CHAPTER 1

FUNCTIONS AND LIMITS

(1.1) DEFINITION, RANGE/DOMAIN

(1.2) COMMON FUNCTIONS

(1.3) LOG (ln), EXPONENTIAL AND HYPERBOLIC FUNCTIONS

(1.4) LIMITS

(1.5) NON-TRIVIAL LIMITS - EXAMPLES

(1.1) DEFINITION

If two variables, x and y, follow a rule:
'When x is given, then y is determined as ....' then y is said to be a FUNCTION of x.

We write $y = f(x)$

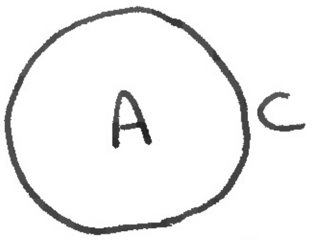
x – INDEPENDENT VARIABLE

y – DEPENDENT VARIABLE

[Note: then writing $\underset{\substack{\uparrow \\ \text{DEPENDENT}}}{x} = g(\underset{\substack{\nwarrow \\ \text{INDEPENDENT}}}{y})$]
$\Rightarrow$ roles reversed

EXAMPLE

CIRCLE of radius r



AREA $A = \pi r^2$

CIRCUMFERENCE $C = 2\pi r$.

[← SAME π! ARCHIMEDES]

DEPENDENT VARIABLES

INDEPENDENT VARIABLE

DOMAIN AND RANGE

For a set of values of x ('DOMAIN') there is a corresponding set of y values ('RANGE')

e.g. $A = \pi r^2$

GIVEN domain $0 \leq r \leq 2$ (say)

$\Rightarrow$ range of A is

$$0 \leq A \leq 4\pi .$$

COMMON NOTATION

we often write $y = f(x)$, but sometimes simply $y = y(x)$

$$\Rightarrow \quad f(x) = x + x^2$$

$$\Longleftrightarrow y = x + x^2 .$$

(1.2) COMMON FUNCTIONS

(a) POLYNOMIALS

$$y = a_0 + a_1 x + a_2 x^2 + \dots a_n x^{n \leftarrow \text{INTEGER}}$$

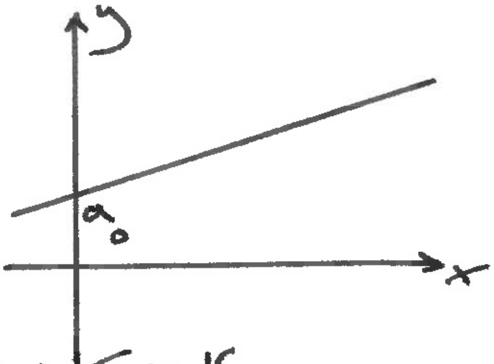
$$= \sum_{r=0}^{n} a_r x^r$$

$n \equiv$ DEGREE of polynomial (NOTE - FINITE).

(b) LINEAR FUNCTIONS

(polynomial of degree 1)

$$y = \underset{\uparrow \atop \text{INTERCEPT}}{a_0} + \underset{\uparrow \atop \text{SLOPE}}{a_1} x$$

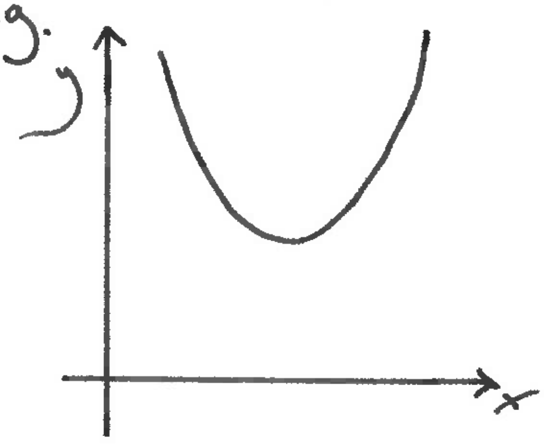


(c) QUADRATIC FUNCTIONS

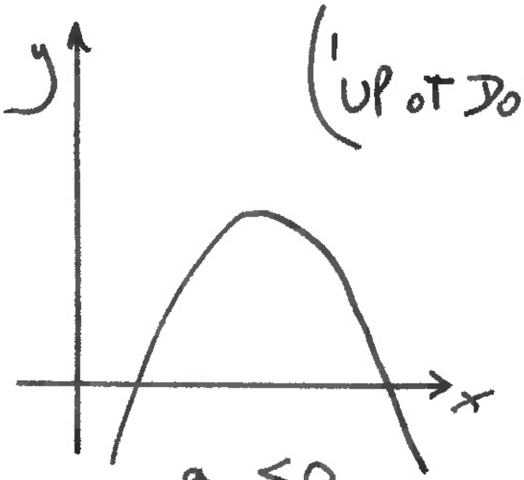
(polynomial of degree 2)

$$y = a_0 + a_1 x + a_2 x^2$$

e.g.



$a_2 > 0$



$a_2 < 0$

('UP or DOWN')

USEFUL TIP: COMPLETING THE SQUARE

$a_2 > 0$ $$a_2x^2 + a_1x + a_0 \equiv \left(\sqrt{a_2}\,x + \frac{a_1}{2\sqrt{a_2}}\right)^2 - \frac{a_1^2}{4a_2} + a_0$$

$a_2 < 0$ $$a_2x^2 + a_1x + a_0 \equiv -\left[|a_2|x^2 - a_1x - a_0\right]$$

[...] METHOD AS ABOVE

NOTE: By this means the properties of a quadratic function can be found – without calculus.

Asides:

(i) $x^2 - 2x\sqrt{b} + b$ with $b \geqslant 0$

$$\equiv (x - \sqrt{b})^2 \geqslant 0$$

EQUALITY only when $x = \sqrt{b}$.

Hence $(\sqrt{a} - \sqrt{b})^2 \geqslant 0$ EQUALITY only when $a = b$.

$a \geqslant 0$

rearrange $$\Rightarrow \frac{a+b}{2} \geqslant \sqrt{ab}$$

ARITHMETIC MEAN GEOMETRIC MEAN

(AM/GM INEQUALITY)

SEE LATER IN CHAPTER 2

(ii) $$\sum_{j=1}^{n} (a_jx - b_j)^2 \geqslant 0$$ Eq

a_j, b_j all $\geqslant 0$

EQUALITY only when $\frac{b_1}{a_1} = \frac{b_2}{a_2} = \dots \frac{b_n}{a_n} (= x)$

$$\Rightarrow \left(\sum_{j=1}^{n} a_j^2\right)x^2 - 2\left(\sum_{j=1}^{n} a_jb_j\right)x + \sum_{j=1}^{n} b_j^2 \geqslant 0$$

$$\Rightarrow \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \geqslant \left(\sum_{j=1}^{n} a_jb_j\right)^2$$

EQUALITY only when

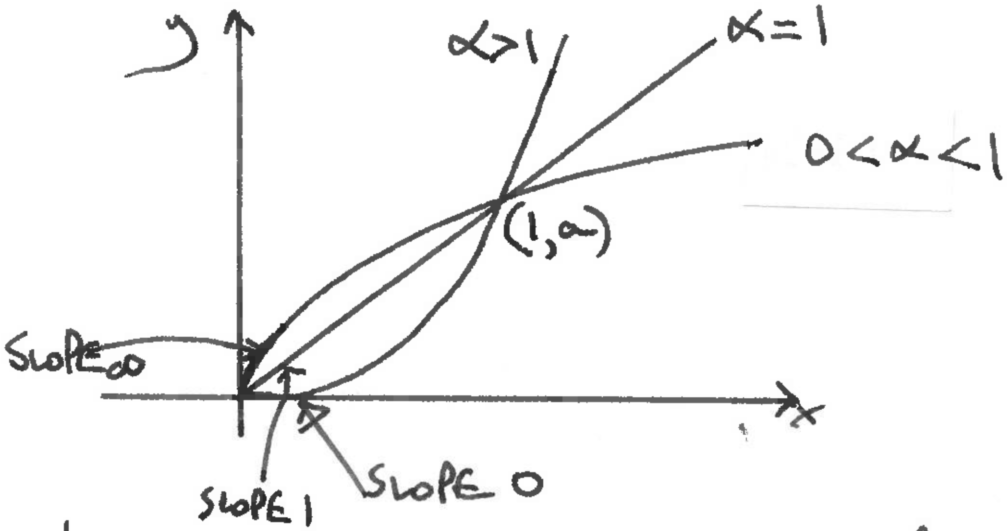
(CAUCHY-SCHWARZ INEQUALITY)

(d) POWER LAWS

$$y = a x^{\alpha}$$ (take $a, \alpha > 0$).

$\alpha \equiv$ POWER, INDEX, EXPONENT

Differentiate $\Rightarrow \frac{dy}{dx} = \alpha a x^{\alpha - 1}$



Many laws in Physics are expressible this way:

(i) n-body problem

Collisions at $t = t_0$ $\quad r_{ij} \propto (t_0 - t)^{2/3}$.

(ii) rowing shell speed $\propto n^{1/9}$ $\quad n \equiv$ number of rowers

(iii) atomic bomb cloud radius $\propto (t - t_0)^{2/5}$.

(iv) biscuit 'dunking' – liquid sucked up by capillary action a distance $\propto (t - t_0)^{1/2}$.

[IG NOBEL Prize (Fisher 1999)] velocity (?) acceleration (?).

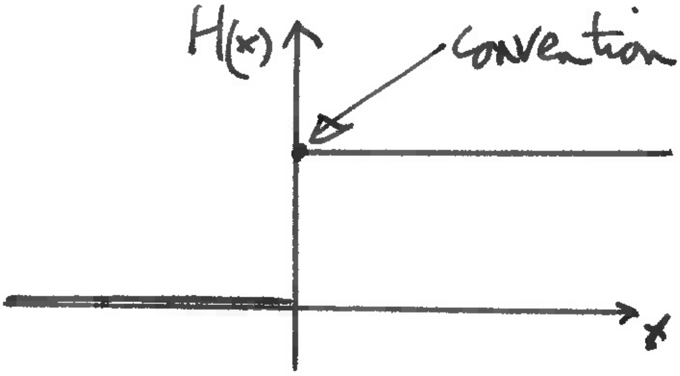
(e) TRIGONOMETRIC FUNCTIONS

$\sin x, \cos x, \ldots$

(f) HEAVISIDE (STEP) FUNCTION

$H(x)$

$$H(x) = \begin{cases} 0 & (x<0) \\ 1 & (x \geqslant 0) \end{cases}$$



although of little practical interest!

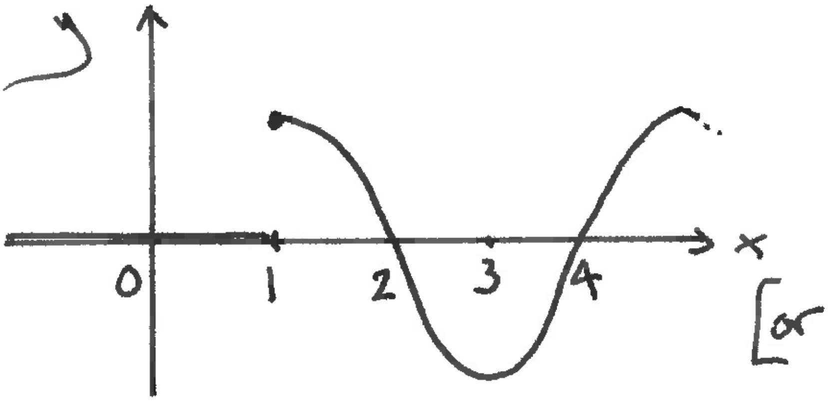
This function is discontinuous at $x=0$.

[NOTE: $\frac{dH}{dx} = 0$ for $x \neq 0$

H(x) NOT differentiable AT $x=0$]

e.g. SWITCH ON

$$y(x) = H(x-1)\sin\left(\frac{\pi x}{2}\right)$$

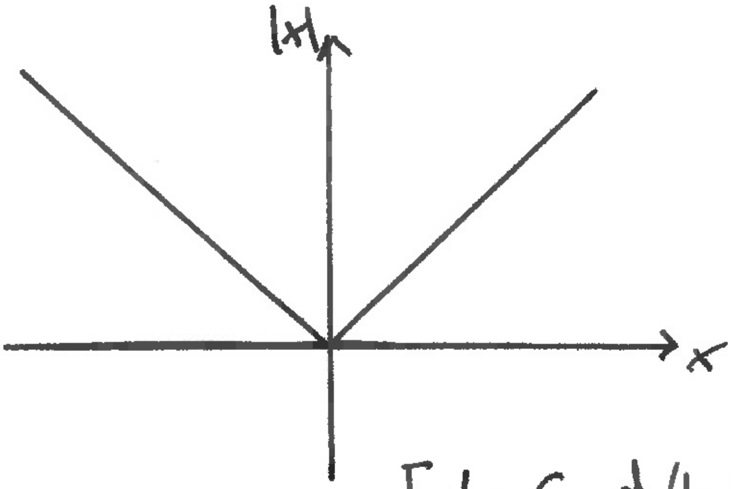


[or SWITCH OFF..]

(g) MODULUS FUNCTION $|x|$

$$|x| = \left| \begin{array}{ll} x & (x>0) \\ -x & (x<0) \end{array} \right.$$

[OFCOURSE $|0| = 0$!



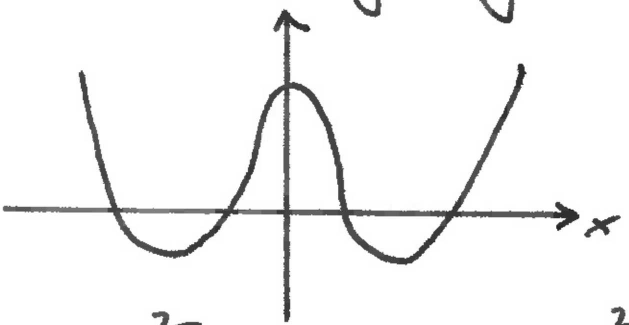
[NOTE $\frac{d}{dx}(|x|) = \left| \begin{array}{ll} +1 & x>0 \\ -1 & x<0 \end{array} \right.$]

discontinuous at $x=0$.

(h) EVEN AND ODD FUNCTIONS

$f(x)$ is EVEN if $f(x) = f(-x)$ for all x

e.g.



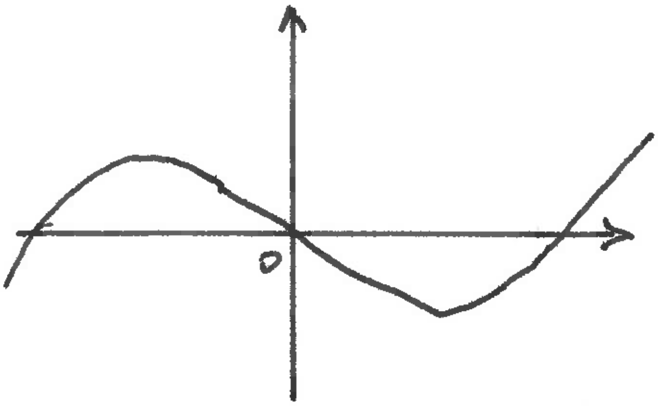
NOTE LEFT/RIGHT REFLECTION

~~Thus~~ e.g. $f(x) = x^2$ since $(-x)^2 \equiv x^2$ for all x

Similarly $f(x) = \cos x$ is even since $\cos(-x) \equiv \cos x$ for all x.

$f(x)$ is ODD if $f(x) = -f(-x)$ for all x

e.g.



NOTE REFLECTION THROUGH THE ORIGIN.

~~Thus~~ e.g. $f(x) = x^3$ since $(-x)^3 \equiv -x^3$ for all x.

Similarly $\sin x$, $\tan x$, x^5, ... are odd.

What about $y = \sin(x^5)$?

Well $\sin(-x)^5 \equiv \sin(-x^5) \equiv -\sin(x^5)$

So this $y(x)$ is odd.

NOTES:

(i) Not all functions are ODD or EVEN.

e.g. $f(x) = x + x^2$.

(ii) (even function)(even function) is EVEN

(odd function)(odd function) is EVEN

(even function)(odd function) is ODD.

(iii) In (i) above

$$f(x) \equiv \text{ODD} + \text{EVEN}.$$

For a general function $g(x)$ we can always express it as the sum of even and odd functions.

Consider the identity

$$g(x) \equiv \underbrace{\frac{1}{2}\left[g(x)+g(-x)\right]}_{\text{EVEN}} + \underbrace{\frac{1}{2}\left[g(x)-g(-x)\right]}_{\text{ODD}}.$$

evidently

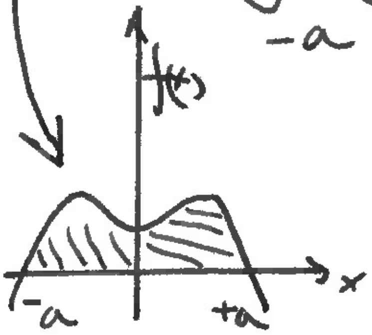
(iv) For an EVEN function $f(x)$ we have

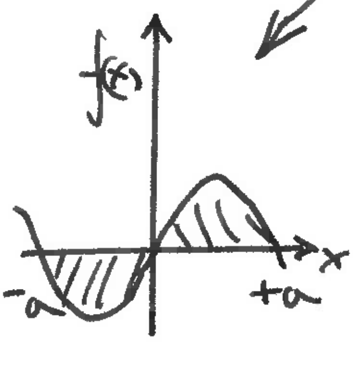
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

For an ODD function $f(x)$ we have

$$\int_{-a}^{a} f(x)\,dx = 0$$

[Consider the integrals as areas





(i) INVERSE FUNCTIONS

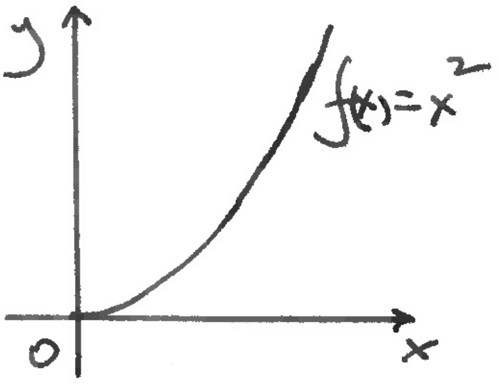
A function $y = f(x)$ can sometimes be INVERTED to get x in terms of y.

e.g. $x = g(y)$

↖ g is the INVERSE of f.

[always possible in principle – not always in practice!]

e.g. $y = x^2$ with $x \geq 0$.



$$x^2 = y$$

$$\Rightarrow \quad x = \pm\sqrt{y}$$

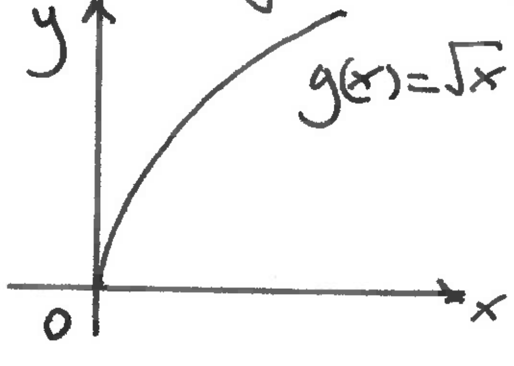
↖ +ve ROOT RELEVANT HERE

$$\therefore \quad x = \sqrt{y}$$

↖ +ve ROOT SURD CONVENTION

$\therefore$ The inverse $g(y) = \sqrt{y}$.

So the inverse function $g(x) = \sqrt{x}$.



NOTE: AXES FLIP OF $f(x)$ VERSUS x GRAPH.

NOTATION : the inverse function $g(x)$ of $f(x)$ is often written $f^{-1}(x)$.

Thus $f(x) = x^2$ (x +ve)

$$\Longleftrightarrow \text{ inverse function } f^{-1}(x) = \sqrt{x}.$$

The notation should not be confused with

$$\frac{1}{f(x)} \equiv \left(f(x)\right)^{-1}$$

[e.g. $\sin^{-1}(x)$
$\equiv \arcsin(x)$
NOT $\operatorname{cosec} x$

Since f and f^{-1} OPERATE on x above we have a general result

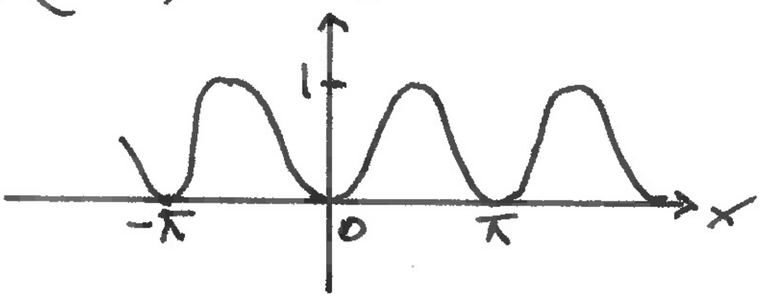
$$f\left(f^{-1}(x)\right) \equiv f^{-1}\left(f(x)\right) \equiv x \quad \text{for all } x.$$

(j) FUNCTION OF A FUNCTION

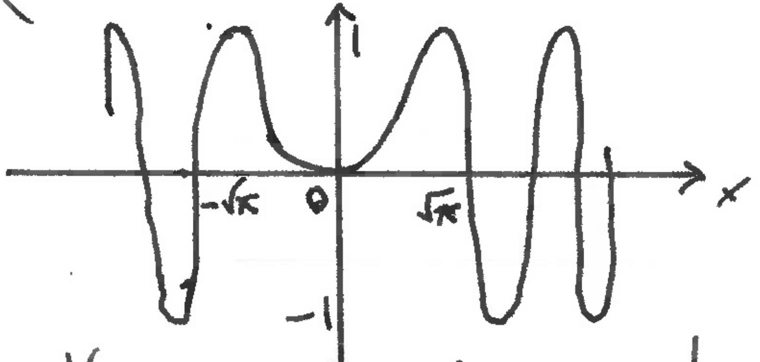
Continuing the operator concept above – given two functions $f(x)$, $g(x)$ we can calculate functions of a function

e.g. $f(x) = x^2$, $g(x) = \sin x$

Then $f(g(x)) = (\sin(x))^2 \equiv \sin^2 x$.



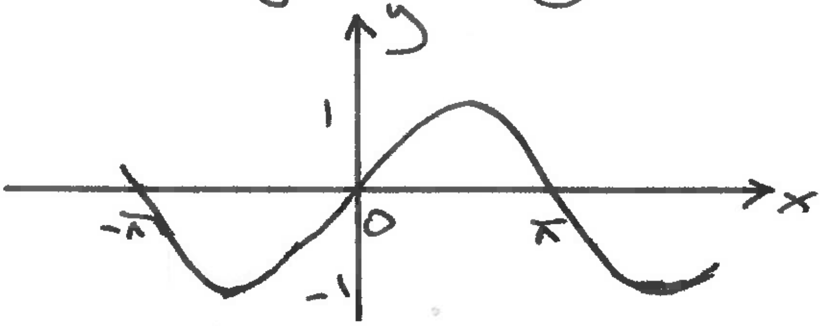
but $g(f(x)) = \sin(x^2)$



Evidently these are NOT the same! – though both EVEN.
We say that this composition does <u>not</u> COMMUTE.
– order matters!

(k) MANY-VALUED FUNCTIONS

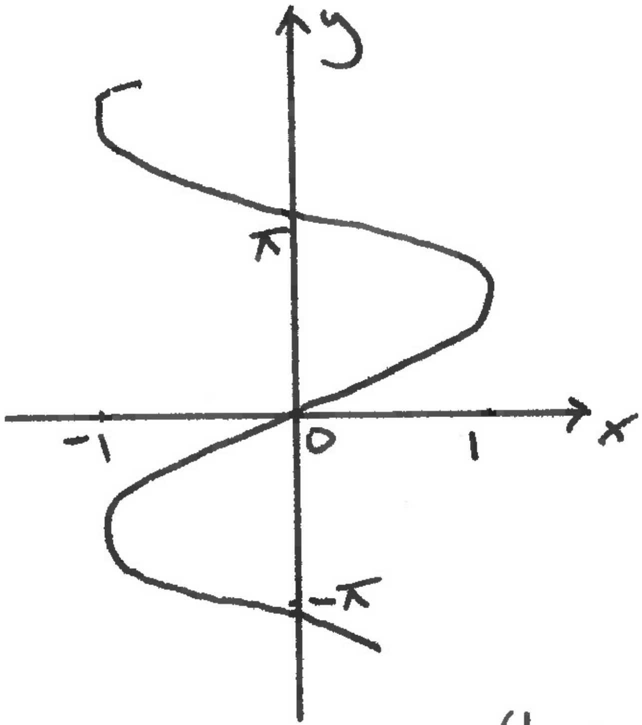
Consider the function $y = \sin x$.



For each x, $\exists$ only one y.

Now consider the inverse function

$$y = \sin^{-1} x \equiv \arcsin x$$



For each value of x (between -1 and $+1$) there are many (∞ty) values of y.

Hence, to clarify, we define the PRINCIPAL VALUE of $\sin^{-1}x$ to be in the domain $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$ for $-1 \leq x \leq 1$

[Similarly for $y = \cos^{-1}x$ we define

P.V. : $0 \leq \cos^{-1}x \leq \pi$ for $-1 \leq x \leq 1$].

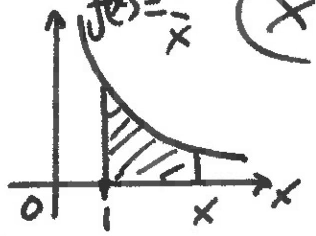
(1.3) LOGARITHM, EXPONENTIAL AND HYPERBOLIC FUNCTIONS

(a) THE LOGARITHM

The NATURAL LOGARITHM $\ln(x)$ or $\log_e(x)$ is defined as

$\uparrow$ CONSTANT - SEE LATER

$$\ln(x) = \int_1^x \frac{dt}{t} \quad (x>0).$$



NOTES:

(i) It follows that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

(ii) Also $\ln 1 = 0$.

(iii) $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$.

