Lecture 1 Current, Potential Difference and Power

1.1 Conduction in Metals

The atoms in a copper wire have a regular arrangement known as **face-centred cubic**. This can be visualised as a copper atom at each corner of a cube, plus one at the centre of each face (it's worth sketching this). The distance between corners of the cube is just 0.361 nm. Each copper atom has 29 electrons arranged in orbits or shells; the first 28 electrons occupy complete shells and are strongly-bound to the nucleus. The outermost valence electron is not so strongly-bound (it is shielded from the nucleus by all the other electrons). In copper, the atoms are so close together that the outer valence orbits overlap and so the valence electrons are not strictly bound to an individual atom and are able to wander freely within the solid. We can think of this as a rigidly-bound structure of positive ions infused with a highly-mobile 'liquid' of **conduction electrons**. Electrical current can be thought of as a fluid-like flow of negative electronic charge, rather like water through a pipe. Note that, like an individual copper atom, this piece of copper wire is electrically neutral since there are an equal number of positive and negative charges. Though the conduction electrons are highly mobile, Coulomb forces prevent them from 'bunching-up', since they all repel each-other and try to find an equilibrium where they are all, on-average, equally spaced. Conceptually, we can imagine forcing some additional electrons into the right-hand side of a short length of wire. This will result in the same number of electrons being pushed out on the left side. On a larger scale, electrons moving from right to left constitute a **current**.

It is important to remember that, even as a current flows, the copper remains overall electrically neutral; the copper 'ions' will be balanced by an equal number of electrons.

1.2 Conventional Current

Current is rate of flow of charge. Looking at the left-hand end of the wire, we could measure the electrons emerging per second, multiply by e and we would have the electron current. It's important to note that the positive charges (the copper 'ions') have a fixed position. However, it is a *convention* in electronics to think of current as a flow of positive charge. Here, this would be equal and opposite to the flow of conduction electrons. An electron current flowing right-to-left is exactly equivalent to a **conventional current** of the same magnitude flowing left-to-right.

> The use of conventional current is ubiquitous in electronics and we have to remember that currents in electronics are taken to be the flow of notional positive charges. Rarely, you may need to think about what the electrons are actually doing, and always it is the opposite direction to conventional current.

It is conventional to use the letter I to represent current and q for charge so we can write

$$I = \frac{dq}{dt}$$

The unit of current is the ampère, or **amp** for short. It's symbol is 'A'. When one coulomb of charge flows past a point in one second then the current is one amp

$$1\,\mathrm{A} = 1\,\mathrm{C/s}$$

The amp is an SI base-unit and the definition is based on the force associated with the magnetic field generated by moving charges. You will cover this further in the E&M course, so we won't consider this further.

A positive conventional current measured in a left-to-right sense is *the same as* a negative conventional current measured right-to-left.

1.3 Drift Speed

Consider a short section of wire with length dL, cross-sectional area A, and carrying a steady current I. The wire is made from a metal where the conduction electron density is n, defined as the number of electrons per unit volume. Hence, the conduction electron charge in this piece of wire is $dq = nAdLe$ and assuming it moves right to left at an average **drift speed** v_d then the time taken for all this charge to leave the volume is $dt = dL/v_d$. Using $I = dq/dt$ we can re-arrange to get

$$v_d = \frac{I}{nAe} \tag{1.1}$$

Since a good conductor has a very large value for n, the drift speed is typically very small. For example, copper has a conduction electron density $n = 8.45 \times 10^{28}\,\mathrm{m^{-3}}$, so a wire with a cross-sectional area of just $1\,\mathrm{mm}^2$ where the electrons are moving right-to-left with an average drift speed of just $1\,\mathrm{mms^{-1}}$, would carry a current just over 13 amps! (exceeding the maximum current-rating for a UK mains-plug).

Note that this right-to-left electron current would be properly described as a positive conventional current $I = 13.5\,\mathrm{A}$ defined in a left-to-right sense.

Sense of the Current Measurement

To properly describe a current we need to define both amplitude and direction (though current itself is not a vector). This is usually done with an arrow on a circuit diagram which defines the direction associated with a positive conventional current. In the previous example, if we draw the current arrow in the left-to-right sense, then the current is $I = 13.5\,\mathrm{A}$. We could instead draw the arrow in the opposite sense, in which case the result would be $I = -13.5\,\mathrm{A}$.

Resistance

In an ideal world, the electrons would flow without hindrance. In such a 'superconductor', it would be impossible to maintain an electric field since the electrons would instantly re-organise their locations to cancel-out any field we may try to apply. However, in the real-world, our electronics is typically operated at room temperature which means that the ionic-lattice and the electrons have a random thermal motion. For the highly mobile electrons, this can be something like $10^6\,\mathrm{m/s}$, which is about ten orders of magnitude greater than the drift speed. We have to think of the drift speed as being an extremely slow movement of the *average* position of the electrons, superimposed on a much faster random motion. The electrons keep bumping into the atomic ions (also in random thermal motion about their lattice location). Each time an electron impacts an ion it shares some of it's small extra energy; the electron slows down and the ion heats up. This means that, in order to maintain a constant current, we need to keep putting energy into the system. This we do by maintaining an electric field across the conductor such that the electrons always feel some small acceleration to maintain the average drift speed.

The drift speed of electrons is analogous to the terminal velocity of an object falling in air. The object experiences a constant force due to gravity, but this is balanced by an equal and opposite force due to the air-drag. The object loses potential energy without gaining kinetic energy, since energy is dissipated as heat.

1.4 Potential Difference

Potential energy.

Any useful circuit will have charges moving under the influence of electric fields, and as a consequence the charges will have some potential energy as a result of their location within the field. The electric field is a conservative field, so the energy U associated with a charge q is a function of just the charge's location; it doesn't matter what path the charge took to get there. U is the energy, in joules, required to bring that charge from infinity to it's location within the circuit. If the charge moves through any electric field, then it's energy U will change. In this respect, electrical PE is quite analogous to gravitational PE. In the same way that the energy associated with a mass rises as it moves to a higher gravitational potential (or falls if it moves to a lower gravitational potential), so a (positive) charge gains/loses electrical PE as it moves to a higher/lower potential.

Electrical potential.

Electrical potential is defined as electrical potential energy *per unit charge*

$$V = \frac{U}{q} \tag{1.2}$$

with the dimensions of energy per charge, or **volts** (symbol 'V'). Consequently, one volt is one joule per coulomb.

$$1\,\mathrm{V} = \frac{1\,\mathrm{J}}{\mathrm{C}}$$

Like potential energy U, the electrical potential V is an absolute quantity. However, rather than calculate the absolute value, it's usually sufficient to know just the *change* in these quantities as a result of charge moving between two locations.

Potential difference.

Often abbreviated to PD, the electrical potential difference is the change in electrical potential between two well-defined locations in the circuit.

$$\Delta V = \frac{\Delta U}{q}$$

Typically, we might define two points a and b in a circuit, in which case we would describe the electrical potential difference between the two as

$$V_{ab} = V_a - V_b$$

If a is at the higher electrical potential then V_{ab} will be positive. Of course, it is also possible to measure the potential difference in the opposite sense, in which case

$$V_{ba} = -V_{ab}$$

The interpretation of electrical potential difference is the energy, per charge, required to move between two points in the circuit. Potential difference, as it is a difference in potentials, also has units of volts.

Voltage.

Electrical potential (an absolute quantity) and *electrical potential difference* (a relative quantity) are well-defined. However, in electronics, we will typically come across the term **voltage**. Unfortunately, voltage is rather loosely-defined and can refer to either the electric potential at a point in a circuit *or* the electrical potential difference between two points. For example, we might see written 'The voltage at point a is V_a' or 'The voltage across ab is V_{ab}'. In the first usage, it is clear that the quantity V_a refers to the absolute value of the potential at some point a, while the second implies that V_{ab} is the potential difference between two points a and b. Usually, the context makes it clear what is meant. However, in this course we will try to use the correct terms, i.e. 'potential' or 'potential difference' (or p.d.), in order to be clear. Nevertheless, the use of the term 'voltage' is so common that it is impossible to avoid and we just have to get used to it. So long as we understand the context, this should not cause any problems.

1.5 Ohm's Law

Returning to our idea of a wire with cross-sectional area A we can define the **current density** J as being the current per unit area at any point inside the conductor.

$$J = \frac{I}{A} = nv_d e \tag{1.3}$$

From the rather simplified conceptual description of resistance in section 1.3, it should be intuitive that $J \propto E$, the electric field inside the conductor. In fact, this is experimentally true for a wide range of 'ohmic' conductors such as metals. The constant of proportionality is the **conductivity** but it's more usual to use it's inverse ρ, known as the **resistivity** such that **ohm's law** is

$$J = \frac{E}{\rho} \tag{1.4}$$

The electric field is, in general, the derivative of the **electric potential** with respect to distance. Again, you will cover this in great detail during E&M, so for now it will be sufficient to make some assumptions: if the conductor has length L and uniform cross-sectional area A then the electric field (and current density) will be uniform throughout, and we can simply write

$$E = \frac{V}{L} \tag{1.5}$$

where V is the difference in electric potential between the two ends of the wire. Combining (1.3), (1.4) and (1.5) gives

$$\frac{V}{L} = \frac{I\rho}{A}$$

The ratio V/I is *defined* to be the **resistance** of the conductor

$$R = \frac{V}{I} = \frac{\rho L}{A} \tag{1.6}$$

Note that while the resistivity is an intrinsic property of the ohmic material, the resistance depends on the dimensions of the material. Hence, while two copper wires may have the same resistivity, the thick, short wire has a lower resistance than the thin, long one.

The unit of resistance is the ohm (Ω). A wire with a resistance $1\,\Omega$ carrying a current of $1\,$A requires a difference in electric potential of 1 volt across its ends.

Temperature-dependence

The resistivity of the conductor ρ has units of $\Omega\,$m. By the description of resistance in section 1.3, it should be clear that resistivity will be a function of temperature, as increasing the thermal motion of the ionic lattice will further impede electron flow. This is modelled as being linear for practical temperatures encountered in real-world electronics and we can write that the resistivity at temperature T is given by

$$\rho(T) = \rho_0[1 + \alpha(T - T_0)] \tag{1.7}$$

where ρ_0 is the resistivity at T_0 (typically 20 °C) and α is the dimensionless temperature coefficient. Table 1.1 gives some values.

Table 1.1: Resistivities of Conductors

Conductor	ρ_0 (Ωm) at 20 °C	α (1/°C)
Copper	1.7×10^{-8}	4.0×10^{-3}
Aluminium	2.7×10^{-8}	3.9×10^{-3}
Tungsten	5.6×10^{-8}	4.5×10^{-3}
Carbon	7.8×10^{-6}	-5×10^{-4}

Interestingly, carbon (a non-metal) has a negative temperature coefficient, since heating releases additional conduction electrons.

1.6 Resistors and Wires

Resistors.

For many circuit applications we need a component with a known value of resistance called a Resistor. These are mostly made by depositing a thin film of metal (or carbon) on the outside of an insulating cylinder, between two metal leads. They are available in values from fractions of an ohm up to tens of MΩ. For most circuit problems we will assume that our resistors are 'ideal', that is to say their value is fixed. In reality, the resistivity of the material will change with temperature. We need to be aware of this, however for the rest of this course, unless otherwise stated, resistances are assumed to be fixed-values.

Ideal wires.

A 60 W mains bulb has a thin coiled tungsten element just a few cm long. It has a resistance (at operating temperature) of about 960 Ω. Contrastingly, a very-long 100 m length of copper cable used in lighting circuits has a resistance of only half an ohm. Consequently, the wire used in electrical circuits is assumed to have negligible resistance when solving circuit problems.

> For the rest of this course, we will assume that all wires - unless specifically stated otherwise - are **ideal wires** with zero resistance. This means that, according to equation (1.6) there will be zero potential difference across our wires. Equivalently, all points along the wire will be at the same electrical potential.

1.7 Ideal Circuit

The properties of the **ideal circuit** are illustrated in figure 1.1.

Ideal voltage source This is a voltage source (section 1.9) which provides an electromotive force $\mathcal{E}$ (section 1.8). Since it is 'ideal' it has zero resistance and it represents an everlasting source of energy.

Ideal resistor R which is fixed in value (section 1.6) and doesn't vary with temperature ($\alpha = 0$).

Ideal wires with zero resistance (section 1.6) such that the electrical potential is the same at all points along each length of wire.

Ground establishes the zero of electrical potential (section 1.10).

With these properties we can derive

- $V_b = 0$ volts electrical potential, since point b is connected to ground.
- $V_{ab} = \mathcal{E}$ volts electrical potential difference, since the action of the voltage source is to raise the electrical potential by $\mathcal{E}$ volts across its terminals.
- $V_a = \mathcal{E}$ volts electrical potential.
- $V_{ac} = 0$ since the wires are ideal.
- $V_c = \mathcal{E}$ volts electrical potential.
- $V_d = 0$ volts electrical potential, since point d is also connected to ground (ideal wires).
- $V_{cd} = \mathcal{E}$ volts electrical potential difference.
- $V_{dc} = -\mathcal{E}$ volts electrical potential difference.
- $I_{cd} = \mathcal{E}/R$ by ohm's law (section 1.5).
- $I_{dc} = -I_{cd} = \mathcal{E}/R$

For practical circuit diagrams we usually omit the letters a, b etc. since they clutter the diagram. The potential difference across the resistor is indicated by V_R with the arrow indicating the sense of the potential difference by convention:

$$V_R = V_{cd} = V_c - V_d$$

I_{cd} is the conventional current *through* R in the sense *from c to d*. The current in the circuit is indicated by the arrow labelled I, with the direction of the arrow showing the direction of conventional current. In this case, the current is shown as circulating in a clockwise sense. Since the current is the same everywhere in the circuit (charge is neither created nor destroyed, section 1.8)

$$I = I_{ac} = I_{cd} = I_{db} = I_{ba}$$

In figure 1.1 we have drawn a positive voltage source with EMF $\mathcal{E}$. Since resistance is always a positive quantity, if we have values for $\mathcal{E}$ and R we would find positive numerical values for V_R and I. There are two important points to note:

1. Were we to reverse to polarity of the source (rotate source $\mathcal{E}$ by 180 degrees) then we would either

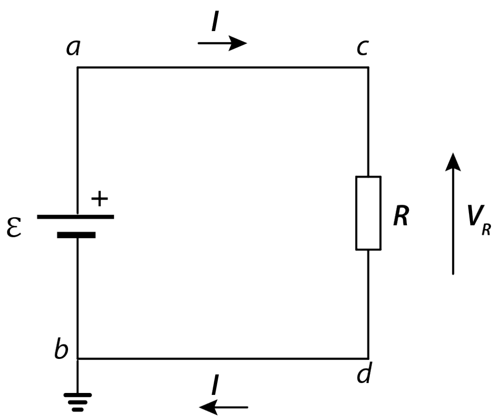

Figure 1.1: Ideal Circuit

(a) Reverse the *directions* of the current and potential difference arrows (keeping the numerical values the same);

(b) Keep the directions of the arrows as drawn, but find the numerical values become negative.

2. A consequence of ohm's law is that a positive (or negative) conventional current in the sense c to d results in a positive (or negative) potential difference V_{cd} respectively. The importance of this is that the **power** associated with the resistor is *always* positive, as we will see in section 1.11. The resistor always dissipates energy when current flows through it.

1.8 Electromotive Force

From section 1.3 we know that maintaining a constant current in a resistive conductor R requires a continuous input of energy. This source is called the **Electromotive Force** or EMF, symbol $\mathcal{E}$. EMF causes charge to flow, and an important point is that charge is a conserved quantity.

Conservation of Charge

When we construct a circuit we have a fixed number of conduction electrons which we can cause to move round in loops when current flows. The circuit neither makes nor destroys charge, it just moves it round the loop. Remember that the circuit (wires and components) remains overall neutral.

Water Flow Analogy

Consider a pump raising water to a height, which then flows back down through a narrow constriction before going back into the bottom of the pump. The PE gained by being raised through the pump is lost as heat through the constriction. Without the constriction, there would be nothing to limit the flow-rate. The equivalence is given in table 1.2.

Clearly, the pump requires some effort to drive, and the work done by the pump results in heating of the resistance. The water flows round the loop, it is neither created nor destroyed, and simply serves as a medium for moving energy from the pump to the resistance.

Table 1.2: Water Flow Analogy

Water	Electrical
Mass (kg)	Charge (C)
Flow rate (kg/s)	Current (A = C/s)
PE per mass (J/kg)	Electrical potential (V = J/C)
PE difference per mass (J/kg)	Potential Difference (V = J/C)
Resistance (Js/kg^2)	Resistance (Ω = V/I = Js/C^2)

Sources of EMF

In the electrical case, the EMF can be thought of as a 'pump' for charge. At a fundamental level, there are a number of methods for generating an EMF, both of which involve converting energy from some external source into an electrical form to propel the charges around the circuit:

Electromagnetic Moving a conductor through a magnetic field will generate an EMF across the terminals of the conductor. Mains power is (mostly) generated from a dynamo - a rotating coil which generates an oscillating EMF. A bench power-supply will convert this to the constant-valued EMF indicated in figure 1.1.

Photovoltaic Solar cells will generate an EMF by the photoelectric effect.

Chemical An electrochemical cell, or a series of cells combined to make a **battery** will generate an EMF through chemical reactions between the materials of the electrodes and the electrolytes of the cells.

> The EMF gives the number of joules of energy gained by each 1 C of charge passing through the source, and consequently it has units of volts, and in the case of an ideal voltage source (section 1.9), the EMF is identical to the potential difference measured across the terminals.

Note: The term electromotive force is mis-leading. EMF is not actually a force; it has units of volts!

1.9 Voltage Source

In a circuit, the source of EMF is called the **voltage source**. A voltage source of $\mathcal{E}$ volts will do $\mathcal{E}$ joules of work on each coulomb of charge it drives through its terminals. The voltage source is like a pump for charge, raising the electrical potential of charge passing through it. Each 1 C of charge which passes through the source gains $\mathcal{E}$ J of energy in the process, hence the electrical potential is increased by $\mathcal{E}$ volts in the process and the p.d. across the terminals is also $\mathcal{E}$ volts.

Ideal voltage source.

The ideal source has zero resistance and represents a limitless and inexhaustible supply of energy. Due to its zero resistance, the ideal voltage source will always have a p.d. of $\mathcal{E}$ volts across its terminals, independent of any resistance connected to it, such that the current it supplies can be anything from zero to ∞. The circuit symbol on the left of figure 1.1 indicates an ideal DC voltage source, with a fixed EMF $\mathcal{E}$. DC stands for Direct Current. A similar symbol with an arrow through it represents a variable source (such as a lab power supply) with the arrow indicating that we can set the EMF to some value we want. Later, we will look at AC (Alternating Current) sources, which are usually shown as a circle with a wavy line inside.

For the rest of this course, you may assume that - unless otherwise stated - all types of voltage sources are ideal. However, it is important to understand how real sources deviate from ideal behaviour.

Real voltage source.

A real source will have a (hopefully small) non-zero resistance called the **internal resistance** r. The real source acts identically to an ideal source $\mathcal{E}$ in series with a resistance r. If we need to accurately represent the real source in a circuit calculation, we can just include a resistor of value r in series with the EMF $\mathcal{E}$. We will come back to this later in section 2.3.

The battery An AA battery (strictly speaking an individual cell, as a battery is a series of cells connected together) has an EMF $\mathcal{E} = 1.5\,\mathrm{V}$. The battery stores energy in chemical form and releases it into electrical form as charge passes through. Each coulomb of charge passing through results in 1.5 J being converted from chemical to electrical form. $\mathcal{E}$ is determined by the chemistry of the cell, with different types having different EMF (for example, a NiCd rechargeable cell only generates 1.2 V). The battery is quite a good approximation to an ideal voltage source, but it does have an internal resistance of about $r = 0.1\,\Omega$.

We would describe a battery as a 'DC voltage source'. Perhaps confusingly, DC stands for **Direct Current** but this simply implies a source with a fixed (non-oscillating) EMF. By convention, DC quantities are often (not always) given as upper-case variables e.g. I, $\mathcal{E}$ or V, while time-varying quantities are represented by lower-case variables e.g. $I = dq/dt$ for a constant current or $i(t)$ if time-varying.

For the first week of this course we'll only be looking at **DC Circuits**, but it's important to note that all of the principles introduced can also be applied to time-varying circuits, also described as **AC circuits** (for Alternating Current) which will be covered later in the course.

1.10 Ground Potential

In simple problems with mechanics, we usually take the zero of gravitational potential energy to be 'ground level', such that $U = mgh$ at some height h. Similarly in electrical circuits, it is *convenient* to define the zero of electrical potential energy at some point in the circuit using the **ground** symbol connected at point b on figure 1.1. The electric potential is therefore defined to be zero at b and we know $V_b = 0$, and from this we can work-out the absolute potentials at all other points in the circuit.

> The ground connection establishes the zero of electrical potential for the circuit. No current flows into the ground connection; we should not confuse it with a circuit component.

