Vectors 8

Gaussian Elimination

Cramer's rule for n-dimensions

n equations in n dimensions:

$$a_{11}x_1 + a_{12}x_2 \dots a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 \dots a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 \dots a_{nn}x_n = b_n$$

in general, $x_1 = \frac{\Delta_1}{\Delta}$ $\quad x_2 = \frac{\Delta_2}{\Delta} \dots \quad x_n = \frac{\Delta_n}{\Delta}$

where,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

if $\Delta = 0$; $\Delta_1, \Delta_2 \dots \Delta_n \neq 0 \Rightarrow$ no solutions
if $\Delta = 0$; $\Delta_1, \Delta_2 \dots \Delta_n = 0 \Rightarrow$ infinite no. solutions

But, this is an inefficent method for large n.

Gaussian Elimination

first form an argumented matrix.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right)$$

Three gaussian commandments: Thou mayst:

I) interchange any two rows

II) multiply / divide any rows by a constant ($\neq 0$)

III) add / subtract any multiple of one row from another.

Aim:

I) subtract multiples of rows until obtain a triangular matrix.

II) remove known variables from other rows

Unique solution Example

$$\left.\begin{aligned} x - 2y + 4z &= 1 \\ -x + y - z &= 2 \\ 2x + 3y - z &= 3 \end{aligned}\right\} \text{ Solve}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ -1 & 1 & -1 & 2 \\ 2 & 3 & -1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & 7 & -9 & 1 \end{array}\right)\begin{array}{l} \\ \text{top+middle} \\ -(2\times\text{top})+\text{bottom} \end{array}$$

bottom + 7(middle)

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 12 & 22 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 12 & 22 \end{array}\right)\begin{array}{l} \\ \text{middle}\times -1 \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 4 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 11/6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 4 & 1-44/6 \\ 0 & 1 & -3 & -3+33/6 \\ 0 & 0 & 1 & 11/6 \end{array}\right)\begin{array}{l} R2+3R3 \\ R1-4R3 \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & -2 & 0 & -38/6 \\ 0 & 1 & 0 & 15/6 \\ 0 & 0 & 1 & 11/6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -38/6+30/6 \\ 0 & 1 & 0 & 15/6 \\ 0 & 0 & 1 & 11/6 \end{array}\right)$$

$x = -4/3 \quad y = 5/2 \quad z = 11/6$

Graphical representation is that there is a unique point of intersection.

Non-unique solutions

$$\left.\begin{aligned} x - 2y - 3z &= 2 \\ x - 4y - 13z &= 14 \\ -4x + 3y - 13z &= 2 \end{aligned}\right\} \text{Solve}$$

$$\Delta = \begin{vmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -4 & +3 & -13 \end{vmatrix} = 0$$ $\therefore$ cramer's rule wont work.

$$\Delta_3 = \begin{vmatrix} 1 & -2 & 2 \\ 1 & -4 & 14 \\ -4 & +3 & 2 \end{vmatrix} = 40 \neq 0.$$ $\therefore$ no solutions

However, lets do a gaussian elimination and see what we get.

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 1 & -4 & -13 & 14 \\ -4 & 3 & -13 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & -2 & -10 & 12 \\ 0 & -5 & -25 & 10 \end{array}\right) \begin{array}{l} \\ R2-R1 \\ R3+4R1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -6 \\ 0 & 1 & 5 & -2 \end{array}\right) \begin{array}{l} \\ \div -2 \\ \div -5 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 4 \end{array}\right) \begin{array}{l} \\ \\ R3-R2 \end{array}$$

bottom line implies $0=4$, obv. not true which implies no solutions. graphically, all three planes never intersect at any one point.

Infinity of Solutions

$$\left.\begin{aligned} x - 2y - 3z &= 2 \\ x - 4y - 13z &= 14 \\ -3x + 5y + 4z &= 0 \end{aligned}\right\} \text{Solve}$$

$$\Delta = \begin{vmatrix} 1 & -2 & -3 \\ 1 & -4 & -13 \\ -3 & 5 & 4 \end{vmatrix} = 0$$ $\therefore$ cramers rule wont work

$$\Delta_3 = \begin{vmatrix} 1 & -2 & 2 \\ 1 & -4 & 14 \\ -3 & 5 & 0 \end{vmatrix} = 0 \quad \therefore \text{ infinity no. of solutions}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 1 & -4 & -13 & 14 \\ -3 & 5 & 4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & -2 & -10 & 12 \\ 0 & -1 & -5 & 6 \end{array}\right) \begin{array}{l} \\ R2-R1 \\ R3+3R1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -6 \\ 0 & 1 & 5 & -6 \end{array}\right) \begin{array}{l} \\ \div -2 \\ \times -1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -3 & 2 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ R3-R2 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 7 & -10 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} x+7z=-10 \\ y+5z=-6 \end{array}$$

Set $z=\lambda$ $\quad x=-10-7\lambda \quad y=-6-5\lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}$$

This is the equation of a line with infinitley many solutions.