If we consider the set of complex exponentials,

$$\frac{1}{\sqrt{2\pi}} e^{in\theta} \quad \text{with } n \in \mathbb{Z}$$

stated without proof, these form a complete set of square integrable functions where

$$\langle f, f \rangle = \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \int_{-\pi}^{\pi} f(\theta) f^*(\theta) d\theta < \infty$$

this is a very large set which have a finite square.

Fourier Series

Consider the periodic functions $f(x)$ which are periodic over 2π. The fourier series can be defined as:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

N.B. the set of functions $\{e^{inx}\}$ form a complete orthogonal set. It is not orthonormal. This is a choice. some textbooks use orthonormal sets.

we can rearrange the fourier series as:

$$f(x) = c_0 + \sum_{n=1}^{\infty} \underbrace{\left[c_n e^{inx} + c_{-n} e^{-inx}\right]}$$

$$= \frac{c_n + c_{-n}}{2}\left(e^{inx} + e^{-inx}\right) + \frac{c_n - c_{-n}}{2} i \left(e^{inx} - e^{inx}\right)$$

$$= [c_n + c_{-n}] \sin(nx) + [c_n - c_{-n}] i \cos(nx)$$

define: $a_n = c_n + c_{-n}$ $\quad b_n = i[c_n - c_{-n}]$

$$= a_n \cancel{\sin(nx)}\ \cos(nx) + b_n \cancel{\cos(nx)}\ \sin(nx) \quad !$$

$$a_n = c_n + c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx + \frac{1}{2\pi}\int_{a}^{b} f(x) e^{inx}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \frac{e^{inx} + e^{-inx}}{2}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\, dx$$

$$= \frac{-i}{\pi}\int_{-\pi}^{\pi} f(x)\, \frac{e^{inx} - e^{-inx}}{2}\, dx = \frac{-i^2}{\pi}\int_{-\pi}^{\pi} f(x)\, \frac{e^{inx} - e^{inx}}{2i}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Now we can rewrite our fourier series.

$a_n = c_n + c_{-n}$ $\quad a_0 = c_0 + c_{-0} = c_0 + c_0 = 2c_0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right) dx$$

Exercise 3.1

Exercise 3.1 *Work out the Fourier series of the square wave function defined as*

$$f(x) = \begin{cases} 0 & \textit{if } -\pi \le x < 0 \\ 1 & \textit{if } 0 < x < \pi \, , \end{cases}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \cancel{\frac{1}{\pi}\int_{-\pi}^{0} 0\,dx}^{\,0} + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin(nx)\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{n}\sin(\pi n) - \frac{1}{n}\sin(0n)\right) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \cancel{\frac{1}{\pi}\int_{-\pi}^{0} 0\,dx}^{\,0} + \frac{1}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} = \frac{-1}{\pi n}\cos(n\pi) + \frac{1}{n\pi}\cos(0n)$$

$$= \frac{1}{n\pi}\left[1 - \cos(n\pi)\right]$$

Δ n is odd: $b_n = 2/n\pi$

Δ n is even: $b_n = 0$

$$b_n = \{b_1, b_3, b_5, \dots\} = \left\{\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \dots\right\}$$

$$= \frac{2}{\pi}\{1, 1/3, 1/5 \dots\}$$

$$a_0 = 2c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(0x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx = 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right)$$