Why? Consider $\ln(x_1) = \int_1^{x_1} \frac{dt}{t}$

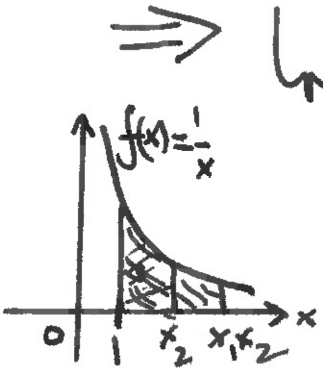
Put $s = t x_2$ with x_2 fixed.
so $ds = x_2 \, dt$

↖ t
s 'DUMMY' VARIABLES ↙

$$\Rightarrow \ln(x_1) = \int_{x_2}^{x_1 x_2} \frac{\cancel{x_2}\, ds}{\cancel{x_2}\, s}$$

$$= \int_1^{x_1 x_2} \frac{ds}{s} - \int_1^{x_2} \frac{ds}{s}.$$

FROM THE DEFINITION



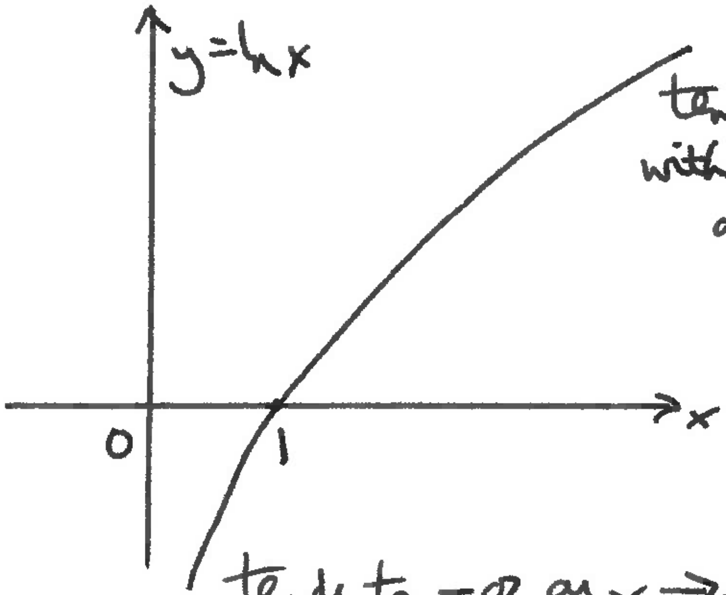
So that $\ln(x_1) = \ln(x_1 x_2) - \ln(x_2)$
the required result.

(iv) Since $\ln\left(\frac{1}{x}\right) + \ln(x) = \ln\left(\frac{1}{x} \cdot x\right)$

$= \ln 1 = 0.$

$\therefore$ we have $\ln\left(\frac{1}{x}\right) = -\ln x.$

(v) Evidently
$$\ln(x^2) = \ln(x) + \ln(x)$$
$$= 2\ln(x)$$
$$\cdots\cdots$$
$$\ln(x^n) = n\ln(x)$$
for n an integer (+ve, −ve, or zero).



tends to $+\infty$ with vanishing slope as $x \to \infty$.

tends to $-\infty$ as $x \to 0^+$

[(iii) above 'ln (PRODUCT) = SUM of the ln's'.
+ INVERSE FUNCTION $\Longrightarrow$ PRACTICAL TOOL 'SLIDE RULE' and TABLES of LOGARITHMS .... (BASE e or BASE 10)].
− (b) BELOW

(b) THE EXPONENTIAL FUNCTION

Consider $x = \ln y$. What is the inverse function $y = f(x)$?

Let $x_1 = \ln y_1$ so that $y_1 = f(x_1)$

$x_2 = \ln y_2$ so that $y_2 = f(x_2)$

Then

$$x_1 + x_2 = \ln y_1 + \ln y_2 = \ln(y_1 y_2)$$

$$\Rightarrow \quad y_1 y_2 = f(x_1 + x_2).$$

and function f must satisfy

$$f(x_1 + x_2) = f(x_1) f(x_2).$$

This implies that $f(x)$ has to be of the form $f(x) = a^x$ since only this satisfies $\left[a^{x_1} a^{x_2} = a^{x_1 + x_2}\right]$.

We note of course that

$$\underset{\text{integer } n}{f(nx) = \left[f(x)\right]^n}, \quad f(n) = \left[f(1)\right]^n,$$

$$f(0) = 1$$

So what is a?

If $x = \ln y$ then $\frac{dx}{dy} = \frac{1}{y}$.

$$\Rightarrow \frac{dy}{dx} = y.$$

So what value of a gives

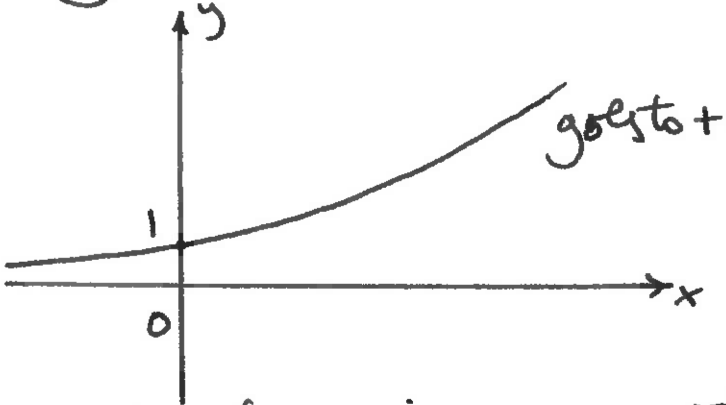
$$\frac{d}{dx}(a^x) = a^x \quad ?$$

The UNIQUE number that satisfies this is found to be $e = 2.7182818284\ldots\ldots$

[IRRATIONAL $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots$

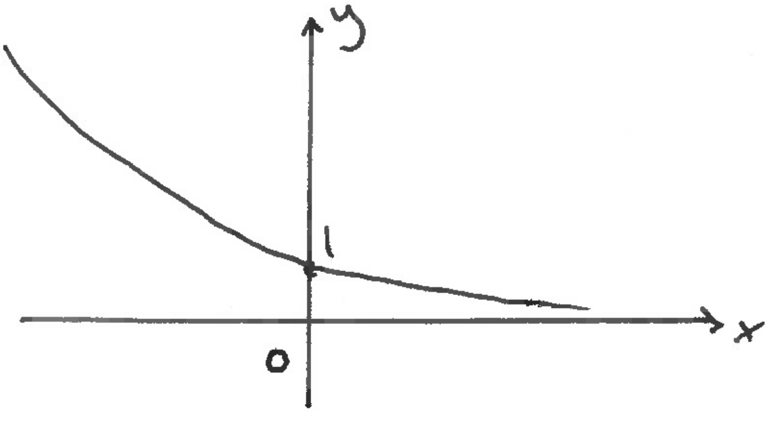
see CHAPTER 6.

so $y = e^x$ is the inverse of $y = \ln x$.



goes to $+\infty$ [faster than any power of x]

Of course: the function $y = e^{-x}$ looks like

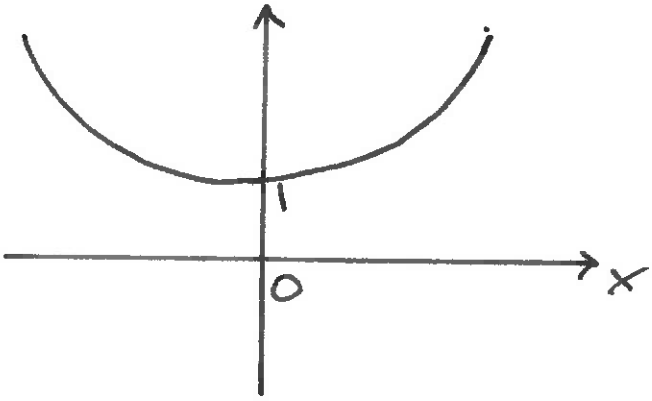


[reflection of e^x in the y axis]

(c) HYPERBOLIC FUNCTIONS

$$\cosh x \equiv \frac{1}{2}\left(e^{x}+e^{-x}\right)$$

'COSH'

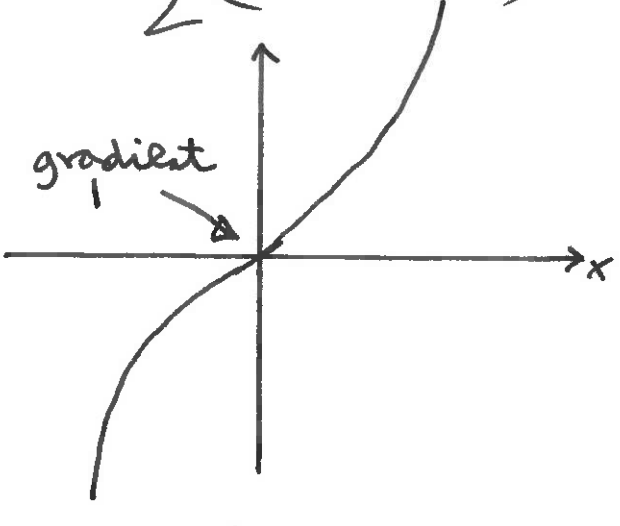


EVEN

NOTE EXPONENTIAL BEHAVIOUR AT $|x| \to \infty$

$$\sinh x \equiv \frac{1}{2}\left(e^{x}-e^{-x}\right)$$

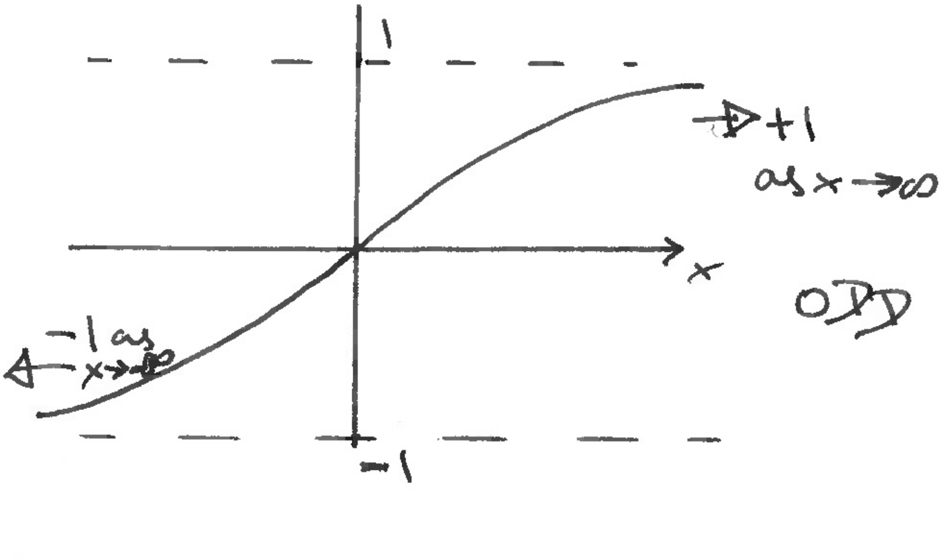
'SHINE' (or 'SINCH')



ODD

$$\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

'TANCH' (or 'THAN')



ODD

NOTES

(i) Hyperbolic functions are similar to trig. functions $\Rightarrow$

$$\cosh^2 x \overset{\downarrow}{-} \sinh^2 x \equiv 1.$$

$$\sinh(x_1+x_2) \equiv \sinh x_1 \cosh x_2 + \cosh x_1 \sinh x_2.$$

$$\cosh(x_1+x_2) \equiv \cosh x_1 \cosh x_2 \underset{\uparrow}{+} \sinh x_1 \sinh x_2$$

and

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

where $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\operatorname{cosech} x = \dfrac{1}{\sinh x}$, $\coth x = \dfrac{1}{\tanh x}$

(ii) The particular sign changes in some corresponding formulae are not incidental (or random!)

e.g. $\cosh(\underset{\underset{\sqrt{-1}}{\uparrow}}{i}x) \equiv \cos x$

$\sinh(ix) \equiv i \sin x$

[SEE LATER COURSES .....]

['PRODUCT of SINH functions' $\Rightarrow (i)^2 = -1$.
OSBORNE'S RULE ....]

(iii) INVERSE FUNCTIONS are related to logarithms.

e.g. $y = \sinh^{-1} x \Rightarrow x = \sinh y$.

[arc sinh x – NOT $\frac{1}{\sinh x}$!]

So $x = \frac{1}{2}(e^y - e^{-y})$

$\Rightarrow e^{2y} - 2xe^y - 1 = 0$ QUADRATIC for e^y.

$\rightarrow e^y = \dfrac{2x \pm \sqrt{(4x^2+4)}}{2}$

So $e^y = x + (x^2+1)^{1/2}$ Take +ve root since $e^y > 0$.

$\therefore y = \ln\cdot\left[x + (x^2+1)^{1/2}\right]$

and $\sinh^{-1} x \equiv \ln\left[x + (x^2+1)^{1/2}\right]$.

Similarly

$\cosh^{-1} x \equiv \ln\left[x \pm (x^2-1)^{1/2}\right]$.

FOR $x \geq 1$ ONLY!

$\equiv \pm \ln\left[x + (x^2-1)^{1/2}\right]$

since $\left[x + (x^2-1)^{1/2}\right]\left[x - (x^2-1)^{1/2}\right] \equiv 1$ of course.

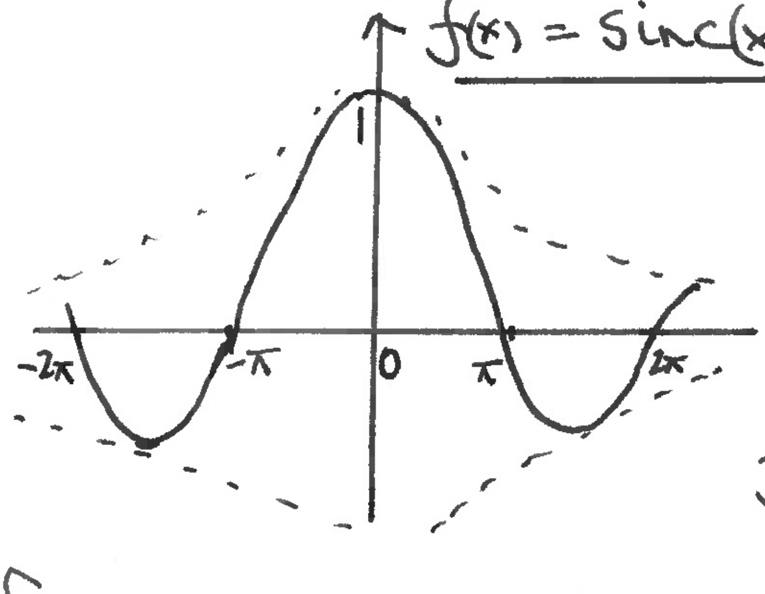
(1.4) LIMITS OF FUNCTIONS

Example Consider

$$f(x) = \frac{\sin x}{x} \quad (\text{for } x \neq 0).$$

$f(x)$ is NOT DEFINED at $x = 0$ since it has the form "$\frac{0}{0}$".

But plotting the function numerically shows that $f(x)$ gets closer and closer to 1 as x gets closer to 0.



$f(0.1) = \cdot 99833\ldots$

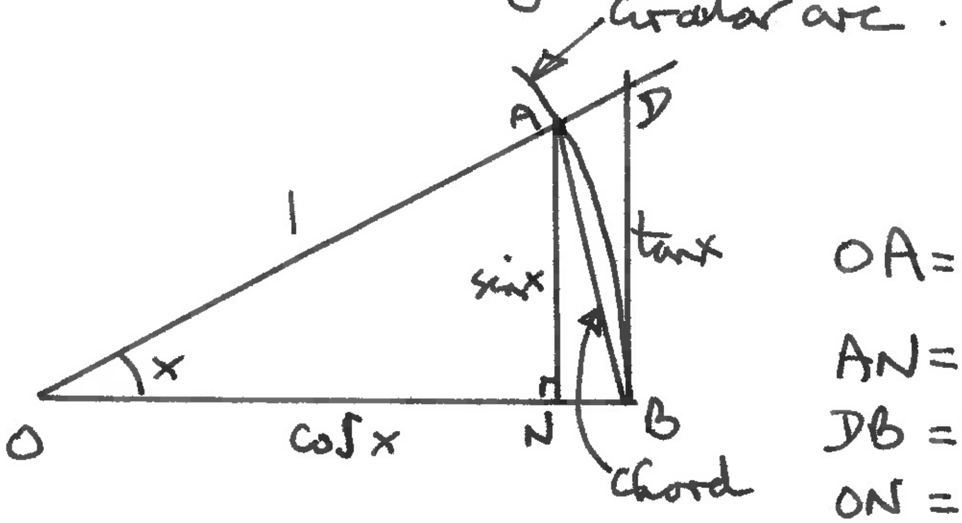
$f(0.01) = \cdot 9999833\ldots$

$f(0.001) = \cdot 99999983\ldots$

So can we confirm what happens to $\frac{\sin x}{x}$ as $x \to 0$?

GEOMETRICAL 'PROOF'

Consider a sector of the unit circle:



OA = OB = 1

AN = sin x

DB = tan x

ON = cos x

NB = 1 − cos x

Evidently

$$\text{AREA of } \Delta \text{ OAB} < \text{AREA sector OAB} < \text{AREA of } \Delta \text{ ODB}$$

$$\therefore \quad \frac{1}{2}\sin x(1) < \frac{x}{2\pi}\pi(1)^2 < \frac{1}{2}\tan x(1).$$

Divide through by $\frac{1}{2}\sin x \Rightarrow$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

$$\Rightarrow \cos x < \frac{\sin x}{x} < 1. \quad \text{taking reciprocals.}$$

Since $\cos x \to 1$ as $x \to 0$ we have

$$\frac{\sin x}{x} \to 1 \text{ as } x \to 0$$

[By the SQUEEZE PRINCIPLE!].

Notation we write

$$\frac{\sin x}{x} \to 1 \text{ as } x \to 0.$$

or

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right) = 1.$$

[Our 'proof' was for $x > 0$; the limit process requires that $x \to 0$ through +ve and through -ve values of x. We can modify our 'proof' to achieve this].

Mathematically a more formal approach is needed to turn 'proof' into proof. (VIA A FORMAL LIMIT DEFINITION)

More generally we write the limit F of a function $f(x)$ at the point $x = x_0$ as

$$f \to F \text{ as } x \to x_0$$

or

$$\lim_{x \to x_0} f(x) = F.$$

Notes

(i) There is no real issue if our function is 'well-behaved' at x_0

e.g. $f(x) = x^2 + 3 \Rightarrow \lim_{x \to 4} f(x) = 19$ [OF COURSE!

(ii) Simple rules for sums / products :

If $f(x) \to F$ and $g(x) \to G$
as $x \to x_0$.

then SUM $af+bg \to aF+bG$ (a, b constant)

PRODUCT $fg \to FG$

QUOTIENT $f/g \to F/G$ (provided that $G \neq 0$)

e.g. $$\lim_{x \to 2} \left[(x^2+2) \cos\left(\frac{\pi x}{2}\right) \right]$$

$$= \left[\lim_{x \to 2} (x^2+2) \right] \left[\lim_{x \to 2} \cos\left(\frac{\pi x}{2}\right) \right]$$

$$= (6) \ (\cos \pi)$$

$$= -6 .$$

[We need to CHECK if our 'limit' can be found simply using e.g. (i)(ii) above] .

(1.5) NON-TRIVIAL LIMITS

The concept of limit is essential when we encounter combinations like

"$\frac{0}{0}$", "$\frac{\infty}{\infty}$", "$\infty . 0$", "$\infty - \infty$",.

[OF COURSE "$\frac{\infty}{0}$", "$\infty+\infty$", "$\frac{0}{\infty}$", "$\frac{0}{\text{FINITE}}$", "$\frac{\text{FINITE}}{0}$" NOT IN DOUBT!]

(a) TYPE "$\frac{0}{0}$"

For this we can use L'Hôpital's rule (1696)

[Proof later – CHAPTER 5]

For $\lim\limits_{x\to x_0}\left[\frac{f(x)}{g(x)}\right]$ where $f(x_0) = 0 = g(x_0)$

then $\lim\limits_{x\to x_0}\left[\frac{f(x)}{g(x)}\right] = \lim\limits_{x\to x_0}\left[\frac{f'(x)}{g'(x)}\right]$

If BOTH $f'(x_0)$ and $g'(x_0)$ are still zero then differentiate again(!) and

$$\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to x_0}\left[\frac{f''(x)}{g''(x)}\right]$$

ETC.

(i) $\lim\limits_{x \to 0} \left(\frac{\sin x}{x} \right)$ is of "$\frac{0}{0}$" type

L'Hôpital $\Rightarrow \lim\limits_{x \to 0} \left(\frac{\cos x}{1} \right) = 1.$ <u>OK</u>.

(ii) $\lim\limits_{x \to 1} \left(\frac{x^3 - x^2 + 2x - 2}{x^3 + x^2 - 2} \right)$ is of "$\frac{0}{0}$" type

L'Hôpital $\Rightarrow \lim\limits_{x \to 1} \left(\frac{3x^2 - 2x + 2}{3x^2 + 2x} \right) = \frac{3}{5}$

<u>Alternative</u> – equally valid

We can use an expansion – Put $x = 1 + h$ and then allow $h \to 0$.

$$\lim_{x \to 1} \left(\frac{x^3 - x^2 + 2x - 2}{x^3 + x^2 - 2} \right) = \lim_{h \to 0} \left[\frac{(1+h)^3 - (1+h)^2 + 2(1+h) - 2}{(1+h)^3 + (1+h)^2 - 2} \right]$$

$$= \lim_{h \to 0} \left[\frac{(1 + 3h + 3h^2 + h^3) - (1 + 2h + h^2) + 2(1+h) - 2}{(1 + 3h + 3h^2 + h^3) + (1 + 2h + h^2) - 2} \right]$$

$$= \lim_{h \to 0} \left[\frac{3h + 2h^2 + h^3}{5h + 4h^2 + h^3} \right]$$

$$= \lim_{h \to 0} \left[\frac{3 + 2h + h^2}{5 + 4h + h^2} \right] = \frac{3}{5}$$

(iii) $\lim_{x \to 0} \left[\frac{(1+x)^{1/2} - (1+2x)^{1/2}}{x} \right]$ is of "$\frac{0}{0}$" type

l'Hôpital $\Rightarrow \lim_{x \to 0} \left[\frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}\cdot(1+2x)^{-1/2}\cdot 2}{1} \right]$

$= -1/2 .$

[<u>**Alternative**</u> − use binomial expansion/series CHAPTER 6]

(b) <u>**TYPE "$\frac{\infty}{\infty}$"**</u> No l'Hôpital [at least in general!] see CHAPTER 5

e.g. $\lim_{x \to \infty} \left(\frac{2x^5 + 2x^2 - 1}{x^5 - x^3 + 1} \right)$

$\equiv \lim_{x \to \infty} \left[\frac{x^5\left(2 + \frac{2}{x^3} - \frac{1}{x^5}\right)}{x^5\left(1 - \frac{1}{x^2} + \frac{1}{x^5}\right)} \right]$ [factor dominant powers.]

$= \lim_{x \to \infty} \left(\frac{2 + \frac{2}{x^3} - \frac{1}{x^5}}{1 - \frac{1}{x^2} + \frac{1}{x^5}} \right) = 2$

<u>Note</u> We can see that we should expect a finite answer because the dominant powers in numerator and denominator are the same.

Whereas $\lim_{x \to \infty} \left(\frac{x^7+6}{x^6-5} \right)$ is ∞te, $\lim_{x \to \infty} \left(\frac{x^4+3}{x^8+2} \right) = 0.$

and $\frac{x^7+6}{x^6-5} \sin x$ has NO LIMIT − OSCILLATES as $x \to \infty$.

(c) TYPE "$\infty . 0$".

e.g. $$\lim_{x\to\infty}\left\{x^{1/2}\left[(x+1)^{1/2}-(x-1)^{1/2}\right]\right\}.$$

$$=\lim_{x\to\infty}\left\{x^{1/2}\left[x^{1/2}\left(1+\frac{1}{x}\right)^{1/2}-x^{1/2}\left(1-\frac{1}{x}\right)^{1/2}\right]\right\}$$

$$=\lim_{x\to\infty}\left\{x\left[\left(1+\frac{1}{x}\right)^{1/2}-\left(1-\frac{1}{x}\right)^{1/2}\right]\right\}$$

BINOMIAL EXPANSION

$$=\lim_{x\to\infty}\left\{x\left[\left(1+\frac{1}{2x}-\frac{1}{8x^2}+\dots\right)-\left(1-\frac{1}{2x}-\frac{1}{8x^2}\dots\right)\right]\right\}$$

$$=\lim_{x\to\infty}\left[1+O\left(\frac{1}{x^2}\right)\right]$$

ORDER NOTATION

REMAINDER $\equiv \frac{K}{x^2}$ ← CONSTANT

$$=1.$$

(d) TYPE "$\infty - \infty$"

e.g. $$\lim_{x\to\infty}\left[x(x^2+2)^{1/2}-x(x^2-3)^{1/2}\right]$$

$$=\lim_{x\to\infty}\left\{x^2\left[\left(1+\frac{2}{x^2}\right)^{1/2}-\left(1-\frac{3}{x^2}\right)^{1/2}\right]\right\}$$

ACTUALLY "$\infty . 0$" AS ABOVE

$$=\lim_{x\to\infty}\left\{x^2\left[\left(1+\frac{1}{x^2}+\dots\right)-\left(1-\frac{3}{2x^2}+\dots\right)\right]\right\}$$

$$=\lim_{x\to\infty}\left[x^2\left(\frac{5}{2x^2}+O\left(\frac{1}{x^4}\right)\right)\right]=5/2.$$

(e) TYPE "1^{∞}" (!)

e.g. $$\lim_{x \to 0} \left[(1-x)^{1/x} \right]$$

Take logarithms and consider

$$\lim_{x \to 0} \left[\frac{1}{x} \ln(1-x) \right]$$ which is type "$\frac{0}{0}$".

l'Hôpital. $$\Rightarrow \lim_{x \to 0} \left[\frac{-\frac{1}{(1-x)}}{1} \right] = -1.$$

Hence $$\lim_{x \to 0} \left[(1-x)^{1/x} \right] = e^{-1}.$$

[In (slight) disguise this is

$$\lim_{n \to \infty} \left[\left(1 - \frac{1}{n}\right)^{n} \right] = e^{-1}$$

An alternative definition of the exponential function is

$$\lim_{n \to \infty} \left[\left(1 + \frac{x}{n}\right)^{n} \right] = e^{x}$$].

