Lecture 13

Inhomogeneous 2nd Order Linear Diff Eqⁿ

(*) $$ay'' + by' + cy = f(x)$$ a, b, c are constant coefficients

Assume $y_1(x)$ & $y_2(x)$ are solutions of (*)

$$\begin{aligned} &ay_1'' + by_1' + cy_1 = f(x) \\ -\ &ay_2'' + by_2' + cy_2 = f(x) \\ \hline &a(y_1''-y_2'') + b(y_1'-y_2') + c(y_1-y_2) = 0. \end{aligned}$$

∴ $(y_1 - y_2)$ is a solution of the homogeneous equation when $f(x) \neq 0$.

$$y_1(x) - y_2(x) = Ay^{(1)}(x) + By^{(2)}(x)$$

$y^{(1)(2)}(x)$ are the linearly independent solutions of the homogeneous equation $f(x) = 0$.

$$\boxed{y_1(x) = \underset{\uparrow}{A}y^{(1)}(x) + \underset{\uparrow}{B}y^{(2)}(x) + \underset{\uparrow}{y_2(x)}}$$

can be found by I.C. particular inhomogeneous case solution

The general solution of the inhomogeous equation equal to the general solution of the homogeous case plus any particular solution of the inhomogeous case.

Example ‡ $y'' + y' - 6y = 5$

homogeneous case: $f(x) = 0$.

$$\alpha^2 + \alpha - 6 = 0$$
$$(\alpha + 3)(\alpha - 2) = 0$$
$$\alpha = -3, 2$$

$$y(x) = Ae^{-3x} + Be^{2x}$$

let $y_p(x) = c$, $\therefore y_p'' + y_p' - 6y = 0 + 0 - 6c = 5$

$$\therefore c = -\frac{5}{6}$$

$$y(x) = Ae^{-3x} + Be^{2x} - \frac{5}{6}$$

<u>Example</u> $\quad y'' + y - 6y = \underbrace{x^2 - 1}$ y_p will be a polynomial.

$$y(x) = Ae^{-3x} + Be^{2x}$$

$$y_p(x) = p_n(x) = p_2(x)$$
$$y_p(x) = a_1 x^2 + a_2 x + a_3$$
$$y_p''(x) = 2a_1 \qquad y_p'(x) = 2a_1 x + a_2$$

$$2a_1 + 2a_1 x + a_2 - 6a_1 x^2 + 6a_2 x - 6a_3 = x^2 - 1$$
$$-6a_1 = 1 \qquad 2a_1 - 6a_2 = 0 \qquad 2a_1 + a_2 - 6a_3 = -1$$
$$a_1 = -1/6 \qquad a_2 = -1/2 \qquad a_3 = 1/36$$

$$y(x) = Ae^{-3x} + Be^{2x} - \frac{1}{6}x^2 - \frac{1}{2}x + \frac{1}{36}$$

We call this method of undetermined coefficients. Also works for exponential inhomogeneous.

<u>Example</u> $\quad y'' + y' - 6y = e^{2ix}$

$y_p(x) = ce^{ax}$

$ca^2e^{ax} + cae^{ax} + 6ce^{ax} = e^{2ix}$

$(ca^2 + ca - 6c)e^{ax} = e^{2ix}$

$a = 2i$

$[c(2i)^2 + c(2i) - 6c] = 1$

$-4c + 2ic - 6c = 1$

$c = \frac{1}{10+2i} = -\frac{10+2i}{104} = -\frac{5+i}{52}$

$y(x) = Ae^{-3x} + Be^{2x} - \frac{5+i}{52}e^{2i\lambda}$

Use of $f(z)$ in ODEs

$x(t) = Ae^{\alpha t} + Be^{\alpha^* t}$ (case 3)

$\downarrow$ I.C.

$x(t) = x_0 \frac{\omega_0}{\omega} e^{-\frac{\gamma t}{2}} \cos(\omega t + \phi)$

Now, the other way round:

$\rightarrow$ $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F}{m}\cos(\omega t)$ $\mathbb{R}$

$\mathrm{Re}(x)$

$\rightarrow$ $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = \frac{F}{m}e^{i\omega t}$ $\mathbb{C}$

$z(t) = x(t) + iy(t)$

Trail Solution $z(t) = Ae^{i\omega t}$

$(-\omega^2 + i\gamma\omega + \omega_0^2)Ae^{i\omega t} = \frac{F}{m}e^{i\omega t}$

$A = A_0e^{i\varphi} = \frac{F/M}{-\omega^2 + i\gamma\omega + \omega_0^2}$

$$A_0 = \frac{F/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \qquad \tan\varphi = \frac{-\omega\gamma}{\omega_0^2 - \omega^2}$$

$$x(t) = Re\left[z(t)\right]$$

$$= \frac{F/m}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\cos(\omega t + \varphi)$$

should go here in notes.

$$\boxed{z(t) = Ae^{i\omega t} = A_0 e^{i\omega t + \varphi}}$$

Example

$$m\frac{d\underline{v}}{dt} = q\underline{v} \times \underline{B} \qquad \underline{B} = (0,0,B)$$

$$m\frac{dv_x}{dt} = qv_xB \qquad m\frac{dv_y}{dt} = -qv_yB$$

Introduce, $V = V_x + iV_y$

$$\frac{dv}{dt} = \frac{dv_x}{dt} + i\frac{dv_y}{dt}$$

$$= \frac{1}{m}\left[qV_xB + i(-qV_yB)\right] \qquad \left(\omega_c = \frac{qB}{m}\right)$$

$$= -i\omega_c V$$

$$V(t) = V(0)e^{-i\omega_c t}$$

$$V_x(t) = V_x(0)\cos(\omega_c t) + V_y(0)\sin(\omega_c t)$$

$$V_y(t) = -V_x(0)\sin(\omega_c t) + V_y(0)\cos(\omega_c t)$$