A ground connection is not essential - for example a torch has no ground connection - but it allows us to know that, by definition, the electrical potential at point b is zero volts. We therefore know that the electrical potential is at point a is $\mathcal{E}$. If we removed the ground, the circuit would still work, and like a torch or a mobile phone we would know there are *potential differences* across components, but we would know nothing about the absolute values of potentials.

In practice, a circuit's ground is connected via the ground-pin of a mains plug to a spike in the real, physical ground somewhere below the building, and hence what we call **ground** is the potential of planet earth. We *choose* to define planet earth as the zero of electrical potential, for convenience.

Note that for complicated circuits it is often easier to include multiple ground symbols to identify all the points which we want to be at zero potential. This is easier than drawing a wire connecting all the 'grounded' points. The circuit interpretation, however, should be that the grounds are all connected together.

1.11 Energy and Power

As introduced in section 1.3, an important result of Ohm's law is that it requires a source of energy to drive a current through a resistance, and as a result we observe a difference in potential across the resistance. In our ideal circuit, the source of energy is the EMF of the voltage-source, which provides the power to pump charge round the circuit.

Energy is Conserved

Consider a charge $+q$ at b. It has zero potential energy, by definition, as this point is **grounded**.

$$U_b = 0$$

As it moves to a it gains a PE of $\mathcal{E}q$. It has the same PE at c because the wires are ideal.

$$U_a = U_c = \mathcal{E}q$$

It loses this PE as heat into R as it falls to d, since returning via d back to b it must again have zero PE.

$$U_d = 0$$

Importantly, the charge gains no net energy as it passes round the loop (otherwise we would have created an unlimited source of power!). Overall, the system conserves energy, as well as charge.

The source of energy is the stored-energy in the voltage source. In an ideal circuit this stored energy is limitless and current will flow for ever. However, if our source is a real-world battery, this energy is stored in chemical form. Thus, as charge flows continuously round the circuit, the battery becomes depleted and, eventually, the current will cease.

In any given time, the same amount of energy delivered to the circuit from the source must be dissipated in the resistor. In a real-world example, such as a torch, the bulb acts as a resistor which gets hot enough to glow brightly. For each component, we would describe the energy per unit time as the power, and clearly, the power delivered by the source must equal the power dissipated by the resistor.

Power

When a charge dq passes through a constant potential difference V then we can state equation 1.2 in terms of the change in energy dU as

$$dU = V dq$$

from which we get the power

$$P = \frac{dU}{dt} = V\frac{dq}{dt} = VI$$

Power is *always* the product of potential difference and current

$$P = VI$$

P will have units J/s (watts).

Considering the resistor, we can note that the current arrow is anti-parallel to the p.d. arrow. Consequently, the potential of the charges falls as they move through the resistor; dU/dt will be negative in respect of the charge as it moves through the resistor. However, this energy must be deposited into the resistor, and as the resistor gets hot, we can be confident that this indicates positive power dissipated in the resistor. It is convention to define the power from the point of view of the component itself and hence we can write

$$P_R = V_R I = \mathcal{E} I$$

At the source, notice that the current arrow and the potential difference arrow point in the same direction: positive current through an increasing potential means that the charge is gaining energy; dU/dt will be positive which we can interpret as energy delivered from the source into the circuit, by the raising of the potential of the charges. The source itself loses energy.

$$P_S = -\mathcal{E} I$$

Consequently, the net power is zero; the circuit conserves energy. Charge is the medium by which energy is moved from the source to the resistor.

$$P_S + P_R = 0$$

For our figure 1.1 circuit, we can write some alternative formulations for the resistor power, using ohm's law

$$P_R = V_R I = \frac{V_R^2}{R} = I^2 R$$

The interpretation of component power as positive or negative is usually clear from the context. As seen in section 1.7, the resistor power - proportional to the square of the p.d. across it - will *always* be positive whenever current is driven through it, and the source power will always be negative when it delivers current into the circuit.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 25: Current, Resistance and Electromotive Force

For a more general introduction from an electronics point of view, see also

Principles of Electronic Instrumentation (*Diefenderfer and Holton*)

Chapter 1: Direct Current Circuits

Lecture 2 Circuit Analysis

In lecture 1 we have seen that energy is conserved in a circuit. It is also the case that charge is conserved: the circuit neither creates nor destroys charge, it simply circulates around the circuit.

2.1 Kirchhoff's Laws

Two laws follow from conservation of energy and charge; they are very useful in solving circuit problems.

Kirchhoff's Voltage Law

Conservation of energy requires that there is no net energy gain around any loop of a circuit, and by the definition of potential difference in section 1.4, this means that the sum of the p.d's around a loop must be zero. Formally, this can be stated as **Kirchhoff's Voltage Law**: around any loop consisting of n components where the potential difference across component k is V_k then

$$\sum_{k=1}^{n} V_k = 0 \tag{2.1}$$

where the potential differences *are all measured in the same sense around the loop.*

For this formulation of Kirchhoff's Voltage Law (often abbreviated to KVL), we have to be very careful about the sense of the measurements. In figure 2.1 we can choose to work clockwise around the left-side loop. Sitting in the centre, with a meter (black lead in left hand, red in right) we would measure a positive EMF $\mathcal{E}$ across the source and rotating clockwise we'd measure the potential differences across the resistors as negative quantities.

$$\mathcal{E} - V_{R_1} - V_{R_2} = 0$$

Consequently, KVL is is more easily remembered as

Around any circuit loop, the sum of the potential gains due to EMFs must equal the sum of the potential drops across the components.

In this 'easier' form

$$\mathcal{E} = V_{R_1} + V_{R_2}$$

We can also do the same for the right hand loop

$$0 = V_{R1} + V_{R2} - V_{R3}$$

We can see that the sum of the potential differences across R_1 and R_2 is equal to the p.d. across R_3, as it must be, since both terms must be equal to V_{cd}.

Kirchhoff's Current Law

Conservation of charge requires that charge flows in a uniform and 'smooth' manner through the circuit so the current *into* a component must be the same as the current flowing *out*. This is most usefully applied at a junction such as point c, where **Kirchhoff's Current Law** (KCL) states that,

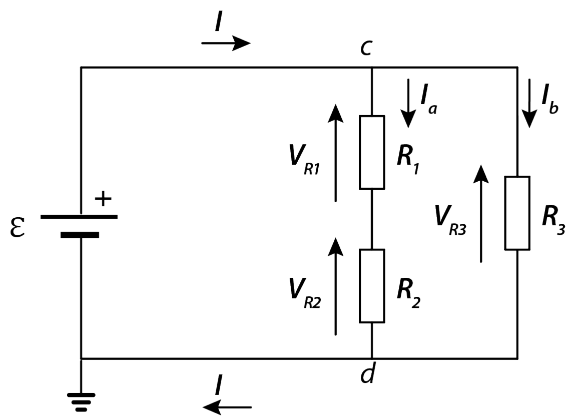

Figure 2.1: Resistors in Series and Parallel

at any junction, the sum of all currents - *defined in a sense pointing into the junction* - must be zero.

$$\sum_{k=1}^{n} I_k = 0 \tag{2.2}$$

In figure 2.1, at junction c

$$I - I_a - I_b = 0$$

As with KVL, there is an easier to remember formulation

At any junction, the sum of currents pointing into the junction must equal the sum of currents pointing out of the junction

$$I = I_a + I_b$$

trivially, also at d

$$I_a + I_b = I$$

At this point it is worth stating some hopefully obvious but key points about **circuits**:

1. Current can only flow when there is a **complete circuit**. If we break a circuit at any point, then no current can flow across a gap. Thus, cutting the wire to the left of point c sets $I = 0$ so there is no current anywhere in the circuit (and $V_c = V_d$).
2. If we remove R$_3$ and leave a gap (described as **open circuit**) then $I_b = 0$, but current will still flow round the left hand loop.
3. If we now close the R$_3$ gap with a wire (described as **short circuit**), then we have created an 'impossible circuit': Our use of ideal wire requires that $V_{cd} = 0$ while the ideal source requires $V_{cd} = \mathcal{E}$. From a circuit analysis point of view, such circuits are not allowed since they do not deliver a solution as $I \to \infty$. At a practical level, connecting a voltage source 'short circuit', while a generally bad thing to do, will result in a current limited only by the internal resistance of the real source plus the resistance of the real wires.

2.2 Equivalent Resistance

Resistors in parallel or series combine to form an 'equivalent' resistance.

Resistors in Series

Since R_1 and R_2 are in series, and the p.d. across them both is $\mathcal{E}$, then the same current must pass through them both.

$$I_a = \frac{\mathcal{E}}{R_{eq}}$$

where R_{eq} is the equivalent resistance of the two resistors in series. Knowing that $V_{R1} = I_a R_1$ and $V_{R2} = I_a R_2$ gives $R_{eq} = R_1 + R_2$ and in the general case the equivalent resistance of n series resistors is

$$R_{eq} = \sum_{k=1}^{n} R_k$$

The overall resistance will always increase, since the current has to pass through a greater quantity of 'resistive' material.

Potential Divider

Series resistances are so useful that the 'divider' rule is worth remembering. This can easily be derived from Kirchhoff's and Ohm's laws, and you should check you can do this.

The **potential divider** rule is

$$V_{R2} = \mathcal{E} \times \frac{R_2}{R_1 + R_2} \tag{2.3}$$

Resistors in Parallel

If R_1 and R_2 were to be in parallel (not shown in the figure!) then each would have *the same* p.d. across them

$$V_{R1} = V_{R2}$$

and (you should check you can derive this)

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

The total current must go up compared to a single resistor on its own. This leads to the general case that the equivalent resistance of n parallel resistors is

$$\frac{1}{R_{eq}} = \sum_{k=1}^{n} \frac{1}{R_k}$$

The overall resistance will always reduce.

For the situation shown in figure 2.1, the combination of the 3 resistors yields an overall equivalent resistance

$$R_{eq} = \left[\frac{1}{R_1 + R_2} + \frac{1}{R_3}\right]^{-1}$$

such that

$$I = \frac{\mathcal{E}}{R_{eq}}$$

2.3 Internal Resistance

While a battery is quite a good voltage source, there are some real-world imperfections that give rise to **internal resistance**. This includes the finite resistance of the materials used to make the contacts on the ends plus effects due to the chemistry of the battery that manifest themselves as equivalent to resistance. This means that a 'real-world' voltage source such as a battery is best modelled as an ideal voltage source with EMF $\mathcal{E}$ in series with a resistance r (figure 2.2).

Real Voltage Source

The reality of the battery is much more complicated, but figure 2.2 does a very good job of modelling the behaviour. Note that we don't see the internal resistance, we just see the two external terminals of the device.

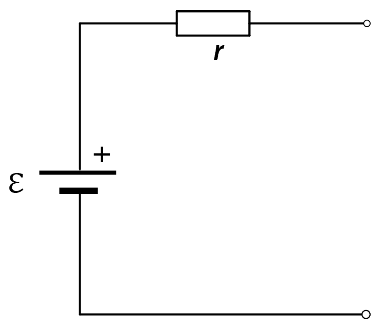

Figure 2.2: Equivalent Circuit for a 'real-world' voltage source such as a battery

The implications are that the battery only delivers the full EMF $\mathcal{E}$ when measured with an ideal meter, such that no current is drawn and hence the potential drop across r is zero.

Loading

For all practical purposes, connecting any real device or resistor (such as a bulb) to the battery puts a resistance in series with r and results in a potential drop over r. If we connect a load resistor R_L across the output (figure 2.3) then the p.d. across the load, using the potential divider formula (2.3), is

$$V_L = \frac{\mathcal{E} R_L}{r + R_L}$$

Equivalently,

$$V_L = \mathcal{E} - Ir$$

The reduction in output depends on the relative values of the resistances and clearly it's best if the internal resistance is small.

> This effect is known as **loading**; whenever a real-world voltage-source is connected to some device which causes it to supply current then the actual p.d. observed across the terminals will be reduced and the source is said to be 'loaded'.

Any source of EMF can be described as a voltage source, so this includes not only batteries but also signal generators, lab power-supplies, sensors, dynamos etc.

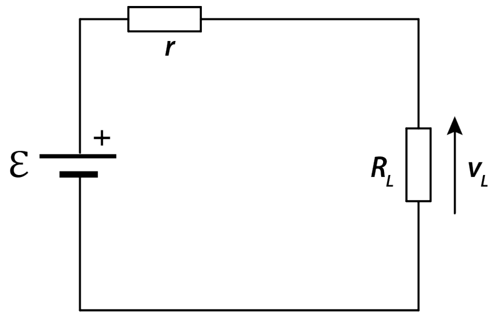

Figure 2.3: Voltage source with internal resistance r connected to a load resistance R_L

2.4 Measuring Potential Difference and Current

In our circuit diagrams so far we have identified potential differences and currents with arrows, but in practice we may need to connect meters to a real circuit to measure these values. Without going into the practical details of how volt-meters and ammeters work, it will be useful to define the properties of both idealised and real meters.

Ideal Voltmeter

The ideal voltmeter might be connected across a resistor in order to measure the 'voltage' (electrical potential difference) across it. So as not to disturb the measurement, the ideal meter has infinite resistance, which means it draws no current into it's terminals.

Real Voltmeter

The real voltmeter has a high (but not infinite) resistance. Some small amount of current flows into its terminals, in order to allow the measurement, which means it slightly modifies the behaviour of the circuit when we connect it. A good modern voltmeter will have an effective resistance of about $R = 1\,\mathrm{M}\Omega$, so the effect is small, but sometimes important to know about. A real voltmeter looks like an ideal voltmeter in parallel with a resistance R.

Ideal Ammeter

The ideal ammeter would be connected in series in a circuit in order to measure the current through that point. So as not to disturb the measurement, the ideal meter has zero resistance, which means it drops no potential difference across it's terminals.

Real Ammeter

The real ammeter will have a small but non-zero resistance in order to function. Modern meters put a deliberate resistance of $R \approx 0.1 \ldots 1\,\Omega$ in series and then measure the p.d. across it. The meter can then calculate the current by ohm's law. As a result, there is a small p.d. across the terminals of a real ammeter. A real ammeter looks like an ideal ammeter in series with a resistance R.

Ideal meters measure potential differences or current with no effect on the circuit's operation. Real meters have finite resistances, and may be said to *load* the circuit. In this course, you can assume that all potential differences and currents are measured with 'ideal' meters, unless specifically stated otherwise.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 26: Direct Current Circuits

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapter 1: Direct Current Circuits

Lecture 3 Capacitors

The capacitor is a device for storing energy by holding charges at a separation. We will often describe a capacitor as 'storing a charge Q' but, as we will see, the capacitor is overall neutral: it holds a charge $+Q$ at separation from a charge $-Q$ such that the net charge is always zero. Any two conductors with an insulator between them will form a capacitor, but the simplest configuration is the **parallel plate capacitor** illustrated in figure 3.1

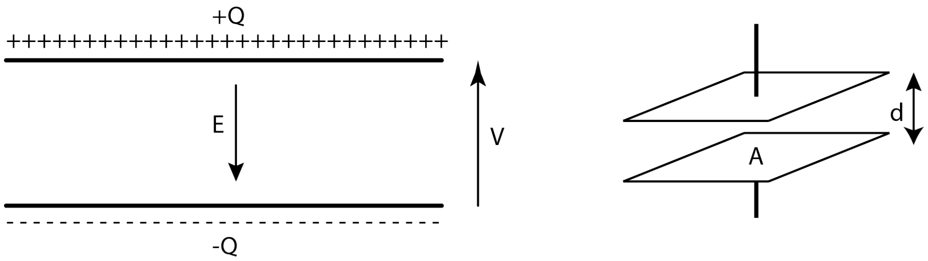

Figure 3.1: The Parallel Plate Capacitor

The top plate has an *excess* of positive charge $+Q$ and we can imagine that this has been taken from the bottom plate such that it has a charge $-Q$. As mentioned above, the net charge is zero, but we would describe this as 'storing charge Q'.

3.1 Capacitance

Capacitors will be covered in more detail during the E&M course, however a key result of E&M is that the electric field between two parallel plates, each carrying a charge equal and opposite to the other, is constant and given by

$$E = \frac{Q}{\epsilon_0 \epsilon_r A} \tag{3.1}$$

Where A is the area of each plate and ϵ_0 is the relative permittivity of any material in-between the plates. Since the electric field is found to be a constant value (so long as the size of the plates is much larger than their separation), we can express this in terms of the potential difference between the plates and the separation between them, d

$$E = \frac{V}{d}$$

Because the electric field between the plates is proportional to the charge-stored, there is also a linear relationship between charge and p.d., and the constant of proportionality is the **capacitance**, with units of **farad** (coulombs per volt).

$$C = \frac{Q}{V} \tag{3.2}$$

Equation 3.2 applies to capacitors of any shape, however for the parallel plate case

$$C = \frac{\epsilon_0 \epsilon_r A}{d} \tag{3.3}$$

Physically, the capacity for charge storage is a function of only the dimensions of the capacitor and the relative permittivity of the insulator between the plates. Some values for ϵ_r are given in table 3.1. Using a material such as polythene between the plates has, compared to air, the effect of *reducing* the electric field (equation 3.1). This seems counter-intuitive, however for a practical capacitor, the charge stored is limited by the maximum electric field which can be supported before we get a 'breakdown' (electrical discharge) between the plates. Consequently, reducing the electric field allows greater charge storage, and hence greater capacitance.

Given the magnitude of ϵ_0 and ϵ_r it should be clear that it's difficult to make a 1 farad capacitor, and even values above a mF are hard to achieve. A typical capacitor consists of two sheets of foil sandwiching a thin plastic insulator. This can be rolled-up into a cylindrical shape. For smaller values, other constructions are possible.

Material	ϵ_r
Vacuum (by definition)	1
Air	1.0005
Polythene	2.25
Glass	4
Water	80

Table 3.1: Relative Permittivity (dimensionless) of common materials

3.2 Equivalent Capacitance

As with resistance we can find the equivalent capacitance of series and parallel capacitors.

Capacitors in Parallel

Consider two capacitors C_1 and C_2 connected in parallel. Both capacitors will see the same potential difference V across their plates. The parallel arrangement increases the charge-storing capability (the area of the plates increases) and the total charge stored is $Q = Q_1 + Q_2 = C_1V + C_2V$. The parallel arrangement is equivalent to a single capacitor of equivalent capacitance is

$$C_{eq} = \frac{Q}{V} = C_1 + C_2$$

From this follows the general rule for capacitors in parallel

$$C_{eq} = \sum_{k=1}^{n} C_k \tag{3.4}$$

Capacitors in Series

Placing two capacitors in series means that the applied p.d. is across both capacitors in series so now we must have $V = V_1 + V_2$. We can write

$$Q_1 = C_1V_1$$

$$Q_2 = C_2V_2$$

However - and it is useful to draw this - the plates joining the two capacitors are isolated from the rest of any circuit we might use them in. There is no way to charge-up this section of circuit, hence it must always remain electrically neutral and this requires

$$Q_1 = Q_2$$

The series arrangement is equivalent to a single capacitor

$$C_{eq} = \frac{Q}{V}$$

where

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

which gives

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

From this follows the general rule for capacitors in series

$$\frac{1}{C_{eq}} = \sum_{k=1}^{n} \frac{1}{C_k} \tag{3.5}$$

3.3 Energy Stored

A capacitor with no net charge has zero p.d. across the plates. The process of 'charging' the capacitor to some charge Q and potential difference V (per figure 3.1) involves moving charge from the bottom plate to the top plate against the force due to the electric field and hence this requires work to be done by some external source. During this process, let q and v represent the charge and p.d. which will always be related by

$$v = \frac{q}{C}$$

The energy required to transfer an additional element of charge dq will be

$$du = v\,dq = \frac{q}{C}dq$$

The total energy required is found by integrating the charge from zero to Q

$$U = \int_0^U du = \frac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C} \tag{3.6}$$

We can also write

$$U = \frac{CV^2}{2} \tag{3.7}$$

This is the work done by the external source to separate the charges, and hence we can interpret this as the energy stored by the capacitor.

Neutrality

In charging the capacitor we have moved charge from one plate to another. If the top plate carries a surplus of positive charge $+Q$ then the bottom plate has a deficit of positive charge and carries $-Q$. A method for achieving this would be to use a source of EMF such as a battery to 'pump' charge from the bottom plate to the top. Overall, the capacitor remains electrically neutral, but as the charge is 'pumped' through the voltage source then the source does work. The energy stored in the electric field of the capacitor must be the same as the work done by the source. Since the capacitor stores energy while the source loses energy, overall energy is conserved.

3.4 Current-Voltage Relationship

We saw that Ohm's law gives a simple linear relationship between the current through a resistor and the potential difference across it. It will be useful to find a similar relationship for the capacitor.

By $q = Cv$ we have

$$i = \frac{dq}{dt} = C\frac{dv}{dt} \tag{3.8}$$

or equivalently

$$v = \frac{1}{C}\int_0^t i(t')dt'$$

assuming we started with an un-charged capacitor at $t = 0$.

