Lecture 6

$$e^{z_1} \cdot e^{z_2} = \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right)$$

we'll remove this temporarily

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \underbrace{\frac{z_1^k}{k!}}_{a_k} \underbrace{\frac{z_2^{n-k}}{(n-k)!}}_{b_{n-k}}\right) \approx \text{binomial expansion ?}$$

We will now add a $n!$ to get the formula for binomial.

$$= \frac{1}{n!}\sum_{k=0}^{n} n! \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!}$$

The binomial expansion formula is:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{r-n} b^r \quad \text{where } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

It is now obvious to see that we get

$$= \frac{1}{n!}(z_1+z_2)^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!}(z_1+z_2)^n$$

$$= e^{z_1+z_2}$$

if you taylor expand this, it's obvious to see they're equal. $\square$

Examples

* $e^{z*} = e^{x+iy} = e^x \times e^{iy} = e^x(\cos(y) + i\sin(y))$

* $\frac{1}{e^z} = \frac{1}{e^x(\cos y + i\sin y)} = e^{-x}\frac{1}{\cos y + i\sin y} \times \frac{\cos y - i\sin y}{\cos y - i\sin y}$

$= \frac{\cos y - i\sin y}{\cos^2 y + \sin^2 y} e^{-x} = e^{-x}(\cos y - i\sin y)$

$= e^{-x-iy} = e^{-(x+iy)} = e^{-z}$

Trig functions (complex)

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{i\theta})$ $\qquad$ $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

We can extend this to the question:

$\cos(z) = ?$ $\qquad$ $\sin(z) = ?$

$$\begin{aligned}
\cos(z) &= \cos(x+iy) \\
&= \tfrac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) \\
&= \tfrac{1}{2}\left(e^{ix}e^{-y} + e^{-ix}e^{y}\right) \\
&= \tfrac{1}{2}\left[(\cos x + i\sin x)e^{-y} + (\cos x - i\sin x)e^{y}\right] \\
&= \cos x \cdot \underbrace{\frac{e^{y}+e^{-y}}{2}}_{\cosh y} - i\sin x \underbrace{\frac{e^{y}-e^{-y}}{2}}_{\sinh y} \\
&= \underline{\underline{\cos x\cosh y - i\sin x \sinh y}}
\end{aligned}$$

Notes on our result:

* it complex - kinda obvious
* our result contain both oscillatory functions (sin/cos) and hyperbolic/unbounded functions.

Example $\qquad x = 0 \iff z = iy$

$\cos(iy) = \cosh y.$

Another proof exists for sin, but we will not cover it here.

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y$$

Example $\quad x = 0 \iff z = iy$

$$\sin(iy) = i\sinh(y)$$

Are $\sin(z)$ & $\cos(z)$ bounded or unbounded?
We can easily see that

$$\boxed{|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

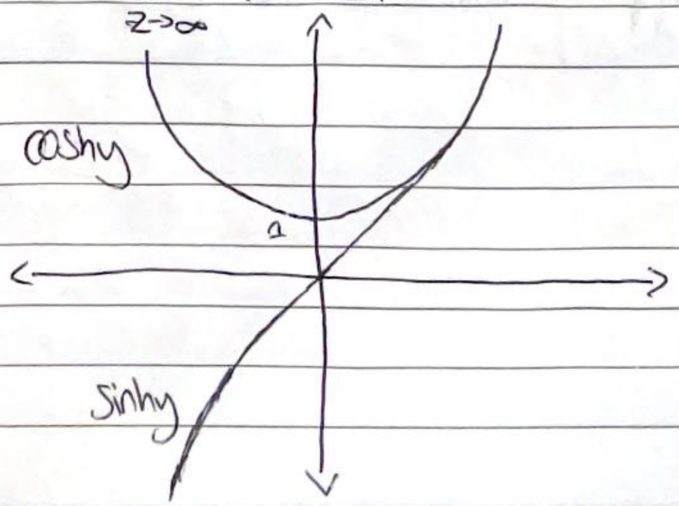
$$= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$
$$= \cos^2 x + \sinh^2 y (\cos^2 x + \sin^2 x)$$
$$= \cos^2 x + \sinh^2 y$$

We can do a similar calculation for sin to get

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

It is easy to see that $\sin z$ and $\cos z$ are unbounded.

$$\lim_{z \to \infty} |\cos z| = \infty .$$



Properties of cos z & sin z

* $\sin^2 z + \cos^2 z = 1$

$$\left[\tfrac{1}{2}(e^{iz} - e^{-iz})\right]^2 + \left[\tfrac{1}{2}(e^{iz} + e^{-iz})\right]^2$$

$$= \tfrac{1}{4}(e^{iz} - e^{-iz})^2 + \tfrac{1}{4}(e^{iz} + e^{-iz})^2$$

$$= \tfrac{1}{4}(2e^{iz}e^{-iz}) + \tfrac{1}{4}(2 \cdot e^{iz}e^{-iz}) = \tfrac{1}{2} + \tfrac{1}{2} = 1 \qquad \square$$

* roots/zeros of cos z & sin z

$$\sin z = 0$$
$$|\sin z|^2 = 0$$
$$\sin^2 x + \sinh^2 y = 0$$
$$(\sin^2 x \geq 0) \quad (\sinh^2 y \geq 0)$$

$\therefore$ they can only be true when both $\sin x$ and $\sinh y$ are equal to zero. $\therefore$ The zeros to $\sin z$ are <u>real</u> $(0, \cancel{2}\pi, 2\pi \ldots)$. $(\mathrm{Im}(z) = 0)$

$$\cos z = 0$$
$$|\cos z|^2 = 0$$
$$\cos^2 x + \sinh^2 y = 0$$

$\therefore$ zeros of $\cos z$ are also real. $(\mathrm{Im}(z) = 0)$

* periodicity

We know that: $\sin(\theta + 2\pi k) = \sin\theta$

$\cos(\theta + 2\pi k) = \cos\theta$

What happens with $\cos z$ & $\sin z$