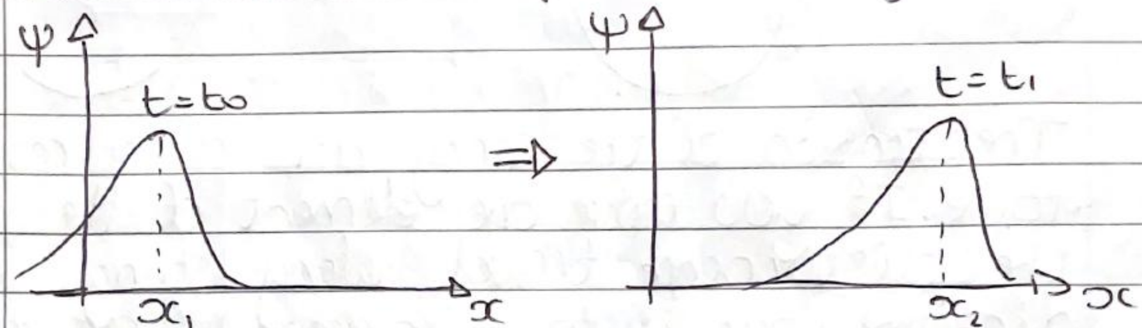
Waves

Waves: Intro

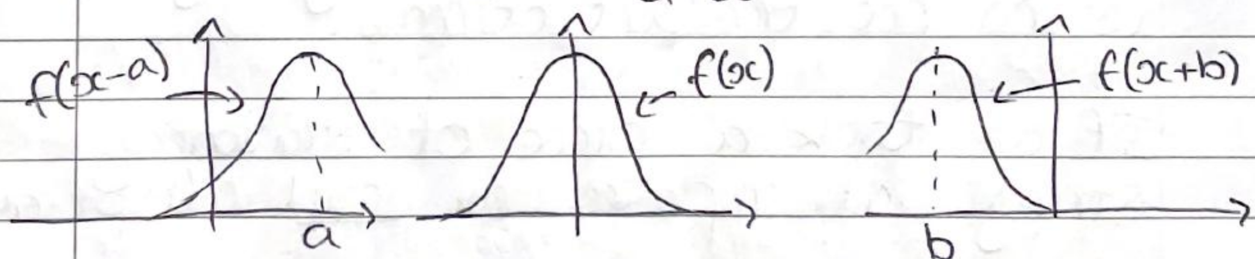
A wave is a disturbance which travels in a certain direction with no net transport of mass. The wave does not need to be periodic.

Let's explore an idealised wave, which has a constant shape and velocity.



ψ = 'disturbance'

v = velocity $= \dfrac{x_2 - x_1}{t_1 - t_0}$



consider $f(x-vt)$. In time Δt, $f(x)$ will move $v\Delta t$ along the x axis in the +ve direction.

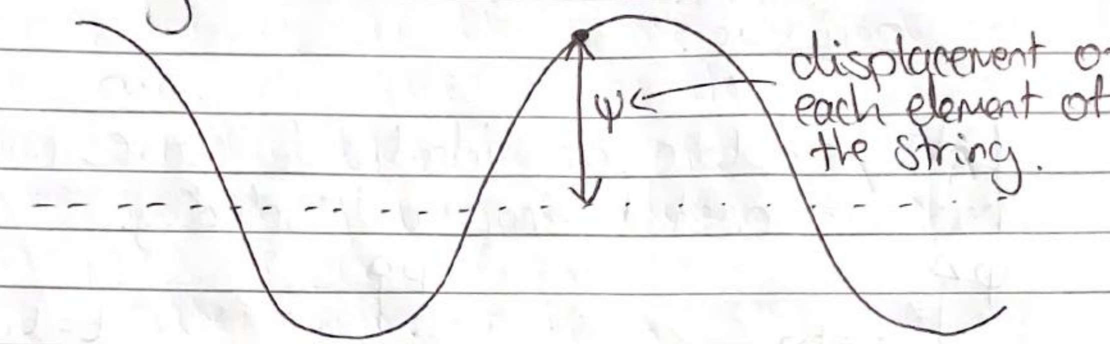
Any function of the form $f(x-vt)$ is the mathematical representation of a wave.

Stretch String

If there was no wave, the string would be stationary along the x axis.

It would be under tension, T_0, and mass per unit length, ρ_0.

With a wave, the string will be displaced latterally.

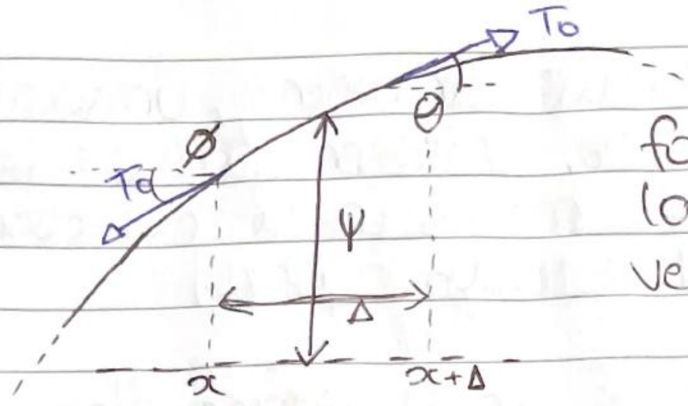


The tension of the string acts as a restoring force. If you move one element of the string, the disturbance travels along string in a direction parallel to displacement (transverse wave).