Charging with a Voltage Source

If we connected a 1 F capacitor to an ideal $\mathcal{E} = 1\,\text{V}$ voltage source then the potential difference across the capacitor would *instantly* rise to 1 V since the ideal source can supply any arbitrary amount of current. The capacitor, carrying 1 C of charge, would remain at 1 V.

Of course, no ideal voltage source exists in reality; there is always some resistance to limit the current.

Current through the Capacitor

Note that while we talk about there being a current *through* the capacitor, no charge actually passes directly between the plates (there is an insulator in between). However, Kirchhoff's Current Law requires that any charge added to the top plate must be balanced by an equal charge removed from the bottom plate (or *vice-versa*). Consequently, a current *appears* to flow between the plates. This is called a **displacement current** - you will cover this in detail during the E&M course. For the purposes of electrical circuits, it is sufficient to know that the displacement current behaves exactly the same as a real current 'through' a capacitor, even though we know that the real charge is simply moving between the plates via the rest of the circuit.

3.5 RC Circuit

Since all real sources have some internal resistance it is useful to study the behaviour of the series RC circuit shown in figure 3.2 where v_S represents the potential difference across the terminals of a variable voltage source. We can think of this as being like an idealised version of a lab power-supply: we can set the value of v_S to be any arbitrary value that we choose, and we can switch the supply on or off at any time we like. Applying Kirchhoff's Voltage Law gives

$$v_S = v_R + v_C$$

where

$$v_R = iR$$
$$v_C = \frac{q}{C}$$

q is the capacitor charge and we can also use this to write the current

$$i = \frac{dq}{dt}$$

We can do this because Kirchhoff's Current Law means that any element of charge dq moving clockwise round the circuit results in an equal element of charge being stored on the capacitor.

This leads to a first-order differential equation for charge which governs the behaviour of the circuit

$$v_S = R\frac{dq}{dt} + \frac{q}{C} \tag{3.9}$$

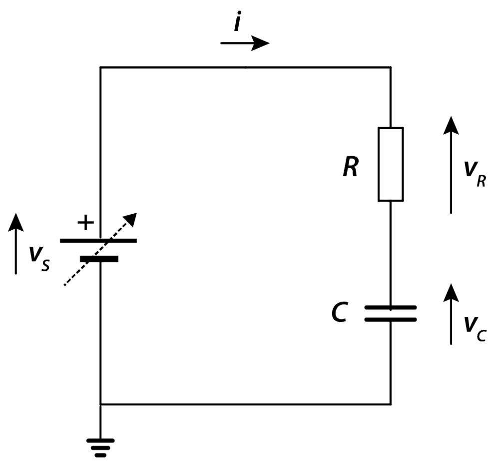

Figure 3.2: Series RC Circuit: The DC source $\mathcal{E}$ is applied at time $t = 0$

Charging

Let's assume that the source has been set to $v_S = 0$ for all time $t < 0$ and then at time $t = 0$ we set it to an EMF $\mathcal{E}$ such that $v_S = \mathcal{E}$ for all time $t > 0$ (as shown in the left panel of figure 3.3). Formally, we would write this as

$$v_S = \begin{cases} 0, & t < 0 \\ \mathcal{E}, & t \geq 0 \end{cases}$$

Since the power supply is off for all time $t < 0$ we can assume that we start at time $t = 0$ with an initially uncharged capacitor. For time $t \geq 0$ we can write the equation to be solved as

$$\mathcal{E} = R\frac{dq}{dt} + \frac{q}{C} \tag{3.10}$$

which re-arranges to

$$\frac{dq}{dt} = \frac{-1}{RC}(q - \mathcal{E}C)$$

which can be solved by separation of variables

$$\frac{dq}{q - C\mathcal{E}} = \frac{-dt}{RC}$$

Now integrate, noting that we must change the integration variables to q' and t' so we can use q and t as the upper limits, and using the initial conditions $q = 0$ at $t = 0$ for the lower limits

$$\int_0^q \frac{dq'}{q' - C\mathcal{E}} = \int_0^t \frac{-dt'}{RC}$$

$$\ln\left[\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right] = \frac{-t}{RC}$$

$$q = C\mathcal{E}(1 - e^{-t/RC}) \tag{3.11}$$

With this expression for q, we can use circuit rules to find all the remaining quantities. The current in the circuit is found by differentiating the charge q

$$i = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC} \tag{3.12}$$

The potential difference across the capacitor is found from $v_c = q/C$

$$v_C = \mathcal{E}(1 - e^{-t/RC}) \tag{3.13}$$

The resistor p.d. comes easily from ohm's law

$$v_R = iR = \mathcal{E}e^{-t/RC}$$

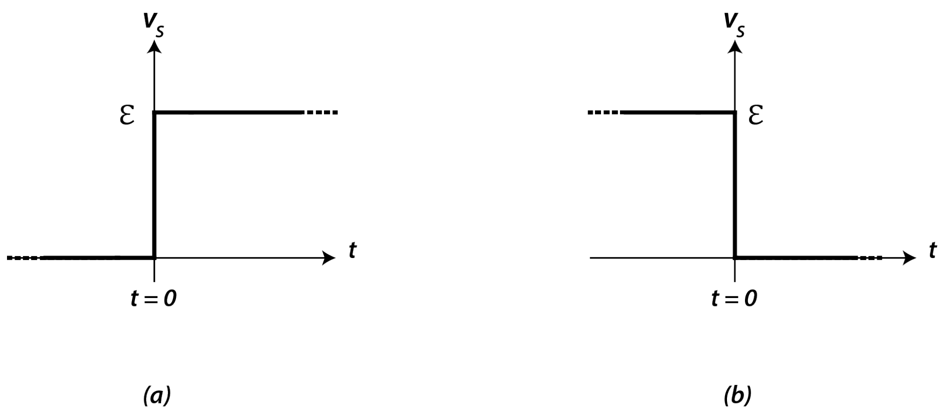

Figure 3.3: Source potential difference profiles: (a) charging and (b) discharging

Time Constant

The circuit's **time constant** τ is the time taken the time taken for an exponentially decreasing quantity, such as the current, to fall by a factor $1/e = 37\%$. Hence for this circuit

$$\tau = RC$$

Equivalently, τ is the time for the capacitor p.d. to rise to a factor $(1 - 1/e) = 63\%$ of the final value. After 3τ, $v_C = 0.95\mathcal{E}$ and after just 10 time constants v_C will have attained 99.995% of its final value. Only for $t \to \infty$ does $v_C \to \mathcal{E}$ but for practical purposes will be adequate to assume that the circuit quickly reaches the *steady-state* condition $v_C = \mathcal{E}$.

Discharging

We can now discharge the capacitor by 'switching off' the voltage source (figure 3.3, right panel). For mathematical convenience, we can choose to do this at a 'new' time $t = 0$ where we can take the voltage supply profile to be

$$v_S = \begin{cases} \mathcal{E}, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

Equation 3.9 becomes

$$0 = R\frac{dq}{dt} + \frac{q}{C}$$

which can be solved again by separation of variables

$$\frac{dq}{q} = \frac{-dt}{RC}$$

To integrate, we need to use the initial condition that $q = C\mathcal{E}$ at time $t = 0$

$$\int_{C\mathcal{E}}^{q} \frac{dq'}{q'} = \int_0^t \frac{-dt'}{RC}$$

giving

$$q = C\mathcal{E}e^{-t/RC}$$

and again by differentiating q

$$i = -\frac{\mathcal{E}}{R}e^{-t/RC}$$

We can see that the current is now negative, which is to say that conventional current is now flowing back from the capacitor and into the (switched-off) source. As before, we find the component potential differences

$$v_C = \frac{q}{C} = \mathcal{E}e^{-t/RC}$$

$$v_R = iR = -\mathcal{E}e^{-t/RC}$$

Note that $v_R = -v_C$. Kirchhoff's Voltage Law requires this, since $v_S = 0$.

3.6 Energy and Power

Since power is always potential difference times current [J/C $\times$ C/s], the power in any component is always $p = vi$, and so the capacitor power is $p_C = v_C i$. The capacitor does not *dissipate* power (it has no resistance) but it does store energy. Positive capacitor power is associated with the capacitor storing energy, and negative power means that the capacitor is returning that stored energy to the circuit.

Charging

When charging the capacitor, the source delivers power to the circuit, part of which is stored in the capacitor, and part of which is dissipated in the resistor. The resistor power is

$$p_R = v_R i = i^2 R = \frac{\mathcal{E}^2}{R}e^{-2t/RC}$$

It is always positive, as it must be for a resistor. Over the complete charging cycle, we can find the energy dissipated in the resistor by

$$U_R = \int_0^\infty p_R dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-2t/RC} dt = \frac{C\mathcal{E}^2}{2} \tag{3.14}$$

The capacitor power is also positive, as the capacitor is storing energy

$$p_C = v_C i = \frac{\mathcal{E}^2}{R}(1 - e^{-t/RC})e^{-t/RC}$$

We could integrate this, but we already know (equation 3.15) that for $t \to \infty$ the capacitor energy stored will be

$$U_C = \frac{C\mathcal{E}^2}{2} \tag{3.15}$$

This is the energy stored in the capacitor.

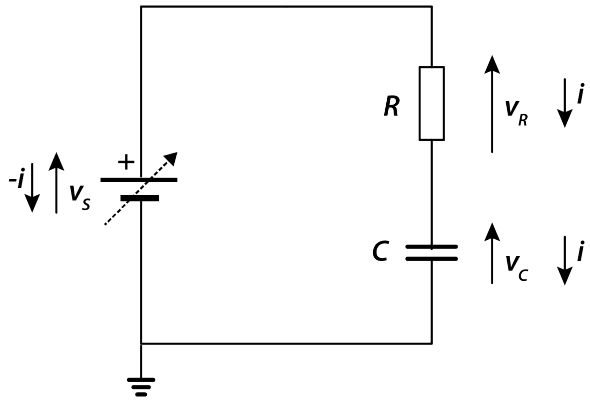

Figure 3.4: Sense of the current in the source, resistor and capacitor

The source will deliver energy into the circuit so we interpret its power as negative. Note the directions of the p.d. and current arrows through the source: they point in the same direction, while for the resistor and capacitor they point in the opposite direction. To write a power equation for the source which is consistent with the other two components, we would need to reverse the direction of the current arrow and label it $-i$, as illustrated in figure 3.4. Hence $p_S = v_S \times -i = -v_S i$ and we can find the source energy during charging as

$$p_S = -v_S i = -\mathcal{E} i$$

$$U_S = \int_0^\infty p_S dt = \frac{-\mathcal{E}^2}{R}\int_0^\infty e^{-t/RC} dt = -C\mathcal{E}^2$$

Overall, energy is conserved

$$U_S + U_R + U_C = 0$$

and also, at any instant in time we have

$$p_S + p_R + p_C = 0$$

Discharging

Proceeding as before, the resistor power

$$p_R = v_R i = i^2 R = \frac{\mathcal{E}^2}{R} e^{-2t/RC}$$

is the same as during charging, and again is a positive quantity indicating energy dissipated in the resistance. The capacitor power, however, is now a negative quantity

$$p_C = v_C i = -\frac{\mathcal{E}^2}{R} e^{-2t/RC}$$

The source power is now zero, since

$$p_S = v_s i = 0$$

so we find that $p_c = -p_R$, which is to say that the capacitor is returning the power back into the circuit, and this power is dissipated in the resistor. We could check the total energy exchanged during this process by integration, but we don't need to. For the resistor, the integral will be the same as equation 3.14, while for the capacitor we know that it starts by storing energy $C\mathcal{E}^2/2$ and ends with zero energy.

Over a complete cycle of charging and discharging, the source overall loses energy, all of which is dissipated by the resistor. The capacitor acts as a 'temporary' store for the energy. This is summarised in table 3.2

Table 3.2: Energy exchanged during charging/discharging

	Charging	**Discharging**	**Overall**
Source	$-C\mathcal{E}^2$	0	$-C\mathcal{E}^2$
Capacitor	$C\mathcal{E}^2/2$	$-C\mathcal{E}^2/2$	0
Resistor	$C\mathcal{E}^2/2$	$C\mathcal{E}^2/2$	$C\mathcal{E}^2$

This might suggest that the charging and discharging cycles are symmetric, however it is interesting to plot the power and energy associated with the capacitor. This is shown in figure 3.5, where the source EMF is 1 volt and we choose component values $R = C = 1$ for convenience. Note that this gives a time-constant $\tau = 1\,\mathrm{s}$.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 24: Capacitance and Dielectrics

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapter 2: Capacitors and Inductors

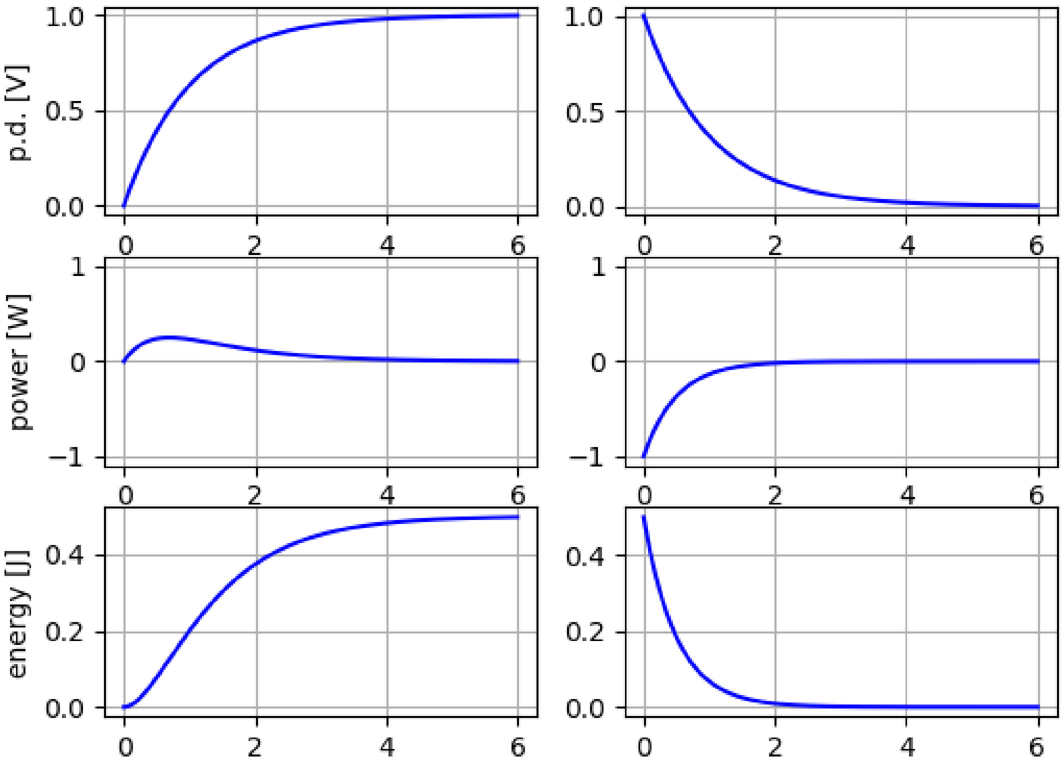

Figure 3.5: Potential difference, power and energy stored by the capacitor, charging (left) and discharging (right). Component values are normalised to $\mathcal{E} = C = R = 1$.

Lecture 4 Inductors

Whenever we see a field (electric, gravitational, magnetic) it is a sign that potential energy has been stored, and that work needed to be done in order to create the field. For example, the arrangement of charges on a capacitor (figure 3.1) stores potential energy, giving rise to an electric field and a force between the separated charges. A magnetic field is a similar indication of potential energy. While an electric field will arise from a *static* arrangement of charges, a circuit needs a *current* to generate a magnetic field. Any current-carrying conductor will generate a magnetic field however - since the field is always perpendicular to the current - we can enhance the field by winding the conductor into loops known as a **coil** or **solenoid** as illustrated in figure 4.1 (left).

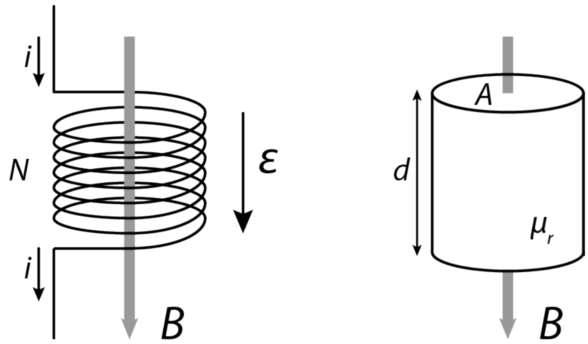

Figure 4.1: A current-carrying coil (left) can be approximated as a cylinder (right) of area A and height d through which the magnetic-field is uniform.

4.1 Inductance

The current i through N turns generates a field B; we can approximate the geometry of the coil as a cylinder (figure 4.1, right) of area A and height d. If the coil has many turns, and it relatively long compared to its diameter, then its magnetic field strength is quite uniform within the coil and approximated by the expression

$$B = \frac{\mu_0 \mu_r N i}{d}$$

where μ_r is the relative permeability of whatever material we place within the volume of the coil. Some values of μ_r are given in table 4.1[1].

Since field strength is magnetic flux per unit area we find that the magnetic flux through the volume is

$$\Phi = BA = \frac{\mu_0 \mu_r N A i}{d}$$

Each of the N turns of the coil will see flux Φ through it, and if this flux changes then each turn will generate an EMF $d\Phi/dt$ according to **Faraday's law**. Since our coil contains N turns one after the other, the total EMF generated by the coil will be

$$\mathcal{E} = -N\frac{d\Phi}{dt} = \frac{\mu_0 \mu_r N^2 A}{d}\frac{di}{dt}$$

[1]The relative permeability values for metals and alloys are influenced by material processing, so these values are approximate.

Material	μ_r
Vacuum (by definition)	1
Air	1.0000004
Nickel	500
Iron	5000
Permalloy (Ni/Fe alloy)	100000

Table 4.1: Relative Permeability (dimensionless) of common materials

The negative sign is due to **Lenz's law**, which requires that EMF act so as to *oppose* the change in current. If the current in the coil is increasing then the flux is increasing. The induced EMF will be negative such that it would generate a current with an opposite sign and hence act so as to reduce the flux.

Since there is a linear relationship between EMF and di/dt we can define the **self-inductance** (L) of the coil by

$$\mathcal{E} = -L\frac{di}{dt} \tag{4.1}$$

where

$$L = \frac{\mu_0 \mu_r N^2 A}{d} \tag{4.2}$$

The unit of self-inductance is the *henry* (H). Compare with equation 3.3 and note that, as with the parallel-plate capacitor, self-inductance is a function only of the geometry and material properties of the coil.

If a coil with N turns carrying a current i generates a flux Φ then we can also write

$$N\Phi = Li$$

Again we can compare with equation 3.2 and we can see that there is a linear relation between flux and current, as there is for charge and potential difference with a capacitor. However, while the charge associated with the capacitor became essential for analysing the RC circuit, we will not need to use the flux and we will not consider Φ further, since you will cover this in more detail during the E&M course. Equations 4.1 and 4.2 will be sufficient for us.

4.2 Equivalent Inductance

Since it is uncommon to see series or parallel arrangements of inductors, we will simply quote the relations for **equivalent inductance**

Inductors in Series

Noting that the same current flows through two inductors in series, the equivalent inductance is the sum (as for resistors).

$$L_{eq} = \sum_{k=1}^{n} L_k$$

Inductors in Parallel

The inductors must share the current, but each has to maintain the same potential difference across its terminals, as for resistors.

$$\frac{1}{L_{eq}} = \sum_{k=1}^{n} \frac{1}{L_k}$$

4.3 Current-Voltage Relationship

A solenoid used as a circuit element is known as an **inductor**, and in this context its self-inductance is simply referred to as **inductance**.

Unlike the resistor and capacitor, the inductor is a source of EMF within a circuit; in this respects it is more similar to the battery, or some other voltage-source. We need to take care when evaluating the sense of the potential difference across the terminals of the inductor. To illustrate this, consider the ideal voltage source connected to an ideal inductor shown in figure 4.2. Note that there is no resistance, so nothing to limit the current. First, note that our voltage source has a current arrow drawn in the same sense as the source EMF $\mathcal{E}_S$. We adopted this convention since we know that positive $\mathcal{E}_S$ drives positive conventional current i clockwise around the circuit. Consequently, when drawing the current arrow in the c to d sense through the inductor, we should define the inductor EMF $\mathcal{E}_L$ in the same sense (as also drawn in figure 4.1).

Terminal c is always going to be more positive than terminal d, due to the ideal source. Consequently $\mathcal{E}_L$, as drawn, must evaluate to a negative quantity. By equation 4.1, i will be increasing. This is entirely consistent with Kirchhoff's Voltage Law:- the inductor will generate an EMF exactly equal and opposite to the source

$$\mathcal{E}_S + \mathcal{E}_L = 0$$

We could continue in this way, always remembering to include the inductor as an EMF source in our KVL expressions. Alternatively, we could consider our inductor to be a circuit component like our resistors and capacitors and simply define a potential difference v_L in the opposite sense to the current

$$v_L = -\mathcal{E}_L = L\frac{di}{dt} \tag{4.3}$$

KVL then gives

$$\mathcal{E}_S = L\frac{di}{dt}$$

This alternative approach tends to be more intuitive for circuit problems. We can immediately see that a positive source $\mathcal{E}_S$ will produce a positive v_L, with an increasing current (positive di/dt). Most importantly, our p.d. and current arrows are drawn in the same sense as we have used for resistors and capacitors, which will make everything much simpler when we start to introduce circuits with multiple components. Consequently, equation 4.3 will be our key equation for circuits containing inductors. We just need to remember that the physics tells us the device actually generates an EMF defined in the opposite sense.