We can be inventive and adaptable!

CHAPTER 2

DIFFERENTIATION

(2·1) FIRST PRINCIPLES

(2·2) PRODUCT, QUOTIENT, CHAIN RULES

(2·3) STATIONARY POINTS

(2·4) CURVE SKETCHING

(2·5) PARAMETRIC REPRESENTATION OF CURVES

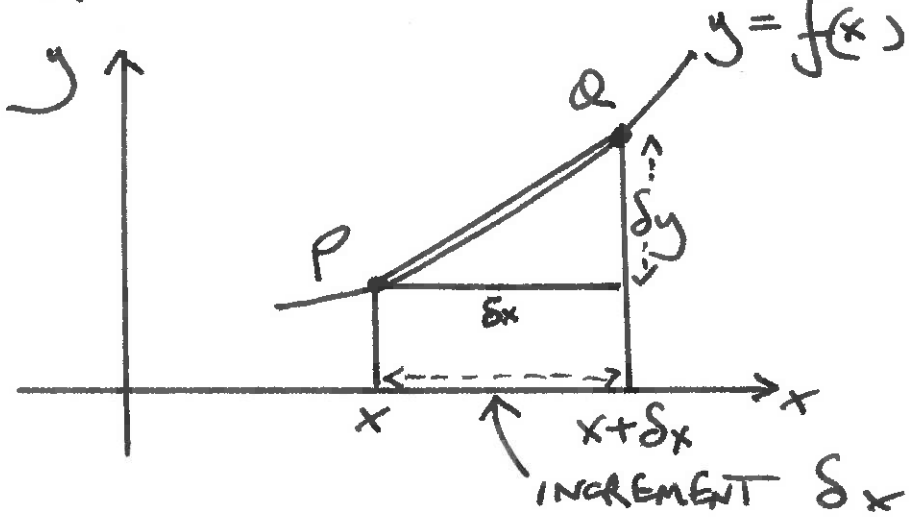
(2·6) POLAR COORDINATES

(2·7) POLYNOMIAL AND SERIES REPRESENTATION OF FUNCTIONS

(2·8) A NOTE ON INEQUALITIES

(2.1) DIFFERENTIATION FROM FIRST PRINCIPLES

Consider the tangent to a curve at point P as the limit of a chord PQ as Q approaches P.



Gradient of chord PQ

$$= \frac{f(x+\delta x)-f(x)}{\delta x} \equiv \frac{\delta y}{\delta x}$$

If P is fixed, i.e. x is fixed, let $\delta x \to 0$ and note that (in the above) δy also $\to 0$

If the limit exists we define

$$\lim_{\delta x \to 0}\left[\frac{f(x+\delta x)-f(x)}{\delta x}\right] = \frac{dy}{dx}$$

[NOTE SAME LIMIT AS $\delta x \to 0^+$ AND $\delta x \to 0^-$

— the derivative of $y = f(x)$ at x

Examples

(i) $y = x^2$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{(x+\delta x)^2 - x^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left(2x + \delta x\right) = 2x.$$

(ii) $y = \sin x$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{\sin(x+\delta x) - \sin(x)}{\delta x}\right]$$

[NOTE : $\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$]

So $$\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\cos\left(x+\frac{\delta x}{2}\right)\right] \cdot \lim_{\delta x \to 0}\left[\frac{\sin(\delta x/2)}{(\delta x/2)}\right]$$

$$= (\cos x)(1) = \cos x.$$

We need to know how this technique works, but normally apply a 'stock' set of derivatives in practice. [FORMULA SHEET +]

(2.2) PRODUCT, QUOTIENT, CHAIN RULES +

(ALL provable from first principles!).

(a) PRODUCT RULE

$$\frac{d}{dx}(fg) = \left(\frac{df}{dx}\right)g + f\left(\frac{dg}{dx}\right).$$

We note that

$$\frac{d^2}{dx^2}(fg) = \left(\frac{d^2f}{dx^2}\right)g + 2\left(\frac{df}{dx}\right)\left(\frac{dg}{dx}\right) + f\left(\frac{d^2g}{dx^2}\right)$$

and there is a general result (LEIBNIZ ~1684) that

$$\frac{d^n}{dx^n}(fg) = \overset{1\searrow}{\binom{n}{0}}\left(\frac{d^nf}{dx^n}\right)g + \overset{n\searrow}{\binom{n}{1}}\left(\frac{d^{n-1}f}{dx^{n-1}}\right)\left(\frac{dg}{dx}\right) + \binom{n}{2}\left(\frac{d^{n-2}f}{dx^{n-2}}\right)\left(\frac{d^2g}{dx^2}\right) + \cdots \binom{n}{n-1}\left(\frac{df}{dx}\right)\left(\frac{d^{n-1}g}{dx^{n-1}}\right) + \binom{n}{n}f\left(\frac{d^ng}{dx^n}\right).$$

note BINOMIAL COEFFICIENTS. $\binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2}$ ETC

This result can be proved by INDUCTION!

(b) QUOTIENT RULE

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \quad \left[\equiv \frac{d}{dx}\left[(f)\cdot\left(\frac{1}{g}\right)\right]\right]$$

note shorthand

$$\text{"f PRIME"} \nearrow f' \equiv \frac{df}{dx},\ f'' \equiv \frac{d^2f}{dx^2} \text{ ETC.}$$

"f DOUBLE PRIME" ↗

(c) FUNCTION OF A FUNCTION

$$y = f(g(x))$$

$$\Rightarrow dy/dx = f'(g(x))\, g'(x)$$ known as the CHAIN-RULE.

e.g. $y = \ln \cos x$

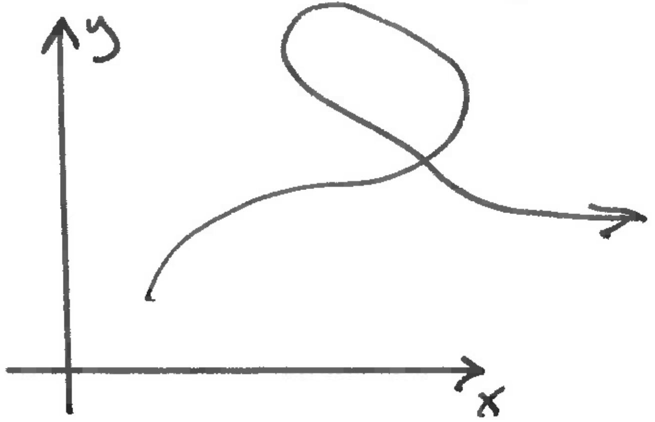
$\Rightarrow y = \ln(g)$ with $g = \cos x$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dg}\right)\left(\frac{dg}{dx}\right) = \frac{1}{g}(-\sin x)$$

$$= -\tan x.$$

(d) PARAMETRIC DIFFERENTIATION

Suppose $y = y(t)$ and $x = x(t)$, which might be the coordinates of a moving point.

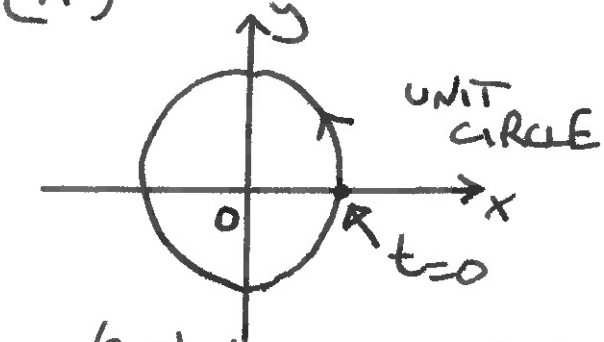


$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \equiv \frac{\dot{y}}{\dot{x}}$$

← NEWTON OVERDOT NOTATION. ESP. IF $t \equiv$ TIME

Similarly $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$\equiv \frac{dt}{dx}\cdot\frac{d}{dt}\left(\dot{y}/\dot{x}\right)$$
$$\equiv \frac{(\ddot{y}\dot{x} - \dot{y}\ddot{x})}{(\dot{x})^3}.$$

e.g. $x = \cos t$
$y = \sin t$



VELOCITY COMPONENTS | $\dot{x} = -\sin t$
$\dot{y} = \cos t \Rightarrow dy/dx = \left(\frac{\cos t}{-\sin t}\right) = -\cot t.$

ACCELERATION COMPONENTS | $\ddot{x} = -\cos t$
$\ddot{y} = -\sin t \Rightarrow \frac{d^2y}{dx^2} = \frac{(\sin^2 t + \cos^2 t)}{(-\sin t)^3} \equiv -\frac{1}{y^3}.$

SPEED | TANGENTIAL
ACCELERATION | RADIAL INWARDS

(e) DIFFERENTIATION OF INVERSE FUNCTIONS

e.g. (i) $y = \sin^{-1} x \Rightarrow x = \sin y.$

So $$\cos y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}}$$
$$\equiv \frac{1}{\sqrt{1-x^2}}.$$

(ii) $y = \tan^{-1}x \Rightarrow x = \tan y.$

so $\sec^2 y \dfrac{dy}{dx} = 1.$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} \equiv \frac{1}{1+\tan^2 y} \equiv \frac{1}{1+x^2}.$$

[SIMILAR results for all INVERSE functions].

(f) IMPLICIT FUNCTIONS

Sometimes the relationship between x and y is implicit:

$$F(x, y)) = 0$$

with an explicit form $y = f(x)$ NOT available.

e.g. $x^2 \sin y + xy = 1$

We can differentiate using the PRODUCT RULE

$$2x \sin y + x^2 \cos y \frac{dy}{dx} + y + x \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(y + 2x \sin y)}{(x^2 \cos y + x)}.$$

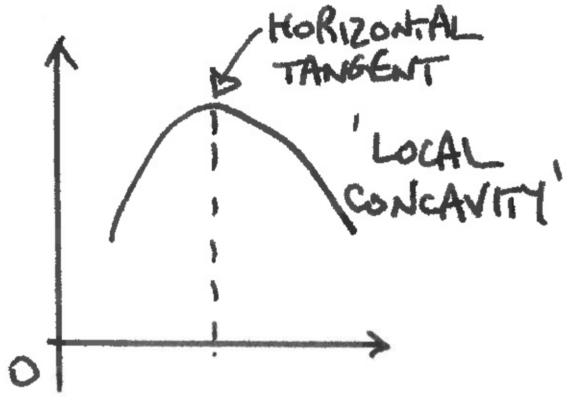
ETC.

(2.3) STATIONARY POINTS

At a STATIONARY POINT $\frac{dy}{dx} = 0$

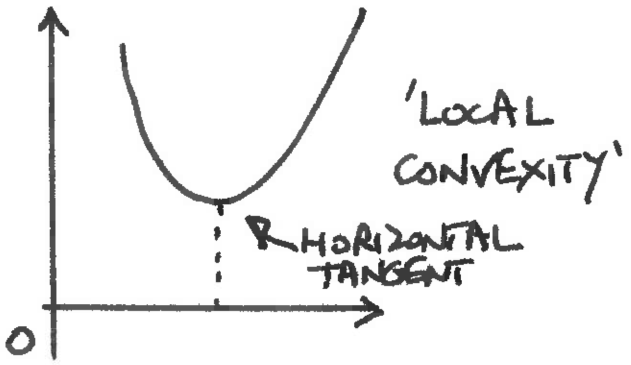
Two basic types:

(LOCAL) MAXIMUM



$y' = 0$ and $y'' < 0$

(LOCAL) MINIMUM



$y' = 0$ and $y'' > 0$

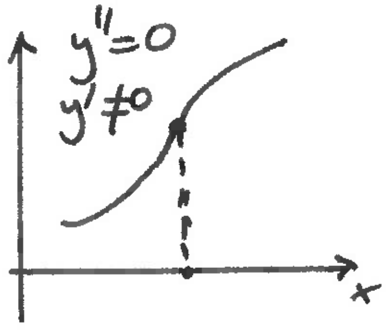
At a POINT OF INFLECTION $\frac{d^2y}{dx^2} = 0$,

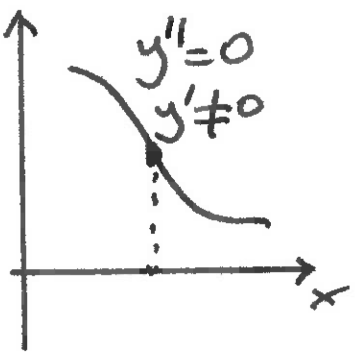
but $\frac{dy}{dx}$ <u>may</u> or <u>may not</u> be zero.

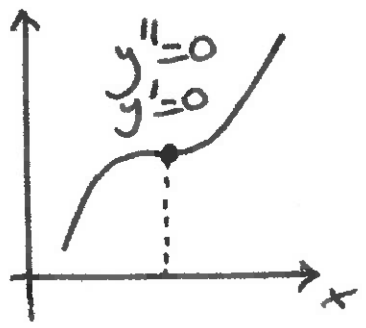
Cases
- y' also $= 0 \Rightarrow$ STATIONARY POINT OF INFLECTION $\equiv$ 'INFLECTION WITH HORIZONTAL TANGENT' or SADDLE POINT
- $y' \neq 0 \Rightarrow$ NON-STATIONARY POINT OF INFLECTION

[An inflection point normally indicates CURVATURE change of sign].

Examples of POINTS OF INFLECTION







e.g. Consider

$$y = x^2(x-1)$$

ZEROS AT
$x=0, 1$.

Derivatives $y' = 3x^2 - 2x \equiv x(3x-2)$

$$y'' = 6x - 2$$

STATIONARY POINTS $(y'=0)$ at $x = 0, \frac{2}{3}$.

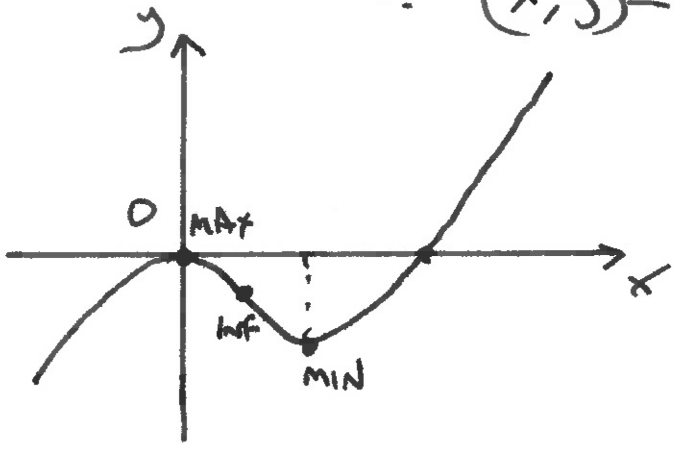
i.e. $(x,y) \equiv (0,0)$ and $\left(\frac{2}{3}, -\frac{4}{27}\right)$

$y'' = -2$
$\Rightarrow$ (LOCAL)
MAXIMUM

$y'' = +2$
$\Rightarrow$ (LOCAL)
MINIMUM.

INFLECTION $y''=0$ at $x = 1/3$

i.e. $(x,y) = \left(\frac{1}{3}, -\frac{2}{27}\right)$



(2.4) CURVE SKETCHING

Main principles (in no particular order):

(a) Examine behaviour for $x \to 0, +\infty, -\infty$.

(b) Look for SYMMETRIES - even, odd.

(c) If $y = \dfrac{P(x)}{Q(x)}$ with P, Q POLYNOMIALS

(i) zeros of P give intersections with x axis

(ii) zeros of Q give infinite DISCONTINUITIES or (vertical) ASYMPTOTES

(d) STATIONARY POINTS / INFLECTION POINTS

$\Rightarrow$ important local detail

['SKETCH' indicates these important features].

e.g. $$y = \frac{x(x-2)}{(x-3)}$$

$y = 0$ when $x = 0, 2$

$y \sim \frac{2}{3}x$ if x is small

'GOES LIKE'

near $x = 3$ we have $y \sim \dfrac{3}{x-3}$

So $y \to +\infty$ as $x \to 3^+$ i.e. FROM ABOVE

$y \to -\infty$ as $x \to 3^-$ ie FROM BELOW

Behavior at large $|x|$:

$$y = \frac{x(x-2)}{(x-3)}$$

$$= \frac{x^2\left(1-\frac{2}{x}\right)}{x\left(1-\frac{3}{x}\right)} \equiv x\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right)^{-1}$$

$$= x\left(1-\frac{2}{x}\right)\left(1+\frac{3}{x}+O\left(\frac{1}{x^2}\right)\right)$$

$$= x\left[1+\frac{1}{x}+O\left(\frac{1}{x^2}\right)\right]$$

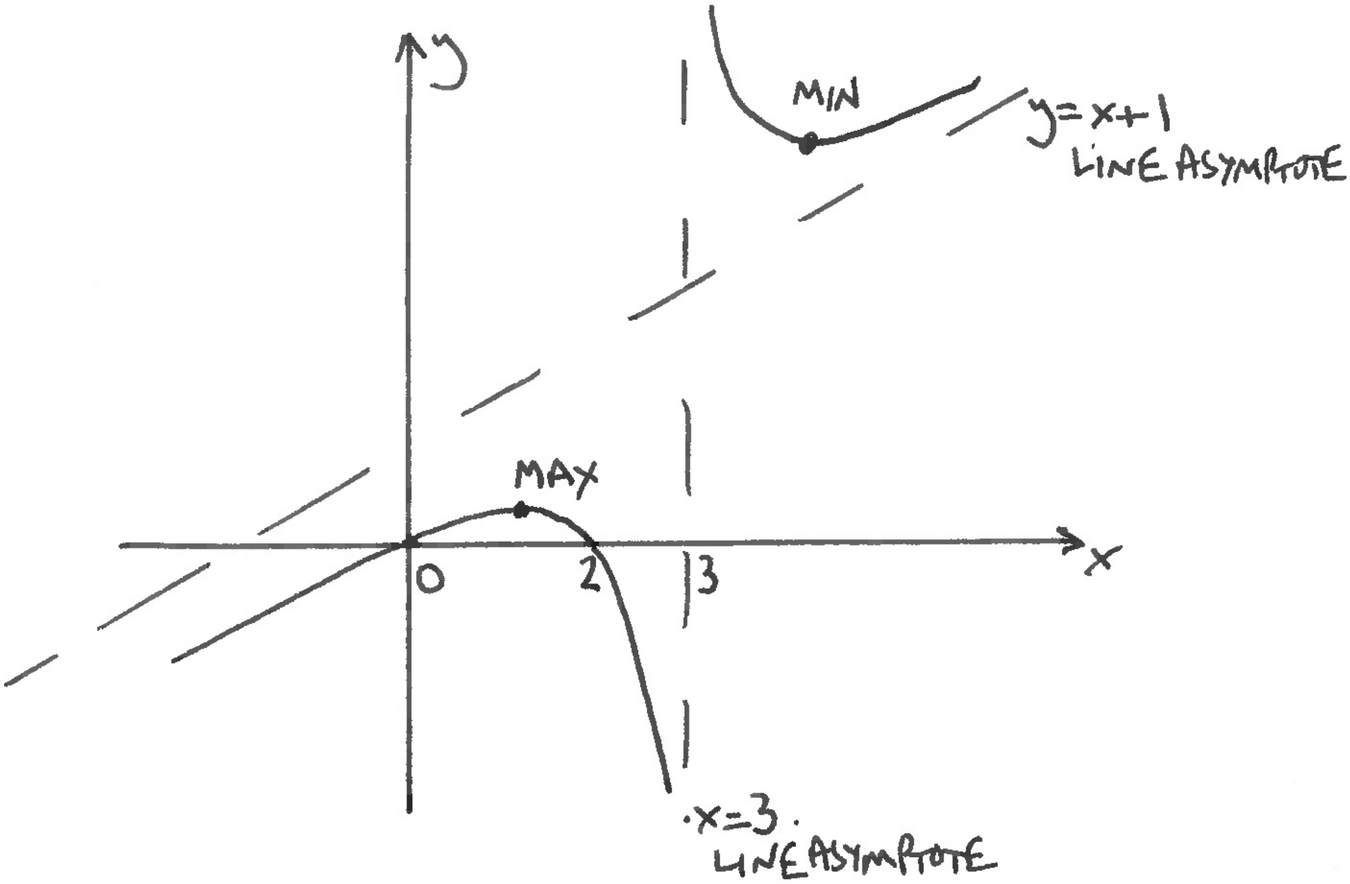
$$= x+1+O\left(\frac{1}{x}\right).$$

So $y(x) \to$ a straight line asymptote as $x \to \pm\infty$.

Since $y' = \frac{(x^2-6x+6)}{(x-3)^2}$ we have stationary points at

$x = 3 \pm \sqrt{3}$.

We can find $y'' = \ldots$ if we wish, but the character of these is already apparent when we start to sketch.....



(2.5) PARAMETRIC REPRESENTATION OF CURVES

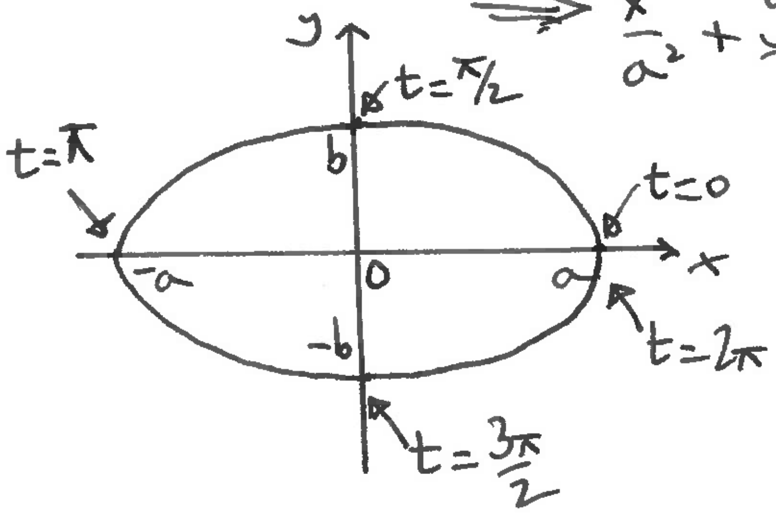
Given $x = x(t)$ and $y = y(t)$

[t may or may not be time!]

e.g (a) $x = a\cos t$, $y = b\sin t$

we can eliminate t here

$$\Longrightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

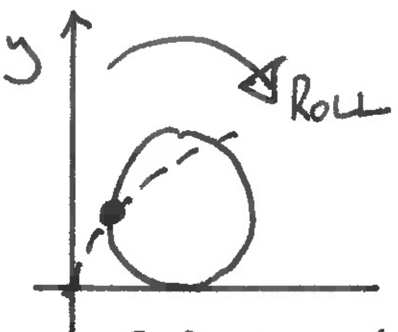


ELLIPSE

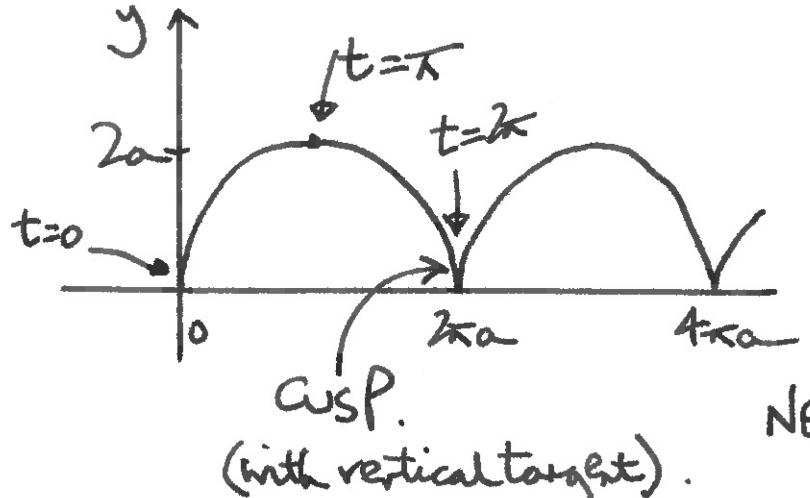
ECCENTRICITY e
SO THAT
$b = a(1-e^2)^{1/2}$.

(b) $x = a(t - \sin t)$

$y = a(1 - \cos t)$



LOCUS OF POINT ON A ROLLING CIRCLE

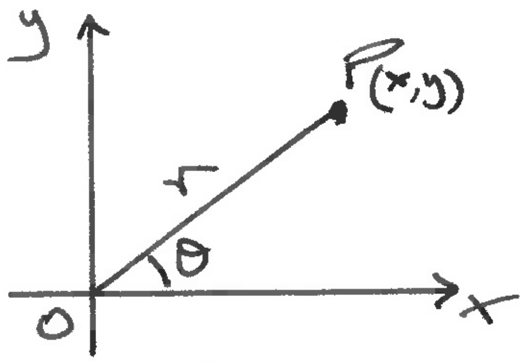


A very important curve - NEWTON, WREN, BERNOULLI ...

[Note: we <u>cannot</u> eliminate $t \Rightarrow y(x)$ explicitly]

(2.6) <u>POLAR COORDINATES</u>

In 2 dimensions it is often useful to employ (plane) POLAR COORDINATES (r, θ) instead of CARTESIANS (x, y)



$r \equiv$ RADIAL DISTANCE to origin O

$\theta \equiv$ ANGLE with respect to +ve x axis.

