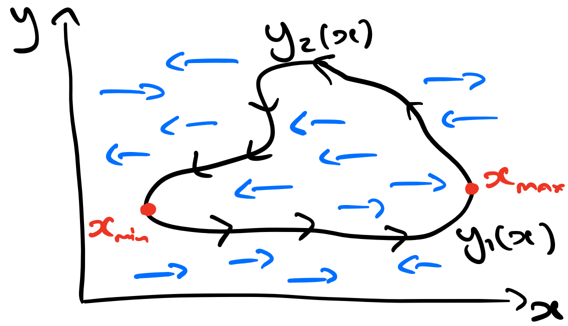
Green's Theorem in a Plane

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

line integral, anti-clockwise looking down on xy plane

double region over the enclosed region

Derivation (Khan)



$\underline{F}(x,y) = P(x,y)\hat{i}$

$d\underline{r} = dx\hat{i} + dy\hat{j}$

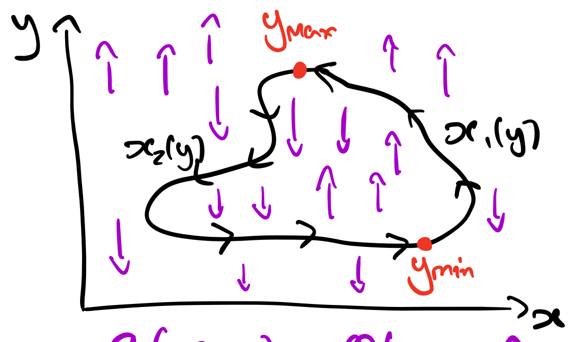
$$\oint_C \underline{F}\cdot d\underline{r} = \oint P(x,y)\,dx$$

$$= \int_{x_{min}}^{x_{max}} P(x,y_1(x))\,dx + \int_{x_{max}}^{x_{min}} P(x,y_2(x))\,dx$$

$$= \int_{x_{min}}^{x_{max}} P(x,y_1(x)) - P(x,y_2(x))\,dx$$

$$= -\int_{x_{min}}^{x_{max}} P(x,y_2(x)) - P(x,y_1(x))\,dx = -\int_{x_{min}}^{x_{max}} P(x,y)\Big|_{y_1(x)}^{y_2(x)}\,dx$$

$$= -\int_{x_{min}}^{x_{max}}\int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$



$\underline{q}(x,y) = Q(x,y)\hat{j}$

$$\oint \underline{q}\cdot d\underline{r} = \oint Q(x,y)\,dy$$

$$= \int_{y_{min}}^{y_{max}} Q(x_1(y),y)\,dy + \int_{y_{max}}^{y_{min}} Q(x_2(y),y)\,dy$$

$$= \int_{y_{min}}^{y_{max}} Q(x_1(y),y) - Q(x_2(y),y)\,dy = \int_{y_{min}}^{y_{max}}\int_{x_2(y)}^{x_1(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

$$\underline{f}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

$$\oint \underline{f}\cdot d\underline{r} = \oint P(x,y)\,dx + \oint Q(x,y)\,dy$$

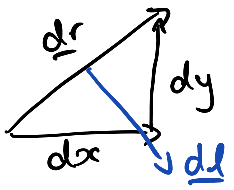
$$= \iint_R \frac{\partial P}{\partial y}\,dy\,dx + \iint_R \frac{\partial Q}{\partial x}\,dx\,dy$$

$$\oint \underline{f}\cdot d\underline{r} = \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\,dy$$

Green's Theorem in a plane!

<u>Example 1</u> if $P(x,y)\,dx + Q(x,y)\,dy$ is an exact differential, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. $\therefore$ $\oint P\,dx + Q\,dy = 0$!

<u>Example 2</u> Consider the 'flux' out of a closed loop in 2D, vector field $\underline{B}$.



$d\underline{l} = dy\,\hat{\imath} - dx\,\hat{\jmath}$ - it defines an <u>outwards</u> facing vector of the same length - for an ACW loop.

$$\underline{B}\cdot d\underline{l} = (B_x\hat{\imath} + B_y\hat{\jmath})\cdot(dy\,\hat{\imath} - dx\,\hat{\jmath}) = B_x\,dy - B_y\,dx$$

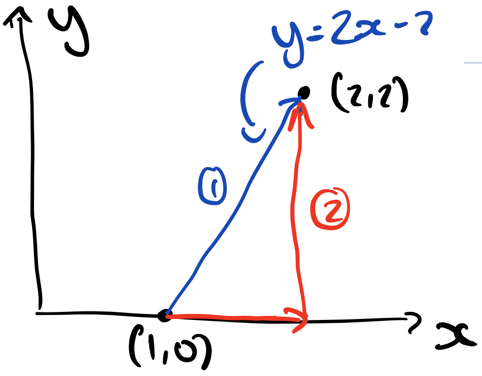
$$\oint \underline{B}\cdot d\underline{l} = \oint -\underbrace{B_y}_{P(x,y)}dx + \underbrace{B_x}_{Q(x,y)}dy = \iint \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\,dx\,dy$$

This is just the divergence theorem in two dimensions!

Example 3 'work done'

$$\oint \underline{F}\cdot d\underline{r} = \oint F_x dx + F_y dy = \iint \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\, dx dy$$

This is stokes theorem in 2D!



Example 4 $\underline{F} = 2xy\hat{i} + x^2\hat{j}$

earlier in the notes we computed the path integrals along paths ① & ②,

① : $\int_A^B \underline{F}\cdot d\underline{r} = -4/3$ ② : $\int_A^B \underline{F}\cdot d\underline{r} = -8$ $\therefore \oint \underline{F}\cdot d\underline{r} = -\frac{20}{3}$

now lets try using green's theorem:

$$\oint \underbrace{2xy}_{P}\,dx - \underbrace{x^2}_{Q}\,dy \qquad \frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = -2x$$

$$\oint \underline{F}\cdot d\underline{r} = \iint \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\, dx dy = -\iint 4x\, dx dy$$

$$= \int_{x=1}^{2} \int_{y=0}^{2x-2} 4x\, dy dx = \int_{x=1}^{2} 8x^2 - 8x\, dx = -\left[\frac{8}{3}x^3 - 4x^2\right]_1^2$$

$$= \left(\frac{64}{3} - 16\right) - \left(8/3 - 4\right) = -\frac{20}{3}$$