# Functions a

**Parametrically**

As above but $x = x(t)$ $y = y(t)$

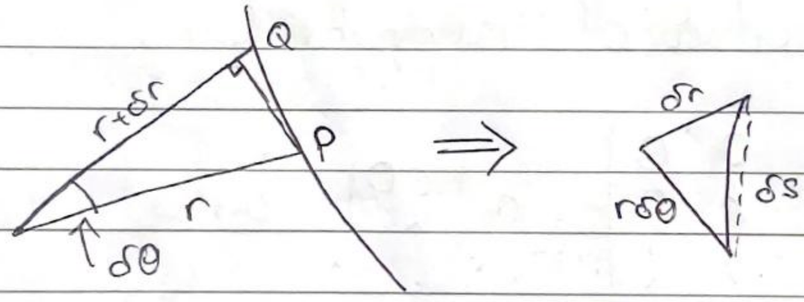
$$\delta s^2 = \delta x^2 + \delta y^2 = \left[\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right] \delta t^2$$

$$ds = (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} dt$$

$$L = \int_{t_0}^{t_1} (\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} dt$$

**Polars**

We look at the local infinitesimal contribution




$$\delta s^2 = \delta r^2 + (r\delta\theta)^2$$

$$\delta s^2 = \left[\left(\frac{\delta r}{\delta\theta}\right)^2 + r^2\right] \delta\theta^2$$

$$L = \int_a^b ds = \int_a^b \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{1/2} d\theta$$

**Example** Arc length of a quarter of a circle

**Cartesian** $x^2 + y^2 = a^2 \iff y = \sqrt{a^2 - x^2}$

$$\Downarrow$$

$$L = \int_0^a \sqrt{1 + y'^2}\, dx \qquad y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$L = \int_0^a \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx = \int_0^a \frac{a}{\sqrt{a^2 - x^2}}$$

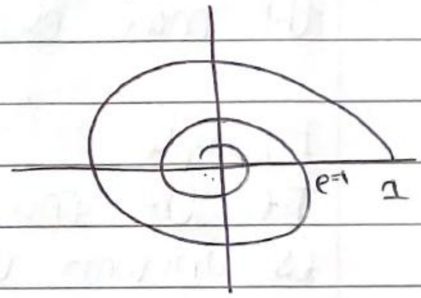
$$= a\left[\arcsin\left(\frac{x}{a}\right)\right]_0^a = \frac{\pi a}{2}$$

## Polar

$$L = \int_0^{\pi/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = a\int_0^{\pi/2} d\theta = \frac{\pi a}{2}$$

Example: Arc length of infinite spiral

$$r = e^{-\theta/2\pi}$$




$$L = \int_0^\infty \sqrt{e^{-\theta/\pi} + \frac{1}{4\pi^2} e^{-\theta/\pi}}\, d\theta$$

$$= \int_0^\infty \sqrt{e^{-\theta/\pi}\left(1 + \frac{1}{4\pi^2}\right)}\, d\theta$$

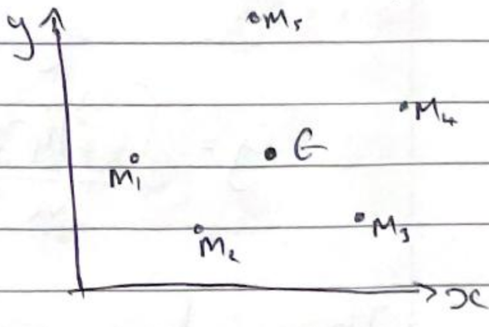
$$r' = -\frac{1}{2\pi} e^{-\theta/2\pi}$$

$$= \int_0^\infty -2\pi e^{-\theta/2\pi}\left(\frac{1}{4\pi^2} + 1\right)^{\frac{1}{2}}$$

$$= \left(1 + 4\pi^2\right)^{\frac{1}{2}}\left[-2\pi e^{-\theta/2\pi}\right]_0^\infty = \left(1 + 4\pi^2\right)^{\frac{1}{2}}$$

## Centre of Mass

Say we have N masses $m_j$ ($j = 1, 2, \dots, N$) at positions $(x_j, y_j)$




The centre of mass

$$G(\bar{x}, \bar{y})$$

is found by

$$\bar{x} = \frac{\sum_{j=1}^{N} m_j x_j}{\sum_{j=1}^{N} m_j} \qquad \bar{y} = \frac{\sum_{j=1}^{N} m_j x_j}{\sum_{j=1}^{N} m_j}$$

