Classical Mechanics 10

Scattering in the CoM frame

The incoming momenta are equal and opposite. $p_1^* = -p_2^*$. Momentum conservation implies that outgoing momentum are also equal and opposite. $q_1^* = -q_2^*$.

If the collision is elastic, KE is conserved so $|q^*| = |p^*|$

As seen in the CoM frame, all collisions are trivial.

Transforming back to the Lab Frame

Given $\underline{p}^*, \theta^*$, how do we find $\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2, \dot{\underline{R}}, \theta, \phi$?

Finding $\underline{p}_2$?

m_2 was initially stationary so $\boxed{\underline{p}_2 = 0}$

Finding $\dot{\underline{R}}$?

- express $\dot{\underline{R}}$ in terms of starred quantities

$$\begin{aligned}\underline{p}_2^* &= m_2\dot{\underline{r}}_2^* \\ &= m_2(\dot{\underline{r}}_2 - \dot{\underline{R}}) \\ &= -m_2\dot{\underline{R}}\end{aligned}$$

($\dot{r}_2 = 0$ particle 2 stationary)

$$\boxed{\dot{\underline{R}} = \underline{p}^*/m_2}$$

$(\underline{p}_2^* = -\underline{p}^* = -\underline{p}_1^*)$

Finding $\underline{p}_1$?

$$M\dot{\underline{R}} = m_1\dot{\underline{r}}_1 + m_2\dot{\underline{r}}_2$$

Using $\underline{\dot{R}} = \underline{p}^*/m_2$

$$\boxed{\underline{p}_1 = M\underline{\dot{R}} = M\frac{\underline{p}^*}{m_2} = \left(\frac{m_1}{m_2}+1\right)\underline{p}^*}$$

We now know the lab frame vectors before the collision!

Finding $\underline{q}_1$?

After the collision: $\underline{\dot{r}}_1 = \underline{\dot{r}}_1^* + \underline{\dot{R}}$

$$m_1\underline{\dot{r}}_1 = m_1\underline{\dot{r}}_1^* + m_1\underline{\dot{R}}$$

$$\left(m_1\underline{\dot{r}}_1^* = \underline{q}^* \qquad \underline{\dot{R}} = \frac{\underline{p}^*}{m_2} \right)$$

$$\boxed{\underline{q}_1 = \underline{q}^* + \frac{m_1}{m_2}\underline{p}^*}$$

Finding $\underline{q}_2$?

$$\underline{\dot{r}}_2 = \underline{\dot{r}}_2^* + \underline{\dot{R}}$$

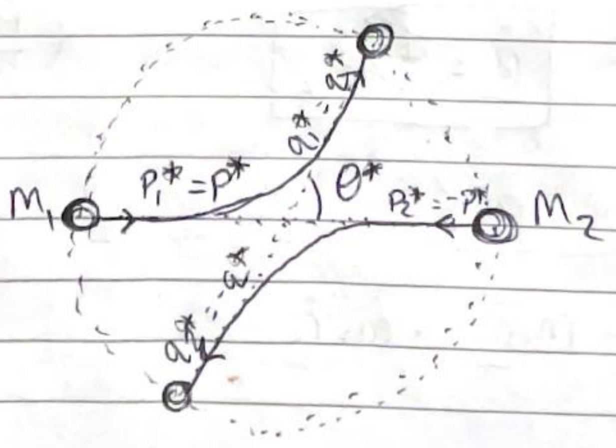
$$m_2\underline{\dot{r}}_2 = m_2\underline{\dot{r}}_2^* + m_2\underline{\dot{R}}$$

$$\left(m_2\underline{\dot{r}}_2^* = -\underline{q}^* \qquad \underline{\dot{R}} = \frac{\underline{p}^*}{m_2} \right)$$

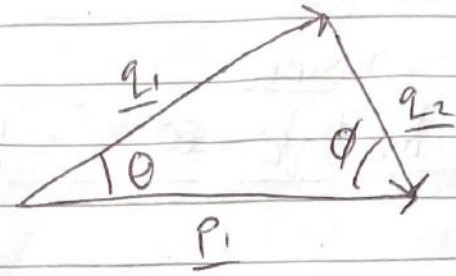
$$\boxed{\underline{q}_2 = -\underline{q}^* + \underline{p}^*}$$

Trivial to see that $\underline{q}_1 + \underline{q}_2 = \underline{p}_1 + \underline{p}_2$.

