Vectors 10

Inverse Matrix

For a system of linear equations, we can write it compactly using matrix notation.

$$A\underline{x} = \underline{b}$$

to find $\underline{x}$, we cannot just divide $\underline{b}$ by A. We need to find the inverse matrix A^{-1}.

$$A^{-1}A = I$$

then, $\underline{A^{-1}A\underline{x} = A^{-1}\underline{b}} \Rightarrow \underline{I}\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$

You can solve simultaneous equations by finding inverse of matrix co-efficients.

2×2 Matrix

Take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$

We want $A^{-1}A = I$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{11}a + \alpha_{12}c & \alpha_{11}b + \alpha_{12}d \\ \alpha_{21}a + \alpha_{22}c & \alpha_{21}b + \alpha_{22}d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\alpha_{11}a + \alpha_{12}c = 1 \qquad \alpha_{11}b + \alpha_{12}d = 0$$
$$a_{21}a + \alpha_{22}c = 0 \qquad \alpha_{21}b + \alpha_{22}d = 1$$

now we use cramer's rule.

$$\alpha_{11} = \frac{\begin{vmatrix} 1 & c \\ 0 & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} \quad \alpha_{12} = \frac{\begin{vmatrix} a & 1 \\ b & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}^T} \quad \alpha_{21} = \frac{\begin{vmatrix} 0 & c \\ 1 & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}^T} \quad \alpha_{22} = \frac{\begin{vmatrix} a & 0 \\ b & 1 \end{vmatrix}}{\Delta}$$

$\Delta^T = \Delta$ transpose of a determinute = determinant

$$\alpha_{11} = \frac{1}{\Delta} d \quad \alpha_{12} = -\frac{b}{\Delta} \quad \alpha_{21} = -\frac{c}{\Delta} \quad \alpha_{22} = \frac{a}{\Delta}$$

$$\boxed{A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

3x3 Matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad A^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$A^{-1}A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{23} & \alpha_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As $A^{-1}A = AA^{-1}$, we can do AA^{-1} and form 3 sets of coupled equations.

$$\left.\begin{aligned} \alpha_{11}a + \alpha_{12}d + \alpha_{13}g &= 1 \\ \alpha_{11}b + \alpha_{12}e + \alpha_{13}h &= 0 \\ \alpha_{11}c + \alpha_{12}f + \alpha_{13}i &= 0 \end{aligned}\right\} \text{1 set}$$

$$\left.\begin{aligned} \alpha_{21}a + \alpha_{22}d + \alpha_{23}g &= 0 \\ \alpha_{21}b + \alpha_{22}e + \alpha_{23}h &= 1 \\ \alpha_{21}c + \alpha_{22}f + \alpha_{23}i &= 0 \end{aligned}\right\} \text{1 set}$$

$$\left.\begin{aligned} \alpha_{31}a + \alpha_{32}d + \alpha_{33} &= 0 \\ \alpha_{31}b + \alpha_{32}e + \alpha_{33}h &= 0 \\ \alpha_{31}c + \alpha_{32}f + \alpha_{33}i &= 1 \end{aligned}\right\} \text{1 set}$$

We can now use cramer's rule to fin α_{ij}.

$$\alpha_{11} = \frac{\begin{vmatrix} 1 & d & g \\ 0 & e & h \\ 0 & f & i \end{vmatrix}}{\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}} = \frac{\begin{vmatrix} e & h \\ f & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{\overbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}^{\text{minor}}}{\Delta} = \frac{\overbrace{A_{11}}^{\text{co-factor}}}{\Delta}$$

$$\alpha_{12} = \frac{\begin{vmatrix} a & 1 & g \\ b & 0 & h \\ c & 0 & i \end{vmatrix}}{\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}} = \frac{-\begin{vmatrix} 1 & a & g \\ 0 & b & h \\ 0 & c & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{-\begin{vmatrix} b & h \\ c & i \end{vmatrix}}{\Delta} = \frac{-\overbrace{\begin{vmatrix} b & c \\ h & i \end{vmatrix}}^{\text{minor}}}{\Delta} = \frac{\overbrace{A_{21}}^{\text{cofactor}}}{\Delta}$$

$$\alpha_{ij} = \frac{A_{ji}}{\Delta}$$

$$\boxed{A^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T}$$

if $\Delta = 0$, there is no inverse matrix.

<u>Gaussian Elimination</u>

eg. for matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$, find A^{-1}.

firstly, lets form an argmented matrix.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

now we can manipulate to find A^{-1}.

$$\xrightarrow[R3-R1]{R2-R1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R2 \div 2]{R2 \Leftrightarrow R3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R3-R2]{R1+R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow[R3 \times (-1)]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \quad \text{So } A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$