What Does $\nabla \cdot \underline{B}$ Mean?

in cartesian

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{j}\frac{\partial}{\partial z}$$

$$\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\nabla \cdot \underline{B} \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$\therefore$ it looks like the dot product of two vectors.

This is not true!

eg. in cylindrical

$$\nabla = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\underline{B} = B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{z}$$

$$\nabla \cdot \underline{B} = \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

$\nabla \cdot \underline{B}$ is not the dot product of two vectors. It is an abuse of notation. It's just shorthand for divegence.

The good news: we can think of $\nabla \cdot \underline{B}$ as 'differentiate first, dot product second'. We can think of curl as 'differentiate first, cross product second'. This works for cylindrical & spherical.

$$\nabla B = \frac{\partial B}{\partial x}\hat{i} + \frac{\partial B}{\partial y}\hat{j} + \frac{\partial B}{\partial z}\hat{k}$$

$$\nabla B \cdot \underline{r} = \frac{\partial B}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial B}{\partial z}$$

Analytic Derivation of $\nabla \cdot B$

$$\nabla = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\underline{B} = B_\rho\hat{\rho} + B_\phi\hat{\phi} + B_z\hat{z}$$

'differentiate first, dot product second'

$$\nabla B = \frac{\partial \underline{B}}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial \underline{B}}{\partial\phi}\hat{\phi} + \frac{\partial \underline{B}}{\partial z}\hat{z}$$

$$\frac{\partial \underline{B}}{\partial\rho} = \frac{\partial}{\partial\rho}\left(B_\rho\hat{\rho}(\phi)\right) + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(B_\phi\hat{\phi}(\phi)\right) + \frac{\partial}{\partial\rho}\left(B_z\vec{z}\right)$$

$$= \hat{\rho}\frac{\partial B_\rho}{\partial\rho} + \hat{\phi}\frac{\partial B_\phi}{\partial\rho} + \hat{z}\frac{\partial B_z}{\partial z}$$

$$\frac{\partial \underline{B}}{\partial\rho}\cdot\hat{\rho} = \frac{\partial B_\rho}{\partial\rho}$$

$$\frac{\partial \underline{B}}{\partial\phi} = \frac{\partial}{\partial\phi}(B_\rho\hat{\rho}) + \frac{\partial}{\partial\phi}(B_\phi\hat{\phi}) + \frac{\partial}{\partial\phi}(B_z\hat{z})$$

$$= \hat{\rho}\frac{\partial B_\rho}{\partial\phi} + B_\rho\hat{\phi} + \frac{\partial B_\phi}{\partial\phi}\hat{\phi} - B_\phi\hat{\rho} + \frac{\partial B_z}{\partial\phi}\hat{z}$$

$$\frac{1}{\rho}\frac{\partial \underline{B}}{\partial\phi}\cdot\hat{\phi} = \frac{B_\rho}{\rho} + \frac{1}{\rho}\frac{\partial B_\rho}{\partial\phi}$$

$$\frac{\partial \underline{B}}{\partial z} = \frac{\partial}{\partial z}(B_\rho\hat{\rho}) + \frac{\partial}{\partial z}(B_\phi\hat{\phi}) + \frac{\partial}{\partial z}(B_z\hat{z})$$

$$= \frac{\partial B_\rho}{\partial z}\hat{\rho} + \frac{\partial B_\phi}{\partial z}\hat{\phi} + \frac{\partial B_z}{\partial z}\hat{z}$$

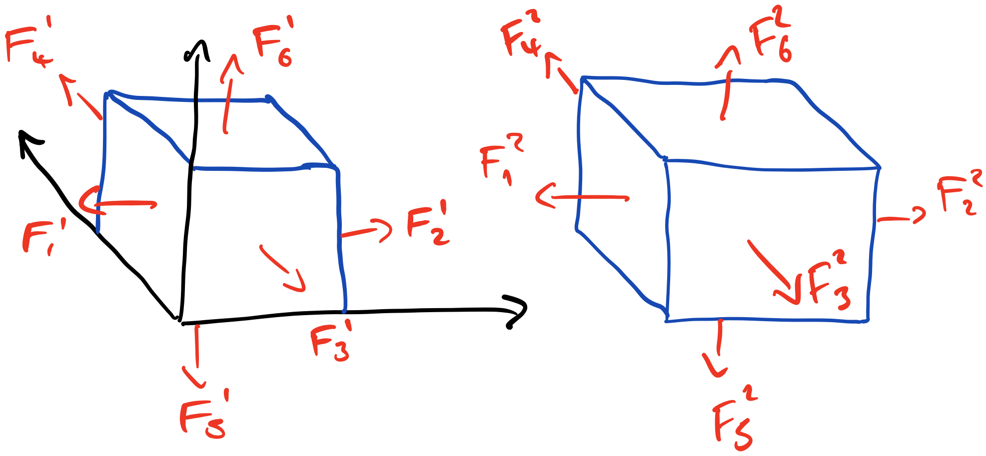
$$\frac{\partial B}{\partial z}\cdot\hat{z} = \frac{\partial B_z}{\partial z}$$

$$\nabla\cdot B = \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial B_\rho}{\partial\phi} + \frac{\partial B_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z}$$

Divergence Theorem (Gauss' Theorem)

Consider two infinitesimal boxes, adjacent along the x axis.