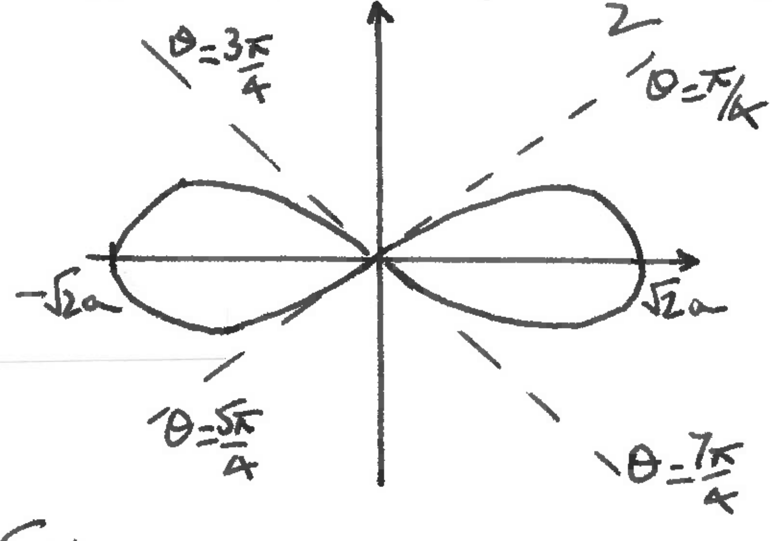
$r = (x^2 + y^2)^{1/2}$ ALWAYS ≥ 0 $\qquad$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ WITH $0 \leq \theta < 2\pi$

$x = r\cos\theta$, $y = r\sin\theta$.

e.g (i) **Lemniscate function** ('figure of eight')

$$r^2 = 2a^2\cos 2\theta$$

Since r is real and non-negative we cannot have $\cos 2\theta < 0$ i.e. $\frac{\pi}{2} < 2\theta < \frac{3\pi}{2}$ etc.

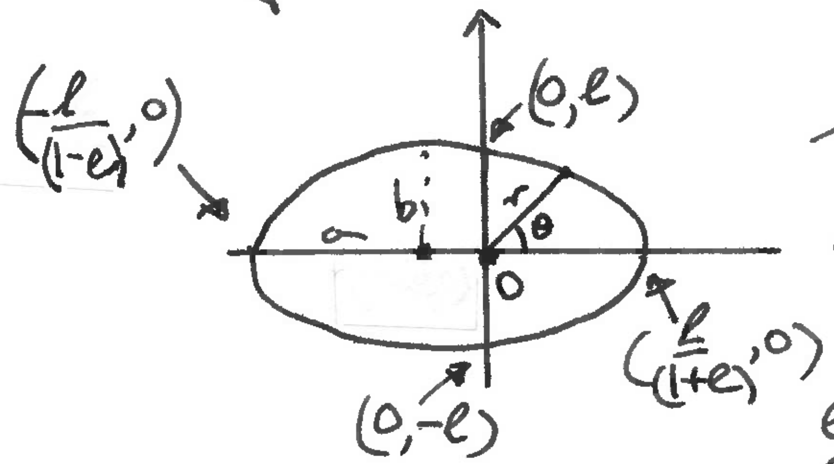


(ii) **Ellipse** [IMPORTANT e.g. for planetary orbits]

$$(*) \qquad \frac{\ell}{r} = 1 + e\cos\theta.$$

ℓ, e positive

$0 < e < 1$.



The origin is at a FOCUS of the ellipse

$e \equiv$ ECCENTRICITY

$\ell \equiv$ 'SEMI-LATUS RECTUM'.

$$a = \frac{\ell}{(1-e^2)}, \quad b = \frac{\ell}{(1-e^2)^{1/2}}$$

$$\ell = b^2/a, \quad e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$$

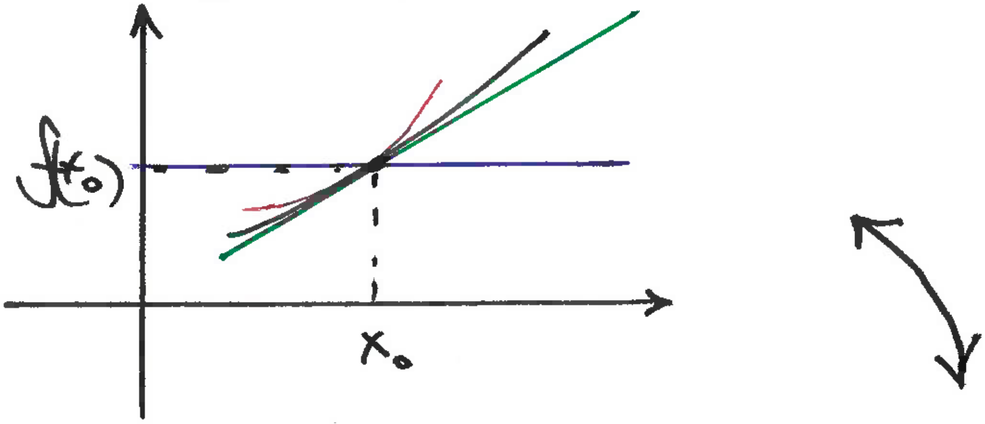
EQ(*)
- $e = 0$ CIRCLE
- $e = 1$ PARABOLA
- $e > 1$ HYPERBOLA

CHANGES!

(2.7) POLYNOMIAL AND SERIES REPRESENTATION OF FUNCTIONS

In (2.3)(2.4) we looked at local behaviour of functions near specific points – zeros and stationary points – with consequences for curve sketching.

Here we note that we can consider local approximations to a smooth function in the neighbourhood of a general point $(x_0, f(x_0))$ as a sequence of polynomials.



$$f(x) \simeq f(x_0)$$ 'BEST' CONSTANT

$$f(x) \simeq f(x_0) + f'(x_0)(x-x_0).$$ 'BEST' LINEAR

$$f(x) \simeq f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$ 'BEST' QUADRATIC

ETC

At a STATIONARY POINT [see (2.3)]
f(x) is (at least) QUADRATIC . . . .

All of these POLYNOMIAL REPRESENTATIONS are *local* - with a trade-off to be expected between *accuracy* over a domain $(x_0-\epsilon, x_0+\epsilon)$ and the *degree* of the polynomial.

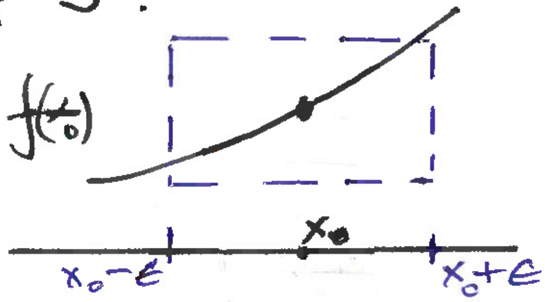
If $f(x)$ has successive derivatives at x_0 we can continue to derive a TAYLOR SERIES EXPANSION at x_0 in the form

$$f(x) = f(x_0)\Big] + f'(x_0)(x-x_0)\Big] + f''(x_0)\frac{(x-x_0)^2}{2!}\Big] + \cdots$$

$$\cdots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \cdots$$

For the moment we consider successive truncations to provide approximations to the local behaviour near $(x_0, f(x_0))$, with successive derivatives providing more and more information.

The validity of the expansion and its domain of accuracy are considered more fully in Chapter 5.

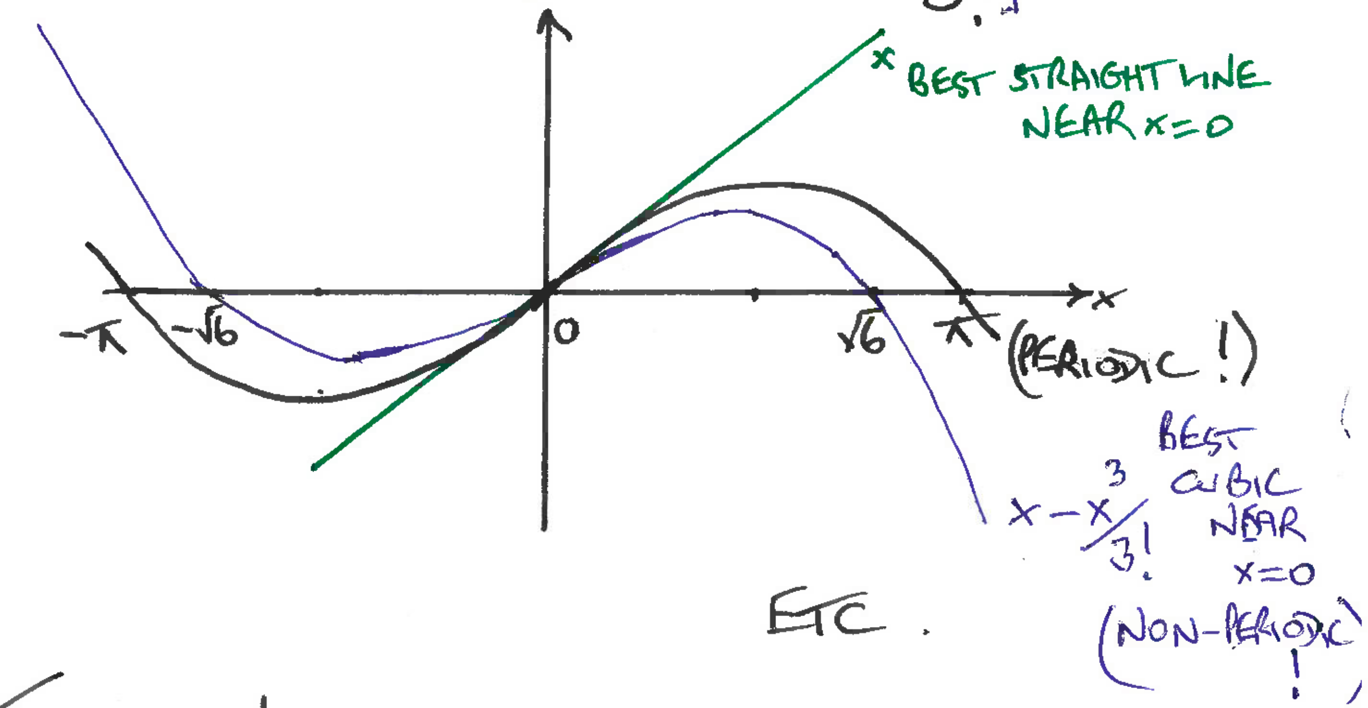


There are several examples given on the formula sheet

'TAYLOR - MACLAURIN'

e.g. $f(x) = \sin x$, $x_0 = 0$.

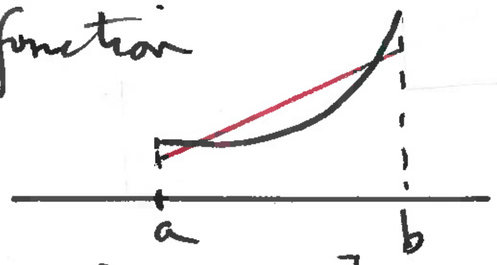
$$\text{Then } \sin x = 0\Big] + x\Big] + 0x^2\Big] - \frac{x^3}{3!}\Big] + \dots$$



ETC.

[These are local approximations about a point. We note in passing that we might alternatively seek a 'best' polynomial approximation over an interval (a, b)

e.g. linear regression → a best straight line fit to data/function



e.g. via 'least squares'

We do not consider this here.]

(2·8) A NOTE ON INEQUALITIES

It is important to realise that the STATIONARY POINTS referred to in (2·3)(2·4) are indicating LOCAL behaviours – a GLOBAL behavior may or maynot follow!

In (1·2) QUADRATIC FUNCTIONS the 'Completion of the square' approach did lead to global results – and in consequence two important INEQUALITIES:

(i) AM/GM for a, b. (+ve)

(ii) CAUCHY-SCHWARZ for $a_j, b_j, j=1,\dots,n$.

With care we can use our calculus of stationary points to arrive at global results and inequalities.

example. The AM/GM inequality is 'easy' to generalise from 2 numbers [i above] to 2^N numbers!

However, generalisation to a set not a power of 2 in number is not so easy!

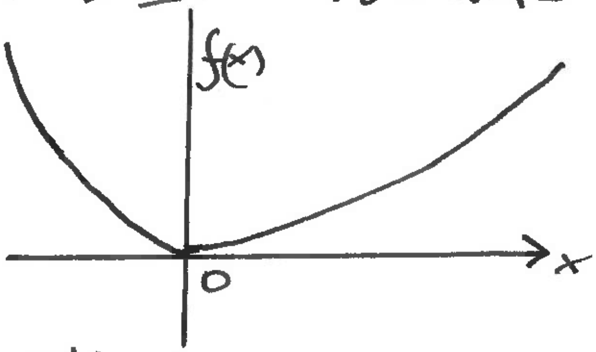
A novel approach [due to POLYA] is to consider the function

$$f(x) = e^x - (1+x)$$

STATIONARY POINT $f'(x) = e^x - 1 = 0$ only when $x=0$.

$f''(x) = e^x = 1$ so a LOCAL MINIMUM

Since $f(x) \to +\infty$ as $x \to \pm\infty$ and there are no singularities it is also a GLOBAL MINIMUM



So $e^x \geq 1+x$ for all x with EQUALITY only at $x=0$.

For a set of n +ve numbers $a_1, a_2, \ldots a_j, \ldots a_n$ we have ARITHMETIC MEAN $A = \frac{1}{n}\sum_{j=1}^{n} a_j$

and GEOMETRIC MEAN $G = (a_1 a_2 \ldots a_n)^{1/n}$

For each a_j write $e^{a_j/A - 1} \geq 1 + \left(\frac{a_j}{A} - 1\right) = \frac{a_j}{A}$. $(j=1,\ldots,n)$.

MULTIPLY(!) $e^{\left[\frac{1}{A}\sum_{j=1}^{n} a_j - n\right]} \geq \frac{(a_1 a_2 \ldots a_n)}{A^n} = \left(\frac{G}{A}\right)^n$

$= e^0 = 1$. $\therefore A \geq G$ EQUALITY ONLY WHEN $a_1 = a_2 = \ldots = a_n$.

GENERAL AM/GM INEQUALITY

CHAPTER 3

INTEGRATION

(3.1) RIEMANN'S DEFINITION

(3.2) THE FUNDAMENTAL THEOREM OF CALCULUS

(3.3) INFINITE AND IMPROPER INTEGRALS

(3.4) SOME USEFUL TECHNIQUES

(3.5) APPLICATIONS - MEAN VALUE, AREA, LENGTH

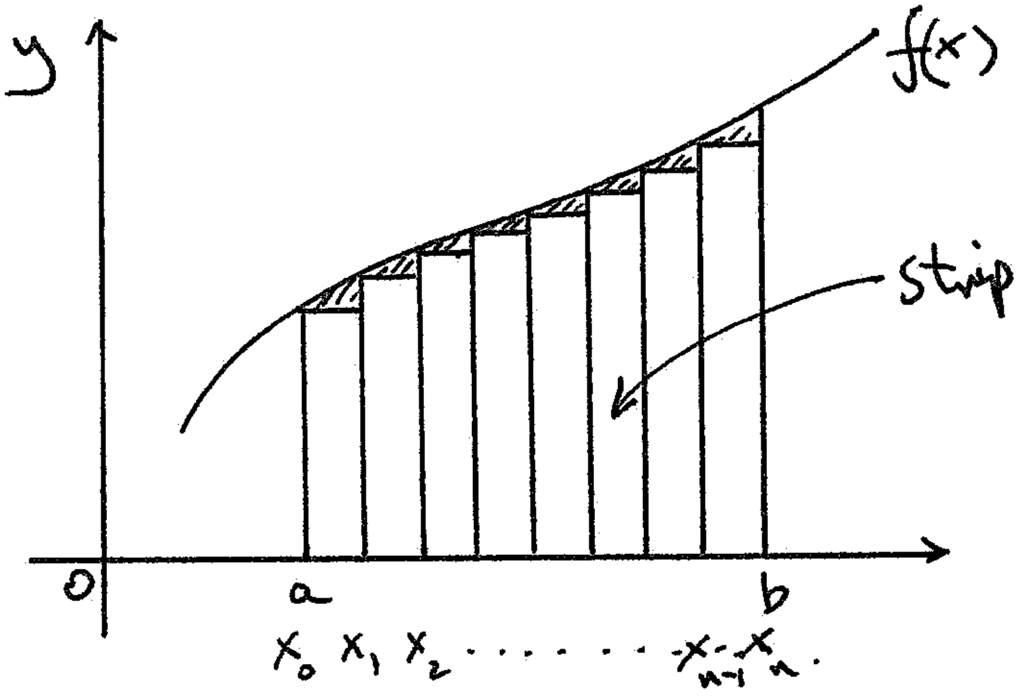
(3.6) APPLICATIONS - CENTRE OF MASS

(3.7) APPLICATIONS - VOLUME/SURFACE AREA OF REVOLUTION

(3.1) RIEMANN'S DEFINITION

Integration arose from intuitive ideas about area and Volume in geometry.

Consider the area A under a curve f(x) between ordinates at $x = a$, $x = b$.



To calculate the area A we imagine a large number of rectangular strips located at $x_0, x_1, \ldots x_n$ with $x_0 = a$, $x_n = b$.

$$\text{AREA of strips} = S_n = \sum_{j=0}^{n-1} f(x_j)\,\delta x_j$$

STRIP HEIGHT

STRIP WIDTH

where $\delta x_j = x_{j+1} - x_j$

NOT NEC. UNIFORM

Intuition leads us to expect $S_n \to A$ as the number of strips $n \to \infty$; we expect that the error (shaded pieces) vanishes in this limit.

RIEMANN generalised the above to show that this intuition is correct:

(i) He used UPPER and LOWER sums (i.e. 'external' and 'internal' rectangles) which have the same limit as $n \to \infty$.

(ii) He used the height of a strip $f(\xi_j)$ where ξ_j is any point on $x_j \leq \xi_j \leq x_{j+1}$

So Riemann's definition is the limit of the SUM OVER STRIPS

$$S_n^* = \sum_{j=0}^{n-1} f(\xi_j)\,\delta x_j \quad \text{with } \xi_j \text{ as above}.$$

and he showed that

$$S_n^* \to S_n \to A \quad \text{as } n \to \infty.$$

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \delta x_j \to 0}} \sum_{j=0}^{n-1} f(\xi_j)\,\delta x_j$$

is the INTEGRAL of f between a and b. [THE <u>DEFINITE</u> INTEGRAL

[The limit has to be independent of the design of the MESH in x_j!].

<u>NOTES</u>

(a) $f(x) \equiv$ INTEGRAND

$$\left.\begin{array}{lcl} a & \equiv & \text{LOWER} \\ b & \equiv & \text{UPPER} \end{array}\right\}\text{LIMIT OF INTEGRATION}$$

x is a DUMMY VARIABLE.

- ANY SYMBOL WILL DO!

$$\int_a^b f(x)\,dx = \int_a^b f(s)\,ds = \int_a^b f(t)\,dt \quad \text{ETC}$$

[and e.g.

$$\int_1^2 \frac{d(\text{cabin})}{(\text{cabin})} = \Big[\ln(\text{cabin})\Big]_1^2 = \ln 2.$$

- the INDEFINITE INTEGRAL $\int(\ \ ) \equiv$ ...... ?!

(b) We also define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

('opposite sense of strip summation').

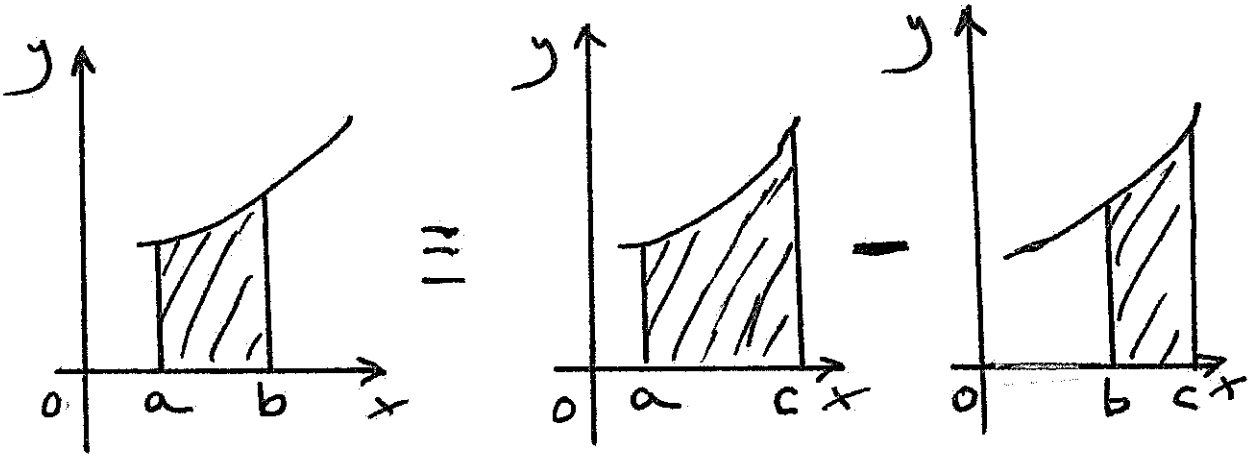
Hence $\int_a^a f(x)\,dx = 0$

('no area under the curve')

(c) $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

valid for any b.

('areas can be added and subtracted).



ADDITION RULES FOR AREAS ⟷ INTEGRALS

(3.2) THE 'FUNDAMENTAL THEOREM OF CALCULUS'

Integration is the inverse of differentiation.

That is to say that:

If $F(x) = \int_a^x f(u)\,du$ INDEFINITE INTEGRAL.

any FIXED a.

then $\frac{dF}{dx} = f(x)$.

Proof:

$$\frac{dF}{dx} = \lim_{\delta x \to 0}\left[\frac{F(x+\delta x) - F(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\int_a^{x+\delta x} f(u)\,du - \int_a^x f(u)\,du}{\delta x}\right]$$

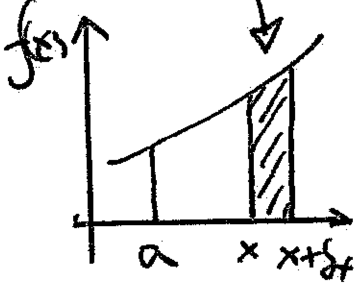
NUMERATOR: DIFFERENCE IN AREAS

$$= \lim_{\delta x \to 0}\left[\frac{\int_x^{x+\delta x} f(u)\,du}{\delta x}\right]$$

NUMERATOR: STRIP AREA

$$= \lim_{\delta x \to 0}\left(\frac{f(x)\,\delta x}{\delta x}\right)$$

$$= f(x)$$



Remarks:

(i) In the above definition of $F(x)$ the lower limit a is ARBITRARY
$\Longrightarrow$ an arbitrary constant can be added to $F(x)$ [Hence 'INDEFINITE INTEGRAL'.

(ii) The DEFINITE INTEGRAL

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(3.3) INFINITE AND IMPROPER INTEGRALS

INFINITE integrals have a $+\infty$ (or $-\infty$) in the upper (or lower) limit.

What is the meaning of e.g. $\int_a^\infty f(x)\,dx$?

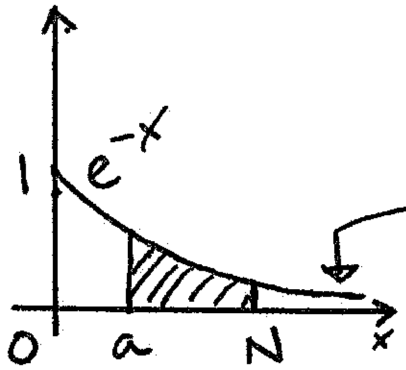
To decide if this is indeed meaningful we write

$$I(N) = \int_a^N f(x)\,dx$$

If this has a FINITE LIMIT as $N \to \infty$ then the infinite integral EXISTS.

ex (i) $$\int_a^\infty e^{-x}\,dx \overset{?}{=} \lim_{N\to\infty}\int_a^N e^{-x}\,dx$$

$$= \lim_{N\to\infty}\left(e^{-a} - \underbrace{e^{-N}}_{\to 0 \text{ as } N\to\infty}\right)$$



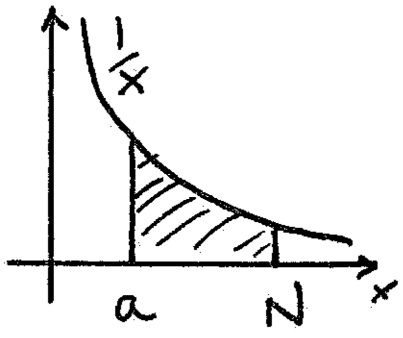
$$= e^{-a}$$ INTEGRAL EXISTS

['TAIL' → 0 as N → ∞
//// stays FINITE]

ex (ii) $$\int_a^\infty \frac{dx}{x} \overset{?}{=} \lim_{N\to\infty}\int_a^N \frac{dx}{x}$$

$$= \lim_{N\to\infty}(\ln N - \ln a)$$

$$\to \infty \text{ as } N\to\infty.$$ INTEGRAL DOES NOT EXIST



In a similar fashion IMPROPER integrals involve a SINGULARITY of the integrand on the range of integration. Of course we need to spot whether this might be <u>at an end point</u> <u>or</u> <u>within</u> this range.

ex (iii) $\int_0^1 x^{-1/2}dx$

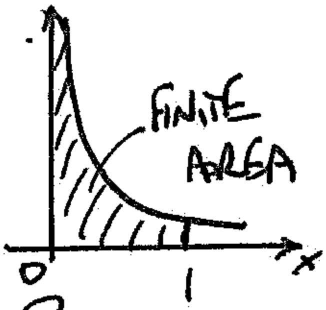
There is a potential problem here because $x^{-1/2}$ is infinite at the end point $x=0$.

To decide the issue we can integrate from ϵ to 1, where $0<\epsilon \ll 1$, then take $\epsilon \to 0$.

$\uparrow$ MUCH LESS THAN!

$$I(\epsilon) = \int_\epsilon^1 x^{-1/2}dx \quad \text{(with No infinities)}$$

$$= \left[2x^{1/2}\right]_\epsilon^1 = 2-2\sqrt{\epsilon}.$$



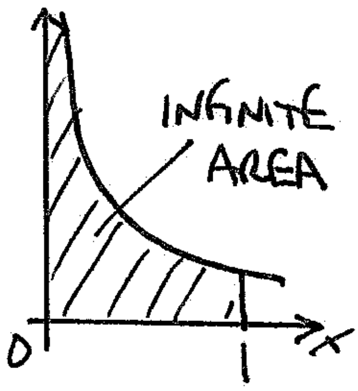
Then $\int_0^1 x^{-1/2}dx = \lim_{\epsilon\to 0} I(\epsilon)$

$$= \lim_{\epsilon\to 0} 2-2\sqrt{\epsilon} = 2.$$

FINE!

ex (iv) $$\int_0^1 \frac{dx}{x^2} = \lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{x^2}$$

$$= \lim_{\epsilon\to 0}\left[-\frac{1}{x}\right]_\epsilon^1$$



$$= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}-1\right) \to +\infty.$$

THIS INTEGRAL DOES NOT EXIST!

ex (v) "$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2$"

BUT integrand is surely positive!

