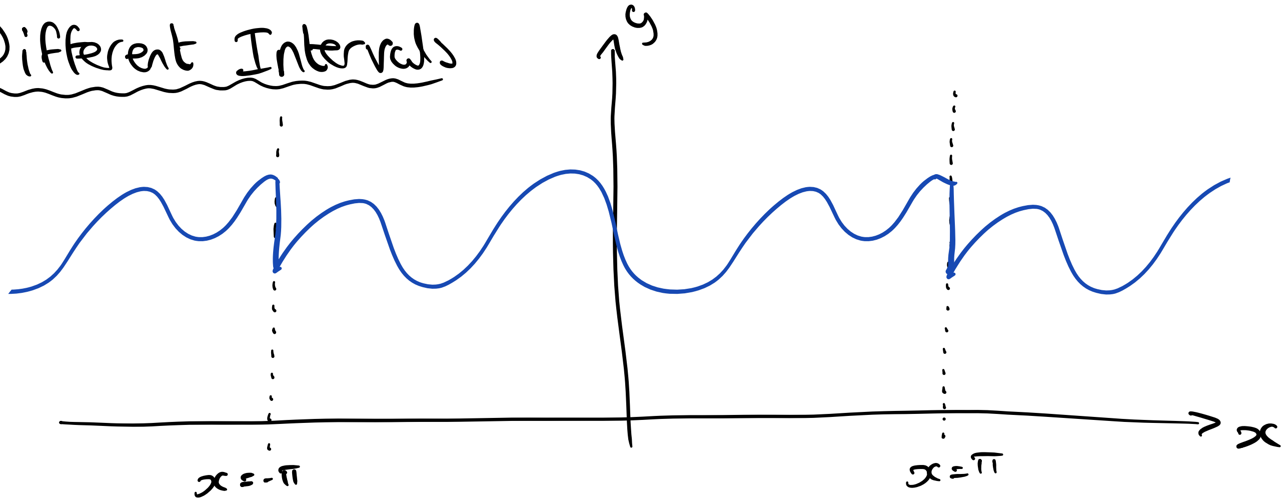
Different Intervals



This function does satisfy the dirichlet conditions, we just pretend the function repeats every 2π.

We want to generalise from $-\pi$ to π to $-l$ to l.
We assume our function is periodic over $2l$.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

Lets define a new variable

$$y = \frac{x}{\pi} l,\ x = \frac{y\pi}{l},\ dx = \frac{\pi}{l} dy$$

$$f(y) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{y\pi}{l}} \qquad c_n = \frac{1}{2\cancel{\pi}} \cdot \frac{\cancel{\pi}}{l} \int_{-l}^{l} f(y) e^{-iny\pi/l}$$

We can now redefine our new series

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-i\left(\frac{n\pi}{l}\right)x} \qquad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi}{l}\right)x}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right) + b_n \sin\left(\frac{n\pi}{\ell}x\right)$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi}{\ell}x\right)dx \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

Same dirichlet conditions, apart from period now 2ℓ instead of 2π.

Power Spectrum

We can rewrite our fourier series as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\alpha_n}_{\substack{\text{amplitude} \\ \alpha_n \in \mathbb{C}}} \cos(nx - \underbrace{\theta_n}_{\text{phase}})$$

$$a_n = \alpha_n \cos(\theta_n) \qquad b_n = \alpha_n \sin(\theta_n)$$

$$\alpha_n^2 = a_n^2 + b_n^2 \qquad \tan(\theta_n) = \frac{b_n}{a_n}$$

α_n : amplitude of the signal.

$|\alpha_n|^2$: Power of the signal

θ_n : Phase of the signal.

Parseval's Identity

lets consider $f(x) \in \mathbb{C}$,

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

if we calculate the integral of the square of both sides.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \underbrace{\int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} c_n e^{inx} \right|^2 dx}_{RHS}$$

$$RHS = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} c_n e^{inx} \times \sum_{m=-\infty}^{\infty} c_m^* e^{-imx} \, dx$$

$$= \int_{-\pi}^{\pi} \sum_{n,m=-\infty}^{\infty} c_n c_m^* e^{i(n-m)x} \, dx$$

$$= \sum_{n,m} c_n c_m^* \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx$$

N.B.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \, dx = \delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

$$= \sum_{n,m} c_n c_m^* 2\pi \delta_{nm}$$

$$= \sum_{n=-\infty}^{\infty} c_n c_n^* 2\pi = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

'power' of the series

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2$$

'power' of $f(x)$

This is known as parseval's identity!

We can form an inequality, known as Parseval's

Inequality:

'truncated'

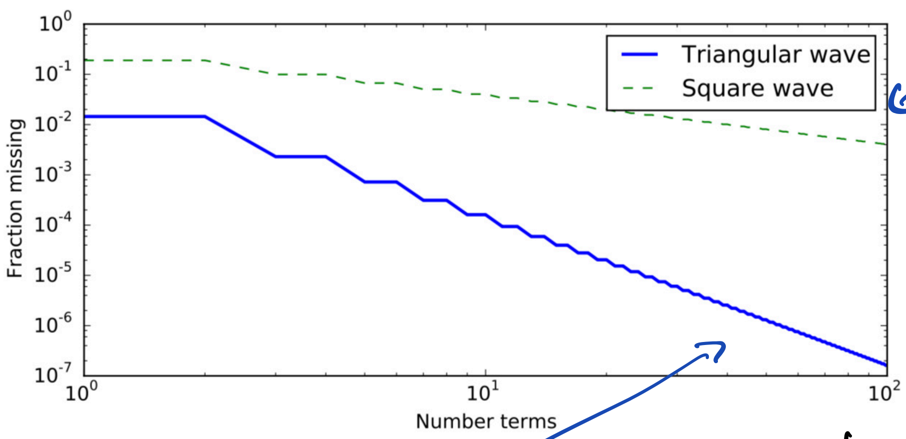
$$\sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

If we look at a trig series:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(\underbrace{a_n^2 + b_n^2}_{=\alpha_n^2})$$

We can use this to see how good our approximation for our fourier series is, with a given no. of terms.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \approx \sum_{n=-N}^{N} c_n e^{inx}$$



~10% missing

$$y_N = 1 - \frac{\sum_{n=-N}^{N} |c_n|^2}{\sum_{n=-\infty}^{\infty} |c_n|^2}$$

'ratio of power'. If the truncated series was similar to the original function, $y_N \to 0$.

lots of terms give good approximation for a triangular wave.

How closely the truncated fourier series matches the original is very important for signal analysis.

(See JPEG compression)