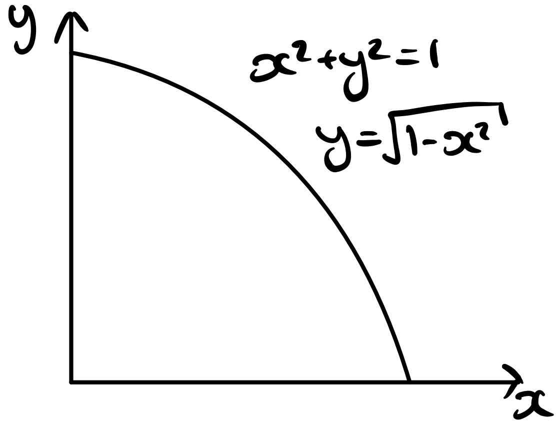
Change of Variables in 1D Integration

lets find the area of a quadrant of a curve, radius 1.

$$A = \int_0^1 y\,dx$$

$$= \int_0^1 \sqrt{1-x^2}\,dx$$



We can now use change of variables.

REMEBER: 3 Things

- ☐ the intergrand: $x = \sin\theta \quad \sqrt{1-\sin^2\theta} = \cos\theta$
- ☐ the limits: $0 < x < 1 \equiv 0 < \theta < \pi/2$
- ☐ the differential: $dx = \frac{dx}{d\theta}\,d\theta$

$x = \sin\theta \quad \frac{dx}{d\theta} = \cos\theta$

$$A = \int_0^1 y\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos\theta\cos\theta\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}(\cos 2\theta + 1) = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$dx = \underbrace{\frac{dx}{d\theta}}_{\text{1D Jacobian}}\,d\theta$$

is 'how much x changes as θ change by $d\theta$'.

$d\theta$ does not have to be evenly spaced, see earlier notes on the Reimann sum.

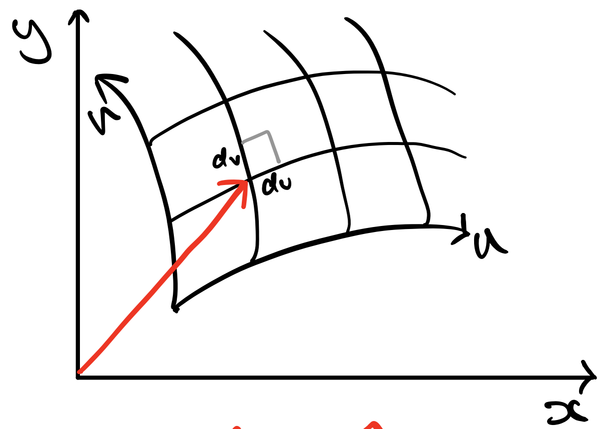
Change of Variables in 2D

$$\iint_R f(x,y)\,dx\,dy \rightarrow \iint_{R'} f(x(u,v),y(u,v))\,\underbrace{|J|}_{\text{Jacobian}}\,du\,dv$$

Change 3 things:

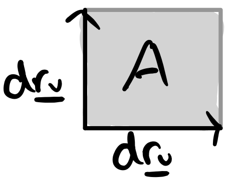
- ☐ integrand $f(x,y) \rightarrow f(u,v)$
- ☐ the limits (always draw)
- ☐ the differential $dx\,dy \rightarrow |J|\,du\,dv$

there isn't an analytic way to do step 2 - you do really need to draw it.



The Jacobian

What is the area of the element $du\,dv$?



$$d\underline{r} = \frac{\partial \underline{r}}{\partial u}du + \frac{\partial \underline{r}}{\partial v}dv = d\underline{r}_u + d\underline{r}_v \qquad \underline{r} = x\hat{i} + y\hat{j}$$

We can use the cross product to find the area.

$$d\underline{A} = d\underline{r}_u \times d\underline{r}_v = \left[\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right] du\,dv$$

$$\frac{\partial \underline{r}_u}{\partial u} = \frac{\partial}{\partial u}(x\hat{i}+y\hat{j}) = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} \qquad \frac{\partial \underline{r}_v}{\partial v} = \frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}$$

$$d\underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\,dv = \hat{k}\underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\text{Jacobian}} du\,dv$$

Example: $\displaystyle\iint_R (x+y)\,dx\,dy$

Boundries:

$x+y=1 \quad x+y=3$

$y-x=-1 \quad y-x=1$

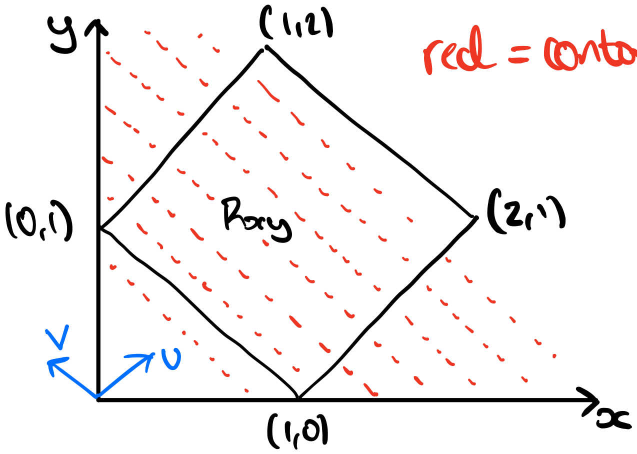
Our new variables will be:

