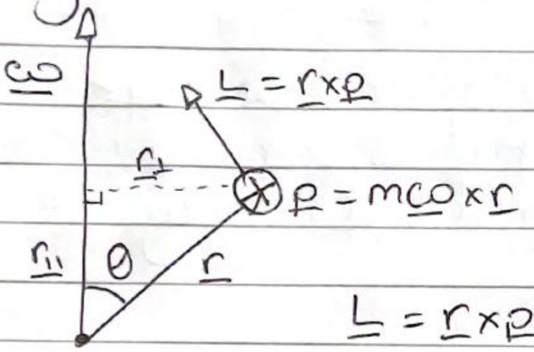
Classical Mechanics 1a

This course only covers:
- bodies with $\underline{L}$ parallel to $\underline{\omega}$
- rotations about axes.

Working out $\underline{L}_{\parallel}$

Suppose m is a particle at $\underline{r}$ in a rigid body rotating ab $\underline{\omega}$.



$$\underline{v} = \underline{\omega} \times \underline{r} = \omega r \sin\theta = \omega r_{\perp} \hat{\phi}$$

$$\underline{L} = \underline{r} \times \underline{p} = (\underline{r}_{\parallel} + \underline{r}_{\perp}) \times (m\omega r_{\perp} \hat{\phi})$$
$$= m\omega r_{\perp} (\underline{r}_{\parallel} \times \hat{\phi} + \underline{r}_{\perp} \times \hat{\phi})$$
$$= m\omega r_{\perp} (\underbrace{-r_{\parallel} \hat{r}_{\perp}}_{\text{not of interest to us}} + r_{\perp} \hat{\omega})$$

$(= \underline{L}_{\perp})$

$$\underline{L}_{\parallel} = m r_{\perp}^2 \omega = m r_{\perp} v_{\phi} \quad \leftarrow \text{from orbits section}$$

Adding the contributions from every mass gives:

$$L_{\parallel} = \left(\sum_i m_i r_{i\perp}^2 \right) \omega$$

We define the moment of inertia to be

$$\boxed{I \triangleq \sum_i m_i r_{i\perp}^2}$$

The expression for the parallel angular momentum is found by:

$$\boxed{L_{\parallel} = I\omega}$$

Since $\underline{G}_{\parallel} = \frac{d\underline{L}_{\parallel}}{dt} = G_{\parallel} = \frac{dL_{\parallel}}{dt}$, we get that:

$$\boxed{G_{\parallel} = I\frac{d\omega}{dt}}$$

The rotation KE can be found by:

$$\boxed{K = \frac{1}{2}I\omega^2}$$

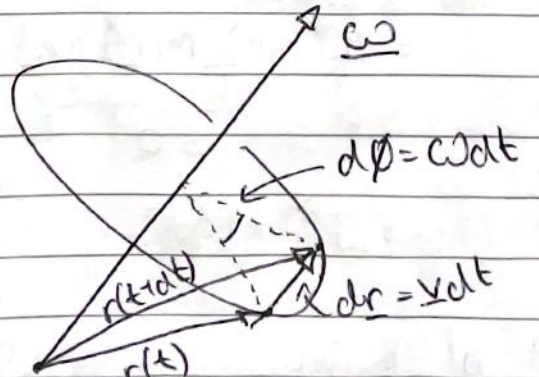
quick proof: $K = \frac{1}{2}\sum_i m_i v_i^2$

$$= \frac{1}{2}\sum_i m_i (r_{i\perp}\omega)^2$$

$$= \frac{1}{2}\left(\sum_i m_i r_{i\perp}^2\right)\omega^2$$

$$= \frac{1}{2}I\omega^2 \quad \blacksquare$$

The rotational work can be found by:



$$\begin{aligned} dW &= \underline{F}\cdot d\underline{r} \\ &= \underline{F}\cdot(\underline{v}dt) \\ &= \underline{F}\cdot(\underline{\omega}\times\underline{r})dt \\ &= \underline{\omega}\cdot(\underline{r}\times\underline{F})dt \\ &= \underline{\omega}\cdot\underline{G}dt \\ &= \omega G_{\parallel}dt \\ &= G_{\parallel}d\phi \end{aligned}$$

cyclic permutation of triple product

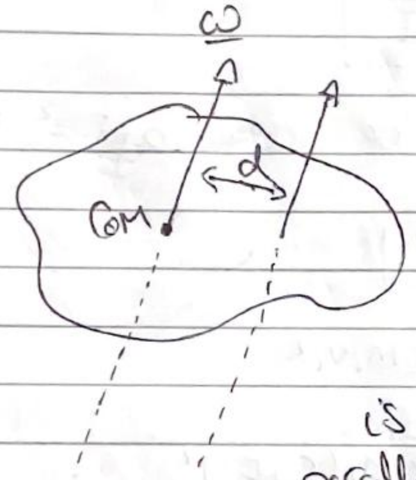
$$\boxed{dW = G_{\parallel}d\phi}$$

Rotational Work Energy Theorem

$$\frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \int_{\phi_i}^{\phi_f} G_{11}(\phi)\,d\phi$$

Proof indentical to non-rotational version.