TROUBLE at $x=0$! WRITE $\int_{-1}^{1} = \int_{-1}^{-\epsilon_1} + \int_{\epsilon_2}^{1}$

and consider $\epsilon_1 \to 0, \epsilon_2 \to 0$.

TRUE AREA is INFINITE!

(3.4) SOME USEFUL TECHNIQUES

Some general principles – we need to be adaptable!

(a) PARTIAL FRACTIONS

e.g. $$\int \frac{dx}{x(x+1)} = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \ln|x| - \ln|x+1| + c$$

$$= \ln\left|\frac{x}{x+1}\right| + c.$$

[more complicated: $\int \frac{(\text{polynomial})\,dx}{(\text{polynomial})}$ ......]

(b) CHANGE OF VARIABLE ('SUBSTITUTION')

(i) $\int x e^{-x^2} dx$

Let $u = x^2 \Rightarrow du = 2x\,dx$

∴ integral is $\frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-u} + c$

$$\equiv -\frac{1}{2}e^{-x^2} + c.$$

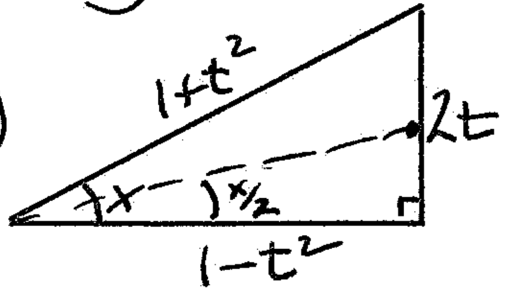
(ii) For trigonometric integrals we can often try $t = \tan\left(\frac{x}{2}\right)$

so that $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$\frac{dx}{dt} = \frac{2}{1+t^2}$, $\tan x = \frac{2t}{1-t^2}$.

The easiest way to see this is via

$\boxed{(1+t^2)^2 = (2t)^2 + (1-t^2)^2.}$



e.g. $\tan\left(2\frac{x}{2}\right) = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

together with $1 = \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\frac{dx}{dt}\right)$

$$\Rightarrow dx/dt = \frac{2}{1+\tan^2\left(\frac{x}{2}\right)} \equiv \frac{2}{1+t^2}$$

ex (i) $\displaystyle\int \frac{dx}{2+\cos x}$

Note: no trouble in the denominator since $|\cos x| \leq 1$.

$$= \int \frac{dx}{dt} \frac{1}{\left(2+\frac{1-t^2}{1+t^2}\right)} dt.$$

$$= \int \frac{2}{\cancel{(1+t^2)}} \frac{\cancel{(1+t^2)}}{(3+t^2)} dt = 2\int \frac{dt}{3+t^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$\equiv \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan\left(\frac{x}{2}\right)\right) + c.$$

ex (ii) $\displaystyle\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{1/2} dx$ Not trig as it stands, but

a good strategy is to consider 'doing something' about square roots....

e.g. let $x = 1-2u^2$ the integral

becomes $4\displaystyle\int_0^1 (1-u^2)^{1/2} du$

and then $u = \sin\theta \implies 4\displaystyle\int_0^{\pi/2} \cos^2\theta \, d\theta$

$$\implies 4\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta) d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \pi.$$

ALTERNATIVE:
Try $x = \cos(2v) \implies$ SAME RESULT

(c) INTEGRATION BY PARTS

From the PRODUCT RULE of differentiation

$$\int u \frac{dv}{dx}\,dx = [uv] - \int v \frac{du}{dx}\,dx$$

ex(i) $$\int \underset{\substack{\uparrow\\ u}}{x}\,\underset{\substack{\uparrow\\ \frac{dv}{dx}}}{e^x}\,dx = xe^x - \int e^x dx$$

$$= (x-1)e^x + c.$$

so $\frac{du}{dx} = 1$, $v = e^x$.

[NOTE: if we had made the main alternative choice: $u = e^x, \frac{dv}{dx} = x \Rightarrow \frac{du}{dx} = e^x, v = \frac{1}{2}x^2$

we get $\int xe^x dx = \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2 e^x dx$ $\nwarrow$

TRUE! but NOT HELPFUL (in fact WORSE)!]

ex(ii) $$\int \ln x\,dx$$

WRITE as $$\int \overset{\swarrow \text{ONE}}{\underset{\substack{\uparrow\\ \frac{dv}{dx}}}{1}}\,\underset{\uparrow}{\ln x}\,dx = [x\ln x] - \int x\frac{1}{x}dx$$

$$= x(\ln x - 1) + c.$$

$\Rightarrow v = x, \frac{du}{dx} = \frac{1}{x}$.

SIM ex(iii) $$\int \tan^{-1}x\,dx = [x\tan^{-1}x] - \int \frac{x\,dx}{1+x^2}$$

$\frac{dv}{dx} = 1, u = \tan^{-1}x$. $$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c.$$

(d) SPECIAL TRICKS *!

ex (i) $$I = \int_0^{\pi/2} \sin^2\theta \, d\theta \qquad \boxed{\phi = \pi/2 - \theta}.$$

$$= \int_0^{\pi/2} \cos^2\phi \, d\phi \equiv \int_0^{\pi/2} \cos^2\theta \, d\theta .$$

[θ, ϕ dummy variables

So $$2I = \int_0^{\pi/2} (\sin^2\theta + \cos^2\theta) \, d\theta$$

$$= \int_0^{\pi/2} 1 \, d\theta = \pi/2 .$$

$$\therefore I = \pi/4 .$$

ex (ii) $$J = \int_0^{\pi/2} \ln(\sin x) \, dx \qquad \boxed{y = \pi/2 - x}$$

[EULER 1769

$$= \int_0^{\pi/2} \ln(\cos y) \, dy = \int_0^{\pi/2} \ln(\cos x) \, dx$$

So $$2J = \int_0^{\pi/2} (\ln(\sin x) + \ln(\cos x)) \, dx = \int_0^{\pi/2} [\ln(\sin 2x) - \ln 2] \, dx$$

$$\boxed{2x = z} \quad = \frac{1}{2} \int_0^{\pi} \ln(\sin z) \, dz - \pi/2 \ln 2 .$$

$$\equiv \int_0^{\pi/2} \ln(\sin x) \, dx - \pi/2 \ln 2 = J - \frac{\pi}{2} \ln 2$$

$$\therefore J = -\pi/2 \ln 2$$

ex (iii) $$K_1 = \int_0^{\pi/2} \frac{dx}{(1 + \tan^2 x)} = \int_0^{\pi/2} \cos^2 x \, dx = \frac{\pi}{4} \left(\begin{matrix} \text{(i)} \\ \text{above} \end{matrix} \right)$$

BUT $$K_2 = \int_0^{\pi/2} \frac{dx}{(1 + \tan^{\sqrt{2}} x)} \qquad \boxed{y = \frac{\pi}{2} - x}$$

$$= \int_0^{\pi/2} \left(\frac{\tan^{\sqrt{2}} y}{1 + \tan^{\sqrt{2}} y} \right) dy \underset{\text{ADD}}{\Longrightarrow} 2K_2 = \int_0^{\pi/2} 1 \, dx = \pi/2$$

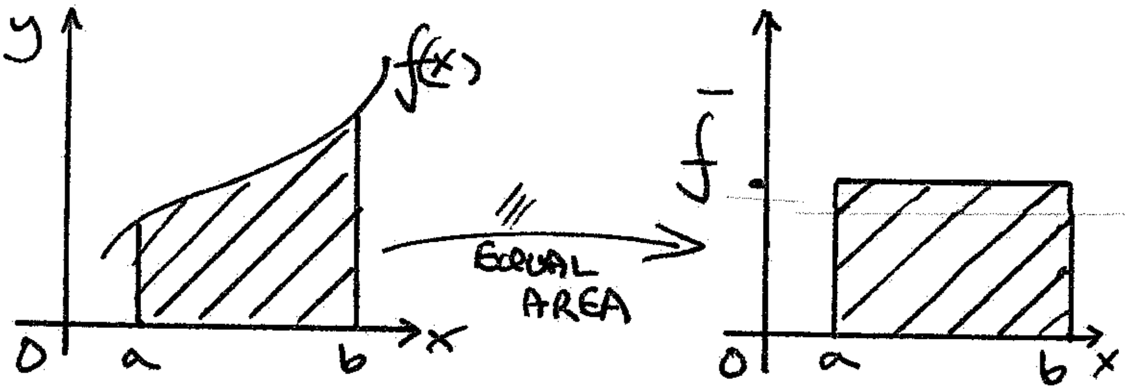
$$\text{So } K_2 = \pi/4$$

$\sqrt{2}$ CAN BE REPLACED BY ANY +ve CONSTANT

(3.5) APPLICATIONS - MEAN VALUE, AREA, LENGTH

(a) MEAN VALUE

Consider a function $f(x)$ and a specified interval $[a, b]$



The MEAN VALUE ('AVERAGE') of $f(x)$ over the interval is

$$\bar{f} \equiv \frac{1}{(b-a)} \int_a^b f(x)\, dx$$

[A 'modern' notation inspired by e.g. Quantum Mechanics (DIRAC) is $\langle f \rangle$].

So $\bar{f}$ is the height of a rectangle whose area is the same as that beneath the curve, over the same x domain.

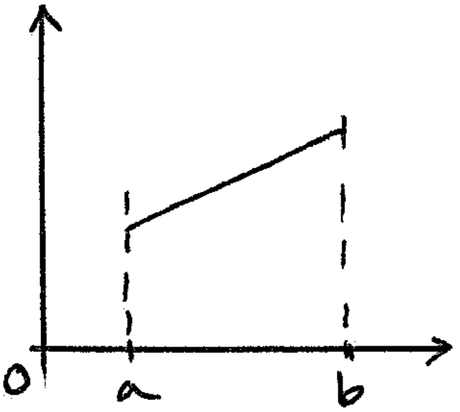
← Note ORDER!

Then the ROOT MEAN SQUARE VALUE of $f(x)$ over $[a,b]$ is

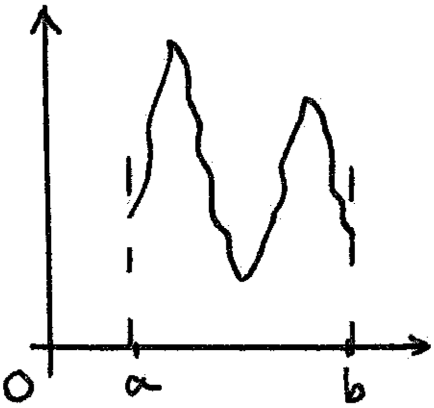
$$f_{r.m.s.} = \left[\frac{1}{(b-a)}\int_a^b f^2(x)\,dx\right]^{1/2}$$

and in DIRAC's notation $\langle f^2\rangle^{1/2}$.

Both $\bar{f}$ and f_{rms} characterise the function over the interval



$f_{rms} - \bar{f}$ SMALL



$f_{rms} - \bar{f}$ LARGE

Note: 'SINE WAVE' $\langle \sin x\rangle = \frac{1}{2\pi}\int_0^{2\pi} \sin x\,dx = 0$

$= \langle \cos x\rangle$.

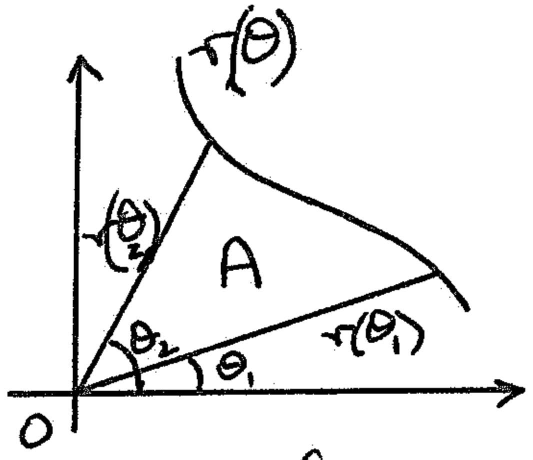
and $\langle \sin^2 x\rangle = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,dx = \frac{1}{2}$.

so $(\sin x)_{rms} = \left(\langle \sin^2 x\rangle\right)^{1/2} = \frac{1}{\sqrt{2}}$.

(b) AREA IN POLARS

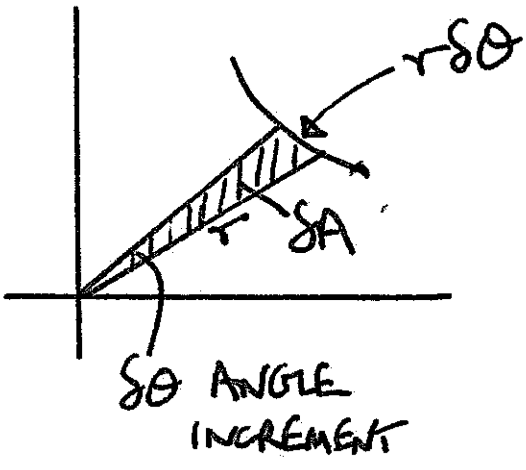
Geometry may lead to using plane polars (r, θ);

Consider



A ≡ area of wedge

We look at an infinitesimal wedge section

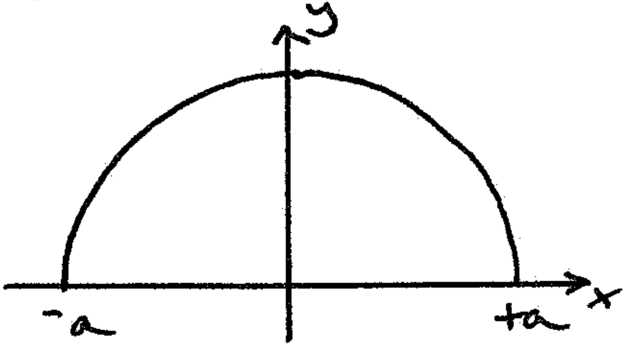


$\simeq$ THIN TRIANGLE

$$\delta A \simeq \frac{1}{2}(r\delta\theta)r$$

$$\Longrightarrow \text{Area } A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2(\theta)\, d\theta$$

ex. Area of semicircle $x^2 + y^2 = a^2$



Contrast:

CARTESIANS

$$A = \int_{-a}^{a} y\,dx = \int_{-a}^{a} (a^2-x^2)^{1/2}\,dx \equiv 2\int_0^a (a^2-x^2)^{1/2}\,dx$$

sub $x = a\sin\theta$

$$= 2\int_0^{\pi/2} a^2\cos^2\theta\,d\theta$$

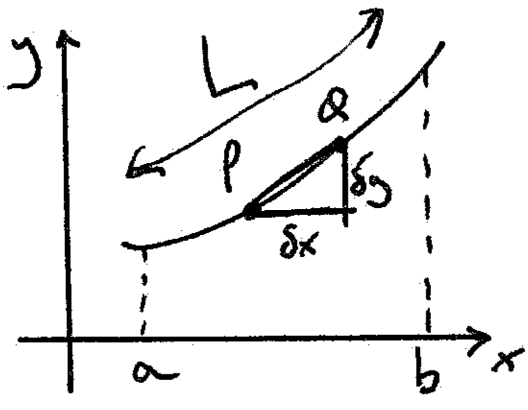
$$= \frac{\pi a^2}{2} \leftarrow \text{(of course)}$$

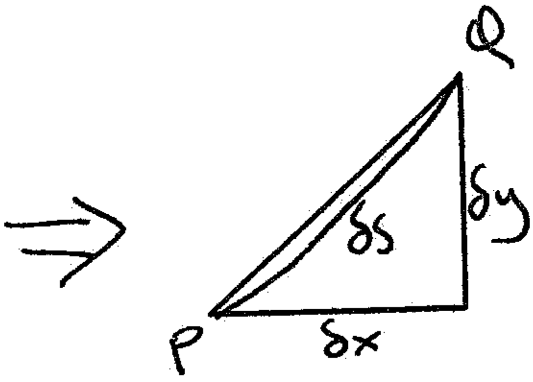
POLARS $r = a$ for $0 \leq \theta \leq \pi$

$$A = \int_0^{\pi} \frac{1}{2} r^2\,d\theta = \frac{1}{2}a^2\int_0^{\pi} d\theta = \frac{\pi a^2}{2}$$

(c) PATH LENGTH

CARTESIANS





S ≡ ARC LENGTH

$$\delta s^2 \simeq \delta x^2 + \delta y^2$$

$$\therefore \delta s \simeq \delta x\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]^{1/2}.$$

So

$$L = \int_a^b ds = \int_a^b \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} dx.$$

PARAMETRICALLY

As above, but $x=x(t)$, $y=y(t)$, say.

$$\delta s^2 \simeq \delta x^2+\delta y^2 = \left[\left(\frac{\delta x}{\delta t}\right)^2+\left(\frac{\delta y}{\delta t}\right)^2\right]\delta t^2$$

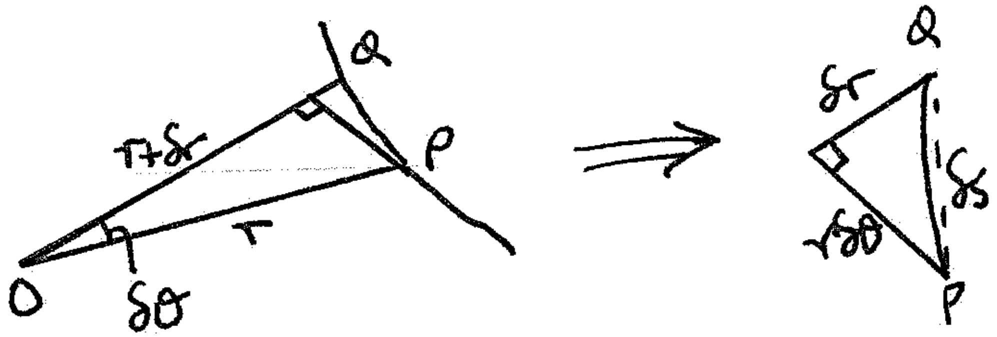
$$\Rightarrow ds = \left(\dot{x}^2+\dot{y}^2\right)^{1/2}dt.$$

SPEED if e.g. t is time.

and $$L=\int_{t_a}^{t_b}\left(\dot{x}^2+\dot{y}^2\right)^{1/2}dt$$

POLARS

We look at the local infinitesimal contribution

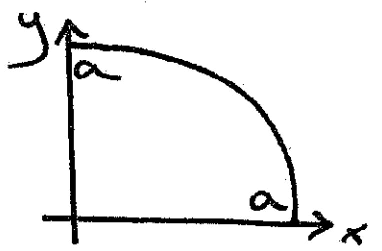


$$\delta s^2 \simeq \delta r^2 + (r\delta\theta)^2.$$

$$= \left[\left(\frac{\delta r}{\delta\theta}\right)^2 + r^2\right]\delta\theta^2$$

So $$L=\int_A^B ds = \int_A^B \left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{1/2} d\theta$$

ex(i) Arc length of a quarter circle



CARTESIANS $x^2+y^2=a^2$

$y=(a^2-x^2)^{1/2}$

$y'=\frac{-x}{(a^2-x^2)^{1/2}}$

$$L=\int_0^a (1+y'^2)^{1/2}dx$$

$$=\int_0^a \left(1+\frac{x^2}{(a^2-x^2)}\right)^{1/2}dx$$

$$=\int_0^a \frac{a\,dx}{(a^2-x^2)^{1/2}} = a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = \frac{\pi a}{2}$$

(of course)

POLARS $r=a,\ 0\le\theta\le\pi/2$

$$L=\int_0^{\pi/2}\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{1/2}d\theta = a\int_0^{\pi/2}d\theta = \frac{\pi}{2}a$$

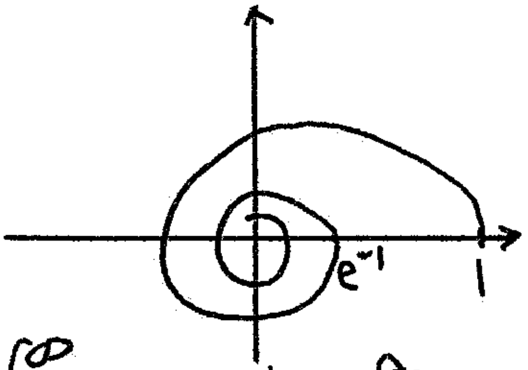
ex (ii) Arc length for an infinite spiral

$r=e^{-\theta/2\pi}$

$r(0)=1$

$r(2\pi)=e^{-1}$

etc



Total length

$0\le\theta<\infty$

$$L=\int_0^\infty \left(1+\frac{1}{4\pi^2}\right)^{1/2} e^{-\theta/2\pi}\,d\theta$$

$\frac{dr}{d\theta}=-\frac{1}{2\pi}e^{-\theta/2\pi}$

$$=\left(1+\frac{1}{4\pi^2}\right)^{1/2}\left[-2\pi e^{-\theta/2\pi}\right]_0^\infty$$

$$=(1+4\pi^2)^{1/2}$$

(3.6) APPLICATIONS - CENTRE OF MASS

Say we have N masses m_j $(j=1,2,\dots N)$ at positions (x_j, y_j)

The CENTRE OF MASS $G(\bar{x},\bar{y})$ is defined by

$$\bar{x} = \frac{\sum_{j=1}^{N} m_j x_j}{M}, \quad \bar{y} = \frac{\sum_{j=1}^{N} m_j y_j}{M}$$

['WEIGHTED' AVERAGE]

where $M \equiv$ TOTAL MASS $= \sum_{j=1}^{N} m_j$.

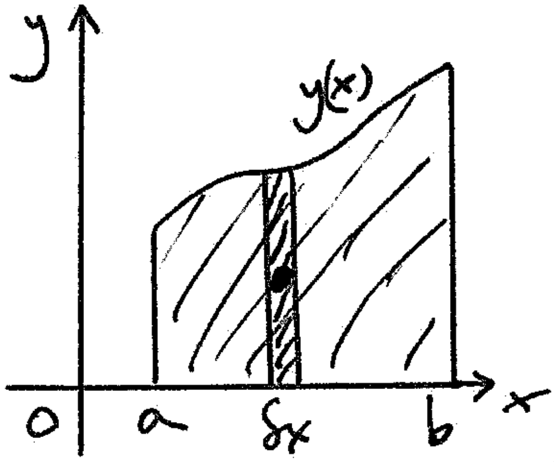
[The numerators of these expressions for $\bar{x}, \bar{y}$ are often called FIRST MOMENTS OF MASS].

The concept is very useful in Mechanics.

To generalise to a 2-dimensional plate ('LAMINA') we consider a continuous mass distribution with say $(\rho_{ij}\,\delta x_i\,\delta y_j)$ the mass element located at (x_i, y_j) say.

ρ_{ij} ← DENSITY ≡ MASS/UNIT AREA.

e.g.
CASE



If the mass density ρ is uniform then

TOTAL MASS $M = \rho \int_a^b y\,dx$.

Split the area into strips as shown.

The mass of our strip $= \rho y \delta x$.

The centre of mass of the strip is at $\approx (x, y/2)$.

So

$$\bar{x} = \frac{\int x(\rho y \delta x)}{M} = \frac{\int \rho x y\,dx}{\int \rho y\,dx} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}$$

$$\bar{y} = \frac{\int \left(\frac{y}{2}\right)(\rho y \delta x)}{M} = \frac{\int \frac{\rho}{2} y^2\,dx}{\int \rho y\,dx} = \frac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$$

[These ideas GENERALISE:

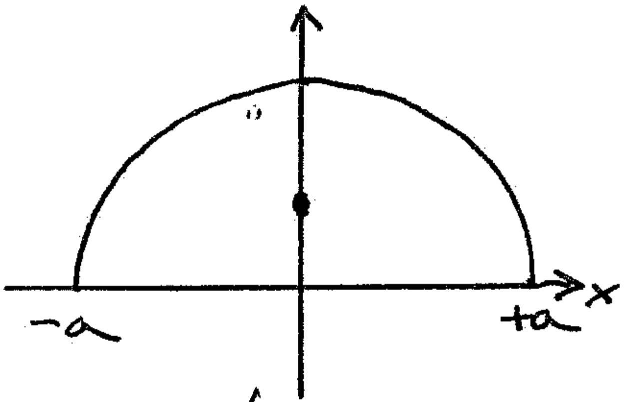
(i) NON-UNIFORM DENSITY.

(ii) THREE DIMENSIONS.

(iii) SECOND MOMENTS OF MASS

['VARIANCE', MOMENTS OF INERTIA...]]

ex (i) Uniform semi-circular plate



$x^2+y^2=a^2$

$$\bar{x} = \frac{\int_{-a}^{a} xy\,dx}{\pi a^2/2} = \frac{\int_{-a}^{a} x(a^2-x^2)^{\frac{1}{2}}\,dx}{\pi a^2/2} = 0$$

of course! (BY SYMMETRY)

$$\bar{y} = \frac{\frac{1}{2}\int_{-a}^{a} y^2\,dx}{\pi a^2/2} = \frac{2}{\pi a^2}\int_{0}^{a}(a^2-x^2)\,dx$$

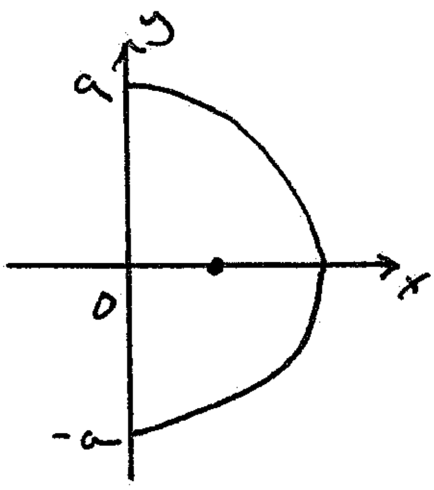
$$= \frac{2}{\pi a^2}\left[a^2x - \frac{1}{3}x^3\right]_0^a$$

$$= \frac{4a}{3\pi}$$

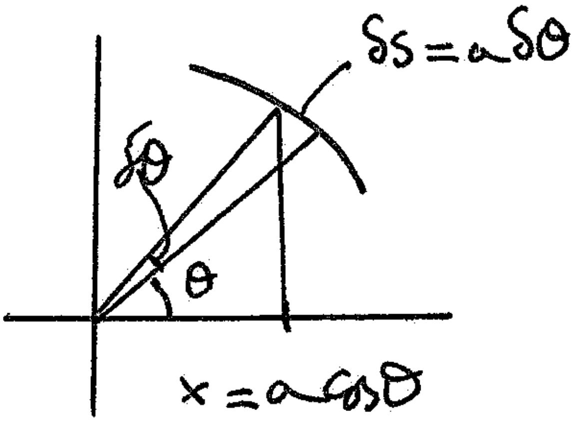
[NOTE: $0<\bar{y}<\frac{1}{2}a$ as expected

ex (ii) Uniform semi-circular $\underline{\underline{\text{wire}}}$

line density $= \frac{M}{\pi a}$ where M is the total mass.