For example, with a source $\mathcal{E}_S = 1\,\text{V}$ and $L = 1\,\text{H}$, the inductor *must* induce a p.d. $v_L = 1\,\text{V}$ ($\mathcal{E}_L = -1\,V$) to keep $V_c = 1\,\text{V}$ ($V_d = 0$ by definition). By equation 4.3, $di/dt = 1\,\text{A/s}$. In the ideal case, the current keeps rising for ever, in order to maintain a steady potential difference across the terminals.

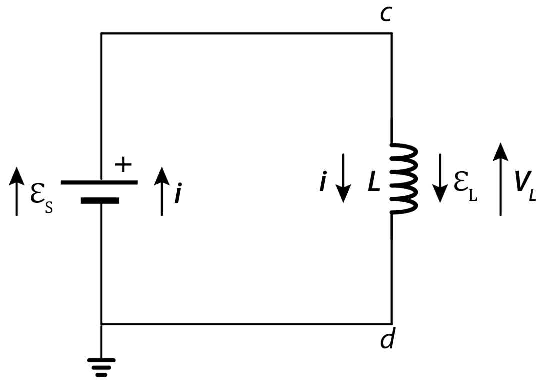

Figure 4.2: Ideal Inductor Circuit

4.4 Energy Stored

The source does work to drive current through a potential difference and conservation of energy requires that this is stored as potential energy in the magnetic field of the inductor. **Energising** an inductor from zero to a current I stores potential energy U. At some voltage v_L and current i the power delivered into the inductor is

$$\frac{dU}{dt} = v_L i = L\frac{di}{dt}i$$

$$dU = Lidi$$
$$U = L\int_0^I idi = \frac{LI^2}{2} \tag{4.4}$$

As with the capacitor, the ideal inductor has no resistance so no energy is dissipated; all the stored potential energy can be returned back to the circuit.

4.5 RL Circuit

Figure 4.2 does not represent a practical circuit since all real-world inductors have some resistance, and are usually used in circuits with resistors. As with the RC circuit, it is useful to study the behaviour of the series RL circuit. In figure 4.3 Kirchhoff's Voltage Law gives

$$v_S = v_R + v_L$$

Energising

We can apply the same source potential profile as we used for the capacitor (figure 3.3):

$$v_S = \begin{cases} 0, & t < 0 \\ \mathcal{E}, & t \geq 0 \end{cases}$$

For time $t < 0$ we can assume steady-state conditions with no current flowing, and for time $t \geq 0$ we can write

$$\mathcal{E} = iR + L\frac{di}{dt} \tag{4.5}$$

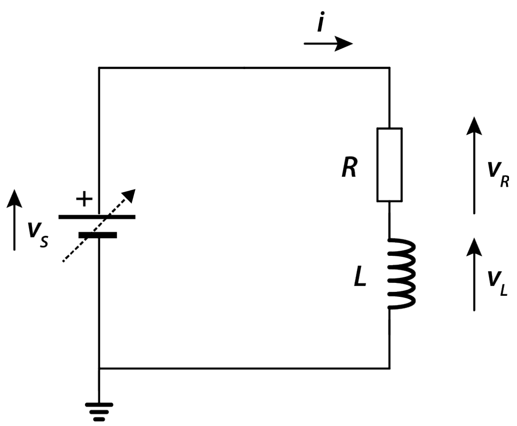

Figure 4.3: Series RL Circuit

Comparing with the RC circuit (equation 3.10) we can see that this is another first-order differential equation which we could solve in the same way. To save time, we could note that the quantities in equation 3.10 map to equation 4.5 by

$$\begin{aligned} q &\rightarrow i \\ R &\rightarrow L \\ 1/C &\rightarrow R \end{aligned}$$

which allows us to take the result of equation 3.11 and directly write the result

$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-tR/L}\right) \tag{4.6}$$

and get the time-constant

$$\tau = \frac{L}{R} \tag{4.7}$$

Note that for time $t \gg \tau$, the current $i \rightarrow \mathcal{E}/R$, i.e. the current tends to a constant value (compare with the capacitor).Consequently, we expect the potential difference across the inductor to tend to zero.

$$v_L = L\frac{di}{dt} = \mathcal{E}e^{-Rt/L} \tag{4.8}$$

De-energising

Switching off the supply, again at a 'new' time $t = 0$

$$v_S = \begin{cases} \mathcal{E}, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

we find that the current decays exponentially

$$i = \frac{\mathcal{E}}{R} e^{-tR/L} \tag{4.9}$$

Note that the sign of the current remains the same during the de-energising (contrast with the capacitor).

$$v_R = iR = \mathcal{E} e^{-tR/L}$$

$$v_L = L\frac{di}{dt} = -\mathcal{E} e^{-tR/L}$$

We can see that $v_L = -v_R$ as it must be, by Kirchhoff's Voltage Law.

4.6 Energy and Power

As with the capacitor, we can find expressions for the power and energy associated with the three circuit components.

Energising

During energising, the current tends to a maximum value $\mathcal{E}/R$ and the inductor stores a maximum potential energy

$$U_L = \frac{Li^2}{2} = \frac{L\mathcal{E}^2}{2R^2}$$

However, unlike the equivalent RC circuit, the source *continues* to deliver power $p_s = -\mathcal{E}i$ even for time $t \gg \tau$. At all times, conservation of energy requires

$$p_S + p_R + p_L = 0 \tag{4.10}$$

$$-\mathcal{E}i + i^2R + Li\frac{di}{dt} = 0$$

Per unit time, the energy delivered by the source is the sum of the energy dissipated in the resistor and that stored in the inductor. It is worth writing out the expressions for p_S, p_R and p_L to check that this is so.

De-energising

Equation 4.10 must still be true but now $p_S = 0$ so $p_R = -p_L$. Again, it is worth checking this. Note that the inductor is now delivering stored energy back into the circuit, where it is dissipated in the resistor.

4.7 LC Circuit

Figure 4.4 shows an ideal capacitor connected to an ideal inductor. Let's impose the initial condition $q = C\mathcal{E}$ for time $t < 0$ which we can do by again using the source profile

$$v_S = \begin{cases} \mathcal{E}, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

Note that, since the initial charge is a constant, the initial current is zero. Kirchhoff gives

$$v_S = v_L + v_C$$

for time $t \geq 0$

$$v_C = -v_L$$
$$\frac{q}{C} = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2}$$
$$\frac{d^2q}{dt^2} = \frac{-q}{LC} \tag{4.11}$$

This is the equation for harmonic oscillation of the charge. Taking a trial solution

$$q = q_0\cos(\omega_0 t + \phi) \tag{4.12}$$

$$\frac{dq}{dt} = -q_0\omega_0\sin(\omega_0 t + \phi) \tag{4.13}$$

$$\frac{d^2q}{dt^2} = -q_0\omega_0^2\cos(\omega_0 t + \phi) = -\omega_0^2 q$$

from which we get the angular frequency of the oscillation

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

Substituting the initial conditions $q = c\mathcal{E}$ into 4.12 and $dq/dt = 0$ into 4.13 yields $\phi = 0$ and $q_0 = C\mathcal{E}$, i.e. q_0 is the initial charge on the capacitor. Hence the solution to equation 4.11, valid

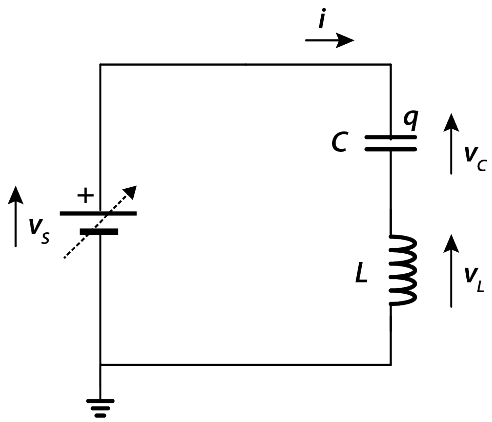

Figure 4.4: LC Circuit

for time $t > 0$, is

$$q = q_0\cos(\omega_0 t)$$

From which the current

$$i = \frac{dq}{dt} = -\omega_0 q_0\sin(\omega_0 t)$$

and the potential differences

$$v_L = L\frac{di}{dt} = -L\omega_0^2 q_0\cos(\omega_0 t) = -\frac{q}{C} = -v_C$$

We can see that

1. potential difference and current are always $\pi/2$ out of phase;
2. p.d. across capacitor and inductor are always π out of phase;
3. the oscillation continues forever, as illustrated in figure 4.5.

Power

The power associated with the inductor

$$p_L = v_L i = \mathcal{E}\omega_0 q_0 \cos(\omega_0 t)\sin(\omega_0 t) = \mathcal{E}\omega_0 q_0 \sin(2\omega_0 t)/2$$

For the capacitor

$$p_C = v_C i = -p_L$$

Energy

At any time $t \geq 0$ the energy stored on the capacitor and inductor is

$$U_C = \frac{q^2}{2C}$$

$$U_L = \frac{Li^2}{2}$$

hence the total energy

$$U = \frac{q^2}{2C} + \frac{Li^2}{2} = \frac{q_0^2 \cos^2 \omega_0 t}{2C} + \frac{L\omega_0^2 q_0^2 \sin^2 \omega_0 t}{2} = \frac{q_0^2}{2C}$$

The total energy of the circuit is constant with time. Since q and i are $\pi/2$ out of phase, when the energy of one component is at the maximum, then the energy of the other is minimum; the energy cycles from one component to the other.

The behaviour is illustrated in figure 4.5 for a circuit with $L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$ and $\mathcal{E} = 10\,\mathrm{V}$. At time $t = 0$, all the energy is stored in the capacitor, however as the capacitor starts to discharge (negative capacitor power) the inductor is energised (positive inductor power) until all the energy is stored in the inductor, at which point the cycle repeats. Current and potential difference oscillate at the natural frequency ω_0, however since energy is the square of current (or p.d.) it varies as $2\omega_0$, and it is always a positive quantity. Note that power is the derivative of energy.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 30: Inductance

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapter 2: Capacitors and Inductors

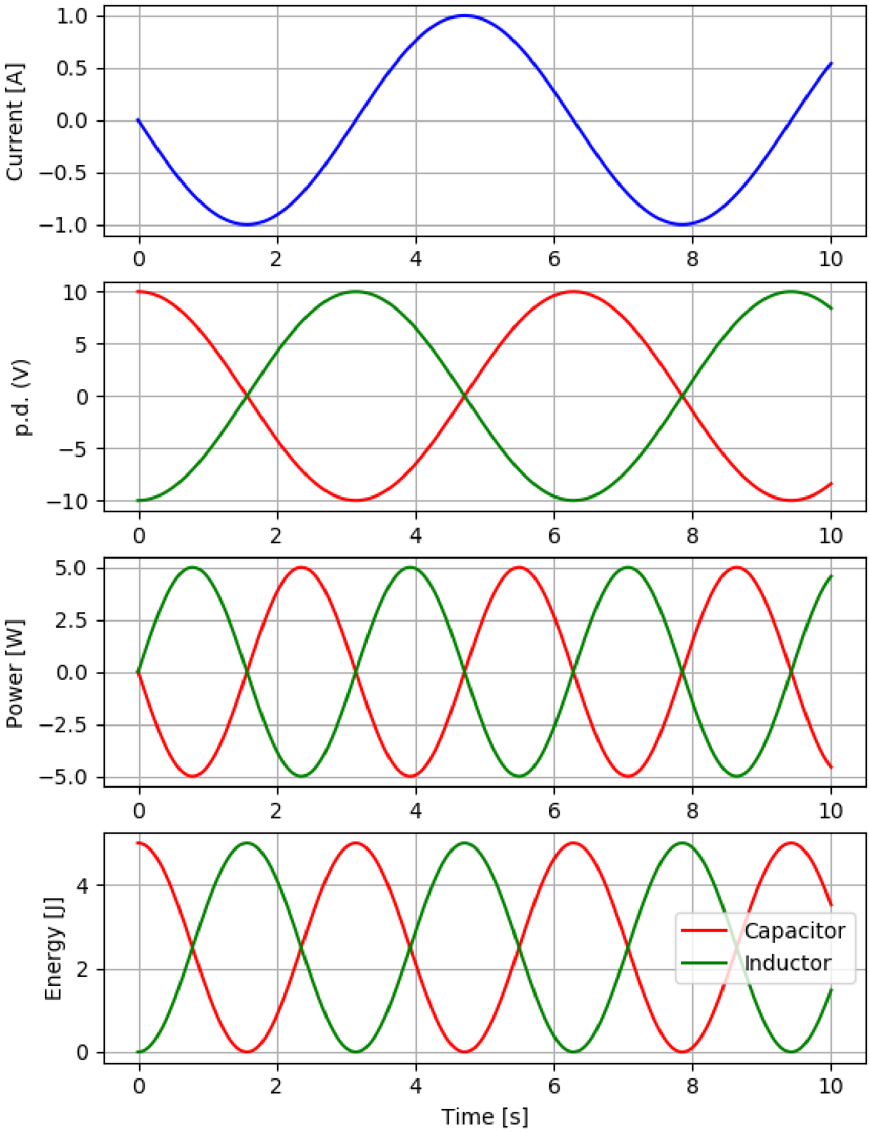

Figure 4.5: Oscillation of the LC Circuit ($L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$, $\mathcal{E} = 10\,\mathrm{V}$). From top to bottom: current, potential difference, power and energy.

Lecture 5 LCR Circuit

The LC circuit of figure 4.4 is physically unrealistic since all real-world circuits have some resistance, hence it is interesting to study the LCR circuit of figure 5.1. As seen in section 3.5, introducing some resistance into the circuit will result in some energy dissipated as current passes through the resistance; we can expect that the oscillation will be **damped**. The LCR circuit is an example of a **damped harmonic oscillator**.

5.1 Damped Harmonic Oscillator

Kirchhoff's voltage law gives

$$v_L + v_R + v_C = v_S$$

As before we can impose the initial conditions

$$v_S = \begin{cases} \mathcal{E}, & t < 0 \\ 0, & t \geq 0 \end{cases}$$

Such that the capacitor initial charge is $q(t = 0) = q_0 = C\mathcal{E}$ and the initial current $i(t = 0) = 0$. For time $t \geq 0$ we can write

$$L\frac{di}{dt} + iR + \frac{q}{C} = 0$$

and since $i = \dot{q}$

$$\ddot{q} + \dot{q}\frac{R}{L} + q\frac{1}{LC} = 0$$

The equation can be written as

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0 \tag{5.1}$$

5.2 Equivalence to the Mechanical Oscillator

In equation 5.1, $\gamma = R/L$ and $\omega_0^2 = 1/LC$ is the natural frequency of oscillation. Note that this is exactly the same form as the governing differential equation for the damped mechanical oscillator; the comparison is summarised in table 5.1.

The LC circuit of section 4.7 is an electrical equivalent to the un-damped mechanical mass-spring oscillator, where energy cycles between P.E. in the spring and K.E of the mass, and for which the frequency of oscillation is

$$\omega_0 = \sqrt{\frac{k}{m}}$$

We can draw an equivalence between mass and inductance, and spring constant and inverse capacitance.

The LCR circuit is equivalent to the damped mechanical oscillator, with the resistor providing the damping.

As studied in the *Oscillations and Waves* and *Complex Analysis* courses, the solution to equation 5.1 depends on the value of $\gamma^2 - 4\omega_0^2$.

Table 5.1: Mechanical and Electrical Oscillator Equivalences

Mechanical	**Electrical**
Displacement x	Charge q
Velocity $\dot{x}$	Current $\dot{q}$
Mass m	Inductance L
Spring constant k	Capacitance $1/C$
Damping b	Resistance R
Natural frequency $\omega_0 = \sqrt{k/m}$	Natural frequency $\omega_0 = 1/\sqrt{LC}$
$\gamma = b/m$	$\gamma = R/L$

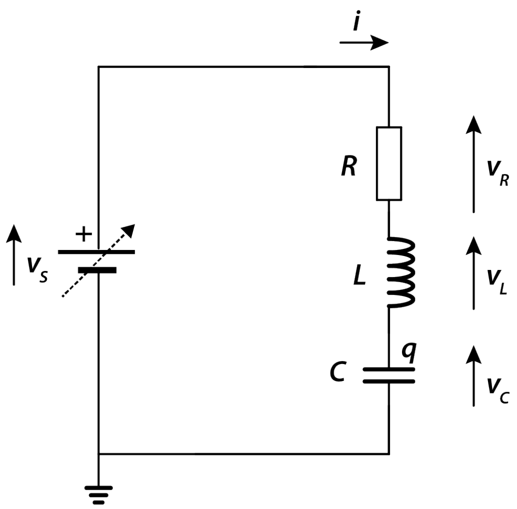

Figure 5.1: Series LCR circuit. The charge on the capacitor is q.

5.3 Under-damped Solution

The system will be **under-damped** for

$$\omega_0 > \gamma/2$$

For the mechanical oscillator the imposed initial conditions were a displacement x_0 and zero initial speed

$$\begin{aligned} x(t=0) &= x_0 \\ \dot{x}(t=0) &= 0 \end{aligned}$$

resulting in a solution of the form

$$x(t) = \frac{x_0\omega_0}{\omega_d} e^{-\gamma t/2} \cos(\omega_d t + \phi) \tag{5.2}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{5.3}$$

and

$$\tan\phi = -\frac{\gamma}{2\omega_d} \tag{5.4}$$

Since our imposed initial conditions are equivalent

$$\begin{aligned} q(t=0) &= q_0 \\ \dot{q}(t=0) &= 0 \end{aligned}$$

we can use table 5.1 to write the solution for $q(t)$ directly

$$q(t) = \frac{q_0\omega_0}{\omega_d} e^{-\gamma t/2} \cos(\omega_d t + \phi) \tag{5.5}$$

where ω_d, the damped frequency of oscillation is as per equation 5.3 and the phase ϕ is given by equation 5.4.

As anticipated, $q(t)$ exhibits exponentially-decaying oscillation. Note that the frequency of the damped oscillation ω_d is slightly less than the natural frequency ω_0, which we can interpret by understanding that the damping slows the oscillation. For the lightly-damped case, with $R = C = L = 0.1$, the response is shown in the top-panel of figure 5.2. Increasing the value of R increases the damping. The dashed line gives the envelope of the exponential term.

As before, we can calculate the component potential differences by

$$\begin{aligned} v_C &= \frac{q}{C} \\ i &= \frac{dq}{dt} \\ v_R &= iR \\ v_L &= L\frac{di}{dt} \end{aligned}$$

These results are left as an exercise. We can also find expressions for the component powers and the energy stored in the capacitor and inductor. The behaviour is illustrated in figure 5.3, with a similar presentation as for the LC circuit (figure 4.5): Here, we again take the values $L = 10\,\text{H}$, $C = 0.1\,\text{F}$ and $\mathcal{E} = 10\,\text{V}$, but we add a resistance $R = 1\,\Omega$.

As with the LC circuit, at time $t = 0$, all the initial 5 J energy is stored in the capacitor, however as energy is cyclically exchanged with the inductor, the current required for this causes dissipation in the resistor. The integrated resistor power is given by the blue trace in the bottom panel; as $t \to \infty$ the oscillation amplitude tends to zero and we can see that all the stored potential energy has been dissipated.