Each element moves vertically only so there is no net transport of mass.

If we took a piece of stationary string and imparted a <u>small</u> displacement.

- ☆ element is stretched, but extra extension is much smaller ≪ than the original extension so we can assume the tension is unchanged
- ☆ the element does not move in the x direction



for now, we'll look at the vertical tension only.

The restoring force (vertical) can be found via:

$$F = T_0 \sin\phi - T_0 \sin\theta$$

$$m \frac{\partial^2 \psi}{\partial t^2} = T_0(\sin\phi - \sin\theta)$$

We can use small angles to assume that

$$\sin\alpha \simeq \tan\alpha \simeq \alpha$$

we can also sub μ (mass per unit length) in.

$$\mu\Delta \frac{\partial^2 \psi}{\partial t^2} = T_0(\tan\phi - \tan\theta)$$

We know that $\frac{\partial \psi}{\partial x} = \tan\theta$ (both equal to rise over run).

$$\mu \frac{\partial^2 \psi}{\partial t^2} = T_0\left(\frac{\left.\frac{\partial \psi}{\partial x}\right|_{x=x+\Delta} - \left.\frac{\partial \psi}{\partial x}\right|_{x=x}}{\Delta}\right)$$

$$\mu \frac{\partial^2 \psi}{\partial t^2} = T_0 \frac{\partial^2 \psi}{\partial x^2}$$

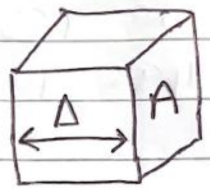
$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = \frac{T_0}{\mu} \frac{\partial^2 \psi}{\partial x^2}}$$

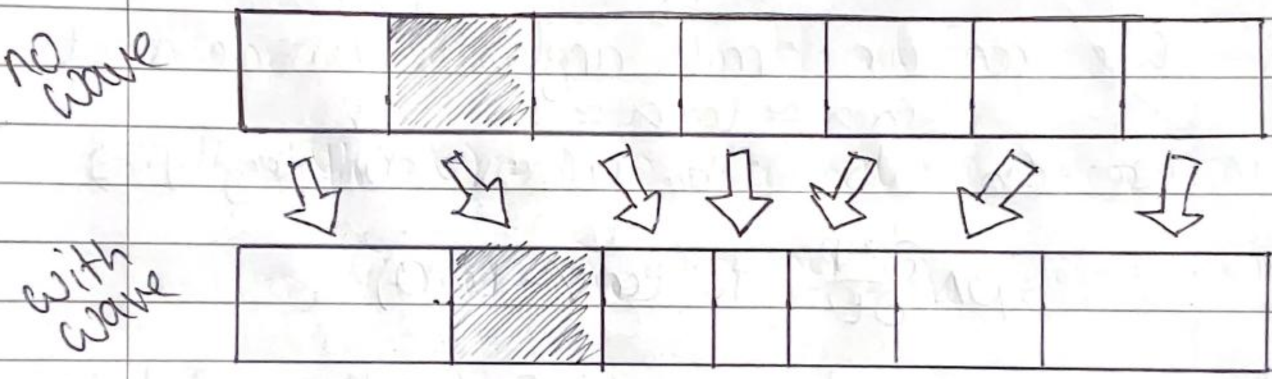
Sound: Wave Equation

In sound waves, pressure provides the restoring force

No wave: air stationary, uniform density ρ_0, uniform pressure P_0.
Wave: air moves in x direction, ρ & P vary with x & t.

Consider a small volume element (parcel) of air. The waves moves the parcel and changes its volume. The restoring force is due to pressure on the parcels on either side.





Looking at the shaded parcel, the volume of the parcel to the left has increased, pressure decreased. The volume of the parcel to the right has decreased, pressure increased.

This pressure imbalance produces a net force on the shaded parcel in the $-x$ direction

With no wave; length Δ_0, cross-sectional area A.

With wave; parcel displaced ψ in x direction, length Δ & density ρ change. A fixed. mass ($m = \rho_0 \Delta_0 A$) fixed.

The wave equation for air (not included here) is

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = \frac{\gamma P_0}{\rho_0} \frac{\partial^2 \psi}{\partial x^2}}$$

almost indentical to stretched string eqⁿ.

$(z = x - vt)$ $\quad \psi(x,t) = f(z) = f(x - vt)$

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$ product rule

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{df}{dz} = \frac{\partial}{\partial x}\left(\frac{df}{dz}\frac{\partial z}{\partial x}\right)$$

$$= \frac{d^2 f}{dz^2}$$

$$\frac{\partial f}{\partial t} = \frac{df}{dz}\frac{\partial z}{\partial t} = -v\frac{df}{dz}$$ product rule.

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{df}{dz}\right) = \frac{\partial}{\partial t}\left(\left[-v\frac{df}{dz}\right]\frac{\partial z}{\partial t}\right)$$

$$= v^2 \frac{d^2 f}{dz^2}$$

$$\Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = \frac{\frac{\partial^2 f}{\partial t^2}}{v^2}$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}}$$

where v is the speed of the wave.

$v = \sqrt{\frac{T}{\rho}}$ for strings $\quad v = \sqrt{\frac{\gamma P}{\rho}}$ for air