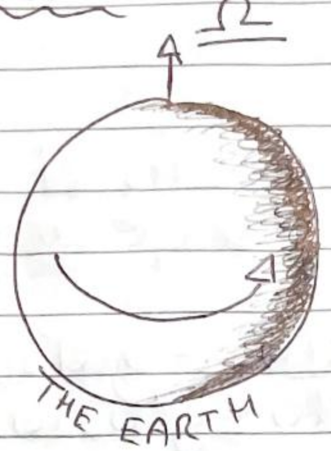
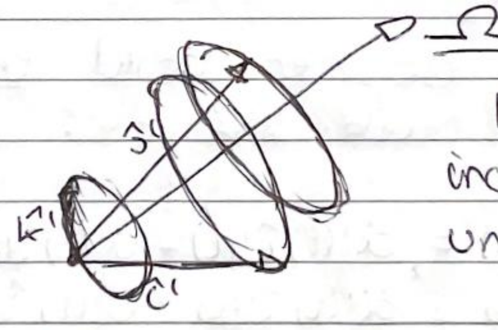
Classical Mechanics 21

Rotating Frames



Inertial & Rotating Coordinate Systems

Compare measurements made in two different coordinate systems: an interial frame with fixed unit vectors $\hat{i}, \hat{j}, \hat{k}$ and a frame with rotating unit vectors $\hat{i}'(t), \hat{j}'(t), \hat{k}'(t)$. The angular velocity of rotation is $\underline{\Omega}$.



N.B. this diagram incorrect, should be unit length.

Rotating Unit Vector

$$\frac{d\underline{a}}{dt} = \underbrace{\dot{a}\hat{a}}_{\text{change in length of } a} + \underbrace{a\underline{\Omega} \times \hat{a}}_{\text{change in angular length of } a}$$

For fixed vectors of unit length such as $\hat{i}', \hat{j}', \hat{k}'$, this becomes

$$\frac{d\hat{i}'}{dt} = \underline{\Omega} \times \hat{i}' \qquad \frac{d\hat{j}'}{dt} = \underline{\Omega} \times \hat{j}' \qquad \frac{d\hat{k}'}{dt} = \underline{\Omega} \times \hat{k}'$$

The position vector

The position vector $\underline{r}(t)$ of a mass, m, could be written in two different ways.

$$\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$
$$\underline{r}(t) = x'(t)\hat{i}'(t) + y'(t)\hat{j}'(t) + z'(t)\hat{k}'(t)$$

- ☆ $x(t)$, $y(t)$ & $z(t)$ are the coordinates described by an inertial observer
- ☆ $x'(t)$, $y'(t)$ & $z'(t)$ are the coordinates described by a rotating observerect
- ☆ both observers draw the same arrow from the (shared) origin but describe it differently.

The rotating observer would think that the velocity and acceleration are:

$$\underline{v}'(t) = \dot{x}'(t)\hat{i}'(t) + \dot{y}'(t)\hat{j}'(t) + \dot{z}'(t)\hat{k}'(t)$$
$$\underline{a}'(t) = \ddot{x}'(t)\hat{i}'(t) + \ddot{y}'(t)\hat{j}'(t) + \ddot{z}'(t)\hat{k}'(t)$$

However, this is not true as the rotating observer has failed to account for the time dependence of $\hat{i}'(t)$, $\hat{j}'(t)$ & $\hat{k}'(t)$.

We will try to relate $\underline{v}'$ to $\underline{v}$ and $\underline{a}'$ to $\underline{a}$.

$$v = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{d}{dt}(x'\hat{i}' + y'\hat{j}' + z'\hat{k}')$$

$$= \left(\dot{x}'\hat{i}' + \dot{y}'\hat{j}' + \dot{z}'\hat{k}'\right) + \left(x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}\right)$$

$$= v' + x'(\underline{\Omega}\times\hat{i}') + y'(\underline{\Omega}\times\hat{j}') + z'(\underline{\Omega}\times\hat{k}')$$

$$\underline{v} = \underline{v}' + \underline{\Omega}\times\underline{r}$$

$$\boxed{\underline{v}' = \underline{v} - \underline{\Omega}\times\underline{r}}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\underline{v}' + \underline{\Omega}\times\underline{r}\right)$$

$$= \frac{d\underline{v}'}{dt} + \underline{\Omega}\times\frac{d\underline{r}}{dt}$$

$$= \frac{d\underline{v}'}{dt} + \underline{\Omega}\times(\underline{v}' + \underline{\Omega}\times\underline{r})$$

$$= \left(\ddot{x}'\hat{i}' + \ddot{y}\hat{j}' + \ddot{z}\hat{k}'\right) + \left(\dot{x}\frac{d\hat{i}'}{dt} + \dot{y}\frac{d\hat{j}'}{dt} + \dot{z}\frac{d\hat{k}'}{dt}\right) + \underline{\Omega}\times(\underline{v}' + \underline{\Omega}\times\underline{r})$$

