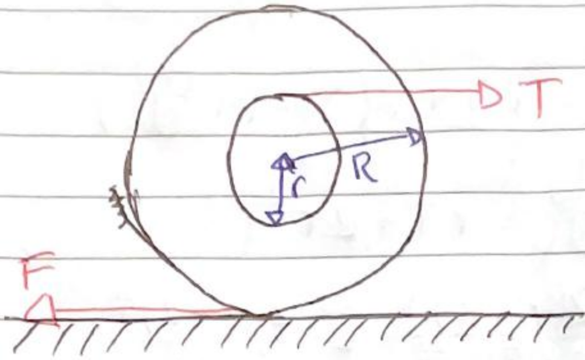
Classical Mechanics 20

Unrolling a Spool of Thread



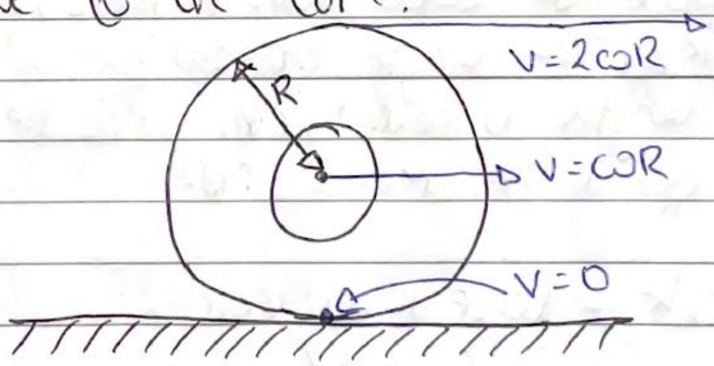
- spool is accelerating so $T > F$.
- mass of M
- Moment of Inertia $\approx \frac{1}{2}MR^2$
- rolls without slipping

Speed of CoM

A spool of outer radius R rolling at angular velocity ω moves a distance $2\pi R$ in time $T = 2\pi/\omega$. Hence

$$v = \frac{\text{distance}}{\text{time}} = \frac{2\pi R}{2\pi/\omega} = \omega R$$

The linear speed of the CoM is the same as the speed of a point on the rim relative to the CoM.



Method 1

CoM motion:

$$M\frac{dv}{dt} = T - F$$

pretend all forces are acting on the CoM. See earlier lectures.

Rotational motion:

$$I\frac{d\omega}{dt} = rT + RF$$

as $v = \omega R$, the torque equation becomes

$$\frac{I}{R}\frac{dv}{dt} = rT + RF$$

$$\frac{I}{R}\frac{dv}{dt} = rT + R\left(T - M\frac{dv}{dt}\right)$$

$$\left(MR + \frac{I}{R}\right)\frac{dv}{dt} = rT + RT$$

$$\frac{dv}{dt} = \frac{R(r+R)T}{MR^2 + I} = \frac{2(r+R)T}{3MR}$$

Assuming T is constant, then we can use SUVAT to solve ...

Method 2 Work-Energy Theorem

Work:

The spool is stationary at the point in contact with the ground, so the frictional force does no work.

Suppose the spool rolls ϕ rads, moving a distance $x = R\phi$. The length of string pulled in is the length unrolled, $r\phi$, plus the distance the spool has moved, $R\phi$.

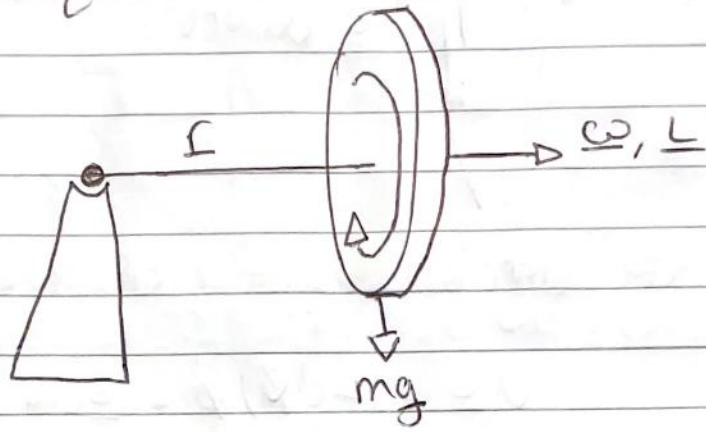
$$\begin{aligned} \text{Work} &= \text{force} \times \text{distance} \\ &= T(r\phi + R\phi) \\ &= T\left(\frac{r}{R} + 1\right)x \end{aligned}$$

$$\begin{aligned} \left(\frac{r}{R} + 1\right)Tx &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}\left(M + \frac{I}{R^2}\right)v^2 \end{aligned}$$

$$v^2 = 2\left(\frac{R(r+R)T}{MR^2 + I}\right)x$$

Gyroscopes

A gyroscope is a spinning flywheel subject to a torque.



As the flywheel is heavy and the angular ~~momentum is large~~ velocity is high, the angular momentum $\underline{L}$ is large. It is also true that angular momentum is parallel to angular velocity.

$$\underline{L} = I\underline{\omega}$$

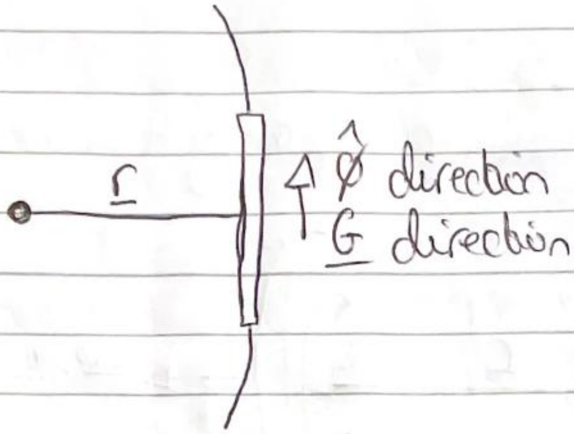
The torque about the pivot point at the top of the stand is

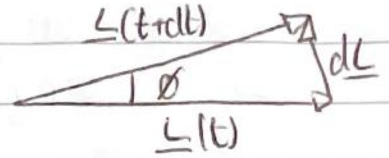
$$\underline{G} = \underline{r} \times m\underline{g} = mgr \text{ into page} = mgr\,\hat{\otimes}.$$

If you released the gryoscope, you would expect it to fall down. i.e. $d\underline{L}$ points down.

But since $d\underline{L} = \underline{G}\,dt$ and G is into the page, $d\underline{L}$ must also be into the page.

This means the gyroscope must precess about its vertical axis.





from the angular version of N2,

$$d\underline{L} = (L\,d\phi)\,\hat{\phi} = \underline{G}dt = mgr\,\hat{\phi}\,dt$$

now the directions of $d\underline{L}$ and $\underline{G}dt$ agree. Therefore it appears as though the precessing solution is the correct one. Equating the lengths:

$$L\,d\phi = mgr\,dt$$

$$\boxed{\frac{d\phi}{dt} = \frac{mgr}{L} = \frac{mgr}{I\omega}}$$

As friction reduces the spin rate, ω, the precession frequency will increase.

The precession creates a small vertical component of angular momentum. The torque has no vertical component & the vertical component of the angular momentum cannot change so the gyroscope must drop a little so the vertical component of $\underline{L}$ exactly matches that from the drop. As the gyro slows, $\underline{L}$ magnitude decreases, the gyro has to drop more and more to ensure they are equal