Parallel Axis Theorem

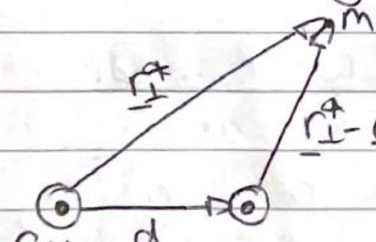


If I^* is the moment of inertia about an axis through the CoM then

$$I = I^* + Md^2$$

is the moment of inertia about a parallel axis distance 'd' away.

Proof looking down the rotational axis



$$I = \sum_i m_i |\underline{r}_\perp^* - \underline{d}|^2$$

$$= \sum_i m_i \left[(\underline{r}_\perp^*)^2 - 2\underline{d}\cdot\underline{r}_\perp^* + d^2\right]$$

$$= I^* + \underbrace{\left(\sum_i m_i \underline{r}_\perp^*\right)\cdot 2\underline{d}}_{=0} + d^2 \sum_i m_i$$

$$= I^* + Md^2$$

Centre of Mass & Moment of Inertia as Integrals

lets introduce a mass density $\rho(r)$ which is defined as.

$$dm = \left\{ \begin{matrix} \text{mass element in volume} \\ \text{element } dV \text{ at } \underline{r} \end{matrix} \right\} = \rho(\underline{r})dV$$

Mass:

$$M = \sum dm = \sum \rho(\underline{r})dV \to \int \rho(\underline{r})dV$$

CoM:

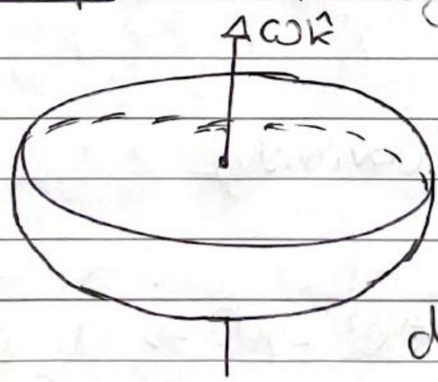
$$M\underline{R} = \sum \underline{r}\,dm = \sum \underline{r}\rho(\underline{r})dV \to \int \underline{r}\rho(\underline{r})dV$$

Moment of Inertia:

$$I = \sum r_\perp^2 dm = \sum r_\perp^2 \rho(\underline{r})dV \to \int r_\perp^2 \rho(\underline{r})dV$$

N.B. There are 3D integrals, which we haven't covered yet.

<u>Example</u> Uniform cylinder. mass density ρ; radius a; height h.



Split into rings, width $dr = dr_\perp$, area

$$dA = \left\{ \begin{matrix} \pi(r+dr)^2 - r^2\pi = 2\pi r dr \\ \frac{dA}{dr}dr = \frac{d(\pi r^2)}{dr}dr = 2\pi r dr \end{matrix} \right.$$

$dV = \cancel{}\, h dA$ N.B. ignoring $O[(dr)^2]$ terms

<u>Mass</u> $M = \sum dm = \sum \rho 2\pi r h dr \to \int_0^a \rho 2\pi r h dr$

$$= \left[\rho \pi r^2 h\right]_0^a = \rho \pi a^2 h$$

<u>CoM</u>

$\underline{R} = \underline{0}$ (obviously)

Moment of Inertia

$$I = \sum r^2 dm = \sum r^2 \rho 2\pi r h dr \to \int_0^a \rho 2\pi r^3 h \, dr$$

$$= \left[\rho \tfrac{1}{2}\pi r^4 h\right]_0^a = \rho \tfrac{1}{2}\pi a^4 h = \tfrac{1}{2} M a^2$$

↑ mass of object.

Example Uniform sphere

- split into discs, height dz
- radius of disc at height z is $\sqrt{a^2 - z^2}$

Mass

$$M = \sum dm = \sum \rho \pi (a^2 - z^2) dz$$

$$\to \int_{-a}^{a} \rho \pi (a^2 - z^2) dz = \rho \pi \tfrac{4}{3} a^3$$

CoM

$\underline{R} = \underline{0}$ (obviously)

Moment of Inertia

office hours

$$I = \sum dI$$

$$= \sum \rho \tfrac{1}{2}\pi (a^2 - z^2)^2 dz \to \int_{-a}^{a} \rho \tfrac{1}{2}\pi (a^2 - z^2)^2 dz$$

$$= \frac{8\pi\rho a^5}{15} = \tfrac{2}{5}\left(\rho \tfrac{4}{3}\pi a^3\right) a^2 = \tfrac{2}{5} M a^2$$

why do I have to do it this way.