Lecture 7

$$\cos(z+2\pi k) = \frac{1}{2}\left[e^{i(z+2\pi k)} + e^{-i(z+2\pi k)}\right]$$

$$= \frac{1}{2}\left(e^{iz} \times \cancel{e^{i2\pi k}} + e^{-iz} \times \cancel{e^{2\pi k}}\right)$$

$$= \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Complex Logarithm

$$z = e^{\omega} \iff \ln z = \omega$$

$$z_1 z_2 = e^{\omega_1} e^{\omega_2} = e^{\omega_1 + \omega_2}$$

$$\ln(z_1 z_2) = \omega_1 + \omega_2 = \ln(z_1) + \ln(z_2)$$

Polar Representation

$$z = re^{i\theta}$$

$$\omega = \ln z = \ln(re^{i\theta}) = \ln(r) + \ln(e^{i\theta})$$

$$e^{i\theta} = e^{i(\theta + 2\pi k)}$$

$$\omega = \ln z = \ln r + \ln(e^{i\theta + i2\pi k}) = \ln r + i(\theta + 2\pi k)$$

Imposing single-valuedness

We will do this by restricting the value that angle can take.

$$\ln z = \ln r + i\theta \qquad -\pi \leq \theta < \pi$$

Logarithm of a negative argument

$$\ln(-|x|) \qquad x \neq 0.$$

$$-|x| = (-1)|x| = e^{-i\pi}|x|$$

$$\ln(|x|e^{-i\pi}) = \ln|x| - i\pi$$ ← logs of -ve numbers are complex.

logs for evaluation of powers

$$z^a = e^{\ln\{z\}^a} = e^{a\ln z} = e^{a}\cdot z \qquad \ln(z) \Rightarrow z \neq 0.$$

Example $\quad z = 1+i \quad a = 2 \quad (1+i)^2$

$$(1+i)^2 = 1 + 2i - 1 = 2i$$

$$(1+i)^2 = e^{2\ln(1+i)} \qquad 1+i = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$= e^{2\ln(\sqrt{2}e^{i\frac{\pi}{4}})}$$

$$\ln(\sqrt{2}e^{i\frac{\pi}{4}}) = \ln\sqrt{2} + i\frac{\pi}{4}$$

$$= e^{2\ln\sqrt{2} + i\frac{\pi}{2}}$$

$$= e^{\ln 2} \times e^{i\frac{\pi}{2}}$$

$$= 2i$$

Example $\quad z = i \quad a = i \quad i^i$

$$i^i = e^{\ln(i^i)} = e^{i\ln i}$$

$$i = e^{i\frac{\pi}{2}}$$

$$= e^{i\ln(e^{i\frac{\pi}{2}})} = e^{i^2\frac{\pi}{2}} = \underline{e^{-\frac{\pi}{2}}}$$

We only get this result as we've restricted θ. What happens when we remove that restriction.

$$i^i = e^{i\ln(i)} = e^{i(i(\frac{\pi}{2}+2\pi k))} = e^{-\frac{\pi}{2}-2\pi k}$$

$\therefore$ All solutions of i^i are real, but of vastly different magnitudes.

Inverse Trig Functions (complex).

$$z = \sin(\omega) \qquad \omega = \arcsin(z)$$

$$z = \frac{e^{i\omega} - e^{-i\omega}}{2i} \qquad 2iz = e^{i\omega} - e^{-i\omega}$$

$$2i \cdot z e^{i\omega} = e^{2i\omega} - 1$$

$$e^{2i\omega} - 2iz e^{i\omega} - 1 = 0 \qquad \text{quadratic eq}^{n}. \quad y = e^{i\omega}$$

$$y^2 - 2izy - 1 = 0$$

$$y = \frac{2iz \pm \sqrt{4z^2+4}}{2} = iz \pm \sqrt{z^2+1}$$

$$e^{i\omega} = iz \pm \sqrt{z^2+1}$$

$$\omega = \frac{1}{i} \ln\left[iz \pm \sqrt{z^2+1}\right] = -i\ln\left[iz \pm \sqrt{z^2+1}\right]$$