Evidently $\bar{y}=0$ BY SYMMETRY



$$\delta m = \rho\,\delta s = \rho a\,\delta\theta.$$

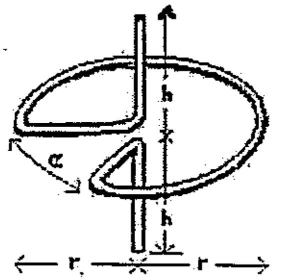
$$\bar{x} = \frac{1}{M}\int x\,dm$$

$$= \frac{1}{M}\int_{-\pi/2}^{\pi/2} a^2\rho\cos\theta\,d\theta$$

$$= \frac{a^2\rho}{M}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{2a^2\rho}{M} = \frac{2a}{\pi}$$

ex(iii) 'Practical' SAKAI TOP *

Paper clip of mass M and wire length l.



Top ≡ bent clip into a circular arc of radius r and angle $(2\pi - \alpha)$

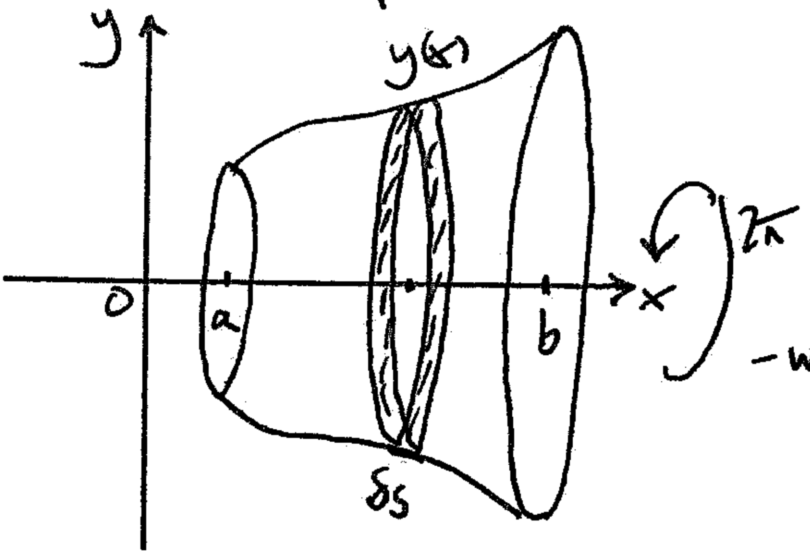
It turns out that the centre of mass of the top is on the axis only if $\tan\left(\frac{\alpha}{2}\right) = \frac{1}{2}$.

[NOTE: $l = 2h + 2r + r(2\pi - \alpha)$]

$\Rightarrow \alpha = 0.9273$ radians
≡ 53.13 degrees.

(3.7) APPLICATIONS - VOLUME/SURFACE AREA OF REVOLUTION

Take function $y(x)$ in 2-dimensional plane and rotate it about the x-axis to create a 3-dimensional shape.



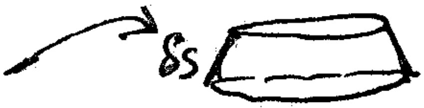
'FRUSTRUM'
- with ⊙ cross-sections

$$\text{Volume } \delta V \simeq \pi y^2 \delta x$$

$$\Rightarrow V = \pi\int_a^b y^2\,dx.$$

TOTAL VOLUME

$$\text{Surface area } \delta S \simeq 2\pi y\,\delta s$$

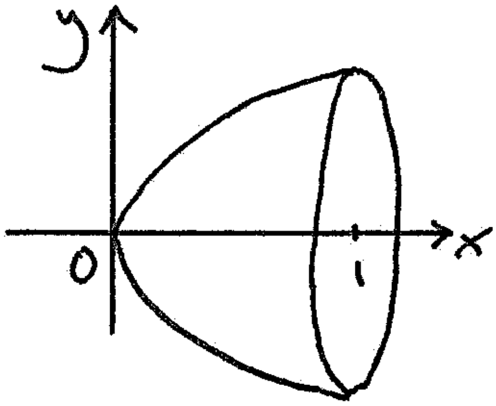


$$= 2\pi y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx.$$

$$\Rightarrow S = 2\pi\int_a^b y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx.$$

TOTAL SURFACE AREA

ex (i) Paraboloid

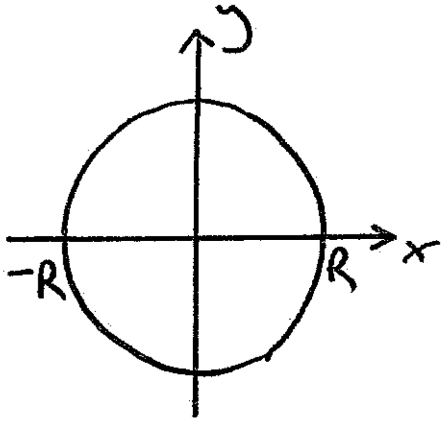


e.g. $y = x^{1/2}$. rotated with (say) $0 \leqslant x \leqslant 1$.

$$V = \pi \int_0^1 y^2 dx = \pi \int_0^1 x \, dx = \pi \left[\frac{1}{2}x^2\right]_0^1$$
$$= \pi/2 .$$

$$S = 2\pi \int_0^1 y \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$
$$= 2\pi \int_0^1 x^{1/2} \left[1 + \frac{1}{4x}\right]^{1/2} dx$$
$$= 2\pi \int_0^1 \left(x + \frac{1}{4}\right)^{1/2} dx$$
$$= \frac{4\pi}{3} \left[\left(x + \frac{1}{4}\right)^{3/2}\right]_0^1$$
$$= \frac{4\pi}{3} \left[\left(\frac{5}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right]$$
$$= \frac{\pi}{6}\left(5^{3/2} - 1\right) .$$

ex (ii) Sphere



$y = +(R^2 - x^2)^{1/2}$

rotated about the x axis through 2π

$$V = \pi\int_{-R}^{R}(R^2-x^2)dx$$

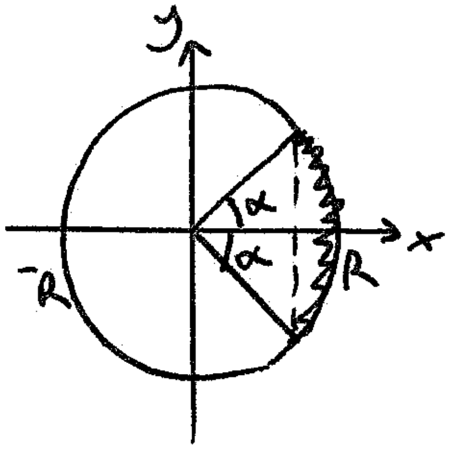
$$= \pi\left[R^2x-\frac{1}{3}x^3\right]_{-R}^{R} = \frac{4}{3}\pi R^3$$

$$S = 2\pi\int_{-R}^{R}(R^2-x^2)^{1/2}\left[1+\left\{\frac{x}{(R^2-x^2)^{1/2}}\right\}^2\right]^{1/2}dx$$

$$= 2\pi\int_{-R}^{R} R\,dx = 4\pi R^2$$

of course.

ex (iii) Spherical cap-area

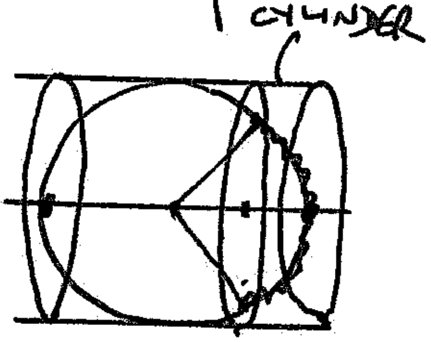


as in (ii) above but with $R\cos\alpha \le x \le R$.

$$S = 2\pi\int_{R\cos\alpha}^{R} R\,dx$$

$$= 2\pi R^2(1-\cos\alpha)$$

An important result



AREA cutoff by 2 parallel planes on SPHERE surface

= AREA cutoff by the planes on the CIRCUMSCRIBING CIRCULAR CYLINDER

$$= [R(1-\cos\alpha)][2\pi R].$$

ARCHIMEDES!

→212 BCE

CHAPTER 4

PARTIAL DIFFERENTIATION

(4.1) DEFINITIONS

(4.2) THE TOTAL DIFFERENTIAL

(4.3) FUNCTION OF A FUNCTION - THE CHAIN RULE

(4.4) CARTESIANS → POLARS

(4.5) IMPLICIT FUNCTIONS

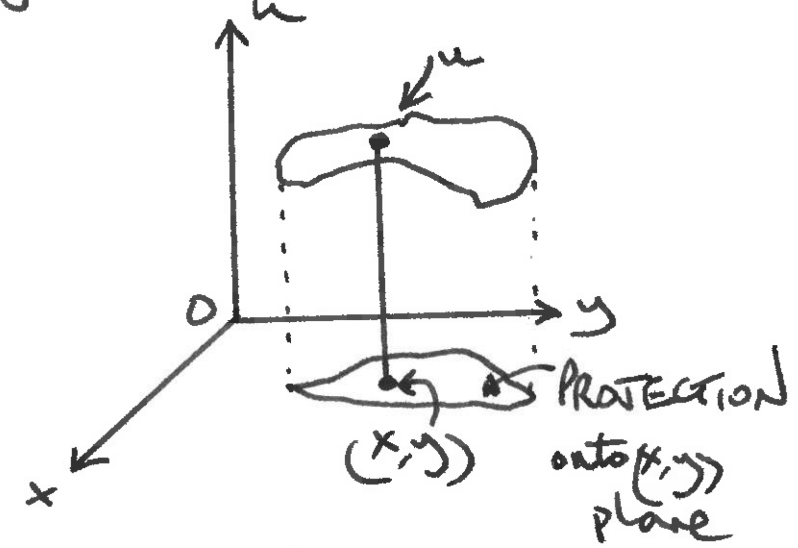
(4.6) EXACT DIFFERENTIALS

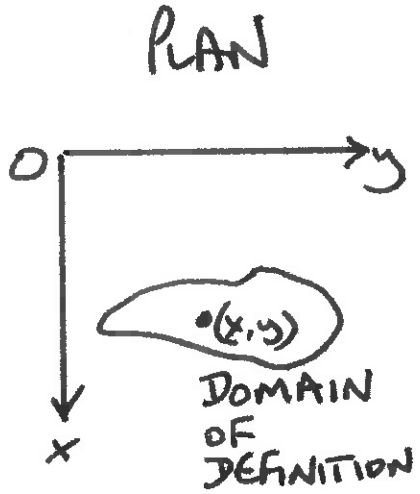
(4.7) STATIONARY POINTS

(4.1) DEFINITIONS

[IDEAS WILL GENERALISE]

Consider a function $u = u(x,y)$ of 2 INDEPENDENT variables x, y. We can think of u as being the height of a surface above the (x,y) plane.



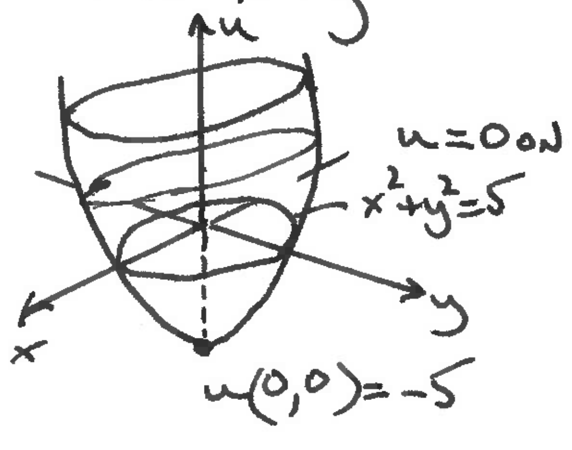


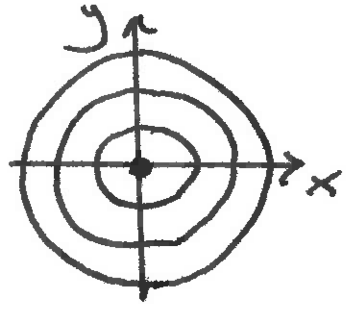
It is often helpful to visualise the surface using CONTOUR LINES

$$u(x,y) = c \text{ (constant)}$$

for different values of c.

ex $u = x^2 + y^2 - 5$ has circular contours





PHYSICALLY u could represent a geometrical object or temperature or pressure or.....

We now look at (SPATIAL) RATES OF CHANGE

First - start at $P(x,y)$ and move a small distance $\delta x \equiv h$ in the x direction to $Q(x+h,y)$ i.e. KEEPING y FIXED.

Then we define (if this limit exists)

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\left[\frac{u(x+h,y)-u(x,y)}{h}\right]$$

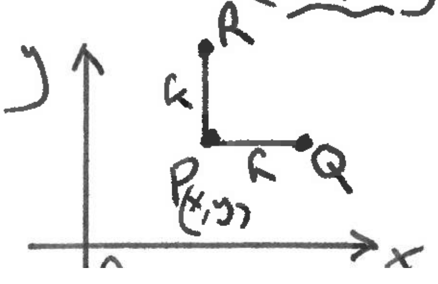
$\equiv$ RATE of CHANGE of u with respect to x at P (keeping y fixed).

NOTATIONS: $\frac{\partial u}{\partial x}, \left(\frac{\partial u}{\partial x}\right)_y, u_x, \ldots$! MEANING

Similarly $P(x,y) \rightarrow R(x,y+k)$:

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\left[\frac{u(x,y+k)-u(x,y)}{k}\right]$$ (if this limit exists).

$\equiv$ RATE of CHANGE of u with respect to y at P (keeping x fixed).



NOTATIONS: $\frac{\partial u}{\partial y}, \left(\frac{\partial u}{\partial y}\right)_x, u_y, \ldots$ MEANINGS ex (i) $\quad u = x^2 \sin y + y^3$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 2x \sin y \quad , \quad \frac{\partial u}{\partial y} = x^2 \cos y + 3y^2 .$$

We can, of course, consider <u>HIGHER DERIVATIVES</u>

$$\frac{\partial^2 u}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \equiv u_{xx} \quad , \quad \frac{\partial^2 u}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \equiv u_{xy}$$

NOTE ORDER. ETC

For the ex. above we have

$$\frac{\partial^2 u}{\partial x^2} = 2 \sin y \qquad \frac{\partial^2 u}{\partial y^2} = -x^2 \sin y + 6y$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2x \cos y \qquad \frac{\partial^2 u}{\partial x \partial y} = 2x \cos y .$$

We NOTE that

$$\underline{\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}}$$

in this case. This is a <u>GENERAL RESULT</u>

[requiring only CONTINUITY of LHS and RHS - usually the case!]

Similarly, we generally have

$$u_{xxyyx} = u_{yxyxx} = \text{ETC}!$$

ex (ii) $u(x,t) = a\sin(x-ct)$ +ve a, c CONSTANTS.

$$\frac{\partial u}{\partial x} = a\cos(x-ct) \qquad \frac{\partial^2 u}{\partial x^2} = -a\sin(x-ct)$$

$$\frac{\partial u}{\partial t} = -ac\cos(x-ct) \qquad \frac{\partial^2 u}{\partial t^2} = -ac^2\sin(x-ct)$$

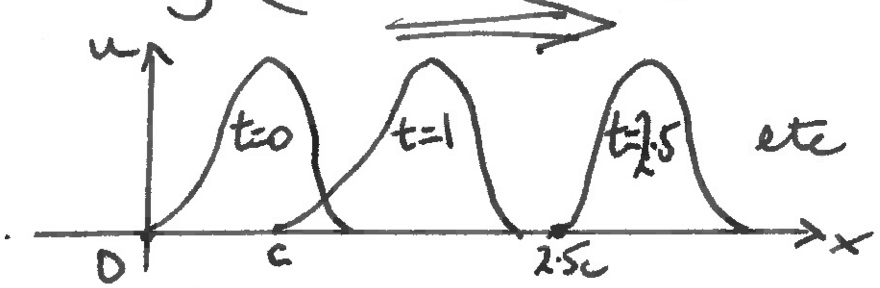
$\Rightarrow$ $u(x,t)$ satisfies

$$\underline{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$$

THE (1Dim) <u>WAVE EQUATION</u>

2ND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

[In fact any reasonable function $f(x-ct)$ will satisfy this equation! It represents a WAVE FORM moving (with $c>0$ here) to the right



]

ex (iii) $u = \tan^{-1}(y/x)$

$\Rightarrow$ ... satisfies

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2ND ORDER LINEAR P.D.E.

(2Dim) <u>LAPLACE'S EQUATION</u>

(4.2) THE TOTAL DIFFERENTIAL

When we have a function of a single variable $f(x)$ and we make a small change $x \to x + \delta x$, so that $f \to f + \delta f$ then

$$\delta f \simeq \frac{df}{dx}\,\delta x \qquad \text{'INCREMENTS'}$$

In the limit ('small → 0')

$$df = \frac{df}{dx}\,dx \qquad \text{'DIFFERENTIALS'}$$

Now for a function of two variables, small changes $x \to x + \delta x$, $y \to y + \delta y$ lead to $u(x,y) \to u + \delta u$ with

$$\delta u \simeq \frac{\partial u}{\partial x}\,\delta x + \frac{\partial u}{\partial y}\,\delta y \qquad \text{'INCREMENTS'}$$

In the limit ('small → 0') we get

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \qquad \text{'DIFFERENTIALS'}$$

This is called the TOTAL DIFFERENTIAL of $u(x,y)$

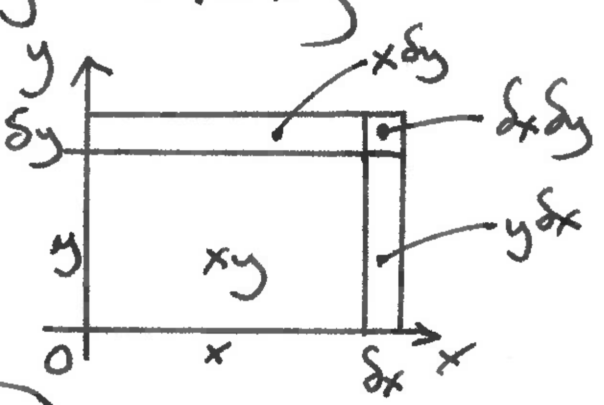
[NOTE: CONVENTIONS! d, δ, ∂ .... IMPORTANT!]

ex (i) $u = x^2 \sin y + y^3$

$$\Rightarrow \delta u \simeq (2x \sin y)\delta x + (x^2 \cos y + 3y^2)\delta y$$

AND $du = (2x \sin y)dx + (x^2 \cos y + 3y^2)dy$

ex (ii) Area of a rectangle $A = xy$



See

EXACT!

$$\delta A \overset{\downarrow}{=} (x+\delta x)(y+\delta y) - xy$$

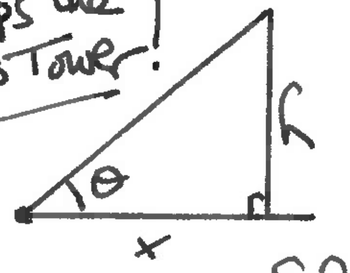
of course

$$= \underbrace{\boxed{y\delta x + x\delta y}}_{\text{1ST ORDER SMALL}} + \underbrace{\boxed{\delta x \delta y}}_{\text{2ND ORDER SMALL}}$$

$$\delta A \simeq y\delta x + x\delta y \iff dA = y\,dx + x\,dy.$$

ex (iii) Height of a building! $h = x \tan\theta$

perhaps the Queen's Tower!



COULD EXPRESS AS %AGE

$x = 200\text{m}$ with error $\pm 2\text{m}$

$\theta = 20°$ with error $\pm \frac{1}{2}°$.

CAREFUL!

$$\delta h \simeq (\tan\theta)\delta x + (x\sec^2\theta)\delta\theta.$$

CENTRAL ESTIMATE $200 \tan(\pi/9) = 72.8\text{m}$

$$\Rightarrow \delta h \simeq (.36)\,\delta x + (226.5)\,\delta\theta$$

$|\delta x| \leq 2$

$|\delta\theta| \leq \pi/360 = 0.0087$

$$\therefore |\delta h| \leq \underbrace{(.36)(2) + (226.5)(0.0087)}_{\text{EXTREMES!}} = 2.7\text{m} \Longrightarrow h = 72.8 \pm 2.7\text{m}. \boxed{\pm 3.7\%}$$

(4.3) FUNCTION OF A FUNCTION
— THE CHAIN RULE

When we have had previously

$u = f(x)$ and $x = g(t)$ (say)

we have as a consequence

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} = f'(x)\,g'(t).$$

$$\equiv f'(g(t))\,g'(t).$$

Now consider

$u = u(x,y)$ where $x(t)$ and $y(t)$.

In section (4.2) we saw that

$$\delta u \simeq \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

Dividing by δt and taking the limit SMALL $\to 0$

we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

THE CHAIN RULE

$\nwarrow \quad \nearrow$

here x and y are functions ONLY of t.

[If $x(r,s)$ and $y(r,s)$ then

e.g. $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r}$, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s}$ ETC] ex(i) Volume V of a cylindrical box of radius r and height h. $\rightarrow V = \pi r^2 h$.

If we know that $r = 2t$

and $h = 1 + t^2$ (say)

then $$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= (2\pi r h)(2) + (\pi r^2)(2t)$$

$$\equiv 8\pi(t + 2t^3)$$ after substitution

check $V = \pi(2t)^2(1+t^2) = 4\pi(t^2+t^4)$ SAME

so, of course $$\frac{dV}{dt} = 8\pi t + 16\pi t^3$$

ex(ii) $u = x^2 y$ with $x = st$, $y = s+t$

$$\Rightarrow \frac{\partial \bar{u}}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2xy)(t) + (x^2)(1) = \ldots\ldots = 3s^2t^2 + 2st^3.$$

SEE BELOW!

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2xy)(s) + (x^2)(1) = \ldots\ldots = 3s^2t^2 + 2s^3t.$$

again we can check, since

$$u(x,y) = (st)^2(s+t) \equiv \bar{u}(s,t)$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial s} = 3s^2t^2 + 2st^3.$$

SAME

$$\frac{\partial \bar{u}}{\partial t} = 2s^3t + 3s^2t^2$$

ex(iii) CAUTIONARY EXAMPLE!

$$u(x,y) = xy + y^2.$$

If we have a second relation $y = x + t$,

then of course

$$\bar{u}(x,t) = x(x+t) + (x+t)^2.$$

We have to be careful when we look at the variation of u and $\bar{u}$ with respect to x.

(a) $\dfrac{\partial u}{\partial x} = y \quad (\equiv x+t)$

(b) $\dfrac{\partial \bar{u}}{\partial x} = 2x + t + 2(x+t) \quad (\equiv 4x+3t)$

These are NOT the same – u and $\bar{u}$ have the SAME function values at corresponding points.

BUT in (a) y is being kept constant $\left(\dfrac{\partial u}{\partial x}\right)_y$

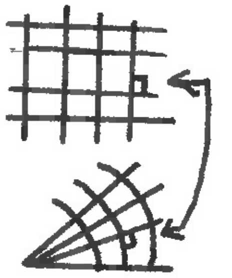
in (b) t is being kept constant. $\left(\dfrac{\partial \bar{u}}{\partial x}\right)_t$

<u>CARE</u> is needed evidently!

Changing variables is crucial for solving the partial differential equations of Physics with different geometries.....

(4.4) CARTESIANS → POLARS

BOTH ARE ORTHOGONAL SYSTEMS



$$u(x,y) \equiv \bar{u}(r,\theta)$$

CARTESIANS PLANE POLARS

$$x = r\cos\theta \qquad r = (x^2+y^2)^{1/2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$$

We need to be careful!

e.g. $\frac{\partial x}{\partial r} = \cos\theta$ keeping θ constant and $\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2)^{1/2}} \equiv \cos\theta$ keeping y constant

Hence we should <u>not</u> be tempted!

$$\left(\frac{\partial x}{\partial r}\right)_\theta \neq \frac{1}{(\partial r/\partial x)_y}$$ here ... ⟸ !