5.4 Critically Damped Solution

Increasing the resistance to a value such that $\gamma = 2\omega_0$ gives rise to a critically-damped solution

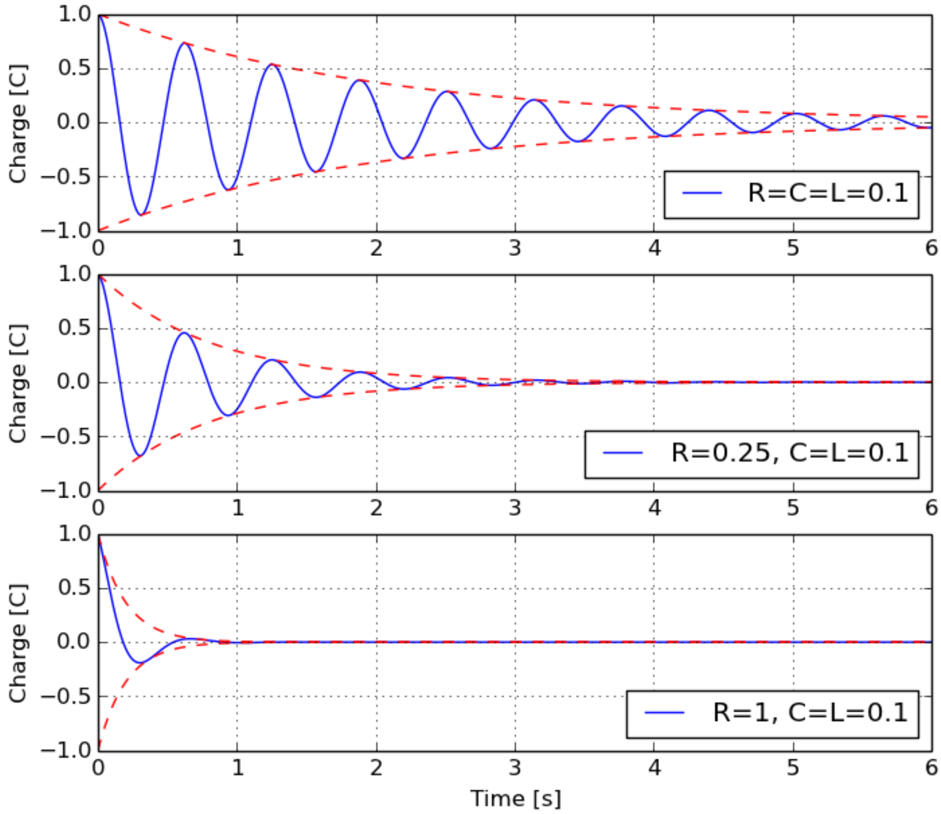

Figure 5.2: Decaying oscillation for the under-damped LCR circuit with $q_0 = 1\,\mathrm{C}$. The dashed-line shows the envelope of the exponential term.

$$q(t) = q_0 \left(1 + \frac{\gamma t}{2}\right) e^{-\gamma t/2} \tag{5.6}$$

with a non-oscillating form for the current as illustrated in figure 5.4. Note the change in time-scale versus figure 5.3; the critically-damped case represents the fastest possible return to equilibrium.

Increasing R still further results in an over-damped solution; we will not investigate this but the results obtained in the *Complex Analysis* course remain applicable.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 30: Inductance

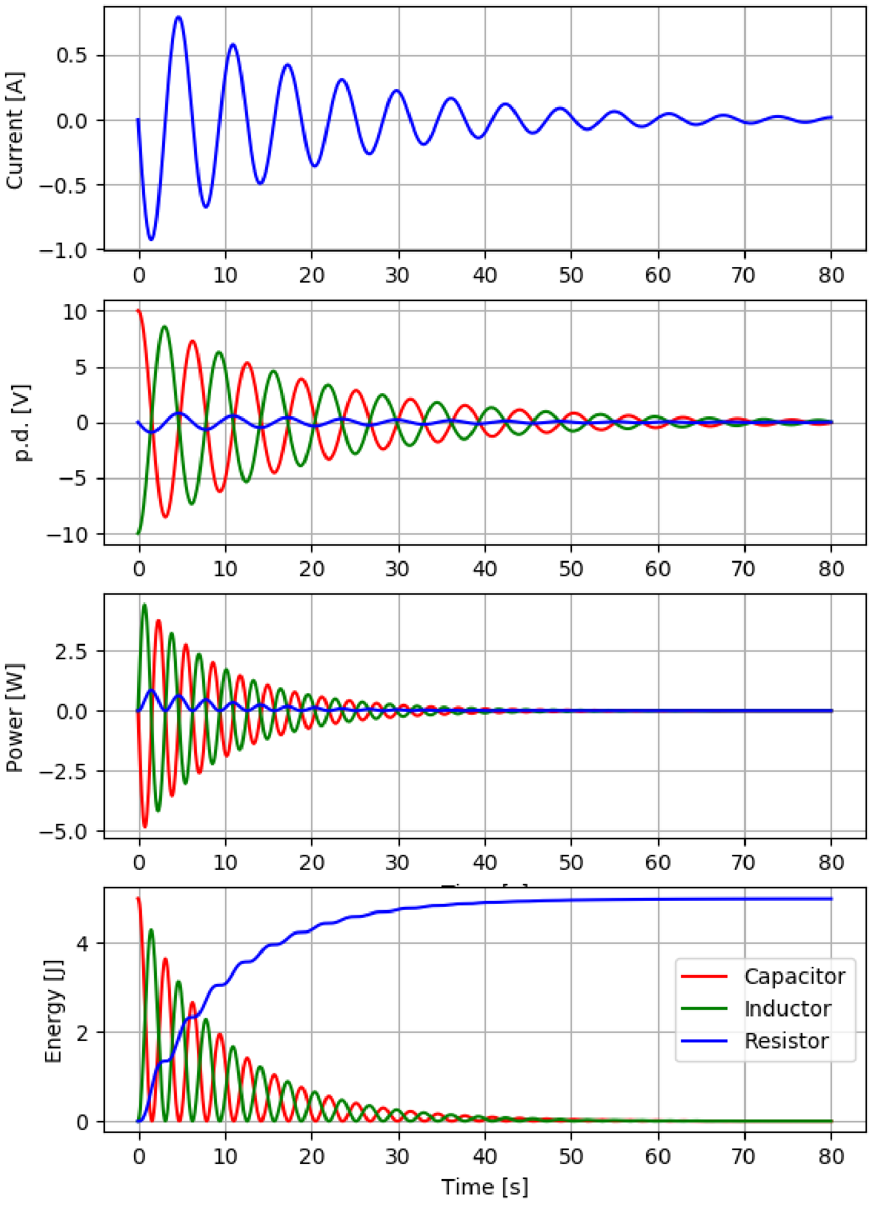

Figure 5.3: Oscillation of the under-damped LCR Circuit ($L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$, $R = 1\,\Omega$, $\mathcal{E} = 10$ V). From top to bottom: current, potential difference, power and energy.

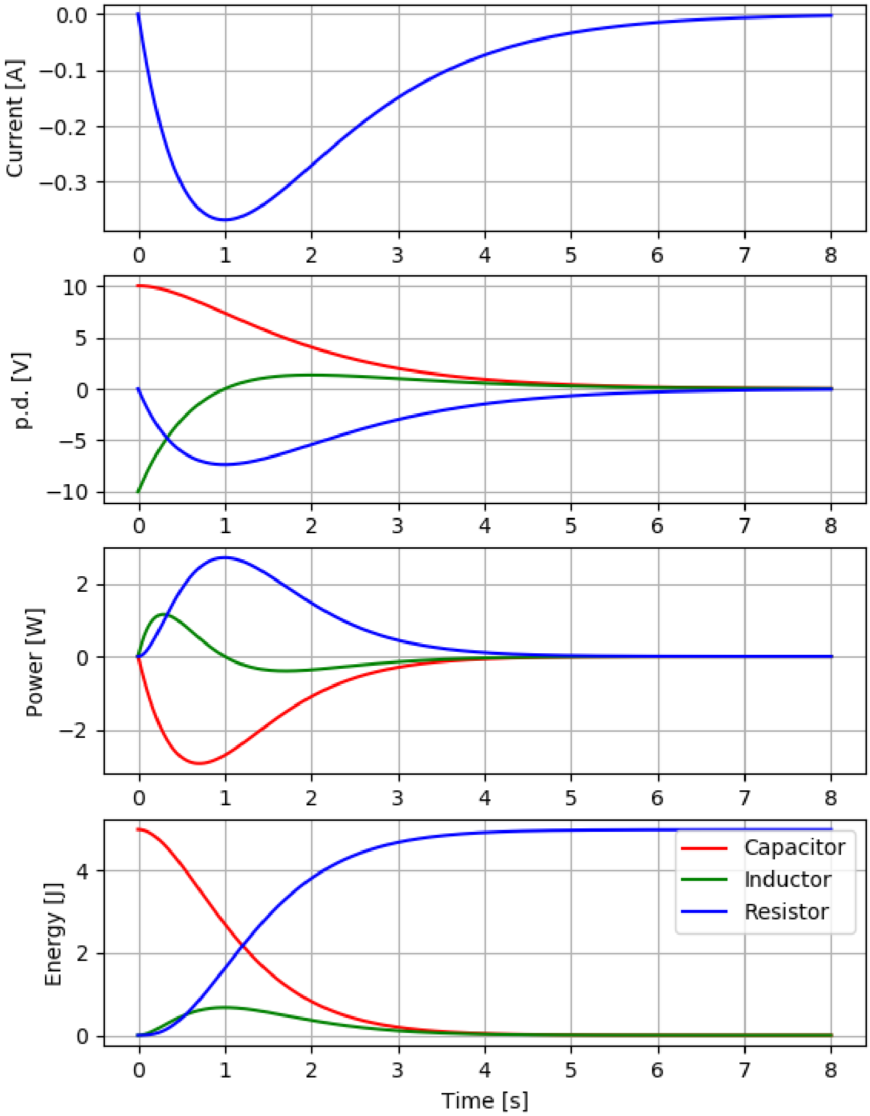

Figure 5.4: Critically-damped LCR Circuit ($L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$, $R = 20\,\Omega$, $\mathcal{E} = 10\,\mathrm{V}$). From top to bottom: current, potential difference, power and energy.

Lecture 6 Forced Oscillations in the LCR Circuit

In section 5 we saw how the LCR circuit responds to an initial condition in the form of a charge q_0 placed on the capacitor for time $t < 0$. We did this by solving the homogeneous equation 5.1. Now we will look at the circuit's response to **forced oscillation** by **driving** the circuit with an AC source. This is equivalent to replacing our DC voltage-source v_S with an AC source (such as a signal generator).

6.1 Alternating Current

AC stands for **Alternating Current** but the term is quite loosely applied to any signal, either current or voltage, which is repetitive or **periodic**. For example, we may talk about the square-wave output from a signal-generator as being an example of an 'AC voltage'. Typically, however, we will drive our circuit with a sinusoidal signal.

AC Signals

We could imagine an almost arbitrary number of repetitive waveforms. A typical signal generator such as the ones used in the lab offer three outputs: sinusoidal, triangle and square-wave. The sinusoidal signal is the simplest case since it represents oscillation at a single frequency, though as you will have seen in the *Fourier* course *any* repetitive waveform can be constructed from a summation of sinusoidal signals at different frequencies. The characteristics of a sinusoidal signal such as equation 6.1 are summarised below. We can interpret v_S as the external manifestation of a voltage source with an oscillating EMF. The circuit symbol for the AC voltage source is shown in figure 6.2.

$$v_S = A\cos(\omega t + \phi) \tag{6.1}$$

Amplitude A, the difference between the maximum and zero.

Peak-to-peak Amplitude $2A$, often written $V_{pp} = 2A$, the difference between maximum and minimum.

Root Mean Square (RMS) Amplitude For a sinusoid this is $A/\sqrt{2}$, the equivalent DC voltage which would result in the same power dissipation as a sinusoid of amplitude A.

Period T, the minimum time between repeat of the waveform.

Frequency $f = 1/T$ with units of hertz (Hz).

Angular Frequency $\omega = 2\pi f$, with units of radians per second, and dimensionally the same as hertz.

Phase ϕ, with units of radians, where $-\pi < \phi \leq \pi$; increasing ϕ shifts the waveform earlier in time.

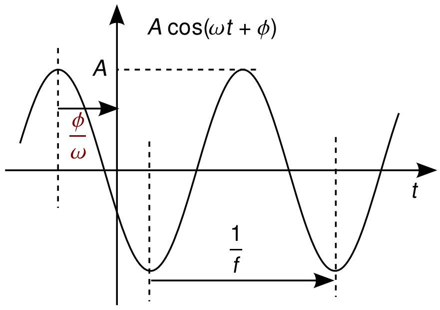

Figure 6.1: Amplitude, Period and Phase for a Sinusoidal Signal [V&W notes]

6.2 Driven LCR Circuit

The series LCR circuit is shown in figure 6.2. Kirchhoff's voltage law gives

$$v_S = v_L + v_R + v_C$$

Since the circuit is driven by a voltage source of the form $v_S = V_0 \cos \omega t$

$$V_0 \cos \omega t = L\frac{di}{dt} + Ri + \frac{q}{C}$$

Rewriting in terms of charge q, $\gamma = R/L$ and $\omega_0^2 = 1/LC$ gives

$$\frac{V_0}{L}\cos \omega t = \ddot{q} + \gamma\dot{q} + \omega_0^2 q \qquad (6.2)$$

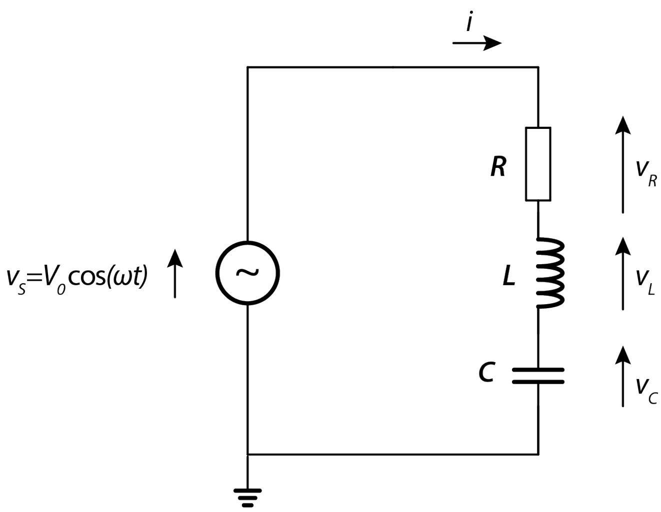

Figure 6.2: Driven Series LCR Circuit

Equivalence to the Mechanical Oscillator

The driven LCR circuit is the exact physical analogue of the forced mass-spring-damper mechanical system. Equation 6.2 is an inhomogeneous linear ordinary differential equation with the same form as the equation for the displacement x of the mass in the mass-spring-damper system:

$$\frac{F_0}{m}\cos\omega t = \ddot{x} + \gamma\dot{x} + \omega_0^2 x$$

Table 5.1 summarises the equivalences. We can see that the amplitude of the force F_0 is equivalent to the amplitude of the source's potential difference V_0.

Steady-state AC Analysis

The general solution to equation 6.2 was covered in the *Oscillations & Waves* course so we will not repeat this but it is useful to remember that the general solution was a superposition of the **transient** (or 'startup') behaviour with the **steady-state**. We will not look at the transient behaviour but will see how we can apply AC circuit analysis to understand the circuit's steady-state behaviour.

> Recall that we define the steady-state as when the energy into the system is balanced by the energy out. For the LCR circuit, this will be when the time-average of the source power is equal to the time-average of the dissipation in the resistor.

The steady-state is reached at a time much after the source has been switched-on, such that any initial transient behaviour has died-down and we are left with a repetitive periodic behaviour where all our AC signals (current, potential differences) have constant *amplitudes*.

It will help to use the complex form for charge $\tilde{q}$

$$\ddot{\tilde{q}} + \gamma\dot{\tilde{q}} + \omega_0^2\tilde{q} = \frac{V_0}{L}e^{j\omega t} \tag{6.3}$$

> Note that we use the letter j in electronics for the complex unit so as to avoid confusion with current i
> $$j = \sqrt{-1}$$

We expect a solution of the form

$$\tilde{q} = \tilde{Q}e^{j\omega t}$$

where ω is the frequency of oscillation and $\tilde{Q}$ is some complex constant

$$\tilde{Q} = Qe^{j\phi}$$

By comparison with the solution for the mechanical system we can directly write

$$\tilde{Q} = \frac{V_0/L}{\omega_0^2 - \omega^2 + j\omega\gamma}$$

which we can resolve into

$$Q = |\tilde{Q}| = \frac{V_0/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$$

$$\tan\phi = \frac{-\omega\gamma}{\omega_0^2 - \omega^2} \tag{6.4}$$

Hence the real physically-observable charge, in the steady-state is

$$q = \Re\{\tilde{q}\} = \Re\{Qe^{j\phi}e^{j\omega t}\}$$

$$q = \Re\{\tilde{q}\} = \frac{V_0/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \cos(\omega t + \phi) \tag{6.5}$$

More typically, we might want to have the potential difference across the capacitor. Using $\tilde{q} = C\tilde{V}$

$$v_C = \frac{q}{C} = \frac{V_0/LC}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \cos(\omega t + \phi) \tag{6.6}$$

6.3 Frequency Response of the Driven LCR Circuit

Equation 6.6 gives us the amplitude and phase of the capacitor potential difference

$$|\tilde{V}_C| = \frac{V_0/LC}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$$

$$\arg(\tilde{V}_C) = \phi = \arctan\left(\frac{-\omega\gamma}{\omega_0^2 - \omega^2}\right)$$

which we can represent in a 'frequency response' plot as illustrated in figure 6.3.

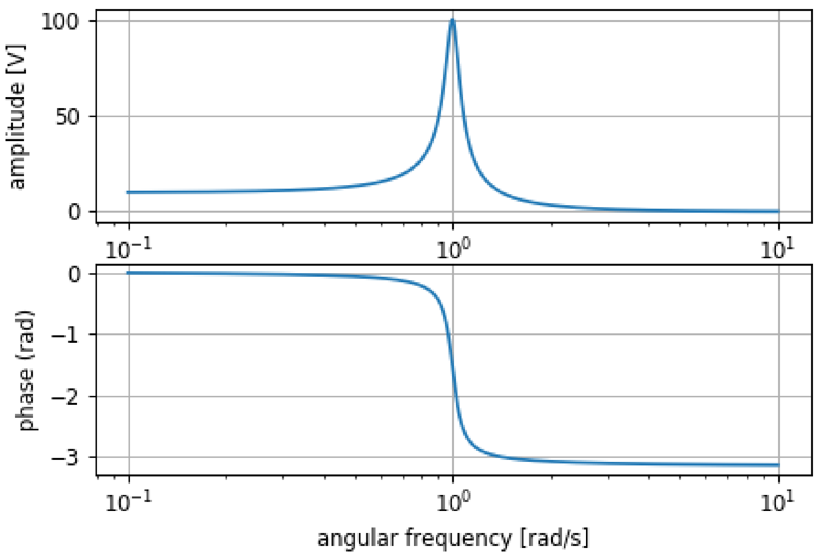

Figure 6.3: Magnitude and phase of $\tilde{V}_C$ versus Frequency. $L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$, $R = 1\,\Omega$ driven with a 10 volt AC source.

6.4 Phasors

It is mathematically convenient to assume a complex solution for the charge $\tilde{q}$ associated with the capacitor in the LCR circuit. There is nothing special about the use of complex notation; we could use trigonometric functions to solve all possible circuit problems, however the use of a complex *representation* makes the maths simpler. For example, if we want to find the current, then we can write

$$\tilde{I} = \dot{\tilde{q}} = j\omega\tilde{q}$$

which is much easier than trying to differentiate equation 6.5. Nevertheless, the approach taken in section 6.2 results in some quite tricky algebra, and in circuits with many components it is clear that it would be hard to keep track of all the current and potential-difference amplitudes and phases. Fortunately, the use of **phasors** (introduced in your *Oscillations & Waves* course) simplifies the analysis.

$\tilde{Q}$ is the phasor representation of the complex oscillating quantity $\tilde{q} = \tilde{Q}e^{j\omega t}$

The phasor approach comes from the **linearity** of the ordinary differential equation: all solutions representing currents and potential differences in our linear circuits will exhibit oscillation at the *same frequency*. Consequently, we can represent a signal (voltage or potential difference) by a vector in the complex plane which represents just the **amplitude** and **phase** of the quantity. When we consider phasors, we just have to remember that all signals are oscillating at the same frequency.

For example, the real physical signal

$$v = V_0 \cos(\omega t + \phi)$$

is the real part of the complex quantity

$$V_0 e^{j\omega t + \phi} = V_0 e^{j\omega t} e^{j\phi}$$

This can be thought of as a vector rotating anti-clockwise in the complex plane (figure 6.4). The projection of this vector onto the real axis describes co-sinusoidal oscillation. The amplitude of the vector is V_0 and the 'real' voltage, i.e. that which we would physically observe and measure, is the projection onto the real axis $V_0 \cos(\omega t + \phi)$.

The term **phasor** (for *phase-vector*) is used to describe this representation. The phasor is a vector in the complex plane with a magnitude representing the amplitude of the oscillation and an angle representing the phase.

Consequently, the signal

$$v = V_0 \cos(\omega t + \phi)$$

is represented by the phasor

$$\tilde{V} = V_0 e^{j\phi}$$

To return the real physical observable associated with this a phasor, we simply multiply by $e^{j\omega t}$ to include the time-dependence, and take the real part.

$$v = \Re\left\{\tilde{V} e^{j\omega t}\right\} = V_0 \cos(\omega t + \phi)$$

We will usually plot several phasors on a single phasor diagram since phasors can be drawn to represent all the oscillating potential differences and currents in our circuit. The phasor diagram gives a simple view of the relative phases and amplitudes of these quantities. The phasor diagram makes it much easier to see which quantity leads/lags which.

