$$

Now,

$$\int_0^\infty t^{q-1} t^{-1} \int_{|f|>t} |f| = \int_{\mathbb{R}^n} |f| \int_0^{|f|} t^{q-2} = \frac{1}{q-1} \int_{\mathbb{R}^n} |f| |f|^{q-1} = \frac{\|f\|_q^q}{q-1}$$

since  $q > p > 1$ , and

$$\int_0^\infty t^{q-1} t^{-r} \int_{|f|\leq t} |f|^r = \int_{\mathbb{R}^n} |f|^r \int_{|f|}^\infty t^{q-1-r} = \frac{1}{r-q} \int_{\mathbb{R}^n} |f|^r |f|^{q-r} = \frac{\|f\|_q^q}{r-q}$$

since  $q < r$ . Altogether,

$$\|Tf\|_q \leq C \|f\|_q$$

□

## Exercise 2.

*Proof.* (a)

$$\int_{\mathbb{R}^n \setminus B_{2r}(0)} |K(x) - K(x-z)| dx = \int_{\mathbb{R}^n \setminus B_{2r}(0)} \left| \int_{x-z}^x DK(t) dt \right| dx \leq$$

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_{2r}(0)} \int_{x-z}^x |DK(t)| dt dx &\leq \int_{\mathbb{R}^n \setminus B_{2r}(0)} B|x|^{-n-1} |x - (x-z)| dx = \\ &B|z| \int_{\mathbb{R}^n \setminus B_{2r}(0)} |x|^{-n-1} dx = C(n)B \end{aligned}$$

(b)

$$\begin{aligned} \int_{\mathbb{R}^n} (K(x) - K(x - x_\xi)) e^{-2\pi i x \xi} dx &= \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx - \int_{\mathbb{R}^n} K(x - x_\xi) e^{-2\pi i x \xi} dx = \\ \hat{K}(\xi) - \int_{\mathbb{R}^n} K(x) e^{-2\pi i (x+x_\xi) \xi} dx &= \hat{K}(\xi) - e^{-i\pi} \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx = \\ \hat{K}(\xi) + \hat{K}(\xi) &= 2\hat{K}(\xi) \end{aligned}$$

(c) By the cancellation condition, we have  $|\int_{B_{\frac{1}{|\xi|}}(0)} K(x)| = 0$ . Therefore,

$$\begin{aligned} \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) e^{-2\pi i x \xi} \right| &= \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) e^{-2\pi i x \xi} - \int_{B_{\frac{1}{|\xi|}}(0)} K(x) \right| = \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x) (e^{-2\pi i x \xi} - 1) \right| \leq \\ &\int_{B_{\frac{1}{|\xi|}}(0)} |K(x)| \cdot |e^{-2\pi i x \xi} - 1| = C(n)A \end{aligned}$$

(d)

$$\begin{aligned} \left| \int_{B_{\frac{1}{|\xi|}}(0)} K(x - x_\xi) e^{-2\pi i x \xi} \right| dx &\leq \int_{B_{\frac{1}{|\xi|} + x_\xi}(x_\xi) \setminus (B_{R+x_\xi}(x_\xi) \cup B_{\epsilon+x_\xi}(x_\xi))} |K(x) e^{-2\pi i (x+x_\xi) \xi}| dx = \\ &\int_{B_{\frac{1}{|\xi|} + x_\xi}(x_\xi) \setminus (B_{R+x_\xi}(x_\xi) \cup B_{\epsilon+x_\xi}(x_\xi))} |e^{-i\pi}| \cdot |K(x) e^{-2\pi i (x+x_\xi) \xi}| dx \leq C(n)A \end{aligned}$$

(e)

$$\begin{aligned} |\hat{K}(\xi)| &= \left| \int_{\mathbb{R}^n} K(x) e^{-2\pi i x \xi} dx \right| = \left| \int_{B_R(0) \setminus B_\epsilon(0)} K(x) e^{-2\pi i x \xi} dx \right| \leq \\ &\int_{B_R(0) \setminus B_\epsilon(0)} |K(x) e^{-2\pi i x \xi}| dx \leq A \int_{B_R(0) \setminus B_\epsilon(0)} \frac{1}{|x|^n} dx = C(n)A \end{aligned}$$

□

**Exercise 3.**

*Proof.* (a) (i)  $|K\mathbf{1}_{B_r \setminus B_\epsilon}(x)| \leq |K(x)| \leq A|x|^{-n}$   
(ii)

$$\int_{\mathbb{R}^n \setminus B_{2r}(0)} |K_{\epsilon,R}(x) - K_{\epsilon,R}(x-z)| dx \leq \int_{\mathbb{R}^n \setminus B_{2r}(0)} |K_{\epsilon,R}(x)| + |K_{\epsilon,R}(x-z)| dx \leq$$

$$\frac{2A}{|\epsilon|^n} \cdot \mu\{B_R(0)\} = C(n)A$$

(iii)  
(b)

$$\|K_{\epsilon,R} * f\|_p^p = \int |K_{\epsilon,R}(x-y)f(y)|^p dy \leq \frac{A}{|\epsilon|^{np}} \int_{B_R(0) \setminus B_\epsilon(0)} |f|^p \leq C(n,p) \|f\|_p^p$$

(c) Since  $f$  is smooth and has compact support, by the Extreme Value Theorem,  $f$  achieves a minimum and a maximum on its domain. Let  $m, M$  be the minimal, and maximal values of  $f$ , respectively. WLOG, suppose  $R \geq 2a$ . Then,

$$\begin{aligned} |(K_{\epsilon,R} * f)(x)| &= \left| \int K_{\epsilon,R}(y)f(x-y)dy \right| = \left| \int_{|x| \geq 2a} K_{\epsilon,R}(y)f(x-y)dy + \int_{B_{2a}(0)} K_{\epsilon,R}(y)f(x-y)dy \right| \leq \\ &= \left| \int_{|x| \geq 2a} K_{\epsilon,R}(y)f(x-y)dy \right| + \left| \int_{B_{2a}(0)} K_{\epsilon,R}(y)f(x-y)dy \right| \leq \\ &= \int_{|x| \geq 2a} |K_{\epsilon,R}(y)f(x-y)|dy + \left| \int_{B_{2a}(0) \setminus B_\epsilon(0)} K(y)f(x-y)dy \right| \leq \\ &= M \left( A \int_{|x| \geq 2a} \frac{1}{2^n |a|^n} + \left| \int_{B_{2a}(0) \setminus B_\epsilon(0)} K(y)dy \right| \right) = C(n)A \cdot M + 0 = C(n)A \cdot M \end{aligned}$$

for every  $\epsilon > 0$ . Thus the integral is absolutely convergent and the limit exists. Now, since  $C_c^\infty(\mathbb{R}^n)$  is dense in  $L^p(\mathbb{R}^n)$ , we can approximate  $g \in L^p(\mathbb{R}^n)$  with a sequence  $C_c^\infty(\mathbb{R}^n) \ni g_j \rightarrow g$ , with convergence in  $L^p$ . Thus

$$K * g = \lim_{\epsilon \rightarrow 0, R, j \rightarrow \infty} K_{\epsilon,R} * g_j$$

since convolution is a continuous operator. □