Lecture 9

$$(af)' = af' \qquad (f/g)' = \frac{f'g - fg'}{g^2}$$
$$(f+g)' = f' + g'$$
$$(f \cdot g)' = f'g + fg' \qquad (f(g))' = \frac{df}{dg}\frac{dg}{dz}$$

if $\frac{df}{dz}$ exists!

Cauchy Reimann Definitions

$$f(z) = u(x,y) + iv(x,y) \qquad z = x + iy$$

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z} \quad \begin{cases} \text{I) } \delta z = \delta z^* \ (\mathbb{R}) \\ \text{II) } \delta z = -\delta z^* \ (\mathbb{C}) \end{cases}$$

Real
$\delta z = \delta z^*$
$\delta z = \delta x$

$$\text{I)} = \lim_{\delta x \to 0} \frac{u(x_0+\delta x, y_0) + iv(x_0 + \delta x, y_0) - (u(x_0,y_0) + iv(x_0,y_0))}{\delta x}$$
$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

complex
$\delta z = -\delta z^*$
$\delta z = \delta y \times i$

$$\text{II)} = \lim_{\delta y \to 0} \frac{u(x_0, y+\delta y) + iv(x_0, y+\delta y) - (u(x_0,y_0) + iv(x_0,y_0))}{i\delta y}$$
$$= \cancel{\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)}$$
$$= \frac{1}{i}\left(\frac{\partial u}{\partial y}(x_0,y_0) + i\frac{\partial v}{\partial y}(x_0,y_0)\right)$$
$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)$$

Conclusion

for df/dz to exist:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x}(x_0,y_0) &= \frac{\partial v}{\partial y}(x_0,y_0) \\ \frac{\partial u}{\partial y}(x_0,y_0) &= -\frac{\partial v}{\partial x}(x_0,y_0) \end{aligned}}$$

These are the cauchy-reimen conditions and are necessary for df/dz to exist.

*) with continunity of
$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$
at (x_0, y_0), these equations become sufficent.

Example $\quad f(z) = z^2$

$$f(z) = (x+iy)^2 = \underbrace{x^2-y^2}_{u(x,y)} + \underbrace{2ixy}_{v(x,y)}$$

$u(x,y) = x^2-y^2$
$v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x \qquad ✓$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \qquad ✓$$

Example $\quad f(z) = |z|^2$

$f(z) = x^2+y^2 \qquad u(x,y) = x^2+y^2 \quad v(x,y) = 0$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 0 \qquad \times$$

no $\frac{df}{dz}$!!

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial x} = 0 \qquad \times$$

But this function is differential only at $z=0$.

Differential Equations

Differential Equations

Imagine a function $F(x,y)$ but where $y = y(x)$.

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0$$

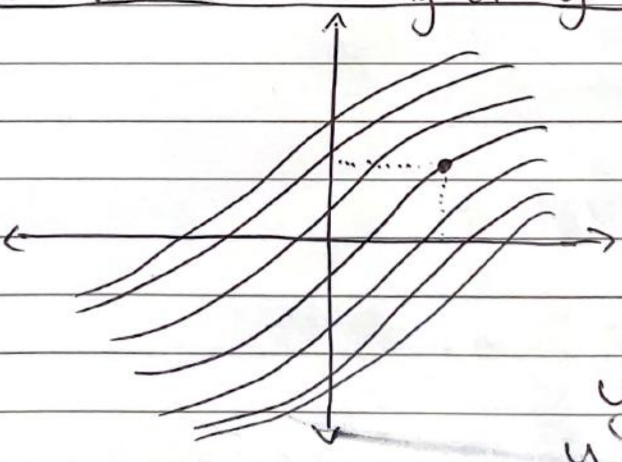
This is a differential equation of n^{th} order. It is ordinary!

1st Order $\quad F(x, y, y') = 0$

we resolve for y' to get an equation in the form

$$y' = \frac{dy}{dx} = f(x,y)$$

Geometrical Meaning of $y' = f(x,y)$?



We get an infinite number of solutions to our equation. Each line fulfills our requirement.

$$y' = f(x)$$
$$y = \int f(x)\,dx + c$$

~~Other solutions are just vertical transformations of each other.~~ To get a unique solution we have to use initial conditions to get a specific solution.

Existence & Uniqueness Theorem

$y' = f(x,y)$ $\quad$ $f(x,y)$ is continuous in R.

$$R = \{(x,y) : |x - x_0| \leq a,\ |y - y_0| \leq b\}$$

if $\frac{\partial f}{\partial y}$ is continous on R, then the solution $y(x)$ of $y' = f(x,y)$ I) exists II) is unique.