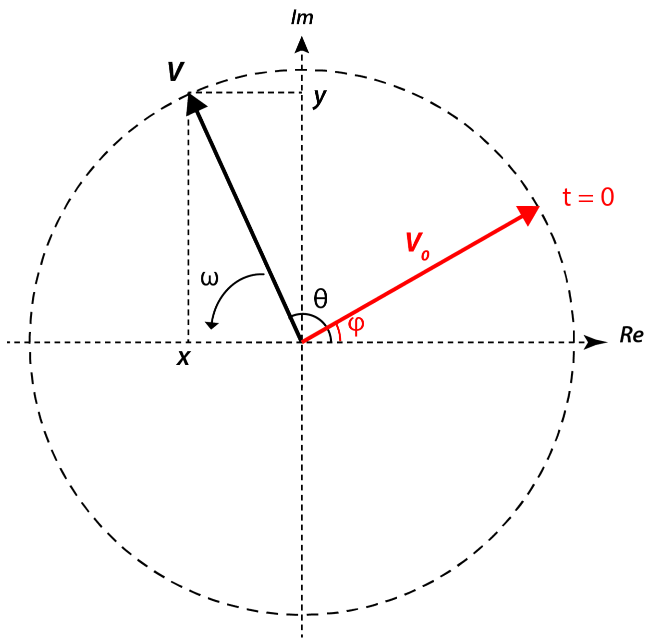

Figure 6.4: Complex representation of an oscillating quantity; for constant ω, the projection onto the x-axis is $V_0 \cos(\omega t + \phi)$

Phasor Manipulation

As phasors are complex quantities we can manipulate them by multiplication and division using the usual rules for complex numbers. Multiplying a phasor $\tilde{V}$ by a complex constant results in scaling and/or rotation in the complex plane. For example:

1. Multiplying $\tilde{V}$ by -1 results in $-\tilde{V}$, equivalent to a **rotation** by an angle π radians
2. Multiplying by j results in anti-clockwise rotation $\pi/2$ radians;
3. Multiplication by $-2j$ results in rotation $-\pi/2$ radians (or $+3\pi/2$ if you prefer) *and* scaling by a factor 2.

To summarise, we **create a phasor** representation of of a signal as follows:

1. Express the signal as a complex exponential.
2. Remove the time-dependent part.
3. The result will be a vector in the complex plane with the length of the signal's amplitude and the angle of the signal's phase. For example

$$v = V_0 \cos(\omega t + \phi) \rightarrow \tilde{V} = V_0 e^{j\phi}$$

Manipulation of the phasor is done by multiplying (or dividing) by complex quantities which both scale the length of the phasor and rotate it in the complex plane. For example

$$\frac{\tilde{V}}{\tilde{Z}} = \frac{V_0 e^{j\phi}}{Z e^{j\alpha}} = \frac{V_0}{Z} e^{j(\phi-\alpha)}$$

As we will see, if $\tilde{Z}$ represents an *impedance* with units of ohms, then the result will be a current.

To **recover the signal as a function of time**

1. Multiply by $e^{j\omega t}$ (all voltages and currents oscillate at the same frequency)
2. Take the real part. For example

$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} \rightarrow i = \frac{V_0}{Z} \cos(\omega t + \phi - \alpha)$$

Table 6.1 gives some useful complex identities.

Operation	Multiply by
Scale by A (without rotation)	A
Rotation by $\pi/2$	$e^{j\pi/2}$ (or j)
Rotation by π	$e^{j\pi}$ (or -1)
Rotation by $-\pi/2$	$e^{-j\pi/2}$ (or $-j$)
Rotation by arbitrary angle θ	$e^{j\theta}$

Table 6.1: Some complex identities useful for for phasor manipulation

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 31: Alternating Current

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapter 3: Alternating Current Circuits

Lecture 7 Complex Impedance

We have a set of linear relations which govern our circuit behaviour

$$i = \frac{dq}{dt} \tag{7.1}$$
$$v_R = iR \tag{7.2}$$
$$v_L = L\frac{di}{dt} \tag{7.3}$$
$$i = C\frac{dv_C}{dt} \tag{7.4}$$

7.1 Complex Impedance Defined

For the resistor the *amplitudes* of the potential difference and current are linearly related, and of course both are always in-phase with each other. The constant of proportionality is **resistance**, with units of ohms.

When our potential difference and current quantities are in phasor form then the complex *equivalent* to resistance is the **Impedance**.

> For any linear component, or network of components, where the current phasor is $\tilde{I}$ and the potential difference phasor is $\tilde{V}$, the complex impedance is
>
> $$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}} \tag{7.5}$$
>
> Since impedance is a ratio of voltage to current, it also has units of ohms.

We can define the impedance for each of our components.

7.2 Impedances of the Circuit Elements

Using the definition of equation 7.5, we can determine impedance values for our circuit components L, C and R.

Inductor

For an inductor carrying a current of the form

$$i = I_0 \cos(\omega t)$$

the potential difference across its terminals will be

$$v_L = L\frac{di}{dt} = -I_0\omega L \sin(\omega t) = I_0\omega L \cos\left(\omega t + \frac{\pi}{2}\right) \tag{7.6}$$

What we see is that the amplitude of the oscillation is scaled by ωL, and the phase of the oscillation rotated by $\pi/2$. Writing these in phasor form, the current phasor is

$$\tilde{I} = I_0 e^{j0} = I_0$$

(The phase angle is zero by choice, but without loss of generality). The potential difference phasor is

$$\tilde{V}_L = \omega L I_0 e^{j\pi/2} = j\omega L I_0$$

Hence the impedance

$$\tilde{Z}_L = \frac{\tilde{V}_L}{\tilde{I}} = j\omega L \tag{7.7}$$

The inductor impedance is both imaginary and frequency-dependent.

Capacitor

We can use the same methods for the capacitor. Let

$$v_C = V_0 \cos(\omega t)$$

The current through the capacitor

$$i = C\frac{dv_C}{dt} = -V_0\omega C \sin\omega t = V_0\omega C\cos(\omega t + \pi/2)$$

Again, in phasor form

$$\tilde{V}_C = V_0$$

and

$$\tilde{I} = j\omega C V_0$$

giving the impedance of the capacitor

$$\tilde{Z}_C = \frac{\tilde{V}_C}{\tilde{I}} = \frac{1}{j\omega C} = \frac{-j}{\omega C} \tag{7.8}$$

The capacitor impedance is also frequency-dependent and imaginary.

Impedance of the Resistor

For completeness, the voltage across the resistor is always in phase with the current (Ohm's law), so

$$\tilde{V}_R = V_0$$
$$\tilde{I} = \frac{V_0}{R}$$

and the complex impedance of the resistor is

$$\tilde{Z}_R = \frac{\tilde{V}_R}{\tilde{I}} = R \tag{7.9}$$

Note that the resistor impedance is entirely real, representing the in-phase nature of the voltage and current signals, while the capacitor and inductor impedances are entirely imaginary, encoding the $\pi/2$ phase shifts between voltage and current.

7.3 Equivalent Impedance

The utility of the complex impedance method is that it allows us to combine multiple components with rules as for simple resistances.

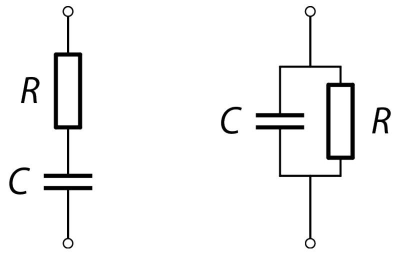

Figure 7.1: Equivalent impedances: series (left) and parallel (right)

Impedances in Series

The equivalent impedance of series impedances is

$$\tilde{Z}_{eq} = \sum_{k=1}^{n} \tilde{Z}_k \tag{7.10}$$

For example, in figure 7.1 (left), the combined total impedance $\tilde{Z}$ is given by

$$\tilde{Z} = \tilde{Z}_R + \tilde{Z}_C = R - \frac{j}{\omega_C}$$

Impedances in Parallel

The equivalent impedance of parallel impedances (e.g. figure 7.1, right) is

$$\frac{1}{\tilde{Z}_{eq}} = \sum_{k=1}^{n} \frac{1}{\tilde{Z}_k} \tag{7.11}$$

Consequently, combining resistances with **reactive** components results in an equivalent impedance with *both real and imaginary* parts.

7.4 AC Analysis

As long as we follow all the rules for manipulation of complex quantities, we can use complex impedances just like resistances with Ohm's law. Using the phasor representation for voltage $\tilde{V}$ and current $\tilde{I}$

$$\tilde{Z} = \frac{\tilde{V}}{\tilde{I}}$$

we can find an unknown voltage from a current by

$$\tilde{V} = \tilde{I}\tilde{Z}$$

Finding the Phase

We can determine the phase shift of the voltage with respect to the current by plotting the result on a phasor diagram. We can also note that the phase shift will be the argument of $\tilde{Z}$. To understand this, it again helps to use the complex exponential form. For example:

$$\tilde{Z}_L = j\omega L = e^{j\pi/2}\omega L$$

Consequently, multiplying a current phasor $\tilde{I}$ by an inductor impedance $\tilde{Z}_L$ results in a voltage phasor $\tilde{V}$ which must lead the current by $\pi/2$ radians.

Finding the Amplitudes

Furthermore, the **amplitudes** of the voltages and currents are related by the **magnitude** (or **modulus**) of the impedance.

$$|\tilde{Z}| = \frac{|\tilde{V}|}{|\tilde{I}|}$$

Circuit Rules

The circuit rules of section 2 can be extended to AC circuits. Kirchhoff's voltage law always applies in that the *instantaneous* sum of voltages around any loop must always be zero. On a phasor diagram, this means that the vector sum of voltages is zero, i.e. both the real and imaginary voltage components sum to zero. We can also plot currents as phasors and Kirchhoff's current law also applies; the vector sum of currents into a junction is zero. We can also use the complex equivalents of the voltage and current divider rules. The use of these principles is best understood by some examples.

Using complex impedances requires care and a bit of practice, but so long as we follow all the rules for complex numbers and the relationships described above, the results fall-out remarkably easily. The expressions for complex impedance (equations 7.7 to 7.9) are usually easiest to remember in the cartesian form, but it's often easiest to convert combined impedances into exponential form in order to simplify the maths.
We will now re-visit the series RC and RL circuits in order to understand their behaviour in response to AC signals.

7.5 Series RC Circuit

The RC circuit of figure 7.2 consists of a resistor and capacitor in series; the input signal v_s is applied to the series combination while v_C is the potential difference across the capacitor alone. Kirchhoff's Voltage Law gives

$$v_S = v_R + v_C$$

and using phasors for the potential differences

$$\tilde{V}_S = \tilde{V}_R + \tilde{V}_C$$

The combined series impedance of the circuit is

$$\tilde{Z} = \tilde{Z}_R + \tilde{Z}_C = R - \frac{j}{\omega C}$$

Hence the current

$$\tilde{I} = \frac{\tilde{V}_S}{\tilde{Z}}$$

The potential difference across the capacitor is

$$\tilde{V}_C = \tilde{I}\tilde{Z}_C$$

and across the resistor

$$\tilde{V}_R = \tilde{I}\tilde{Z}_R$$

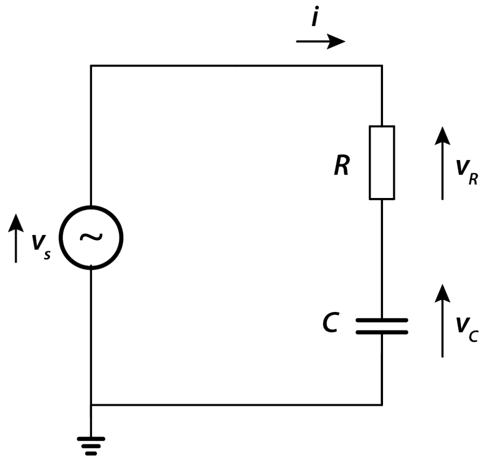

Figure 7.2: Series RC Circuit driven with AC Source

Impedance Divider

R and C form an **impedance-divider** much as two series resistors form a resistive-divider. As with the resistive potential divider, there is a useful formula to give the p.d. across one of the components:

$$\tilde{V}_C = \tilde{V}_S \times \frac{\tilde{Z}_C}{\tilde{Z}_R + \tilde{Z}_C} = \tilde{V}_S \times \frac{-j/\omega C}{R - j/\omega C} \tag{7.12}$$

Frequency Dependence

Since the capacitor's impedance is inversely proportional to frequency, we can make some predictions about the behaviour:

1. For low frequencies, $\omega \to 0$, the capacitor's impedance $\tilde{Z}_C \to \infty$ and hence $\tilde{V}_C \to \tilde{V}_s$.
2. For high frequencies, $\omega \to \infty$, the capacitor's impedance $\tilde{Z}_C \to 0$ so $\tilde{V}_C \to 0$.

Consequently, figure 7.2 is described as a **filter** since it's frequency-dependent behaviour allows us to selectively pass (or block) signals depending on their frequency. We will come back to this topic in the next lecture.

AC Analysis

For an input signal

$$v_S = V_0 \cos \omega t$$

we may wish to find the potential difference across the capacitor (or resistor). We start by writing the input as a phasor

$$\tilde{V}_S = V_0$$

The p.d. across the capacitor is

$$\tilde{V}_C = V_0 \times \frac{-j/\omega C}{R - j/\omega C} = V_0 \times \frac{1 - j\omega CR}{1 + (\omega CR)^2}$$

Expressing $\tilde{V}_C$ in polar form

$$\tilde{V}_C = Ae^{j\phi}$$

This resolves into

$$A = \frac{V_0}{\sqrt{1 + (\omega CR)^2}} \tag{7.13}$$

$$\tan\phi = -\omega CR \tag{7.14}$$

We can write the actual physical signal as

$$v_C(t) = \Re\{\tilde{V}_C e^{j\omega t}\} = A\cos(\omega t + \phi)$$

For example, we could choose to drive the RC circuit with a 10 volt AC source at a frequency $\omega = 1/RC$ such that

$$\begin{aligned} v_S(t) &= 10\cos\omega t \\ A &= \frac{10}{\sqrt{2}} \\ \phi &= -\frac{\pi}{4} \\ v_C(t) &= \frac{10}{\sqrt{2}}\cos(\omega t - \pi/4) \end{aligned}$$

We can visualise the steady-state solution with an argand diagram, as shown in figure 7.3, which also illustrates the phasor for the resistor potential difference. Kirchhoff's laws hold for phasors so

$$\tilde{V}_R = 10 - (5 - 5j) = 5 + 5j\,\text{volts}$$

which will give

$$v_R(t) = \frac{10}{\sqrt{2}}\cos(\omega t + \pi/4)$$

Recalling that we can consider the phasor diagram as a 'snapshot' at time $t = 0$, with the physical signals being the projection onto the real axis, the diagram shows

1. $v_R(t = 0) = 5\,\text{V}$.
2. $v_C(t = 0) = 5\,\text{V}$.
3. The phase of v_R leads the phase of v_C by $\pi/2$ rad.
4. The phase of v_R leads v_S by $\pi/4$ rad.

We can also illustrate this with a time-series plot for the steady-state solution as shown in figure 7.4

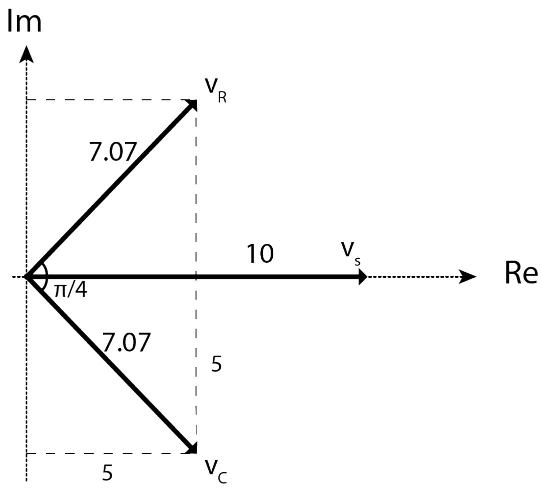

Figure 7.3: Argand Diagram for the Series RC Circuit with $R = 10\,\mathrm{k}\Omega$, $C = 100\,\mu\mathrm{F}$, driven with an AC source $V_0 = 10\,\mathrm{V}$ at angular frequency $\omega = 1\,\mathrm{rad/s}$. Note that all phasors have units of volts.

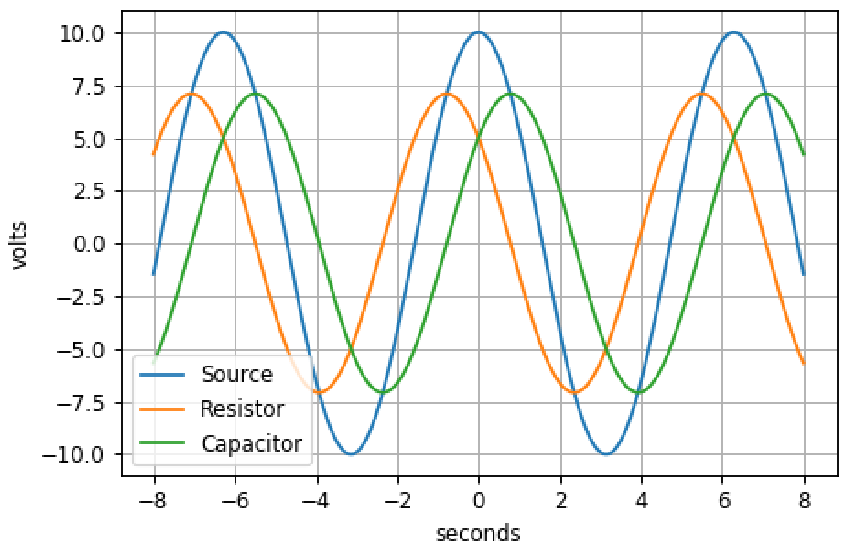

Figure 7.4: Series RC Circuit ($R = 10\,\mathrm{k}\Omega$, $C = 100\,\mu\mathrm{F}$), steady-state solution for $V_0 = 10\,\mathrm{V}$ $\omega = 1$ rad/s.

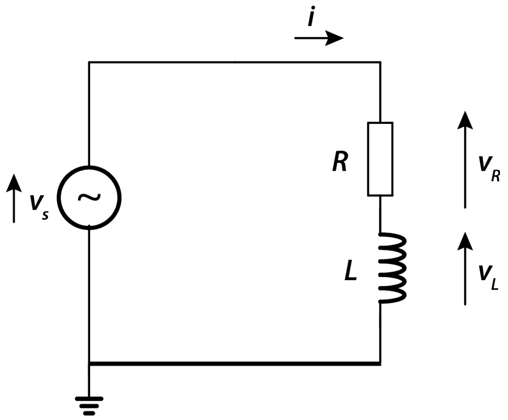

Figure 7.5: Series RL Circuit driven with AC Source

7.6 Series RL Circuit

We can replace the capacitor with an inductor as shown in figure 7.5.
The combined impedance is

$$\tilde{Z} = \tilde{Z}_R + \tilde{Z}_L = R + j\omega L$$

The potential difference across the inductor is

$$\tilde{V}_L = \tilde{V}_S \times \frac{\tilde{Z}_L}{\tilde{Z}_R + \tilde{Z}_L} = \tilde{V}_S \times \frac{j\omega L}{R + j\omega L}$$

1. For high frequencies, $\omega \to \infty$, the inductor's impedance $\tilde{Z}_L \to \infty$ and hence $\tilde{V}_L \to \tilde{V}_s$.
2. For low frequencies, $\omega \to 0$, the inductor's impedance $\tilde{Z}_C \to 0$ so $\tilde{V}_L \to 0$.

Note that this is the opposite behaviour to the capacitor.

The AC analysis of section 7.5 can be repeated for the example case $\omega = R/L$.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 31: Alternating Current

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapter 3: Alternating Current Circuits

Lecture 8 Filters

The frequency-dependent behaviour of RL and RC circuits makes them useful for a type of circuit known as a **filter** which is used to select (or remove) a range of frequencies from a signal. From your *Fourier* course you will be familiar with the idea that a simple signal such as a square-wave contains frequency components from the fundamental to infinity. Real-world signals have particular frequency characteristics too. However, real-world signals are always subject to noise and interference, which may have different frequency characteristics to the signal. We can use a filter to help separate the signal from the noise.

8.1 Low-pass RC Filter

An example of a simple filter is shown in figure 8.1. Note that since the signal is assumed to come from some source (e.g. a sensor of some kind) and assumed to then go on to some measurement device (e.g. a 'scope) then typically the source and destination are not shown. Of course, in reality there will always be complete circuits, however for the filter circuit we are mainly interested in the relation between the signal input to the filter and the output from the filter. The use of complex quantities will make the frequency-dependence of this input/output relationship much easier to visualise.

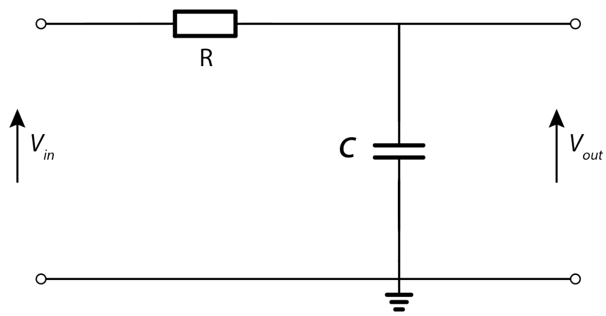

Figure 8.1: Low Pass RC Filter Circuit.