$$= \underline{a}' + \left(\dot{x}(\underline{\Omega}\times\hat{i}') + \dot{y}(\underline{\Omega}\times\hat{j}') + \dot{z}(\underline{\Omega}\times\hat{k}')\right) + \underline{\Omega}\times\underline{v}' + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$

$$= \underline{a}' + \underline{\Omega}\times\underline{v}' + \underline{\Omega}\times\underline{v}' + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$

$$= \underline{a}' + 2\underline{\Omega}\times\underline{v}' + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$$

$$m\underline{a}' = m\underline{a} + \underbrace{(-2\underline{\Omega}\times\underline{v}')}_{\text{coriolis force}} + \underbrace{(-\underline{\Omega}\times(\underline{\Omega}\times\underline{r}))}_{\text{centrifugal force}}$$

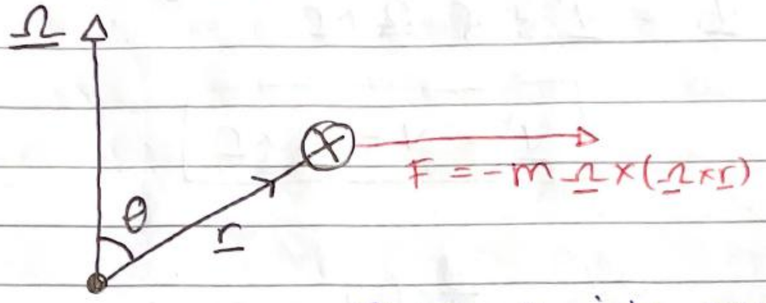
$m\underline{a}'$: force that rotating observer "sees".

$m\underline{a}$: true force

$$\boxed{\underline{F}' = \underline{F} + (-2m\underline{\Omega}\times\underline{v}') + (-m\underline{\Omega}\times(\underline{\Omega}\times\underline{r}))}$$

Centrifugal Force

$$\underline{F} = -m\underline{\Omega} \times (\underline{\Omega} \times \underline{r})$$

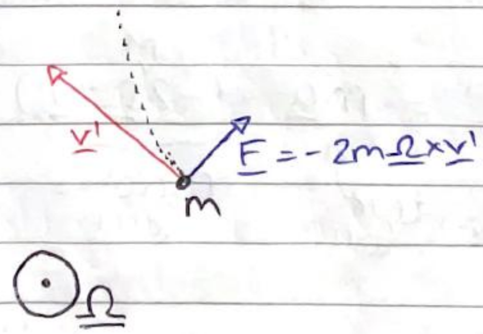


$\underline{\Omega} \times \underline{r}$ is into page!

The centrifugal force points radially outwards from the axis of rotation. For the rotating observer, the centripetal force would be balanced by the centrifugal force so no acceleration was acting on the particle.

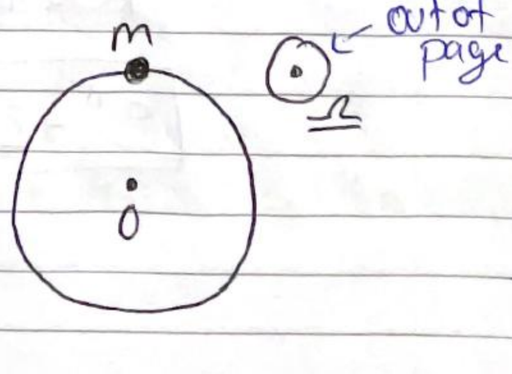
Coriolis Force

$$\underline{F} = -2m\underline{\Omega} \times \underline{v}'$$



Example

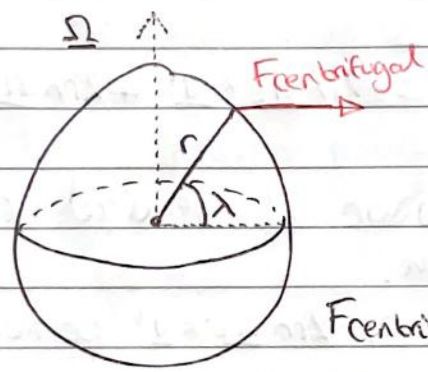
the centrifugal force is found by



$\underline{F} = -m\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = m\Omega^2 r\hat{r}$ and acts away from the origin. The coriolis force $\underline{F} = -2m\underline{\Omega}\times\underline{v}' = -2m\Omega^2 r\hat{r}$ acts towards the origin. Summing them gives the centripetal force.