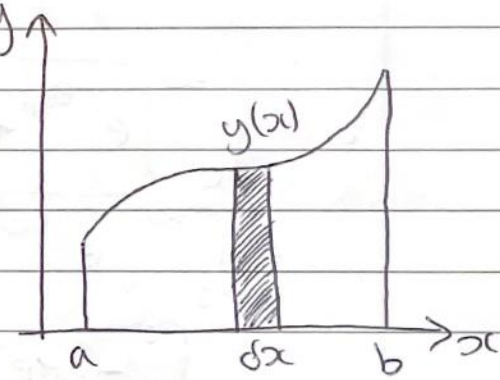
The numerators of these ~~eq~~ expressions for $\bar{x}, \bar{y}$ are often called the first moment of mass.

To generalise to a two dimensional plate 'lamina', we consider a continous mass distribution with say

density = $\frac{\text{mass}}{\text{unit area}}$

$\rho_{ij} \delta x_i \delta y_j$

the mass element located at $(x_i, y_j)$.




Example

If the mass density $\rho$ is uniform then.

Total Mass $= \rho \int_a^b y \, dx$

Split the area into strips a shown. The mass of each strip is $\rho y \delta x$. The centre of mass is at $\approx (x, y/2)$.

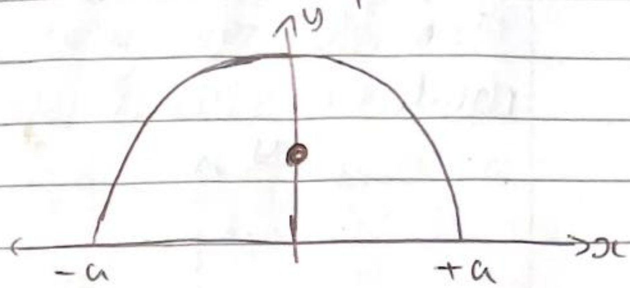
$$\therefore \quad \bar{x} = \frac{\int x(\rho y dx)}{M} = \frac{\int \rho x y dx}{\int \rho y dx} = \boxed{\frac{\int_a^b x y dx}{\int_a^b y dx}}$$

$$\bar{y} = \frac{\int (\frac{y}{2})(\rho y dx)}{M} = \frac{\int \frac{\rho}{2} y^2 dx}{\int \rho y dx} = \boxed{\frac{\frac{1}{2}\int y^2 dx}{\int y dx}}$$

These ideas generalise to:

- non uniform density
- 3 dimensions
- Second moments of mass (moment of inertia)

Example uniform semi-circuler plate




$$\bar{x} = \frac{\int_{-a}^{a} xy\,dx}{\frac{\pi a^2}{2}}$$

$$= \frac{\int_{-a}^{a} x(a^2-x^2)^{\frac{1}{2}}\,dx}{\frac{\pi a^2}{2}}$$

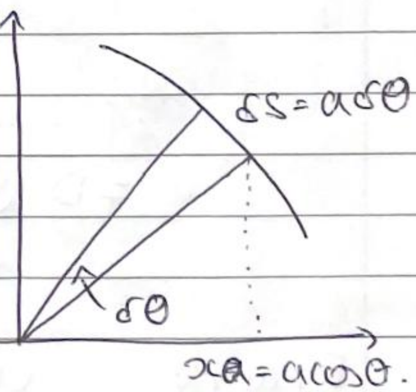
$= 0$ (~~odd~~ even function!) $\left(\int x^2a^2 - x^4\right)$

$$\bar{y} = \frac{\frac{1}{2}\int_{-a}^{a} y^2\,dx}{\frac{\pi a^2}{2}} = \frac{1}{\pi a^2}\int_{-a}^{a} a^2 - x^2\,dx$$

$$= \frac{1}{\pi a^2}\left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a} = \frac{4a}{3\pi}$$