Since the filter is just another example of the impedance-divider, we can use equation 7.12 from section 7.5 to write

$$\tilde{V}_{out} = \tilde{V}_{in} \times \frac{\tilde{Z}_C}{\tilde{Z}_R + \tilde{Z}_C}$$

Equation 7.13 gives the amplitude of the capacitor p.d. as $A = V_0/\sqrt{1+(\omega CR)^2}$; we can consider 2 cases:

Low frequencies, $\omega CR \ll 1$: The capacitor amplitude A will be approximately V_0, the input amplitude.

High frequencies, $\omega CR \gg 1$: The capacitor amplitude A will tend to zero.

Hence the description of this circuit as a **low-pass filter**: low-frequencies *'pass through'* the filter, while high-frequencies are *'blocked'*. We can formalise the input/output relationship by defining the **gain** of the filter.

8.2 Gain

The filter is characterised by the **gain**, which is the ratio of output to input voltage phasors. Hence the gain is a complex quantity $\tilde{G}$ where

$$\tilde{G} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j\omega RC} \tag{8.1}$$

It is useful to know the magnitude and argument of the complex gain so writing

$$\tilde{G} = Ge^{j\phi}$$

we can find

$$G = |\tilde{G}| = \frac{1}{\sqrt{1 + (\omega CR)^2}} \tag{8.2}$$

$$\tan\phi = -\omega RC$$

$$\phi = -\tan^{-1}\omega RC \tag{8.3}$$

8.3 Bode Plot

To visualise $\tilde{G}$ it is convenient to plot the **magnitude** G versus frequency on a log-log plot. The argument or **phase** ϕ is usually shown in degrees (by convention) on a log-linear plot. Together, these two plots are commonly known as the **Bode Plot**. The general form is shown in figure 8.2, which uses $RC = 1\,\mathrm{s}$ for convenience.

Decibels

The logarithm of the gain magnitude is, by convention, expressed in dB (decibels).

$$G_{\mathrm{dB}} = 20\log_{10} G \tag{8.4}$$

The decibel is a dimensionless measure of **relative power** defined as

$$P_{\mathrm{dB}} = 10\log_{10}\frac{P_2}{P_1}$$

P_{dB} expresses the power P_2 relative to P_1 so if the ratio is 100 then $P_{\mathrm{dB}} = 20\,\mathrm{dB}$ and if the ratio is 1/2 then $P_{\mathrm{dB}} = -3.01\,\mathrm{dB}$. Whilst the decibel always expresses relative power, we can consider the power in a signal to be proportional to the square of the amplitude ($P = V^2/R$) so we can express the ratio of two *amplitudes* as

$$P_{\mathrm{dB}} = 20\log_{10}\frac{V_2}{V_1}$$

Since G is a ratio of amplitudes, equation 8.4 expresses the relative power in the output of the filter to the input of the filter.

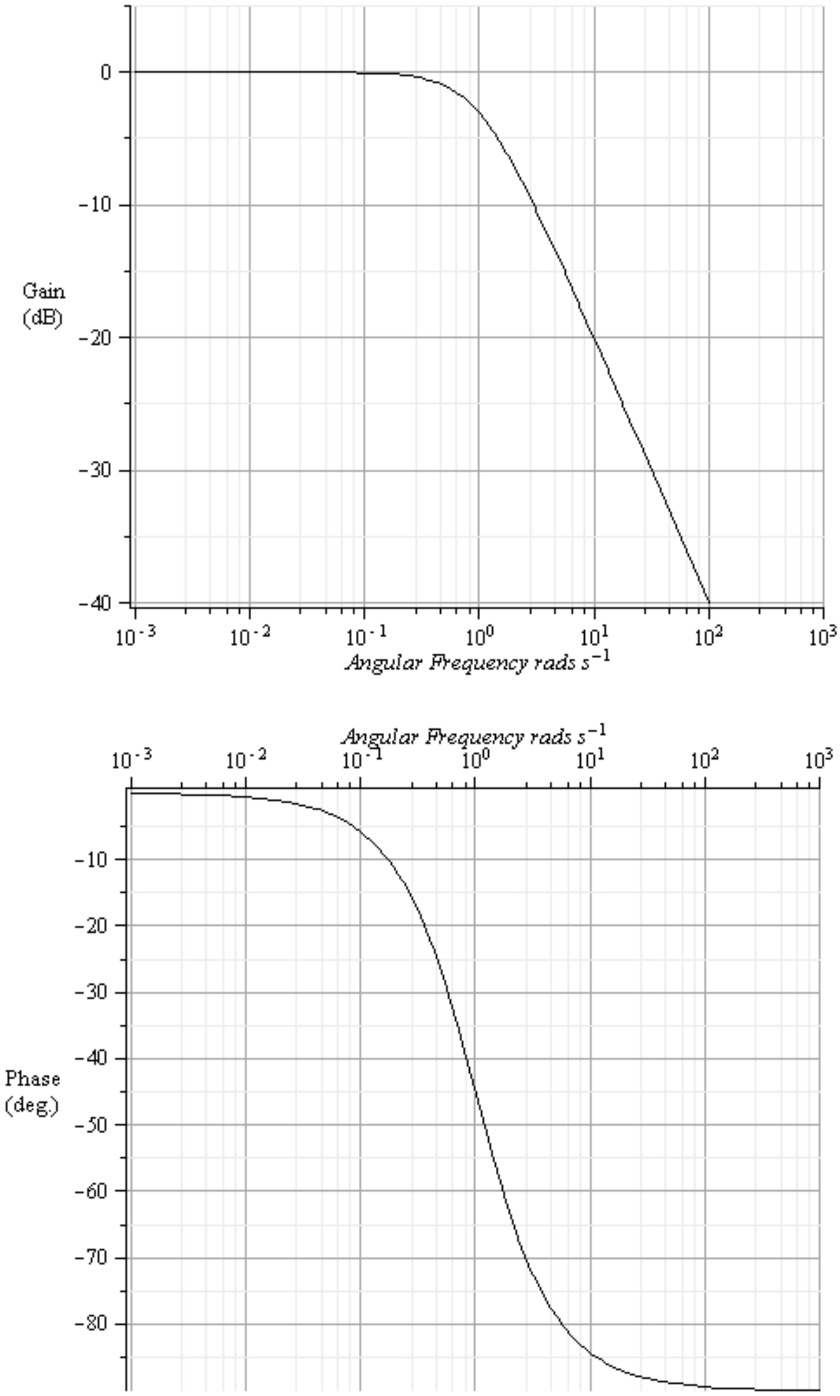

Figure 8.2: Magnitude and Gain of the Low Pass Filter ($RC = 1$).

Cut-off Frequency

All low-pass RC filters will have the same general shape of figure 8.2. For low frequencies, $G \approx 1$ (0 dB), which means that $v_{out} \approx v_{in}$; this region is called the **pass-band** of the filter. The frequency at which the filter starts to attenuate can be adjusted by changing the values R and C. Since there is no sharp transition from the pass-band to the **roll-off**, by convention we choose the frequency where the power of the output signal is half the input which can be found by solving

$$G^2 = \frac{1}{2} = \frac{1}{1 + (\omega_c C R)^2}$$

which gives the **cut-off angular frequency**

$$\omega_c = \frac{1}{RC} = \frac{1}{\tau}$$

where τ is the **time-constant** of the RC circuit.

At the frequency ω_c, the gain is $G = 1/\sqrt{2}$ so $g_{dB} = -3.01\,\text{dB}$, as seen in the top-panel of figure 8.2. Note also that the phase will be $-45°$.

Roll-Off

An ideal low-pass filter would pass all frequencies below the cut-off and block all frequencies above. This is impossible to achieve in real-world filters. For the RC low-pass filter, it can be seen that for $\omega \gg \omega_c$,

$$G \approx \frac{\omega_C}{\omega}$$

which results in a gradient of -20 dB/decade, where a decade is a factor of 10 in frequency. For example, the gain falls by -20 dB between the frequencies 10 and 100 rad/s.

Phase

Conventionally, the phase of the Bode plot is expressed linearly in degrees. The phase of the low-pass filter expresses the phase change between the output versus the input and the negative sign indicates that the phase of the output lags the input. We can see this by re-arranging equation 8.1:

$$\tilde{V}_{out} = \tilde{V}_{in}\tilde{G}$$

8.4 High-Pass RC Filter

The circuit of figure 8.1 is a *low-pass filter*, as shown by equation 8.2. We can turn this into a **high-pass filter** by swapping the resistor and the capacitor (this means we take the output voltage across the resistor) as shown in figure 8.3
Performing the same circuit analysis gives the gain as

$$G = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

$$\phi = \tan^{-1}\frac{1}{\omega RC}$$

The cut-off frequency is again

$$\omega_c = \frac{1}{RC}$$

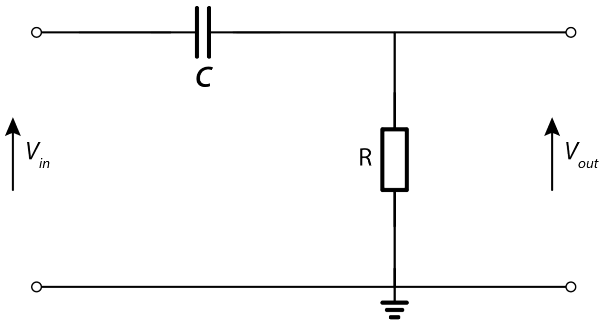

Figure 8.3: High Pass RC Filter Circuit.

and it's often more intuitive to write the gain magnitude in terms of ω_C

$$G = \frac{\omega/\omega_C}{\sqrt{1+(\omega/\omega_C)^2}}$$

For low frequencies, $G \propto \omega$ (the gradient is $+20\,\mathrm{dB/decade}$) while for $\omega \gg \omega_c$, $G \to 1$.

8.5 RL Filters

Since the impedance of the inductor is also frequency-dependent, we can build RL filters with low and high-pass characteristics. In the circuit of figure 8.1 the capacitor can be replaced with an inductor to give figure 8.4. $|Z_L| \propto \omega$ so this is a high-pass RL filter. The same principles, starting with the impedance divider, can be used to derive the gain of the RL filter, and again we can make a low-pass RL filter by swapping the resistor and inductor.

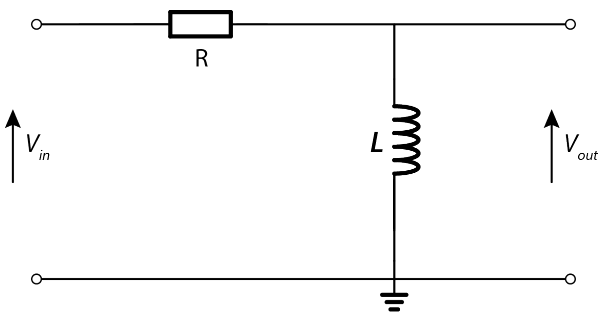

Figure 8.4: High Pass RL Filter Circuit.

8.6 Interpretation of Gain and the Bode Plot

Since the gain is the ratio of output/input phasors, we can use our phasor approach to get the output signal for any given input. For example, for $v_{in}(t) = V_0 \cos \omega t$ we write this as $\tilde{V}_{in} = V_0$ the output will be

$$\tilde{V}_{out} = \tilde{V}_{in}\tilde{G} = V_0 G e^{j\phi}$$

and

$$v_{out}(t) = V_0 G \cos(\omega t + \phi)$$

where G and ϕ are functions of R, C, L and ω, depending on the type of filter.

It will always be the case that

- The amplitude of the output will be scaled by a factor G
- The phase of the output will be shifted by ϕ

For the low-pass RC filter ϕ will be negative, while the high-pass filter always has a positive phase. All of the four RC and RL filter circuits are described as **first order passive filters** since they are composed of *'passive'* components (resistors, capacitors and inductors) and their behaviours are governed by first-order differential equations. For all such filters it will always be true that

- G will be a positive quantity ≤ 1
- ϕ will be in the range $\pm\pi/2$

8.7 Filter Applications

Filters are shown drawn as in figure 8.4 in order to indicate that the input signal V_{in} comes in from some **source** (not shown but assumed to be on the left) and the output signal V_{out} goes on to some **load** (not shown but assumed to be on the right). The purpose of the filter is to modify the frequency content of the signal.

Audio Example

The source could be a microphone, which is a sensor with characteristics of a voltage source. The microphone might be sensitive to frequencies up to 50 kHz, but the human ear can't hear hear much above 16 kHz or so. Consequently, it makes sense to 'filter-out' all frequencies above 16 kHz before amplifying the signal. The amplifier, typically known as a 'pre-amp' in audio applications, would appear as a load resistance. If we wish to study the setup, the circuit would look like figure 8.5.

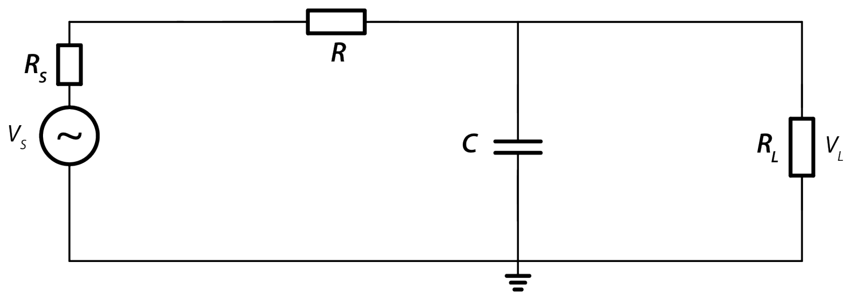

Figure 8.5: A low-pass filter used in an audio example. The source may be a microphone and the load may be a pre-amplifier.

The microphone is represented as a voltage source V_S (note that in practice this may have some small output resistance, typically $R_S = 150\,\Omega$ or so). The pre-amp is represented by a load resistor R_L. A typical pre-amp would have an **input resistance** $R_L = 10\,\text{k}\Omega$. The time-constant $\tau = RC$ would be chosen to give a filter-cut-off frequency about 16 kHz. Typically, we would use a value of R which satisfies $R_S \ll R \ll R_L$ so as to avoid **loading** (section 2.3).

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 31: Alternating Current

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapters 3 & 4: Alternating Current Circuits

Lecture 9 LCR Circuit - AC Analysis

In lecture 6 we looked at the second-order LCR circuit driven by an AC source. We saw that this was analogous to the mechanical mass-spring-damper system and we were able to apply the same solutions to the series LCR circuit. In this lecture, we will apply our *AC analysis* phasor approach to the LCR circuit, and hopefully we will see that this approach greatly simplifies the analysis and gives insights into the underlying physical behaviour. Furthermore, the use of complex impedance will allow us to analyse other circuit topologies without needing to go back to the differential equations.

9.1 Series LCR Circuit

The series LCR circuit is shown again in figure 9.1. Kirchhoff gives

$$v_S = v_L + v_R + v_C$$

in phasor form

$$\tilde{V}_S = \tilde{V}_L + \tilde{V}_R + \tilde{V}_C$$

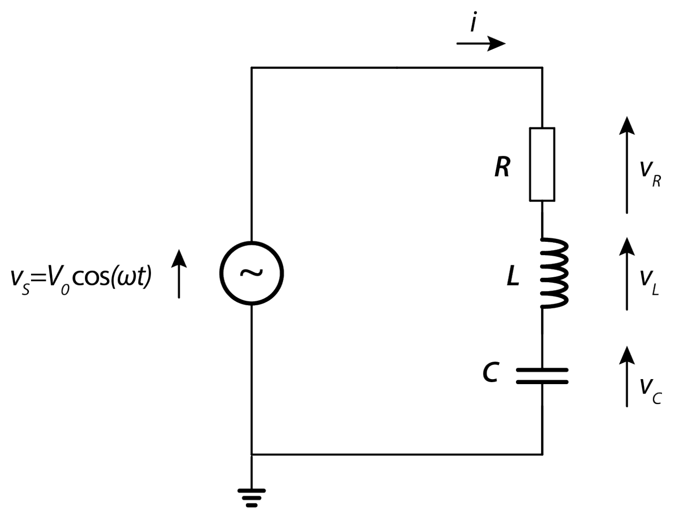

Figure 9.1: Driven Series LCR Circuit

The steady-state is reached at a time much after the source has been switched-on, such that any initial transient behaviour has died-down and we are left with a periodic behaviour at constant amplitude. AC circuit theory provides an easy method to characterise the LCR circuit in the steady-state, where we assume that the circuit is driven with a signal of the form $v_S(t) = V_0 \cos \omega t$ which has existed for all time. As usual, v_S can be be represented by a phasor $\tilde{V}_S = V_0$

The complex impedance of the series LCR circuit is

$$\tilde{Z} = R + j\omega L - \frac{j}{\omega C}$$

The current in the circuit is

$$\tilde{I} = \frac{\tilde{V}_S}{\tilde{Z}} = \frac{V_0}{\tilde{Z}} \tag{9.1}$$

Writing the current in the polar form $\tilde{I} = I_0 e^{j\phi}$ it just requires some algebra to get

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{9.2}$$

$$\tan\phi = \frac{-(\omega L - 1/\omega C)}{R} \tag{9.3}$$

The magnitude and phase of the current are represented in figure 9.2. Note that this is the same result as we found in section 6.3, however it is much easier to obtain this result by the complex impedance method. Furthermore, the potential differences across the three components can easily be found from

$$\tilde{V}_R = \tilde{I}R$$
$$\tilde{V}_L = \tilde{I}\tilde{Z}_L$$
$$\tilde{V}_C = \tilde{I}\tilde{Z}_C$$

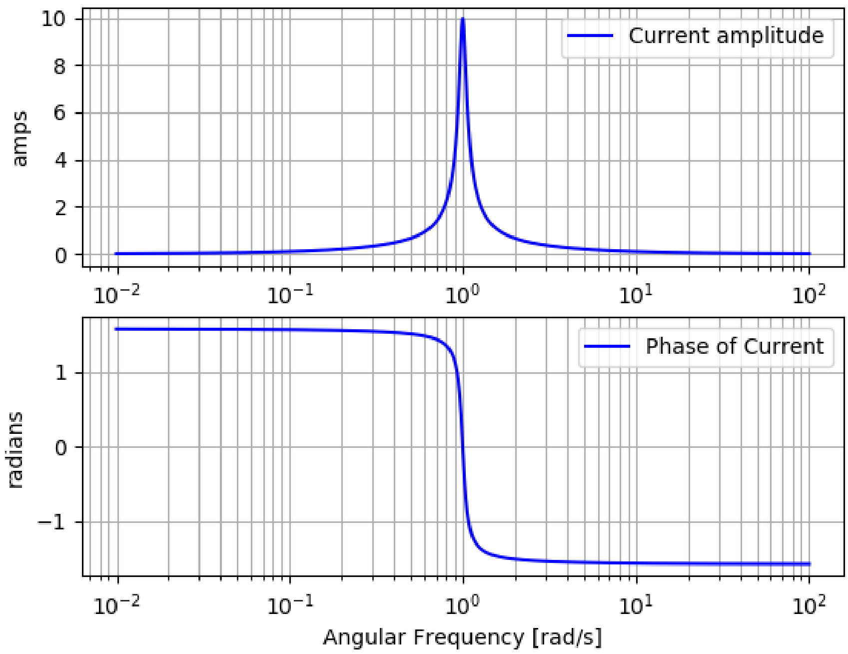

Figure 9.2: Magnitude of the Current versus Frequency, $L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$, $R = 1\,\Omega$, driven with an AC source $V_0 = 10\,\mathrm{V}$

9.2 Resonance

We can interpret the form of figure 9.2 by examining the frequency-dependent behaviour of each component. For $\omega \to 0$ the impedance of the capacitor $\tilde{Z}_C \to \infty$ so the overall circuit impedance

$\tilde{Z} \to \infty$ and $\tilde{I} \to 0$. For $\omega \to \infty$ the impedance of the inductor $\tilde{Z}_L \to \infty$. The real (resistive) part of $\tilde{Z}$ is fixed but the imaginary part is a function of frequency and it is clear that $\tilde{Z}$ will be minimised for the frequency at which $\text{Im}(\tilde{Z})$ is zero:

$$j\omega L = \frac{j}{\omega C}$$

which occurs at the natural frequency

$$\omega = \omega_0 = \sqrt{\frac{1}{LC}}$$

Hence, when the voltage source is oscillating at the natural frequency ω_0

- The impedance is at a minimum.
- The current is at a maximum.
- The impedance is entirely real and equal to the resistance R
- $\tilde{I} = V_0/R$
- The voltage applied across the LCR series circuit will be in phase with the current.
- The voltage across the resistor will be at a maximum.
- The voltage across the inductor and capacitor will be equal in amplitude but opposite in phase.