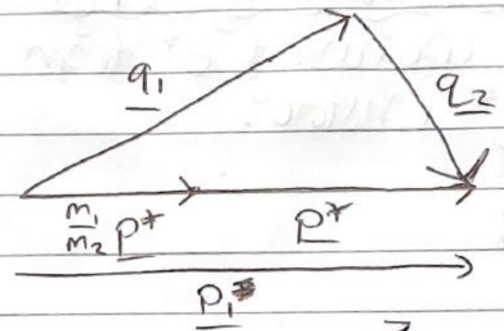
CoM Frame



Step 1.

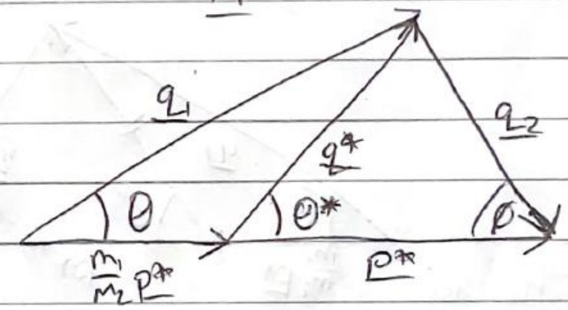


Step 2.



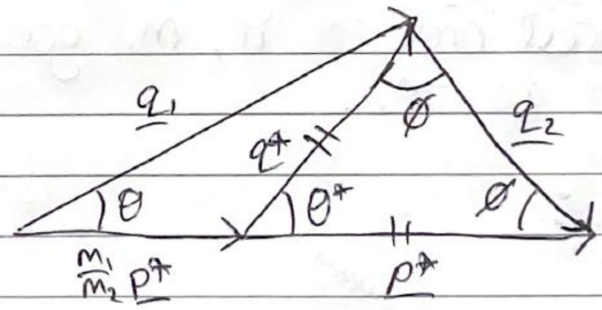
$$p_1 = \left(\frac{m_1}{m_2}+1\right)p^*$$

Step 3.



$$q_1 = \frac{m_1}{m_2}p^* + q^*$$

Step 4.

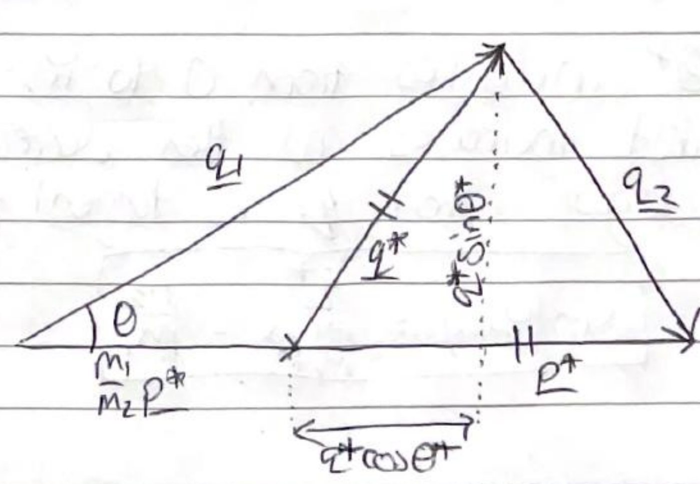


$$|q^*| = |p^*|$$

$\therefore$ recoil angle (ϕ) is related to CoM frame scattering angle by

$$\phi = \frac{1}{2}(\pi - \theta^*)$$

Step 5.

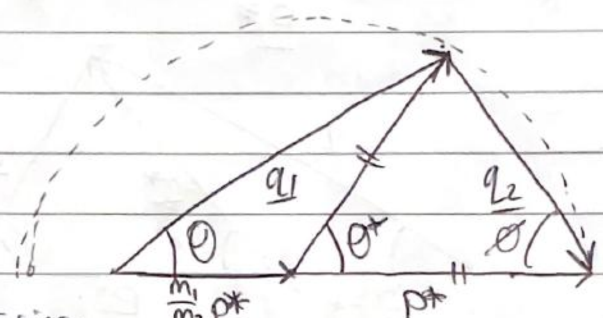


$$\boxed{\tan\theta = \frac{q^* \sin\theta^*}{\frac{m_1}{m_2}p^* + q^*\cos\theta^*} = \frac{\sin\theta^*}{\frac{m_1}{m_2} + \cos\theta^*}}$$

