Vectors 3

How do we multiply two vectors together?

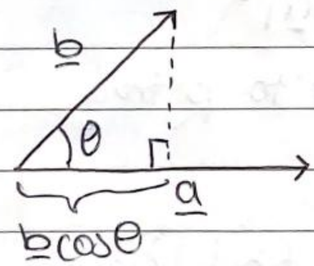
Should the product of two vectors be scaler of vector? Both!

Scaler Product (Dot)

The scaler product is defined as:

$$\boxed{\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta}$$

Geometric Representation



$$\frac{\underline{a} \cdot \underline{b}}{|\underline{a}|} = \underline{b}\cos\theta$$

'the projection of b onto a'

$$b\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|} \cdot \hat{a}$$

Dot Product Rules

I) $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ (commutative)

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta = |\underline{b}||\underline{a}|\cos(-\theta) = \underline{b} \cdot \underline{a}$$

II) $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ (distributive)

$$\frac{\underline{a} \cdot \underline{b}}{|a|} \cancel{\hat{a}} + \frac{\underline{a} \cdot \underline{c}}{|a|} \cancel{\hat{a}} = \frac{\underline{a} \cdot (\underline{b} + \underline{c})}{|a|} \cancel{\hat{a}}$$

$$\underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} = \underline{a} \cdot (\underline{b} + \underline{c})$$

orthogonal.

III) if $\underline{a} \perp \underline{b}$, $\cos\theta = \cos\frac{\pi}{2} = 0$
then $\underline{a} \cdot \underline{b} = 0$

$\hat{\imath} \cdot \hat{\jmath} = 0 \quad \hat{\imath} \cdot \hat{k} = 0 \quad \hat{\jmath} \cdot \hat{k} = 0$
$\hat{\imath} \cdot \hat{\imath} = 1 \quad \hat{\jmath} \cdot \hat{\jmath} = 1 \quad \hat{k} \cdot \hat{k} = 1$

$\underline{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \quad \underline{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$

$$\underline{a} \cdot \underline{b} = (a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k})$$
$$= a_xb_x + a_yb_y + a_zb_z$$

in general, in an n-dimensional space

$$\boxed{\underline{a} \cdot \underline{b} = \sum_{i=1}^{n} a_i b_i}$$

IV) $\underline{a} \cdot \underline{a} = \sum_{i=1}^{n} a_i a_i = |\underline{a}|^2$
link to pythag.

Finding Angles

Dot products are used to find angles between vectors (even when beyond 2D).

eg. $\underline{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{2}{\sqrt{5}\sqrt{25}} = \frac{2}{5\sqrt{5}}$$

$$\theta = \arccos\left(\frac{2}{5\sqrt{5}}\right) = 79.7^\circ$$

Applications of Dot Product

→ work done $W = \underline{F} \cdot \underline{d}$
we're interested in the work done along the dimension we're interested in.

→ Power $\quad P = \underline{F} \cdot \underline{v}$

→ energy of electric dipole $\quad U_E = -\underline{p} \cdot \underline{E}$

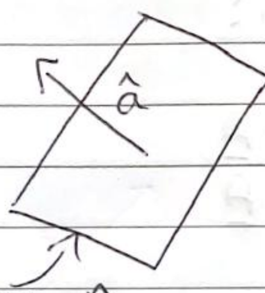
electric dipole moment

→ magnetic dipole $\quad U_B = -\underline{\mu} \cdot \underline{B}$

magnetic dipole moment

→ flux

Flux is the rate of flow of a vector quantity through a surface.



area = A

How to get area in vector form? multiply normal vector by the area (a scalar).

define $\underline{A} = A\underline{\hat{a}}$

during time Δt, a particle would move $v\Delta t$ distance.

every particle within this volume will travel through the surface area.

The volume of particles is found by

$$V = v\Delta t \cos\theta \, A$$

∴ the number of particles can be found by

$$N = \underline{v}\Delta t\cos\theta\,\underline{A}\cdot n$$ ← particle density

$$\frac{\Delta N}{\Delta t} = \underbrace{n\underline{v}A\cos\theta}_{\text{dot product}}$$

$$\frac{\Delta N}{\Delta t} = n\underline{v}\cdot\underline{A}$$ ← vector (area).

scalar

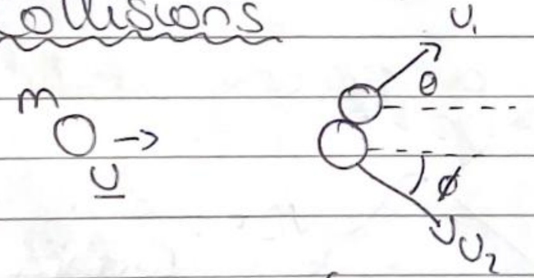
vector (velocity)

Similar to electromagnetism:

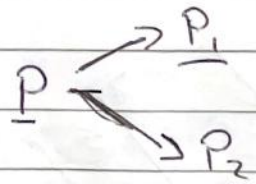
flux of $\underline{E}$, $\phi_E = \underline{E}\cdot\underline{A}$

flux of B, $\phi_B = \underline{B}\cdot\underline{A}$

__Collisions__



what is φ?



__conservation of momentum__

$$\underline{p}_1 + \underline{p}_2 = \underline{p}$$

$$m\underline{U}_1 + m\underline{U}_2 = m\underline{U}$$

$$\Rightarrow \underline{U}_1 + \underline{U}_2 = \underline{U}$$

__conservation of energy__

$$\tfrac{1}{2}mU^2 = \tfrac{1}{2}mU^2 + \tfrac{1}{2}mU_2^2$$

$$U^2 = U_1^2 + U_2^2$$

but $u^2 = \underline{u} \cdot \underline{u} = (\underline{u}_1 + \underline{u}_2) \circ (\underline{u}_1 + \underline{u}_2)$

$= u_1^2 + u_2^2 + 2u_1u_2$

but we know $u^2 = u_1^2 + u_2^2$, so what about $2u_1u_2$. $\Rightarrow$

$$u_1 \circ u_2 = 0 \qquad (u_1 \perp u_2)$$