9.3 Power Dissipation

Over a complete cycle of the oscillation, the inductor and capacitor store no net energy hence the average power in these components is zero. Only the resistor dissipates power according to

$$p_R(t) = v_R(t)i(t) = i(t)^2 R$$

Note that time-domain current $i(t)$ will be of the form

$$i(t) = I_0 \cos(\omega t + \phi)$$

so the resistor power $p_R(t)$ will have a $\cos^2$ form, and will vary at twice the frequency of the current, as we have seen in (for example) figure 5.3. The peak power, each cycle, will be $I_0^2 R$, and the minimum power is zero, hence, by the symmetry of the waveform, the time-average power will be half the peak.

$$P_{av} = \left\langle I_0^2 R \cos^2(\omega t + \phi)\right\rangle = I_0^2 R \left\langle \cos^2(\omega t + \phi)\right\rangle = \frac{I_0^2 R}{2} \tag{9.4}$$

P_{av} will be a function of frequency and is illustrated in figure 9.3.

9.4 Bandwidth

We saw that the LR and RC first-order circuits have a single cut-off frequency ω_C, however the LCR circuit has both lower ω_l and higher ω_h cut-off frequencies. The definition is the same: the cut-off frequencies are where the signal amplitude falls to a factor $1/\sqrt{2}$ of the maximum, or, equivalently, where the power falls to one half maximum. This of course can be the peak power or the average, so figure 9.3 illustrates how the **bandwidth** $\Delta\omega = \omega_h - \omega_l$ depends on R.

The power is maximum at $\omega = \omega_0$ where $p = I_0^2 R / = V_0^2/R$ It is left as an exercise to show that the bandwidth

$$\Delta\omega = \omega_h - \omega_l = \frac{R}{L} = \gamma \tag{9.5}$$

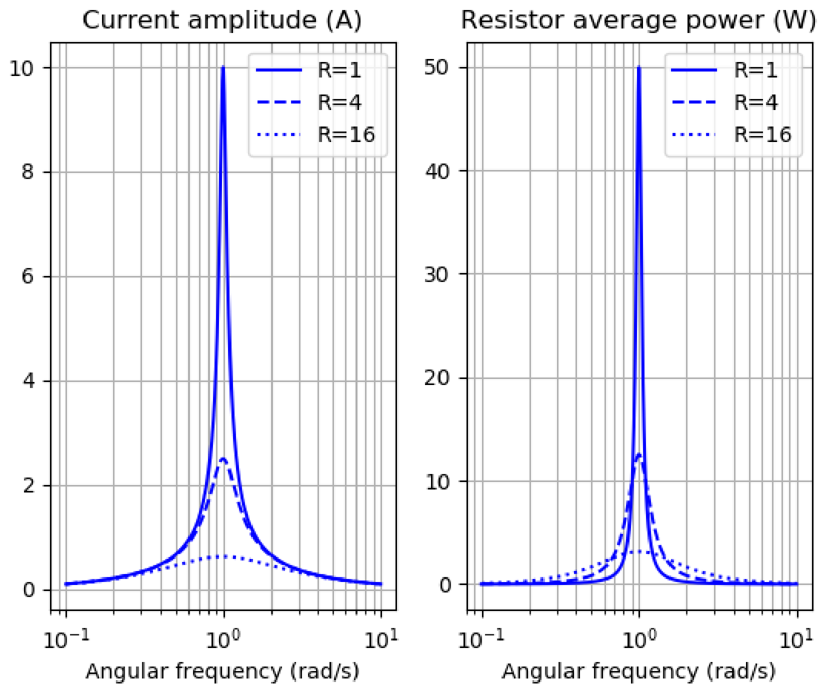

Figure 9.3: Amplitude of the circuit current and average resistor power dissipation, for various R and with $L = 10\,\text{H}$, $C = 0.1\,\mu\text{F}$ and $V_0 = 10\,\text{V}$. The series LCR circuit is under-damped for values of $R < 2\sqrt{L/C}$, here $R < 20\,\Omega$.

Q-factor

The shape of the curve in figure 9.3 is characterised by a dimensionless parameter Q known as the **Q-factor** or **quality-factor**.

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{9.6}$$

It is clear that a high Q-factor is associated with a narrow peak. Consequently, reducing the resistance in an LCR circuit increases the 'sharpness' of the resonance peak, and makes the circuit more 'selective' if we use it as a filter (section 9.7).

9.5 Worked Example

An LCR circuit with $L = 0.1\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$ and $R = 0.1\,\Omega$ has a natural frequency $\omega_0 = 10\,\mathrm{rad/s}$ and $\gamma = 1$. The quality-factor $Q = \omega_0/\gamma = 10$. The circuit is lightly damped so we expect a sharp resonance peak. For this example, the circuit is driven at the resonant frequency, i.e. $\omega = \omega_0$, and we'll take the amplitude of the driving source to be $1\,\mathrm{V}$. As a phasor this is

$$\tilde{V}_S = 1$$

We can find the impedances of the three components

$$\tilde{Z}_R = 0.1\,\Omega$$

$$\tilde{Z}_L = j\,\Omega$$

$$\tilde{Z}_C = -j\,\Omega$$

Therefore the total series impedance is

$$\tilde{Z} = 0.1\,\Omega$$

We can find the current to be

$$\tilde{I} = \frac{\tilde{V}_S}{\tilde{Z}} = 10\,\mathrm{A}$$

The current phasor is entirely real so the current is in phase with the source. We can find the potential differences across the components:

$$\tilde{V}_R = \tilde{I}\tilde{Z}_R = 1\,\mathrm{V}$$

$$\tilde{V}_L = \tilde{I}\tilde{Z}_L = 10j\,\mathrm{V}$$

$$\tilde{V}_C = \tilde{I}\tilde{Z}_C = -10j\,\mathrm{V}$$

We can see that at resonance, the p.d. across the resistor is equal to the source in both magnitude and phase, which we expect as the impedance is entirely real (resistive). The amplitudes of the p.d.'s across the capacitor and inductor are much larger, equal, and π radians out of phase with each other. It is helpful to visualise on the phasor diagram (figure 9.4).

To get the measurable, physical signal, we can add the time-dependence and take the real part

$$v_R = \mathrm{Re}\left\{1e^{j0}e^{j\omega t}\right\} = \cos(10t)$$

$$v_L = \mathrm{Re}\left\{10e^{j\pi/2}e^{j\omega t}\right\} = 10\cos(10t + \pi/2)$$

$$v_C = \mathrm{Re}\left\{10e^{-j\pi/2}e^{j\omega t}\right\} = 10\cos(10t - \pi/2)$$

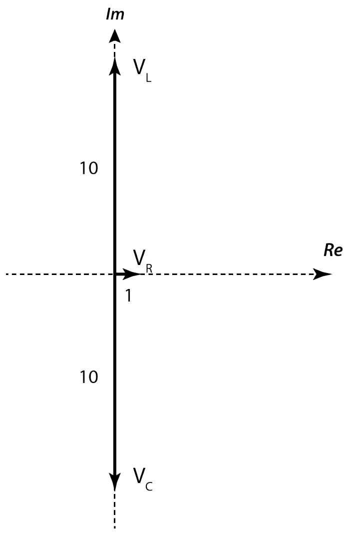

Figure 9.4: Phasor Diagram for Series LCR Circuit at Resonance (Q=10)

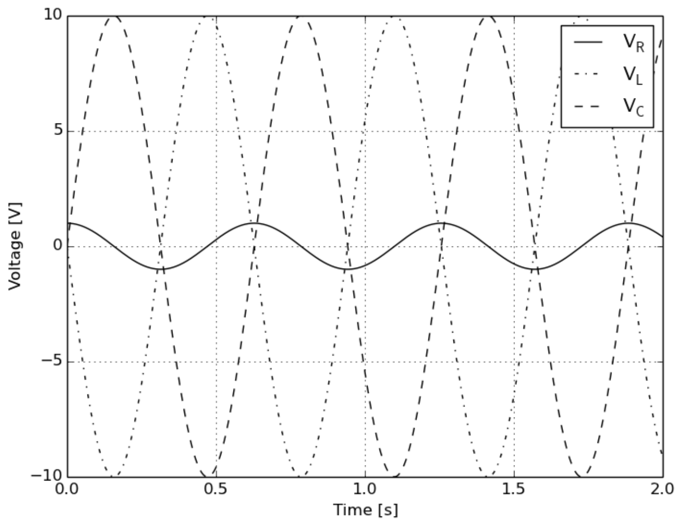

Figure 9.5: Series LCR Circuit potential differences at resonance (applied voltage 1 V)

These potential differences are shown in figure 9.5

At resonance, the average power is

$$P_{av} = \langle p(t) \rangle = \frac{|\tilde{V}_R|^2}{2R} = 5\,\mathrm{W}$$

Theory predicts that the power should drop to *approximately* half this value at a frequency

$$\omega = \omega_0 \pm \frac{\gamma}{2}$$

To test this, let's choose $\omega = 9.5\,\mathrm{rad/s}$.

$$\tilde{Z}_R = 0.1\,\Omega$$

$$\tilde{Z}_L = 0.95j\,\Omega$$

$$\tilde{Z}_C = \frac{-j}{0.95}\,\Omega$$

Total impedance

$$\tilde{Z} = Ze^{j\phi}$$

where $Z = 0.143\,\Omega$ and $\phi = -0.798\,\mathrm{rad}$.

$$\tilde{I} = \frac{\tilde{V}_S}{\tilde{Z}}$$

$$\tilde{V}_R = \tilde{I}\tilde{Z}_R = \frac{0.1e^{-j\phi}}{Z}$$

$$|\tilde{V}_R| = 0.698\,\mathrm{V}$$

$$P_{av} = \frac{|\tilde{V}_R|^2}{2R} = 2.4\,\mathrm{W}$$

This is not exactly half because equations 9.5 tell us that the bandwidth $\Delta\omega$ is not *exactly* centred on ω_0. Completing the maths for the other two potential differences results in the phasor diagram shown in figure 9.6, and for which the physical signals look like figure 9.7.

Finally, note that the vector sum of the three voltage phasors equals the applied voltage

$$\tilde{V}_S = \tilde{V}_R + \tilde{V}_L + \tilde{V}_C$$

Their real components must follow Kirchhoff's voltage law

$$v_S(t) = v_R(t) + v_L(t) + v_C(t)$$

9.6 Parallel LCR Circuit

Figure 9.9 shows a circuit where the resistor, capacitor and inductor are in a parallel arrangement.

The complex impedance of the parallel components is given by equation 7.11 as

$$\frac{1}{\tilde{Z}} = \frac{1}{\tilde{Z}_R} + \frac{1}{\tilde{Z}_C} + \frac{1}{\tilde{Z}_L} = \frac{1}{R} + j(\omega C - 1/\omega L)$$

Taking a source phasor $\tilde{V}_S = V_0$ we find a current

$$\tilde{I} = \frac{\tilde{V}_S}{\tilde{Z}} = I_0 e^{j\phi}$$

The amplitude and phase of the current are shown in figure 9.9.

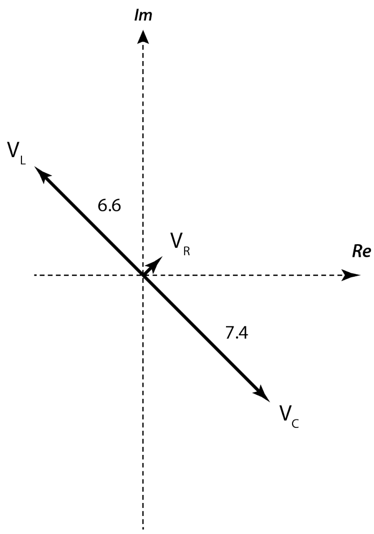

Figure 9.6: Phasor Diagram for series LCR Circuit at half-power

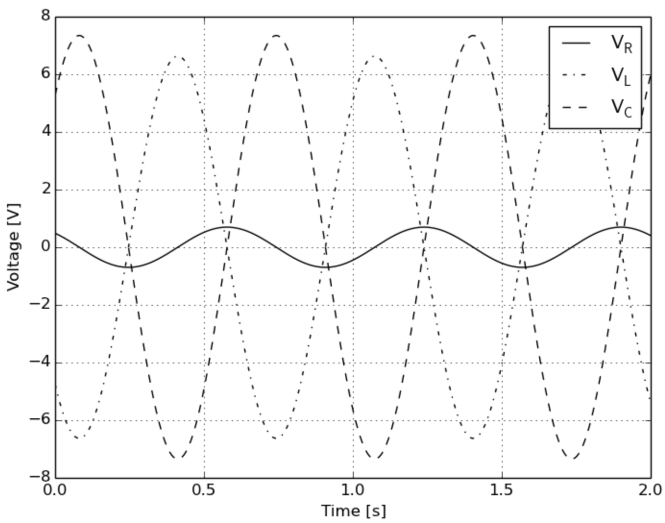

Figure 9.7: Series LCR Circuit potential differences at half-power

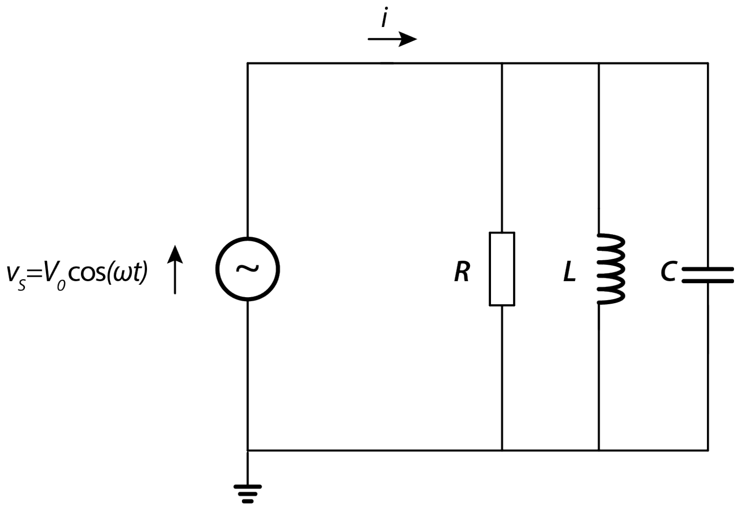

Figure 9.8: Driven Parallel LCR Circuit

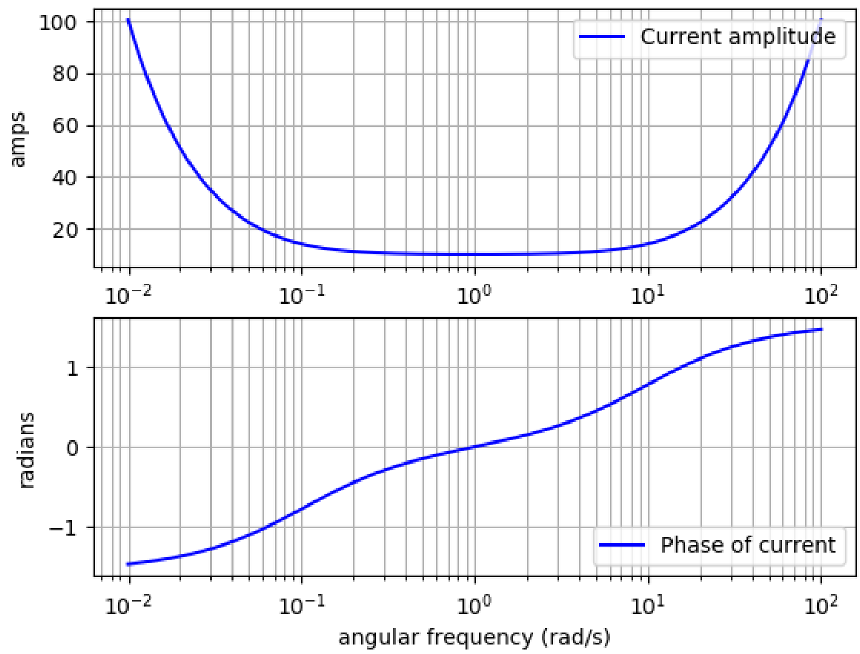

Figure 9.9: Magnitude and phase of the current in the parallel LCR Circuit. $L = 10\,\mathrm{H}$, $C = 0.1\,\mathrm{F}$, $R = 1\,\Omega$, $V_0 = 10\,\mathrm{V}$.

9.7 LCR Filters

As with the RC and RL circuits, we can make filters out of various series and/or parallel combinations of resistors, capacitors and inductors, and due to the resonant-behaviour these **LCR filters** can be highly-selective in terms of the range of frequencies which are passed (or blocked).

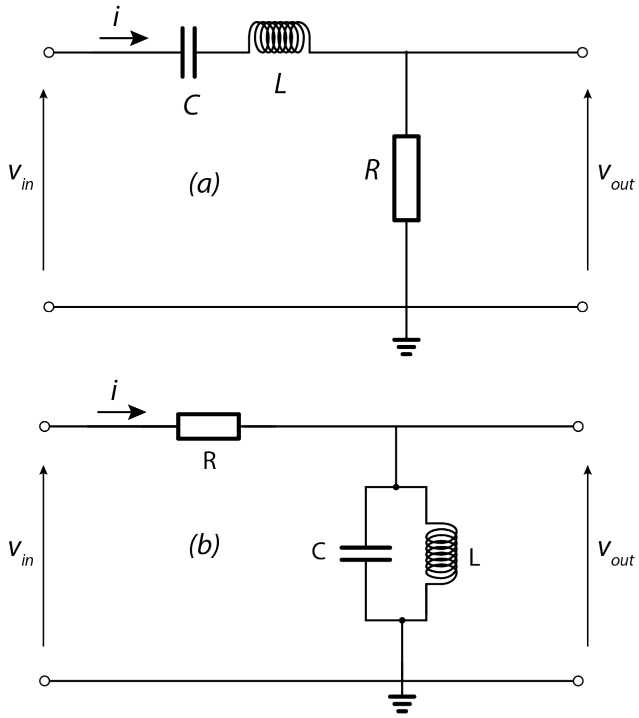

Figure 9.10: Two examples of LCR Filter circuits; (a) is a simple series arrangement while (b) has a series/parallel combination.

Figure 9.10 shows two examples of LCR filters, though it will be clear that there are many more serial/parallel arrangements we could choose. The top panel (a) shows a simple series arrangement which is essentially the same as the the circuit we studied in section 9.1 but here we take the output of the filter to be the potential difference across the resistor. Since we already have an expression for the current flowing, equation 9.1, we can write

$$\tilde{V}_{out} = \tilde{I}R$$

and then find the gain of the filter by

$$\tilde{G} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

We can also see that this circuit is another impedance divider so we could directly write

$$\tilde{G} = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{R}{\tilde{Z}} = \frac{R}{R + j(\omega L - 1/\omega C)} = Ge^{j\phi}$$

where the magnitude of the gain is

$$G = \frac{R}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

and the argument, or phase of the gain is

$$\tan \phi = \frac{-(\omega L - 1/\omega C)}{R}$$

We can see that $G = 1$ and $\phi = 0$ for the case $\omega = \omega_0$; the Bode Plot for this filter is shown in figure

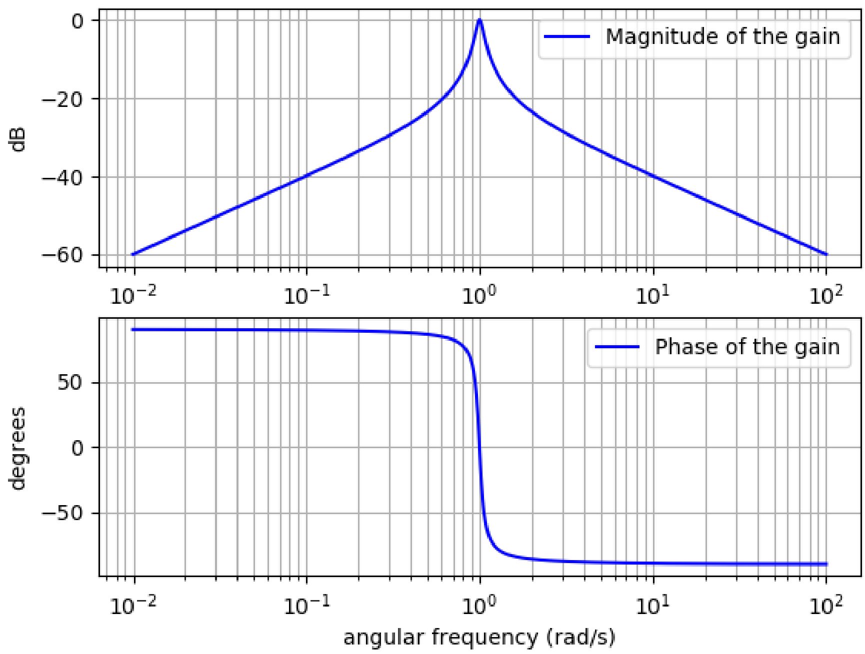

Figure 9.11: Bode Plot for the LCR Series Filter circuits with $\omega_0 = 1$ rad/s.

As before, we can find the lower and higher cut-off frequencies of the filter by solving

$$G^2 = \frac{1}{2}$$

The analysis of circuit (b) in figure 9.10 will be left for a problem sheet question.

Further Reading

The relevant chapter in Sears and Zemansky's **University Physics** is

Chapter 31: Alternating Current

See also also **Principles of Electronic Instrumentation** (*Diefenderfer and Holton*)

Chapters 3 & 4: Alternating Current Circuits