Charge Density

So far we've considered point charges, but it is often more useful to consider charge spread out continuously, we call this charge density.

uniform charge density $\rightarrow$ $$\rho = \frac{Q}{V}$$ $\leftarrow$ total charge, $\leftarrow$ volume

For a non-uniform charge density with charge $\rho = \rho(x, y, z)$

$$dQ = \rho \, dV$$

$$\Rightarrow \quad Q = \int_V \rho \, dV$$ $\leftarrow$ total charge

Surface charge density (σ) is a useful concept if the charge is spread-out over a thin layer.

$$Q = \iint_S \sigma \, dS$$

Gauss's Law

$$\oint \underline{E} \cdot d\underline{S} = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

divergence theorem $\rightarrow$

$$\int_V (\nabla \cdot \underline{E}) \, dV = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

$$(\nabla \cdot \underline{E}) \, dV = \frac{1}{\epsilon_0} \rho \, dV$$

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

Charged Spheres

Lets start by looking at inside a uniformly charged sphere.

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3}$$

The charged enclosed by the sphere $r<a$ is

$$Q = \frac{4}{3}\pi r^3 \rho = \frac{Qr^3}{a^3}$$

Now, applying gauss's law to the sphere of radius a:

$$4\pi r^2 E = \frac{Qr^3}{\epsilon_0 a^3}$$

$$E = \frac{Qr}{4\pi\epsilon_0 a^3}$$

Outside the sphere, the charged enclosed is simply Q:

$$4\pi r^2 E = \frac{Q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

Laplaces Equation

we can rewrite $\underline{E}$ in terms of potential V by

$$\nabla \cdot \underline{E} = -\nabla \cdot (\nabla V) = -\nabla^2 V$$

Therefore we can rewrite gauss's law as

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

This called poissions equation. When there is no charge density

$$\nabla^2 V = 0$$

which is known as laplace's equation.

Uniqueness Theorem

Electrostatics can be summariesed by the two eq^n's:

$$\nabla \times \underline{E} = 0 \qquad \nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

helmholtz theorem!

or equally written as

$$E = -\nabla V \qquad \nabla^2 V = -\frac{\rho}{\epsilon_0}$$