$$F = m\Omega^2 r\hat{r} - 2m\Omega^2 r\hat{r} = -m\Omega^2 r\hat{r}$$

The Earth as a Rotating Frame



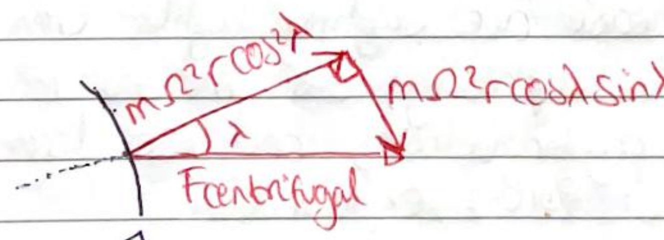
$\lambda - \frac{\pi}{2}$ is lattitude

$\underline{\Omega}\times\underline{r} = \Omega r\cos\lambda$ into page

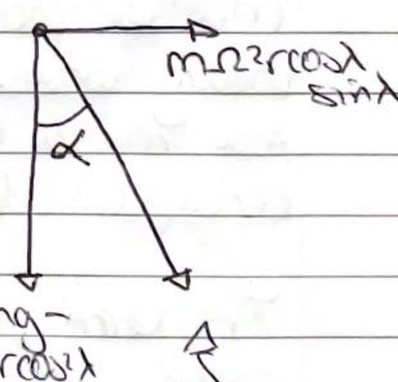
$\underline{\Omega}\times(\underline{\Omega}\times\underline{r}) = \Omega^2 r\cos\lambda$ to left

$F_{centrifugal} = m\Omega^2 r\cos\lambda$ outwards

N.B. centrifugal force is zero at the poles and maximum at equator



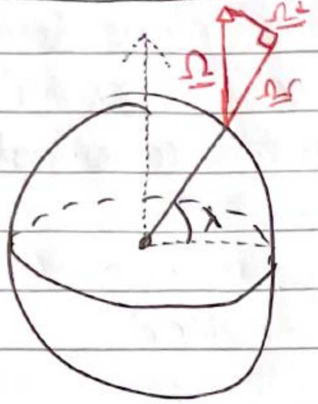
from an astronauts point of view



the radial components affect the force of gravity (mg)

from the point of view of the rotating frame.

Coriolis Force on Earth



As the coriolis force depends upon velocity, it can be difficult to analysis.

To simplfy resolve $\underline{\Omega}$ (angular velocity) into radial and tangental components.

$$F_{coriolis} = -2m(\underline{\Omega}_t + \underline{\Omega}_r) \times \underline{v}' = -2m\underline{\Omega}_t \times \underline{v}' - 2m\underline{\Omega}_r \times \underline{v}'$$

The $-2m\underline{\Omega}_t \times v'$ contribution is vertical compared to the surface of the earth.

- If $\underline{v}'$ points west, $-2m\underline{\Omega}_t \times \underline{v}'$ point down
- If $\underline{v}'$ points east, $-2m\underline{\Omega}_r \times \underline{v}'$ points up
- If $\underline{v}'$ points north/south, $-2m\underline{\Omega}_t \times v'$ is zero.

Masses moving east are slightly lighter than masses moving west. Less suprising; as masses moving east have a larger angular velocity and so larger centripetal force ... further explaination

The term involving the radial component of the Earth's angular velocity is more interesting. The $-2m\underline{\Omega}_r \times \underline{v}'$ will be in the plane of the Earth's surface (perp to $\underline{\Omega}_r$). It is also perp to $\underline{v}'$.

In the northern hemisphere it will push masses to their right, in the southern hemisphere; left.

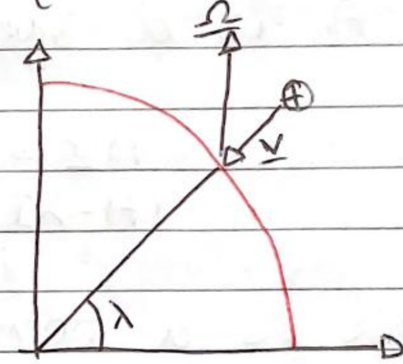
The magnitude of the horizontal coriolis force can be found by:

$$|F^{horizontal}_{coriolis}| = |-2m\underline{\Omega}r \times \underline{v}'| = 2m\Omega r v' = 2m\Omega \sin\lambda \, v'$$

This will be maximum at the poles and a minimum at the equator.

<u>Falling Bodies</u>

The diagram shows a body falling vertically in a rotating frame. (e.g. towards the earth).



The coriolis force points east (into page), so falling objects in the northern hemisphere land a little east of where you would expect.

The magnitude of the coriolis force $|F| = 2mv'\Omega\cos\lambda$, increases as the body falls and v' increases. ∴ Need to use intergration to find total deflection.

N.B. the small horizontal component of velocity produces an additional coriolis force which points radially outwards and has components in the $\hat{r}$ and $\hat{\theta}$ directions.