[Note also the overbar on u above.
The Cartesian and polar versions of our function have the SAME function values, but are described differently!

$$\text{e.g. } u(x,y) = x^2+y^2 \equiv r^2 \equiv \bar{u}(r,\theta)$$

<u>NOT</u> $(r^2+\theta^2)$]

CHAIN RULE

$$\frac{\partial u}{\partial x} = \frac{\partial \bar{u}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \bar{u}}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= (\cos\theta)\frac{\partial \bar{u}}{\partial r} + \left(-\frac{\sin\theta}{r}\right)\frac{\partial \bar{u}}{\partial \theta}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial \bar{u}}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \bar{u}}{\partial \theta}\frac{\partial \theta}{\partial y}.$$
$$= (\sin\theta)\frac{\partial \bar{u}}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial \bar{u}}{\partial \theta}.$$

We have therefore PARTIAL DIFFERENTIAL OPERATORS

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

and $$\frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

which relate rates of change in the two different coordinate systems.

ex (i) $u(x,y) = x^2 - y^2 \equiv r^2(\cos^2\theta - \sin^2\theta) \equiv \bar{u}(r,\theta)$

$$\frac{\partial u}{\partial x} = 2x \equiv 2r\cos\theta$$
$$\frac{\partial u}{\partial y} = -2y \equiv -2r\sin\theta$$
using the above results.

ex (ii) We can express e.g. $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$ in plane polars as

$$\left[(\cos\theta)\frac{\partial \bar{u}}{\partial r} + \left(-\frac{\sin\theta}{r}\right)\frac{\partial \bar{u}}{\partial \theta}\right]^2 + \left[(\sin\theta)\frac{\partial \bar{u}}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial \bar{u}}{\partial \theta}\right]^2$$
$$\equiv \left(\frac{\partial \bar{u}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \bar{u}}{\partial \theta}\right)^2.$$

ex (iii) What about LAPLACE'S EQUATION?

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Well

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial \bar{u}}{\partial r}-\frac{\sin\theta}{r}\frac{\partial \bar{u}}{\partial\theta}\right)$$

ATTRITION!

$$\equiv \quad \cdots\cdots$$

PHEW!

$$\equiv \cos^2\theta\frac{\partial^2\bar{u}}{\partial r^2}-\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2\bar{u}}{\partial r\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial\bar{u}}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2\bar{u}}{\partial\theta^2}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\bar{u}}{\partial\theta}$$

and

$$\frac{\partial^2 u}{\partial y^2}\equiv\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial \bar{u}}{\partial r}+\frac{\cos\theta}{r}\frac{\partial \bar{u}}{\partial\theta}\right)$$

$$\equiv \sin^2\theta\frac{\partial^2\bar{u}}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\bar{u}}{\partial r\partial\theta}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\bar{u}}{\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2\bar{u}}{\partial\theta^2}+\frac{\cos^2\theta}{r}\frac{\partial\bar{u}}{\partial r}.$$

PHEW AGAIN!

Hence(!)

LAPLACE.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\equiv\underbrace{\frac{\partial^2\bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}}{\partial r}}_{\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\bar{u}}{\partial r}\right)}+\frac{1}{r^2}\frac{\partial^2\bar{u}}{\partial\theta^2}=0$$

[LAPLACE in 3 dimensions

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \Rightarrow \left.\begin{array}{l}\text{SPHERICAL POLARS}\\ \text{or}\\ \text{CYLINDRICAL POLARS}\end{array}\right\}\begin{array}{l}\text{GRAVITY}\\ \text{QUANTUM}\\ \text{MECHANICS}\end{array}$$

]

(4.5) IMPLICIT FUNCTIONS

If we have a function defined IMPLICITLY

$$F(x,y) = 0$$

then F does *not* change as x and y do so.

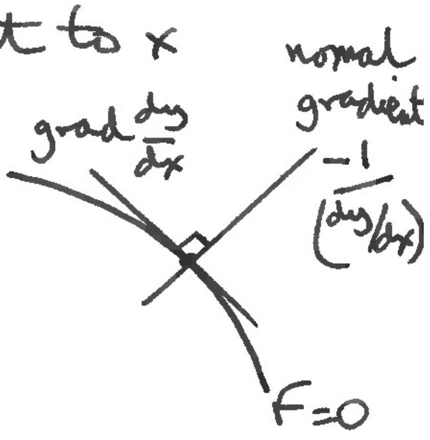
The total derivative then is

(x,y) CONSTRAINED to a curve.

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

So the derivative of y with respect to x is given by

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$



ex(i) from (2.2)(f)

$$F(x,y) = x^2\sin y + xy - 1 = 0$$

$$\frac{dy}{dx} = -\frac{(2x\sin y + y)}{(x^2\cos y + x)}$$ as before.

If we have an IMPLICIT function of 3 variables

$$F(x,y,z) = 0 \qquad \Longrightarrow$$

this CONSTRAINS our point (x, y, z) to be on a particular surface. We can certainly regard, if we wish,

$$x = x(y,z) \ \underline{\text{or}}\ y = y(z,x) \ \underline{\text{or}}\ z = z(x,y)$$

Now, no change in F on the surface

$$\Rightarrow dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

Then AT CONSTANT y: $\left(\frac{\partial z}{\partial x}\right)_y = -F_x/F_z$.

AT CONSTANT x: $\left(\frac{\partial z}{\partial y}\right)_x = -F_y/F_z$.

AT CONSTANT z: $\left(\frac{\partial y}{\partial x}\right)_z = -F_x/F_y$.

NOTE: e.g. $\left(\frac{\partial z}{\partial x}\right)_y \nwarrow = \frac{1}{\left(\frac{\partial x}{\partial z}\right)_y} \swarrow$

here because of course the variable y is being kept constant on both sides.

– we are looking at variation on a constant y SLICE of the $F=0$ surface!

ex (ii) In thermodynamics the equation of *STATE* of a gas/liquid is written

$$F(\underset{\substack{\uparrow\\ \text{pressure}}}{p},\underset{\substack{\uparrow\\ \text{volume}}}{V},\underset{\substack{\uparrow\\ \text{absolute temperature}}}{T})=0$$

$\Rightarrow$ an *implicit* definition of $p = p(V,T)$.

Only in *simple cases* can we express this relation *explicitly* e.g. IDEAL GAS

$$p = \frac{RT}{V}.$$

In any case, from the general relation above we can show that

$$\left(\frac{\partial p}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p\left(\frac{\partial T}{\partial p}\right)_V = -1$$

an example of an *exact thermodynamic identity*.

(4.6) EXACT DIFFERENTIALS

We know that for a function of two variables $u(x,y)$ the TOTAL DIFFERENTIAL (DERIVATIVE) is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{in } (4.2)$$

and of course $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ will both, in general, be functions of x and y.

Now consider the converse problem!

Given $P(x,y)dx + Q(x,y)dy$

(i.e. given P and Q), when is it the case that this is the total differential of some (as yet unknown) function $u(x,y)$? If it is such then $P(x,y) = \frac{\partial u}{\partial x}$ and $Q(x,y) = \frac{\partial u}{\partial y}$ for that function $u(x,y)$.

PROOF NOT GIVEN HERE!

This IMPLIES ✓ and IS IMPLIED BY the CONDITION OF INTEGRABILITY

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}. \qquad \left[\equiv \frac{\partial^2 u}{\partial x \partial y} \text{ OF COURSE!}\right.$$

ex (i) ? $y^2\,dx + (x^2+2y)\,dy$

$P \equiv y^2$, $Q \equiv x^2+2y$

$\frac{\partial P}{\partial y} = 2y \not\equiv \frac{\partial Q}{\partial x} = 2x$ TEST FAIL NOT EXACT

(ii) ? $(\underset{\uparrow P}{2xy+\cos x\cos y})\,dx + (\underset{\uparrow Q}{x^2-\sin x\sin y})\,dy$

$\frac{\partial P}{\partial y} = 2x - \cos x\sin y \equiv \frac{\partial Q}{\partial x}$ SO TEST PASS

so $\frac{\partial u}{\partial x} = 2xy + \cos x\cos y$

'CONSTANT' OF INTEGRATION W.R.T. x

$\Rightarrow u(x,y) = x^2y + \sin x\cos y + f(y)$ $\circledast$

Then EITHER

$\frac{\partial u}{\partial y} = x^2 - \sin x\sin y + \frac{df}{dy}$

$\equiv x^2 - \sin x\sin y$ $\Rightarrow$ so $\frac{df}{dy} \equiv 0$

and $f(y) = K$

CONSTANT W.R.T. x and y.

OR

ALTERNATIVE:

$\frac{\partial u}{\partial y} = x^2 - \sin x\sin y$

'CONSTANT' OF INTEGRATION W.R.T. y.

$\Rightarrow u(x,y) = x^2y + \sin x\cos y + g(x)$ $\circledast$

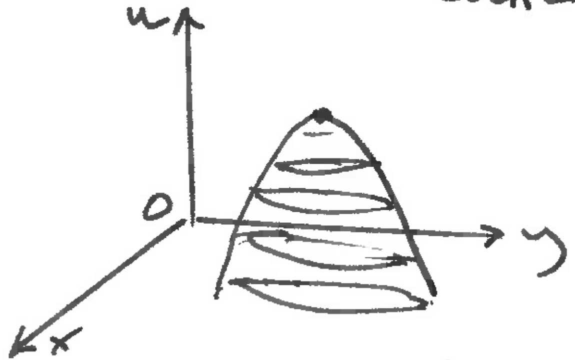
Compare $\circledast$ expressions $\Rightarrow$ $\overline{u(x,y) = x^2y + \sin x\cos y + K}$.

[NOTE $\underbrace{P\,dx + Q\,dy}_{du=} = 0 \Rightarrow \frac{dy}{dx} = -\frac{P}{Q}$ $\Rightarrow$ SOLUTION $u(x,y) = $ constant]

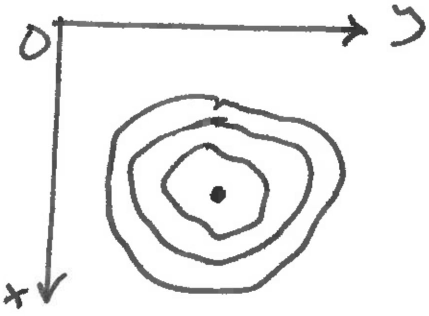
(4.7) STATIONARY POINTS

In Section (2-3) we look at stationary points for functions of 1 independent variable. What happens for 2 independent variables?

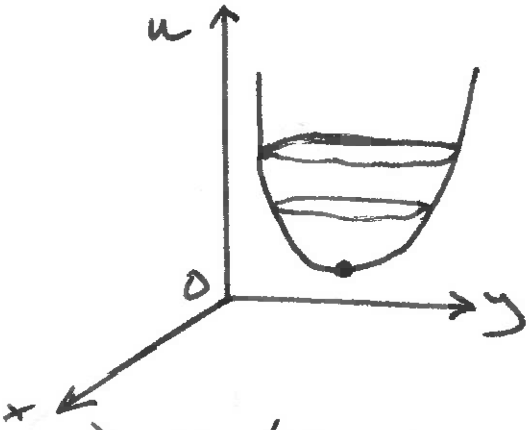
There are 3 types of stationary points → LOCAL HORIZONTAL TANGENT PLANE



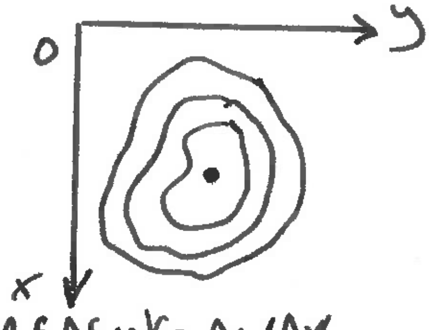
(LOCAL) **MAXIMUM**



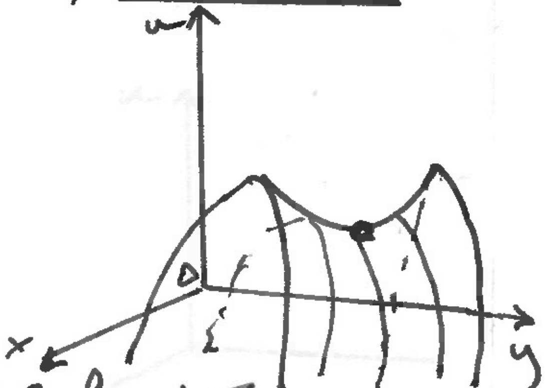
u(x,y) DECREASING AWAY FROM THE POINT

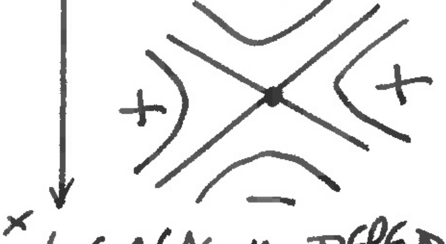


(LOCAL) **MINIMUM**



u(x,y) INCREASING AWAY FROM THE POINT





u(x,y) INCREASING/DECREASING DEPENDING ON DIRECTION

SADDLE POINT

For functions of two variables $u(x,y))$, STATIONARY POINTS are located at simultaneous solution of the two equations

du = 0 LOCALLY ⇒

$$\left|\begin{array}{l} \dfrac{\partial u}{\partial x} = 0 \\ \dfrac{\partial u}{\partial y} = 0 . \end{array}\right.$$

HORIZONTAL TANGENT PLANE

→ (several) (x_0, y_0) POINTS.

Each (x_0, y_0) has CHARACTER determined by

$$E_0 \equiv \left[\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\right]_{(x_0,y_0)} \equiv B^2 - AC$$

via.

$E_0 > 0$ SADDLE

$E_0 < 0$

$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_0,y_0)}^{\swarrow A} < 0$ (LOCAL) MAXIMUM

$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(x_0,y_0)}^{\swarrow A} > 0$ (LOCAL) MINIMUM.

[$E_0 = 0$ → HIGHER ORDER DERIVATIVES determine the issue. NOT considered here].

PROOF of the above CRITERIA in CHAPTER 5 .....

ex (i) $u(x,y) = x^3 + xy^2 - x - yx^2 - y^3 + y$

$$\equiv (x-y)(x^2+y^2-1).$$

$$\frac{\partial u}{\partial x} = 3x^2 + y^2 - 1 - 2xy = 0$$

(!)

$$\frac{\partial u}{\partial y} = 2xy - x^2 - 3y^2 + 1 = 0$$

ADD $\Rightarrow$ $2x^2 - 2y^2 = 0 \Rightarrow y = \pm x$

TO REPLACE ONE OF THESE

⊕ $2x^2 = 1 \rightarrow x = \pm\frac{1}{\sqrt{2}}$

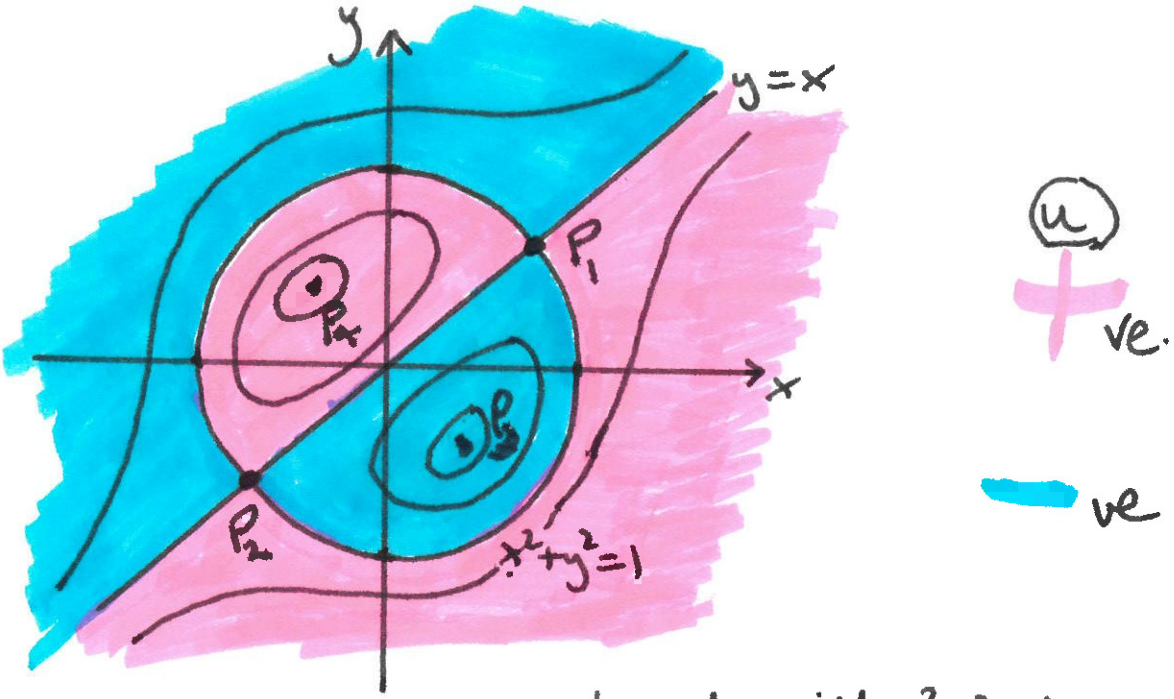
⊖ $6x^2 = 1 \rightarrow x = \pm\frac{1}{\sqrt{6}}$.

So there are 4 STATIONARY POINTS

$P_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), P_2\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), P_3\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), P_4\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.

P_T	$A = \left(\frac{\partial^2 u}{\partial x^2}\right)_0 = (6x-2y)_0$	$B = \left(\frac{\partial^2 u}{\partial x \partial y}\right)_0 = (2y-2x)_0$	$C = \left(\frac{\partial^2 u}{\partial y^2}\right)_0 = (2x-6y)_0$	$B^2 - AC \equiv E_0.$	u_0	TYPE
P_1	$4/\sqrt{2}$	0	$-4/\sqrt{2}$	8	0	SADDLE
P_2	$-4/\sqrt{2}$	0	$4/\sqrt{2}$	8	0	SADDLE
P_3	$8/\sqrt{6}$	$-4/\sqrt{6}$	$8/\sqrt{6}$	-8	$-\frac{2}{3}\sqrt{\frac{2}{3}}$	MINIMUM
P_4	$-8/\sqrt{6}$	$4/\sqrt{6}$	$-8/\sqrt{6}$	-8	$+\frac{2}{3}\sqrt{\frac{2}{3}}$	MAXIMUM

Armed with this information we can sketch the CONTOURS.



The ZERO CONTOUR is $y=x$ together with $x^2+y^2=1$.

WARNING !

When we are faced with a function of several variables and we need to find stationary points (and potential LOCAL MAX and MIN) – we need to make sure that our independent variables ARE indeed INDEPENDENT.

ex (ii) MAXIMISE volume V of a rectangular box given the surface area A (fixed)

max $V = xyz$ given that $A = 2xy + 2yz + 2xz$ FIXED.

x, y, z are NOT INDEPENDENT here.

Simple method WRITE $z = \frac{A-2xy}{2(x+y)} \Rightarrow V = \frac{xy(A-2xy)}{2(x+y)}$.

NOW x, y ARE INDEPENDENT

$\Longrightarrow \ldots\ x_0 = \left(\frac{A}{6}\right)^{1/2} = y_0 \ (\equiv z_0)$ BOX IS CUBICAL!

and $V_{max} = \left(A/6\right)^{3/2}$.

CHAPTER 5

SERIES EXPANSION
(TAYLOR / MACLAURIN)

(5·1) TAYLOR SERIES

(5·2) MACLAURIN SERIES

(5·3) L'HÔPITAL'S RULE (REVISITED)

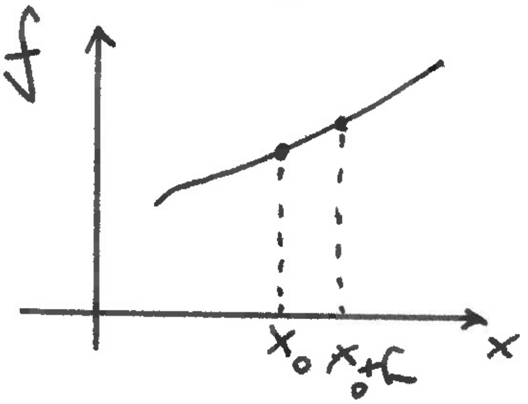
(5·4) DOUBLE TAYLOR SERIES

(5·5) STATIONARY POINTS (REVISITED)

(5.1) TAYLOR SERIES

Consider a function $f(x)$ of a single independent variable, where the value of all its derivatives are known at a point x_0.

Try to evaluate the function at a nearby point x_0+h, using a POWER SERIES



h is 'small'

$$f(x_0+h) = \alpha_0 + \alpha_1 h + \alpha_2 h^2 + \ldots\ldots$$
$$\equiv \sum_{n=0}^{\infty} \alpha_n h^n$$

with coefficients α_n to be determined.

Putting $h=0 \implies f(x_0) = \alpha_0$.

Then, differentiating with respect to $h \implies$

$$f'(x_0+h) = \alpha_1 + 2\alpha_2 h + 3\alpha_3 h^2 + \ldots$$

Then $h=0 \implies f'(x_0) = \alpha_1$.

Continuing in this manner:

$$f''(x_0+h) = 2\alpha_2 + 3.2\alpha_3 h + \dots$$

$$\Rightarrow \quad \frac{f''(x_0)}{2} = \alpha_2$$

$$f'''(x_0+h) = 3.2\alpha_3 + 4.3.2\alpha_4 h + \dots$$

$$\Rightarrow \quad \frac{f'''(x_0)}{3.2} = \alpha_3 .$$

The general result is $\alpha_n = \dfrac{f^{(n)}(x_0)}{n!}$

and we obtain the TAYLOR SERIES of $f(x)$ about x_0 in the form

$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)\, h^n}{n!}$$

ALTERNATIVE EXPRESSION

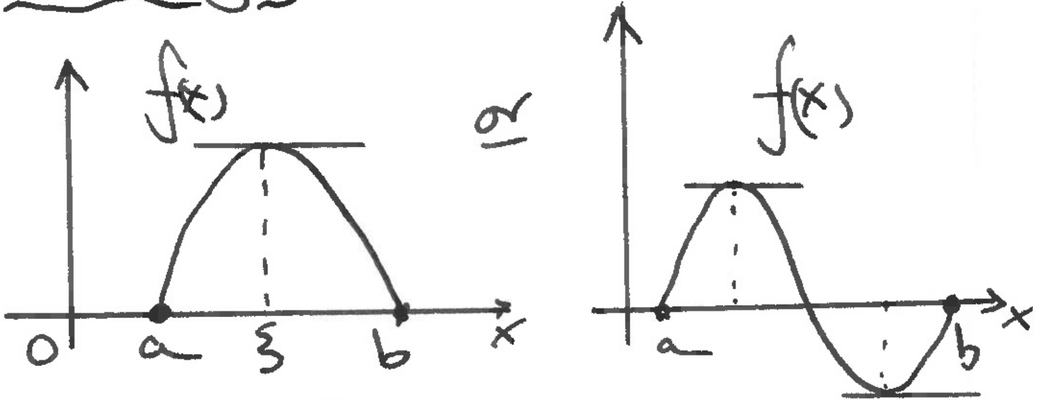
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)\,(x-x_0)^n}{n!}$$

The above is NOT a formal proof, since it does rely on various assumptions about e.g. differentiating an infinite series. There is more to come about series in Chapter 6. For now, just consider the successive terms of the Taylor series as improvements to higher and higher order to a local polynomial approximation

$\Rightarrow$ function value, linear poly, quadratic, cubic, ...

How a proof would go:

Firstly

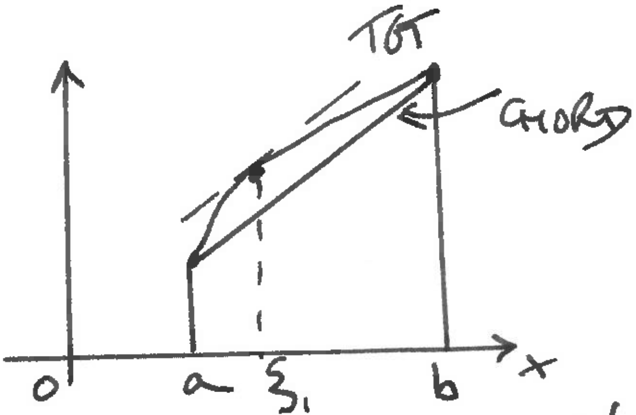


$f(a) = 0 = f(b)$ $\quad f(x)$ CONTINUOUS and DIFFERENTIABLE on (a,b)

$\Rightarrow$ there exists $a < \xi < b$ such that $f'(\xi) = 0$

ROLLE'S THEOREM (maybe more than one such ξ).

Then



there must be ξ on (a,b) such that

$$f'(\xi_1) = \frac{f(b) - f(a)}{b - a}.$$

DERIVATIVE somewhere = GRADIENT OF CHORD

i.e. $$f(b) = f(a) + f'(\xi_1)(b-a).$$

THE (FIRST) MEAN VALUE THEOREM

Now putting $b \equiv x$ and $a \equiv x_0$ we get

$$f(x) = f(x_0) + f'(\xi_1)(x - x_0) \quad \text{with } x_0 < \xi_1 < x$$

[COMPARE 1st 2 TERMS OF TAYLOR SERIES.

We can extend the MEAN VALUE THEOREM (no technicalities given here!) to find

$$f(x) \underset{\nearrow}{=} f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots$$

EQUALITY!

$$\dots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \underset{\uparrow}{R_n}$$

'REMAINDER' TERM

with $R_n = \frac{(x-x_0)^n}{n!} f^{(n)}(\xi_n)$ with $x_0 < \xi_n < x$.

TAYLOR'S THEOREM 1712

The utility of this result depends on: [GREGORY 1670

(a) $f(x)$ having the requisite number of derivatives

(b) the behavior of the REMAINDER TERM as n increases!

ex (i) $f(x) = 1 + x^2$ about $x_0 = 1$.

Here $f'(x) = 2x$, $f''(x) = 2$, $f'''(x) \equiv 0$

and $f(x) = 2 + 2(x-1) + \frac{2}{2!}(x-1)^2$.

the TAYLOR SERIES here TERMINATES.

ex (ii) $f(x) = \sin x$ about $x_0 = \frac{\pi}{4}$.

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, etc.

So $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{(x-\pi/4)^2}{2!} + \dots$

This does not terminate, but performs well for small $\left|x - \frac{\pi}{4}\right|$ values.

[of course the series must be equivalent to

$$\sin\left[\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\right] \equiv \sin\frac{\pi}{4}\cos\left(x-\frac{\pi}{4}\right) + \cos\frac{\pi}{4}\sin\left(x - \frac{\pi}{4}\right)$$

and this is indeed the case!].

(5.2) MACLAURIN SERIES

Simply put, the Maclaurin series of a function $f(x)$ is the Taylor series about the origin – write $x_0 = 0$ and $h = x$ in the (5.1) result.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$ MACLAURIN 1742

with the same remarks about truncation, remainder terms for practical utility.

ex(i) $f(x) = e^x$.