Example uniform semi-circuler <u>wire</u>

Line density $= \frac{M}{\pi a}$




$$\delta m = \rho\,\delta s$$
$$= \rho a\,\delta\theta.$$

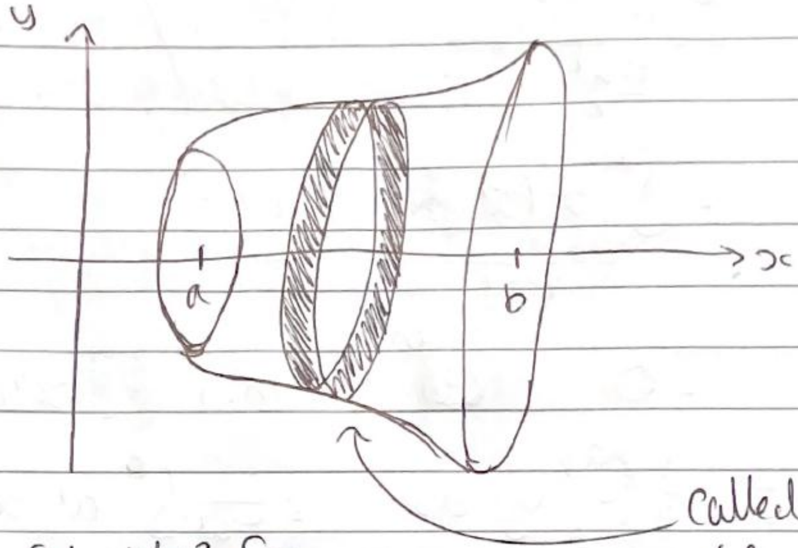
$$\bar{x} = \frac{\int x\,dm}{\int dm}$$

$$= \frac{\int_{-\pi/2}^{\pi/2} a^2\rho\cos\theta\,d\theta}{\int dm} = \frac{a^2\rho}{\int dm}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{2a^2\rho}{M}$$

$$= \frac{2a}{\pi}.$$

## Volumes/Surfaces of Revolution

The function $y(x)$ in a 2-dimensional plane and rotate it about the $x$-axis to create a 3D shape.




Volume $\delta V = \pi y^2 \delta x$

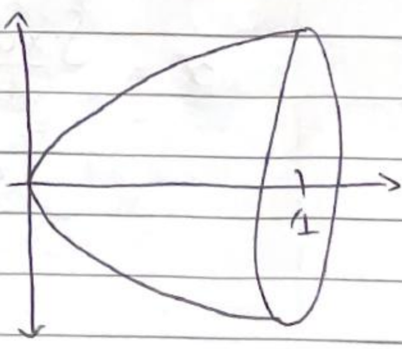
$$\Rightarrow V = \pi \int_a^b y^2 \, dx$$

Surface Area $\delta S = 2\pi y \, \delta s$

$$= 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$\Rightarrow S = 2\pi \int_a^b y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

Example $\quad y = \sqrt{x}$ rotated $0 \leq x \leq 1$




$$V = \pi \int_0^1 y^2 \, dx$$

$$= \pi \left[\tfrac{1}{2}x^2\right]_0^1 = \pi/2.$$

$$S = 2\pi \int_0^1 y \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$

$$= 2\pi \int_0^1 x^{\frac{1}{2}} \left(1 + \frac{1}{4x}\right)^{\frac{1}{2}} dx$$

$$= 2\pi \int_0^1 x + \tfrac{1}{4} \, dx$$

$$= \frac{4\pi}{3} \left[ \left(x + \tfrac{1}{4}\right)^{\frac{3}{2}} \right]_0^1$$

$$= \frac{4\pi}{3} \left[ \left(\tfrac{5}{4}\right)^{\frac{3}{2}} - \left(\tfrac{1}{4}\right)^{\frac{3}{2}} \right] = \frac{\pi}{6}\left(5^{\frac{3}{2}} - 1\right)$$

<u>Example</u> sphere

$y = \sqrt{R^2 - x^2}$ rotated around $x$ axis by $2\pi$

$$V = \pi \int_{-R}^{R} (R^2 - x^2)\, dx = \pi \left[ R^2 x - \tfrac{1}{3} x^3 \right]_{-R}^{R} = \tfrac{4}{3}\pi R^3$$

$$S = 2\pi \int_{-R}^{R} (R^2 - x^2)^{\frac{1}{2}} \left[ 1 + \left( \frac{x}{\sqrt{R^2 - x^2}} \right)^2 \right]^{\frac{1}{2}} dx$$

$$= 2\pi \int_{-R}^{R} R \, dx = \underline{\underline{4\pi R^2}}$$

<u>Example</u> sphereical cap area

$$S = 2\pi \int_{R\cos\alpha}^{R} R \, dx$$

$$= 2\pi R^2 (1 - \cos\alpha).$$