On each face $F = \underline{B} \cdot \underline{ds}$

Box 1: $\sum_{i=1}^{6} F_i^1 = \nabla \cdot \underline{B}_1 \, dV$ Box 2: $\sum_{j=1}^{6} F_j^2 = \nabla \cdot \underline{B}_2 \, dv$

N.B. $F_1^1 = -F_2^1$

Now lets push these two boxes together and then sum $\underline{B} \cdot \underline{ds}$ over the surfaces.

$$\sum F = \sum_{i=1}^{6} F_i^1 + \sum_{j=1}^{6} F_j^2 \underbrace{- F_2^1 - F_1^1}_{=0} = \nabla \cdot \underline{B}_1 dv + \nabla \cdot \underline{B}_2 dv$$

$$\therefore \sum_{\text{new box}} \underline{B} \cdot \underline{ds} = \sum_{j=1}^{2} \nabla \cdot \underline{B}_j \, dv$$

Now lets keep adding boxes to form a macroscopic surface.

$$\oiint \underline{B} \cdot \underline{ds} = \iiint \nabla \cdot \underline{B} \, dv$$

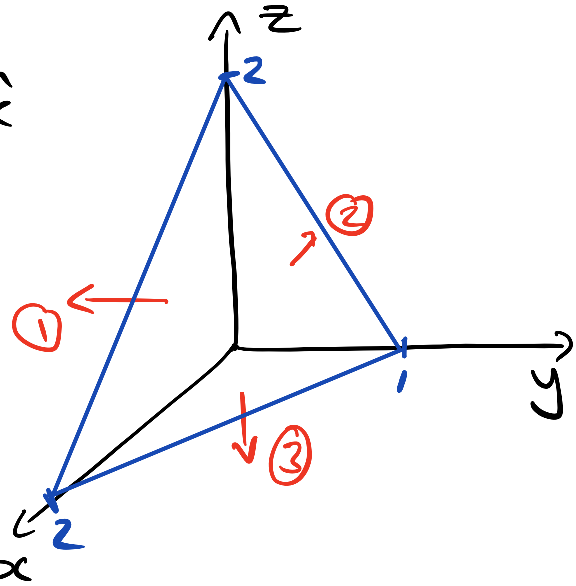
THE DIVERGENCE THEOREM

Example $\underline{B} = x\hat{i} + y\hat{j} + z\hat{k}$

the tetrahedron bounded by

$x=0,\ y=0,\ z=0,\ z=2-x-2y$

$$\oint\!\!\oint \underline{B}\cdot\underline{ds} = \iiint \nabla\cdot\underline{B}\, dv$$



face ① points in the $-\hat{j}$ direction. However $B_y = y = 0$ on the x-z plane. $\therefore\ \underline{B}\cdot\underline{ds} = 0$. This is also true for face ② & ③ by similar argument.

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + y\hat{j} + (2-x-2y)\hat{k}$$

$$\frac{\partial r}{\partial x} = \hat{i} - \hat{k} \qquad \frac{\partial r}{\partial y} = \hat{j} - 2\hat{k}$$

$$\underline{ds} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} dx\,dy = (\hat{i} + 2\hat{j} + \hat{k})\, dx\,dy$$

$$\underline{B}\cdot\underline{ds} = (\hat{i} + 2\hat{j} + \hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = (x + 2y + z)\, dx\,dy$$

$$= (x + 2y + (2-x-2y))\, dx\,dy = 2\,dx\,dy$$

$$\int_{y=0}^{1}\int_{x=0}^{2-2y} 2\, dx\,dy = \int_{y=0}^{1} 4-4y\, dx = \left[4y - 2y^2\right]_0^1 = 2$$

Now if we look at the RHS of the divergence theorem:

$$\nabla\cdot\underline{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 1 + 1 + 1 = 3$$

$$\iiint \nabla\cdot\underline{B}\, dv = \iiint 3\, dv$$

$$\int_{y=0}^{1}\int_{x=0}^{2-2y}\int_{z=0}^{2-x-2y} dz\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{2-2y} 2-x-2y$$

$$\int_{y=0}^{1}\left[2x-\tfrac{1}{2}x^2-2yx\right]_0^{2-2y} = \int_{y=0}^{1} 2(2-2y)-\tfrac{1}{2}(2-2y)^2-2y(2-2y)$$

$$\int_{y=0}^{1} 4-4y-2+4y-2y^2-4y+4y^2\,dy = \left[2y-2y^2+\tfrac{2}{3}y^2\right]_0^1$$

$$= 2/3$$

$3 \times 2/3 = 2$ (same as LHS).

<u>The Laplacian</u>

$$\nabla\Omega = \frac{\partial\Omega}{\partial x}\hat{\imath} + \frac{\partial\Omega}{\partial y}\hat{\jmath} + \frac{\partial\Omega}{\partial z}\hat{k}$$

$$\nabla\cdot(\nabla\Omega) = \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2} = \nabla^2\Omega$$