Here $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$.

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\equiv \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

ex(ii) $f(x) = \sin x \Rightarrow f(0) = 0$.

$f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$.

$f^{(1)}(0) = 1$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = -1$. ....

$$\Rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

ex(iii) $f(x) = \cos x$

in similar fashion

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\equiv \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

[Evidently from (i),(ii),(iii) we get

$$e^{ix} = \cos x + i \sin x$$ $(i^2 = -1)$ EULER 1749]

ex (iv) $f(x) = \frac{1}{1-x} \Rightarrow f^{(n)}(x) = n!(1-x)^{-n-1}$

$$f^{(n)}(0) = n!$$

$$\Rightarrow f(x) = 1 + x + x^2 + \cdots$$

$$\equiv \sum_{n=0}^{\infty} x^n .$$

ex (v) $f(x) = \ln(1+x)$

$$\Rightarrow f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\text{e.g. } \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

NOTES:

(a) We will look at the matter of CONVERGENCE/DIVERGENCE of series in Chapter 6.

However there is an evident need for care (!)

e.g. in (iv) above $f(-1) = \frac{1}{2}$ (TRUE!)

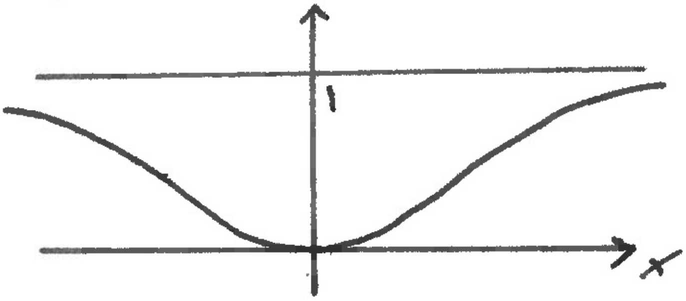
but our series gives

$1 - 1 + 1 - 1 + 1 \ldots\ldots$ which is certainly not convergent!

– to anything, let alone $1/2$.

(b) We normally can rely on our expansion parameter → $h = x - x_0$ (Taylor), x (Maclaurin) being small enough for our expansion to be useful, but the REMAINDER term can contain more information than we might anticipate!

e.g. $f(x) = e^{-1/x^2}$



Here we can show that $f^{(n)}(0) = 0$, so that

$$f(x) = 0 + 0x + 0x^2 + \ldots 0x^{n-1} + R_n .$$

is our Maclaurin expansion.

The function is VERY flat at $x = 0$ and is contained wholly in R_n!

(5.3) L'HÔPITAL'S RULE (REVISITED)

When we considered in Chapter 1 (in (1.5))

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \text{with} \quad f(x_0) = 0, \quad g(x_0) = 0 \qquad \left[\frac{'0'}{0}\right]$$

we used L'Hôpital's rule.

What was the justification for this?

If we can assume that $f(x)$ and $g(x)$ each have a Taylor series expansion in the neighborhood of x_0 we can write the above limit as

VITAL!

$$\lim_{h\to 0}\frac{f(x_0+h)}{g(x_0+h)}$$

$$= \lim_{h\to 0}\left[\frac{\cancel{f(x_0)}+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\dots}{\cancel{g(x_0)}+hg'(x_0)+\frac{h^2}{2!}g''(x_0)+\dots}\right]$$

if $f(x_0)=0=g(x_0)$

$$= \lim_{h\to 0}\left[\frac{f'(x_0)+\frac{h}{2}f''(x_0)+\dots}{g'(x_0)+\frac{h}{2}g''(x_0)+\dots}\right]$$

$$= \frac{f'(x_0)}{g'(x_0)}$$

if at least one of numerator and denominator is non-zero.

If $f'(x_0)=0=g'(x_0)$ then we go to the next terms and get $\frac{f''(x_0)}{g''(x_0)}$ – and so on!

[EVIDENTLY if $\frac{f(x_0)}{g(x_0)}$ is of form $\frac{\infty}{\infty}$ THIS justification FAILS. The RULE does work for SOME LIMITS!]

(5.4) DOUBLE ~~TAYLOR~~ TAYLOR SERIES

We now consider a function $u(x,y)$ of two independent variables x, y in the neighbourhood of (x_0, y_0). That is we seek an expansion in powers of $h = x - x_0$, $k = y - y_0$.

$$u(x,y) = u(x_0+h, y_0+k)$$

TREAT x FIRST!

$$= u(x_0, y_0+k) + h\frac{\partial}{\partial x}u(x_0,y_0+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}u(x_0,y_0+k) + \ldots\ldots$$

THEN TREAT y!

$$u(x_0,y_0) + k\frac{\partial u}{\partial y}(x_0,y_0) + \frac{k^2}{2!}\frac{\partial^2 u}{\partial y^2}(x_0,y_0) + \ldots\ldots$$

$$h\left[\frac{\partial u}{\partial x}(x_0,y_0) + k\frac{\partial^2 u}{\partial y \partial x}(x_0,y_0) + \ldots\ldots\right]$$

$$\frac{h^2}{2!}\left[\frac{\partial^2 u}{\partial x^2}(x_0,y_0) + k\frac{\partial^3 u}{\partial y \partial x^2}(x_0,y_0) + \ldots\right]$$

NOW COLLECT TERMS →

SECOND ORDER

$$= u_0 + \left[h\left(\frac{\partial u}{\partial x}\right)_0 + k\left(\frac{\partial u}{\partial y}\right)_0\right] + \frac{1}{2!}\left[h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + 2hk\left(\frac{\partial^2 u}{\partial x \partial y}\right)_0 + k^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0\right]$$

FIRST ORDER

$+ \ldots\ldots$

SUBSCRIPT ≡ EVALUATION AT $(x_0 y_0)$

There is a straightforward pattern to these terms, where we are assuming that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ etc.

We can write $D \equiv h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$.

$$\Rightarrow u(x_0+h, y_0+k) = u_0 + Du_0 + \frac{D^2 u_0}{2!} + \frac{D^3 u_0}{3!} + \dots$$

so our pattern involves the BINOMIAL COEFFICIENTS explicitly

i.e. $D^3 u_0 \equiv h^3 \frac{\partial^3 u}{\partial x^3}_0 + 3h^2k\frac{\partial^3 u}{\partial x^2 \partial y}_0 + 3hk^2\frac{\partial^3 u_0}{\partial x \partial y^2} + k^3\frac{\partial^3 u_0}{\partial y^3}$.

etc.

ex/ $u(x,y) = e^{2x-y}$ $x_0 = 0 = y_0$

$x = 0+h,\ y = 0+k$

$u_0 = u(0,0) = 1.$

$\frac{\partial u}{\partial x} = 2e^{2x-y},\ \frac{\partial u}{\partial y} = -e^{2x-y} \Rightarrow \left(\frac{\partial u}{\partial x}\right)_0 = 2,\ \left(\frac{\partial u}{\partial y}\right)_0 = -1$

$\frac{\partial^2 u}{\partial x^2} = 4e^{2x-y},\ \frac{\partial^2 u}{\partial x \partial y} = -2e^{2x-y},\ \frac{\partial^2 u}{\partial y^2} = e^{2x-y}$

$$\Rightarrow \left(\frac{\partial^2 u}{\partial x^2}\right)_0 = 4 \quad \left(\frac{\partial^2 u}{\partial x \partial y}\right)_0 = -2,\ \left(\frac{\partial^2 u}{\partial y^2}\right)_0 = 1.$$

$$\text{So } e^{2x-y} \equiv e^{2h-k} = 1 + (2h-k) + \frac{1}{2!}\left[4h^2 - \underset{\uparrow}{4}hk + k^2\right] + \dots$$

CHECK! $e^{2h-k} \equiv 1 + (2h-k) + \frac{1}{2!}(2h-k)^2 + \dots$ OK!

(5.5) STATIONARY POINTS (REVISITED)

We considered stationary points of a function $u(x,y)$ of two independent variables in (4.7) of Chapter 4. We are now in a position to justify the criterion used there to determine their character.

Consider a stationary point (x_0, y_0) where we have (of course) $\left(\frac{\partial u}{\partial x}\right)_0 = 0 = \left(\frac{\partial u}{\partial y}\right)_0$.

Write our Taylor expansion for $u(x,y)$ about (x_0, y_0):

zero at stationary point

$$u(x_0+h, y_0+k) = u_0 + \cancel{h\left(\frac{\partial u}{\partial x}\right)_0} + \cancel{k\left(\frac{\partial u}{\partial y}\right)_0} + \frac{1}{2}\left(Ah^2 + 2Bhk + Ck^2\right) + \ldots$$

where $A = \left(\frac{\partial^2 u}{\partial x^2}\right)_0$, $B = \left(\frac{\partial^2 u}{\partial x \partial y}\right)_0$, $C = \left(\frac{\partial^2 u}{\partial y^2}\right)_0$.

So

$$\delta u = u(x_0+h, y_0+k) - u(x_0, y_0)$$

$$= \frac{1}{2}\left(Ah^2 + 2Bhk + Ck^2\right) + \ldots$$

NOTE: A, B, C ALL ZERO – need Higher Terms!

Evidently :

$\delta u > 0$ for ANY small h, k then $u(x_0, y_0)$ is a (LOCAL) MINIMUM

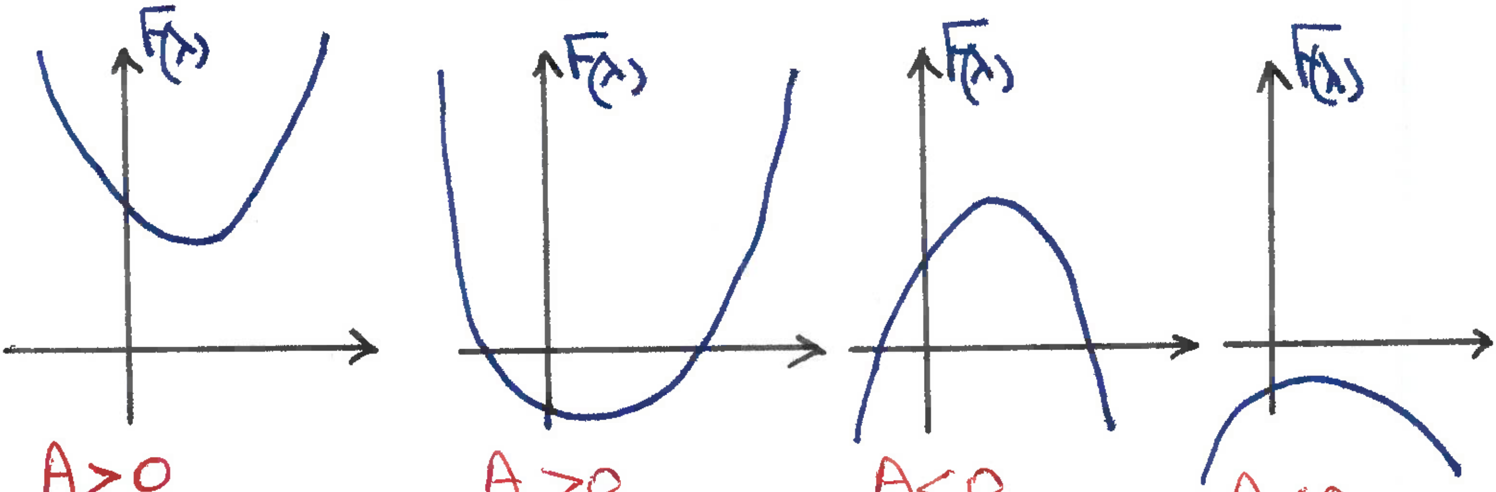
$\delta u < 0$ for ANY small h, k then $u(x_0, y_0)$ is a (LOCAL) MAXIMUM

$\left[\begin{array}{l}\delta u > 0 \text{ for SOME } h, k \\ \delta u < 0 \text{ for SOME } h, k\end{array}\right.$ then $u(x_0, y_0)$ is a SADDLE POINT

How can we tell ?!

Easiest way is via e.g. $\delta u = \frac{1}{2}k^2\left[A\left(\frac{h}{k}\right)^2 + 2B\left(\frac{h}{k}\right) + C\right] + \dots$

and consider $F(\lambda) \equiv A\lambda^2 + 2B\lambda + C$ (say).



$A > 0$	$A > 0$	$A < 0$	$A < 0$
$B^2 - AC < 0$	$B^2 - AC > 0$	$B^2 - AC > 0$	$B^2 - AC < 0$
$\delta u > 0$ for all λ	δu sign depends on λ	δu sign depends on λ	$\delta u < 0$ for all λ.
F HAS COMPLEX ZEROS	F HAS REAL ZEROS		F HAS COMPLEX ZEROS
LOCAL MINIMUM OF u	SADDLE POINT	SADDLE POINT	LOCAL MAXIMUM

CHAPTER 6

SERIES AND CONVERGENCE

(6.1) INFINITE SUMS
(6.2) MORE DEFINITIONS AND THEOREMS
(6.3) CONVERGENCE TESTS
(6.4) RADIUS OF CONVERGENCE OF TAYLOR / MACLAURIN SERIES
(6.5)* THE MYSTERIOUS ZETA FUNCTION

(6.1) INFINITE SUMS

We consider here the infinite sum

$$S = \sum_{n=0}^{\infty} u_n$$

What does this mean?

The approach we take is to consider a TRUNCATION to

$$S_N = \sum_{n=0}^{N} u_n .$$

and to examine whether $\lim_{N\to\infty} S_N$ exists, that is to say, has a finite value.

If so, we say the series S CONVERGES – if not then the series DIVERGES, and the infinite sum then has no meaning.

We note that <u>divergence</u> can involve $|S_N|$ increasing without bound as $N \to \infty$ or that S_N just does not approach a finite limit e.g. it could just oscillate.

eg(i) $\sum_{n=0}^{\infty}(-1)^n$ oscillates $S_0 = 1$, $S_1 = 0$, $S_2 = 1$, etc.

ex (ii) The geometric series

$$S_N = 1 + x + x^2 + \dots + x^N$$

Evidently $xS_N = x + x^2 + \dots + x^N + x^{N+1}$

$$\Rightarrow S_N(1-x) = 1 - x^{N+1}$$

and $S_N = \dfrac{1-x^{N+1}}{1-x}$

So $S = \lim_{N\to\infty} S_N$ exists ONLY for $|x|<1$.

For $|x| \geqslant 1$ the series DIVERGES.

Hence our Maclaurin series expansion of

$f(x) = 1 + x + x^2 + \dots$ is ONLY valid for $|x|<1$.

ex (iii) Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (THE HARMONIC SERIES)

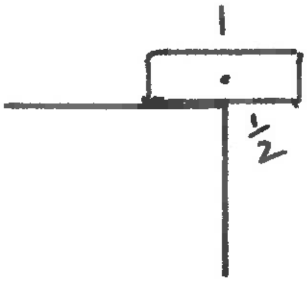
This series DIVERGES.

Proof $1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$

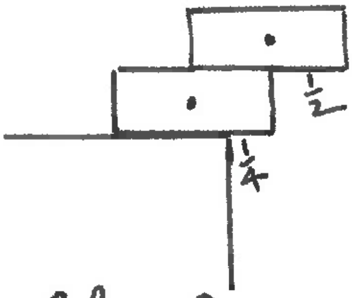
$$\Rightarrow \sum \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

So $\displaystyle\sum_{1}^{N} \frac{1}{n} \to \infty$ as $N \to \infty$.

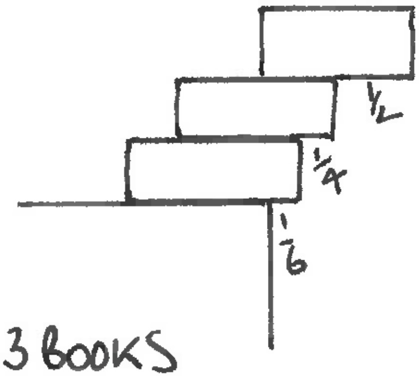
'Book' stacking challenge :



1 BOOK

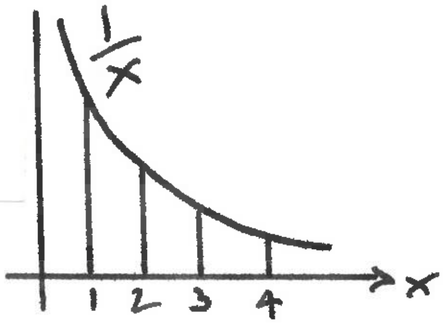


2 BOOKS



3 BOOKS

OVERHANG $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$ DIVERGES



By considering upper and lower sums we can estimate the growth of the harmonic series sum.

$$\Rightarrow \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{n} - \ln N\right] = \gamma \equiv 0.577215\dots$$

1735 1790

EULER-MASCHERONI

N=1000 $\ln N = 6.9077..$

N=1000000 $\ln N = 13.815..$

not even known to be IRRATIONAL

if it is then DENOMINATOR $> 10^{244663}$

(6.2) MORE DEFINITIONS AND THEOREMS

An infinite series $S = \sum_{n=0}^{\infty} u_n$ is said to be ABSOLUTELY CONVERGENT if $\sum_{n=0}^{\infty} |u_n|$ is CONVERGENT.

This is useful because of the theorem:

<u>ABSOLUTE CONVERGENCE</u> $\Longrightarrow$ <u>CONVERGENCE</u>

and can be applied when some u_n may be negative <u>or</u> complex.

<u>ex(i)</u> $S_{(1)} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + - - -$

<u>is</u> absolutely convergent because

$$S_{(2)} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

geometric progression

$$= \frac{1}{2} \Big/ 1 - \frac{1}{2} = 1.$$

Here the convergence of $S_{(2)} \Longrightarrow$ convergence of $S_{(1)}$

[in fact it is easy to see $S_{(1)} = \dfrac{\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = \frac{1}{3}$.

If $\sum_{n=0}^{\infty} u_n$ converges and $\sum_{n=0}^{\infty} |u_n|$ <u>does not</u>

then we say $\sum_{n=0}^{\infty} u_n$ is <u>CONDITIONALLY CONVERGENT</u>

<u>ex(ii)</u> $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - - -$ is <u>conditionally convergent</u> since it converges (to ln 2), but of course $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges.

(6.3) CONVERGENCE TESTS

Four simple ones here (!)

$$\sum_{n=0}^{\infty} u_n$$

(a) NECESSARY but not SUFFICIENT for convergence is that

$$\lim_{n\to\infty} u_n = 0 .$$

If this limit is NOT zero then series DIVERGES

If it is zero then the series may still DIVERGE e.g. $u_n = \frac{1}{n}$.

ex (i) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$ is DIVERGENT since $\cos\left(\frac{1}{n^2}\right) \to 1$ as $n \to \infty$.

(b) COMPARISON TEST

If u_n is given and we can find a converging series $\sum_{n=0}^{\infty} b_n$ with b_n non-negative such that $|u_n| \leqslant b_n$ for all n.

then $\sum u_n$ is absolutely convergent.

— and so is convergent.

Similarly, if we can find a diverging series $\sum_{n=0}^{\infty} b_n$ with b_n non-negative such that $u_n \geq b_n$ for all n, then $\sum u_n$ is divergent.

ex(ii) $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent because $\frac{1}{n^{1/2}} \geq \frac{1}{n}$ for all n and we know $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(c) ALTERNATING SERIES – LEIBNIZ TEST

If we have $\sum_{n=0}^{\infty} (-1)^n a_n$ with

(i) positive a_n

(ii) a_n decreasing $a_{n+1} \leq a_n$.

(iii) $a_n \to 0$ as $n \to \infty$

then our series converges.

ex(iii) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is convergent

– its sum is actually $\tan^{-1}(1) = \pi/4$

[it should be noted that this convergence is VERY slow – there are much better ways of approximating π....]

A natural question to ask is whether condition (ii) is actually needed — as long as particularly (iii) is satisfied.

That it is so is given by

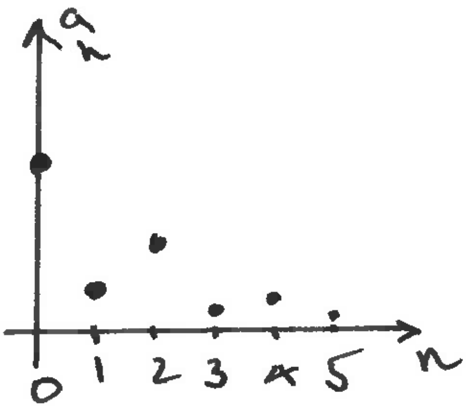
$$S = \underset{a_0}{\frac{1}{\sqrt{2}-1}} - \underset{a_1}{\frac{1}{\sqrt{2}+1}} + \underset{a_2}{\frac{1}{\sqrt{3}-1}} - \underset{a_3}{\frac{1}{\sqrt{3}+1}} + \underset{a_4}{\frac{1}{\sqrt{4}-1}} - \underset{a_5}{\frac{1}{\sqrt{4}+1}} + \dots$$

CONDITIONS

(i) ✓ (ii) × (iii) ✓

$$\frac{1}{\sqrt{4}-1} > \frac{1}{\sqrt{3}+1} \text{ etc}$$

So FAIL for the CONVERGENCE TESTS



BUT of course we have

$$S_{\substack{2k\\ \text{terms}}} = \frac{2}{(\sqrt{2})^2-1} + \frac{2}{(\sqrt{3})^2-1} + \frac{2}{(\sqrt{4})^2-1} + \dots$$

$$= 2\left[1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{k}\right]$$

$\longrightarrow \infty$ as $k \to \infty$ So that S DIVERGES!

(d) THE RATIO TEST

For $\sum_{n=0}^{\infty} u_n$ we suppose $u_n \neq 0$ for all n.

Define $L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$

Then if $L < 1 \Rightarrow$ ABSOLUTE CONVERGENCE $\Rightarrow$ CONVERGENCE

$L > 1 \Rightarrow$ DIVERGENCE

If $L = 1$ we have NOT PROVEN $\equiv$ WE DON'T KNOW!
- at least without further work!

(6.4) RADIUS OF CONVERGENCE OF TAYLOR/MACLAURIN SERIES

The ratio test allows us to determine the range of convergence of Taylor/Maclaurin series.

ex(i) $f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\equiv \sum_{n=0}^{\infty} u_n .$

So $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \; \frac{n!}{x^n}\right| = \frac{|x|}{(n+1)}$.

and we have $L = \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$

$$= \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0 \text{ for all fixed } x.$$

Since $L < 1$, the Maclaurin series for e^x converges for all x [Real and complex!

note RADIUS OF CONVERGENCE

ex (ii) $f(x) = \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

here $\left|\frac{u_{n+1}}{u_n}\right| = \left|-\frac{x^{2n+3}}{(2n+3)!} \; \frac{(2n+1)!}{x^{2n+1}}\right|$

$$= \frac{x^2}{(2n+3)(2n+2)}$$

Thus $L = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0$ for all fixed x.

and again our Maclaurin series for $\sin x$ converges for all x [Real and complex!

note RADIUS OF CONVERGENCE

ex (iii) $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$L = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = |x|$ So convergent for $|x| < 1$

RADIUS OF CONVERGENCE 1

ex (iv) $f(x) = \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)}$

$$L = \lim_{n\to\infty} \left| (-1) x \frac{(n+1)}{(n+2)} \right| = |x|.$$

So convergence if $|x| < 1$.

RADIUS OF CONVERGENCE 1.

WITHIN the 'RADIUS OF CONVERGENCE' we have CONVERGENCE.

AT that value we have NOT PROVEN ≡ DONT KNOW!

ex (v) $\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)} \frac{n}{x^n} \right| = |x|.$$

RADIUS OF CONVERGENCE 1

So the series $\begin{cases} \text{CONVERGES if } |x| < 1. \\ \text{DIVERGES if } |x| > 1. \end{cases}$

At $x = +1$ we have $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ldots$ DIVERGES.

At $x = -1$ we have $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \ldots$ CONVERGES $\equiv -\ln 2$.

[The word RADIUS in our account refers to that of a CIRCLE OF CONVERGENCE in the complex plane].

(6.5) THE MYSTERIOUS ZETA FUNCTION *

The zeta function of RIEMANN (1859) is the function

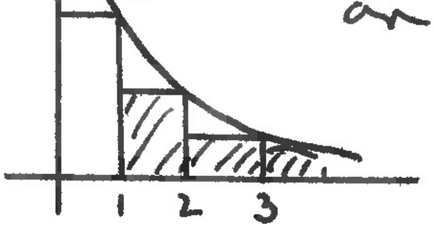
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which has attained a mysterious, indeed almost mythical status.

$\zeta(1)$ is of course the harmonic series which diverges

$\zeta(s)$ can be shown to converge: $\operatorname{Re} s > 1$

[NOT by the tests in (6.3) above, which are inconclusive, but by being bounded by an integral



The 'tail' is bounded above by an integral that is convergent

i.e. $\frac{1}{2^s} + \frac{1}{3^s} + \cdots\cdots$

$$< \int_1^{\infty} \frac{1}{x^s}\,dx$$

$$= \left[\frac{-1}{(s-1)}\frac{1}{x^{s-1}}\right]_1^{\infty} = \frac{1}{s-1}.$$

This argument works for s complex too.

$\therefore \zeta(s) < 1 + \frac{1}{s-1}$

We can find

THE BASEL PROBL

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} = 1.6449\dots$$

EULER 1735

not easily by elementary means

$$\zeta(4) = \pi^4/90 \quad \text{etc}$$

$$\zeta(3) = 1.2020569\dots$$

APÉRY

IRRATIONAL 1979

THE RIEMANN HYPOTHESIS is that ALL non-trivial ZEROS of the ζ function have real part $\frac{1}{2}$ i.e. $s_0 = \frac{1}{2} \pm it$.

[this is known as the CRITICAL LINE in the complex s plane].

$\frac{1}{2} \pm 14.135i$, $\frac{1}{2} \pm 21.022i$, $\frac{1}{2} \pm 25.011i$, ...

A VERY large number have been computed, but so far there is NO PROOF..... ! [10 TRILLION]

MYSTERY

Connections to (a) DISTRIBUTION OF PRIMES

(b) STATISTICAL MECHANICS

(c) QUANTUM CHAOS

WHY? HOW?