$u = x+y \qquad v = y-x$

$x = (u-v)/2$

$y = (u+v)/2$

$$I = \iint_{R_{xy}} f(x,y)\,dx\,dy$$



red = contours

$$= \int_{x=0}^{1}\int_{y=1-x}^{1+x} (x+y)\,dy\,dx \; +$$

$$\int_{x=1}^{2}\int_{y=x-1}^{3-x} (x+y)\,dy\,dx$$

$$\int_{x=0}^{1} \left[xy + \tfrac{1}{2}y^2\right]_{1-x}^{1+x} dx$$

$$= \int_{x=0}^{1} \left[x(1+x) + \tfrac{1}{2}(1+x)^2 - x(1-x) - \tfrac{1}{2}(1-x)^2\right] dx$$

$$= \int_{x=0}^{1} \left[x + x^2 \cancel{+\tfrac{1}{2}} + x \cancel{+\tfrac{1}{2}x^2} \cancel{-x} + x^2 \cancel{-\tfrac{1}{2}} \cancel{+x} \cancel{-\tfrac{1}{2}x^2}\right] dx$$

$$= \int_{x=0}^{1} 2x^2 + 2x = 2\left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 = \frac{5}{3}$$

$$\int_{x=1}^{2} \left[xy + \tfrac{1}{2}y^2\right]_{x-1}^{3-x} dx$$

$$= \int_{x=1}^{2} x(3-x) + \tfrac{1}{2}(3-x)^2 - x(x-1) - \tfrac{1}{2}(x-1)^2$$

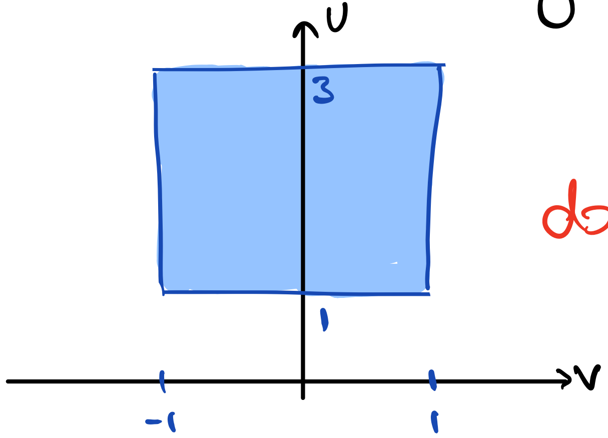
$$= \int_{x=1}^{2} \cancel{3x} - x^2 + \tfrac{9}{2} \cancel{-3x} \cancel{+\tfrac{1}{2}x^2} - x^2 + x \cancel{-\tfrac{1}{2}x^2} + x - \tfrac{1}{2}\,dx$$

$$= \int_{x=1}^{2} -2x^2 + 2x + 4 = -2\int_{x=1}^{2} x^2 - x - 2\,dx = -2\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 2x\right]_1^2 = -2\left[\left(\tfrac{8}{3} - 2 - 4\right) - \left(\tfrac{1}{3} - \tfrac{1}{2} - 2\right)\right] = \frac{7}{3}$$

$$\underline{\underline{I = \frac{5}{3} + \frac{7}{3} = 4}}$$

now lets do the same but using the Jacobian.

We'll do this integral using u & v.



Remember:
- ☐ integrand
- ☐ limits (draw)
- ☐ differential

↑ Jacobian

$dxdy = |J|dudv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$I = \iint_{R_{uv}} u \, \tfrac{1}{2} \, dudv$$

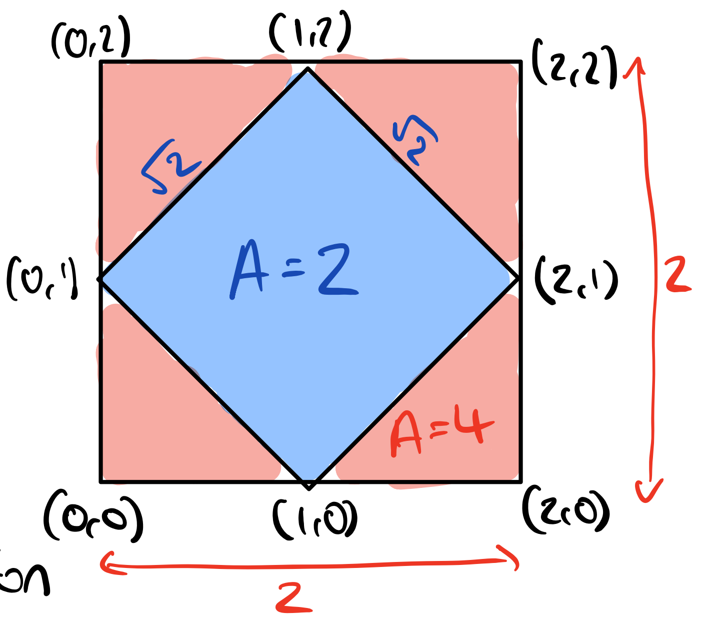
$$= \int_{v=-1}^{1} \int_{u=1}^{3} \tfrac{1}{2} u \, dudv = \int_{v=-1}^{1} \left[\tfrac{1}{4}u^2\right]_1^3 dv = \int_{v=-1}^{1} 2 \, dv$$

$$= [2v]_{-1}^{1} = 2 - -2 = 4 \qquad \underline{\underline{I = 4}}$$

Same solution as before. easier working

Geometrically,

the area of the region in the xy plane is $\sqrt{2} \times \sqrt{2} = 2$.



the area of the region in the uv plane is $2 \times 2 = 4$.

Blue: viewed from uv

Red: viewed from xy

↳ $dxdy = \frac{1}{2} dudv$.