Double-Slit Diffraction Pattern

The aperture function of a double slit can be modelled by two dirac delta functions.

$$A(x) = \delta(x - d/2) + \delta(x + d/2)$$

The diffraction integral becomes

$$E_p(\theta) = C \int_{-\infty}^{\infty} [\delta(x - d/2) + \delta(x + d/2)] e^{-ikx\sin\theta} dx$$

$$= C[e^{-ik\frac{d}{2}\sin\theta} + e^{+ik\frac{d}{2}\sin\theta}]$$

$$= 2C\cos\left(\frac{kd\sin\theta}{2}\right)$$

The intensity pattern is given by

$$I = |E_p|^2 = 4I_0\cos^2\left(\frac{kd\sin\theta}{2}\right)$$

The maxima are found out $d\sin\theta = m\lambda$.

Single Slit Diffraction

$$A(x) = \begin{cases} 1 & |x| < a/2 \\ 0 & |x| > a/2 \end{cases} = \mathrm{rect}\left(\frac{x}{a}\right)$$

$$E_p(\theta) = C\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{x}{a}\right) e^{-ikx\sin\theta} dx = C\int_{-a/2}^{a/2} e^{-ikx\sin\theta} dx$$

$$E_p(\theta) = \left[\frac{e^{-ikx\sin\theta}}{-ik\sin\theta} \right]_{-a/2}^{a/2} = \frac{e^{-ik\frac{a}{2}\sin\theta} - e^{ik\frac{a}{2}\sin\theta}}{ik\sin\theta}$$

Given that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\operatorname{sinc} x = \frac{\sin x}{x}$ ($\operatorname{sinc} 0 = 1$) we get that

$$E_p(\theta) = a \frac{\sin\left(\frac{ka\sin\theta}{2}\right)}{\left(\frac{ka\sin\theta}{2}\right)} = \operatorname{sinc}\left(\frac{ka\sin\theta}{2}\right)$$

The intensity pattern is given by $I_p = a^2 \operatorname{sinc}^2\left(\frac{ka\sin\theta}{2}\right)$. The central peak scales with a and its width $1/a$. As $a \to \infty$ $E_p(\theta) \to \delta(\theta)$. This does not mean a small point in space but that the light is unidirectional ($\theta = 0$).

The first minima is at $\sin\theta = \frac{\lambda}{a}$. Minima occur at $\frac{ka\sin\theta}{2} = \pi a \sin\theta = \pi m \Rightarrow a\sin\theta = m$ ($m \neq 0$).

Cosine Aperture Function

Given the cosine aperture function $A(x) = \cos\left(\frac{2\pi}{d}x\right)$ we get the diffraction pattern.

$$E_p(\theta) = \int_{-\infty}^{\infty} \cos\left(\frac{2\pi}{d}x\right) e^{-ikx\sin\theta} \, dx$$

Given that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ we get

$$E_p(\theta) = \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{i\frac{2\pi}{d}x} + e^{-i\frac{2\pi}{d}x}\right) e^{-ikx\sin\theta} \, dx$$

$$E_p(\theta) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(k\sin\theta - \frac{2\pi}{d})x} + e^{-i(k\sin\theta + \frac{2\pi}{d})x} \, dx$$

$$E_p(\theta) = \frac{1}{2}\left[\delta\left(k\sin\theta - \frac{2\pi}{d}\right) + \delta\left(k\sin\theta + \frac{2\pi}{d}\right)\right]$$

$$\left(k = \frac{2\pi}{\lambda}\right)$$

$$E_p(\theta) = \frac{1}{2}\left[\delta\left(\sin\theta - \frac{\lambda}{d}\right) + \delta\left(\sin\theta + \frac{\lambda}{d}\right)\right]$$

Fourier Optics

The cosine 'aperture' may appear artificial but we can decompose any aperture into cosine functions.

$$A(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(m\frac{2\pi x}{d}\right)$$