Maximum Scattering Angle

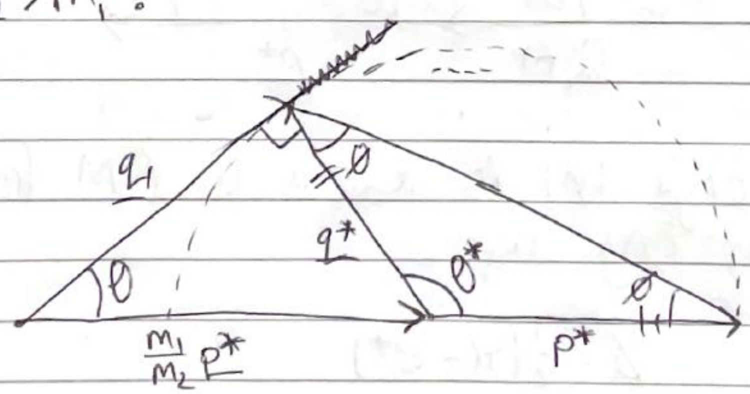
Imagine increasing θ^* from 0 to π holing p^* and q^* constant.

$m_1 < m_2$:



max ~~recoil~~ scattering angle is π, m_1 goes in opposite direction to m_2.

$m_2 > m_1$:

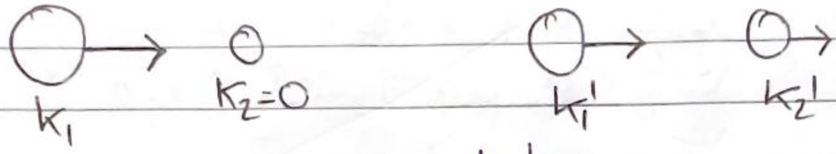


As θ^* increases from 0 to π, the value of θ first increases and then decreases. The value of θ is largest when q_1 is tangent to the circle.

$$\boxed{\sin(\theta_{max}) = \frac{q^*}{\frac{m_1}{m_2}p^*} = \frac{m_2}{m_1}}$$

Kinetic Energy Transfer

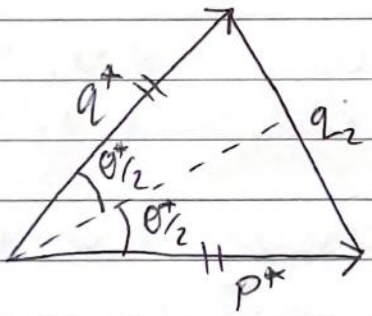
What fraction of the incident KE is transferred to a stationary target in an elastic collision?



We want to find K_2'/K_1.

$$K_1 = \frac{(p_1)^2}{2m_1} = \frac{\left(\frac{m_1}{m_2}+1\right)^2 (p^*)^2}{2m_1}$$

To get K_2', we reuse our momentum diagram.



$$q_2 = 2p^* \sin(\theta^*/2)$$

$$K_2' = \frac{(q_2)^2}{2m_2} = \frac{4(p^*)^2 \sin^2(\theta^*/2)}{2m_2}$$

$$\frac{K_2'}{K_1} = \frac{\left[4(p^*)^2 \sin^2(\theta^*/2)\right]/2m_2}{\left[\left(\frac{m_1}{m_2}+1\right)^2 (p^*)^2\right]/2m_1}$$

$$\boxed{= \frac{4m_1 m_2 \sin^2(\theta^*/2)}{(m_1+m_2)^2}}$$

The transfer of KE is largest when $\theta^* = \pi$.

$$\boxed{\left(\frac{K_2'}{K_1}\right)_{MAX} = \frac{4m_1 m_2}{(m_1+m_2)^2}}$$

→ If $m_1 = m_2$ the $\left(\frac{k_2'}{k_1}\right)_{MAX} = 1$, target picks up 100% of the incident KE.

→ If $m_1 >> m_2$, $(k_2'/k_1)_{MAX} \approx \frac{4m_2}{m_1}$ is very small.

→ If $m_2 >> m_1$, $(k_2'/k_1)_{MAX} \approx \frac{4m_1}{m_2}$ is very small.