Functions 2

13/10/21

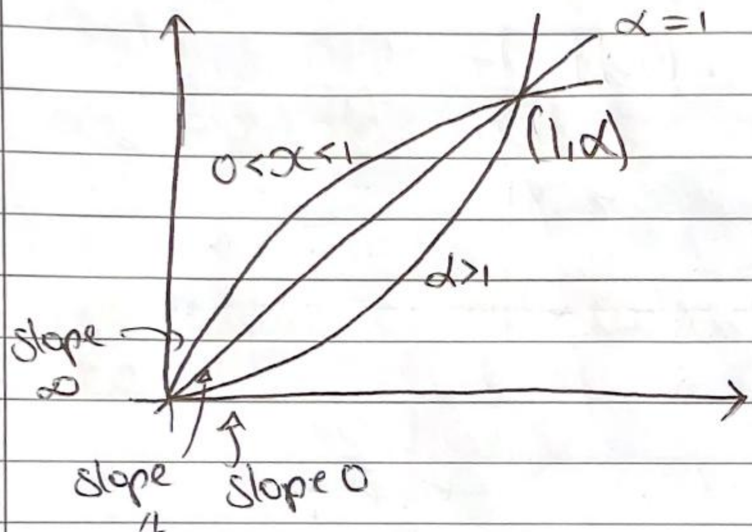
Power laws

$$y = ax^{\alpha}$$

α = power, index, exponent

differentiate $\Rightarrow$

$$y' = \alpha a x^{\alpha - 1}$$



Many laws in physics are expressible via power laws.

eg. 1999 Ig Nobel Prize Biscuit Dunking

height of sogginess in biscuit found by

$$h = \sqrt{t - t_0}$$

$$\therefore \dot{h} = \frac{1}{2}(t - t_0)^{-\frac{1}{2}}$$
$$\therefore \ddot{h} = -\frac{1}{4}(t - t_0)^{-\frac{3}{2}}$$

negative acceleration

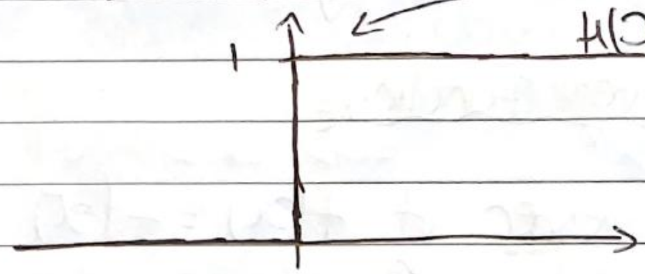
Trig Functions

$\sin x, \cos x \ldots$

Heavyside step function

$$H(x) = \begin{cases} 0 & (x<0) \\ 1 & (x \geq 0) \end{cases}$$

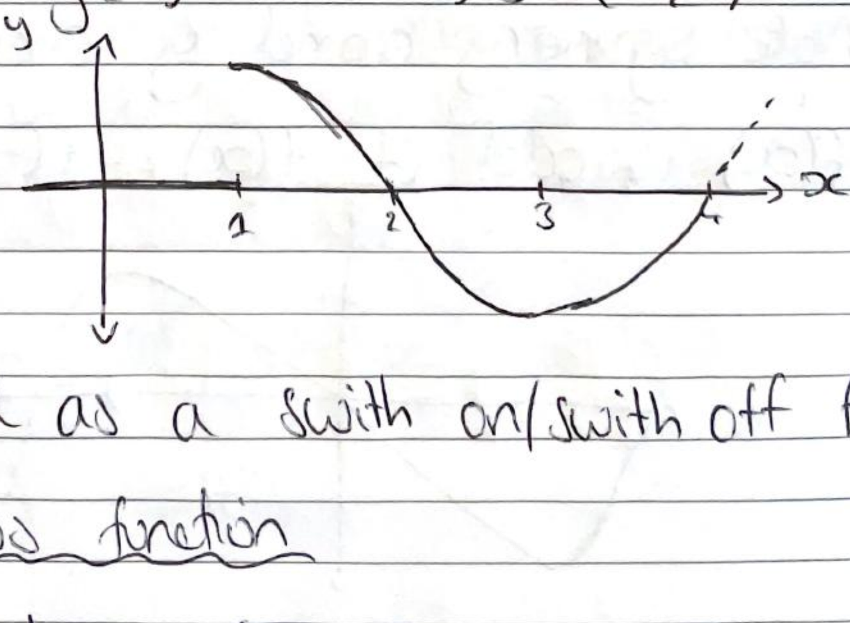
convection that $H(x)=1$ at $x=0$.



* This function is discontinous at $x=0$.
* The derivate of this function is equal to zero apart from $x=0$ where undefined.

eg

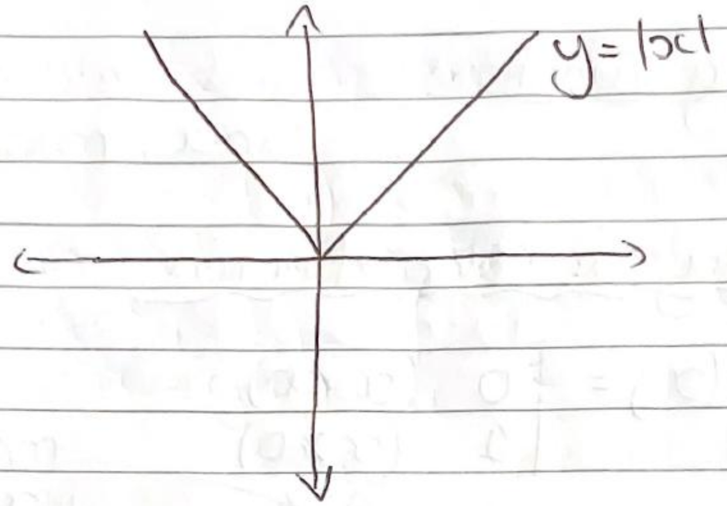
$$y(x) = H(x-1)\sin\left(\frac{\pi x}{2}\right)$$



viewed as a swith on/swith off function.

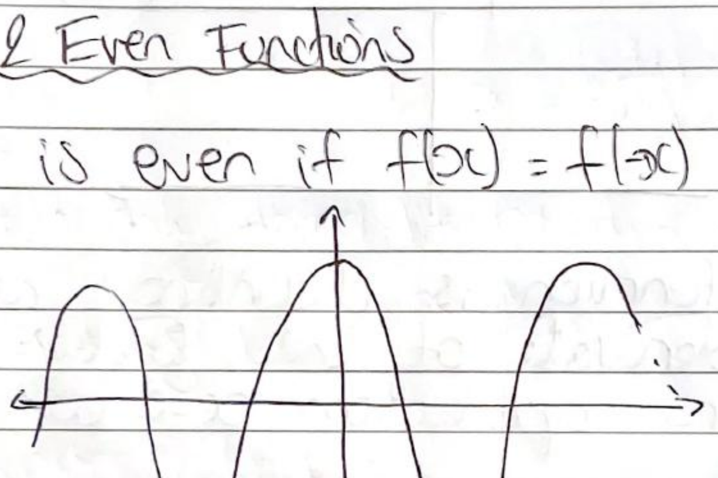
Modulus function

$$|x| = \begin{cases} x & (x>0) \\ -x & (x<0) \end{cases}$$



Odd & Even Functions

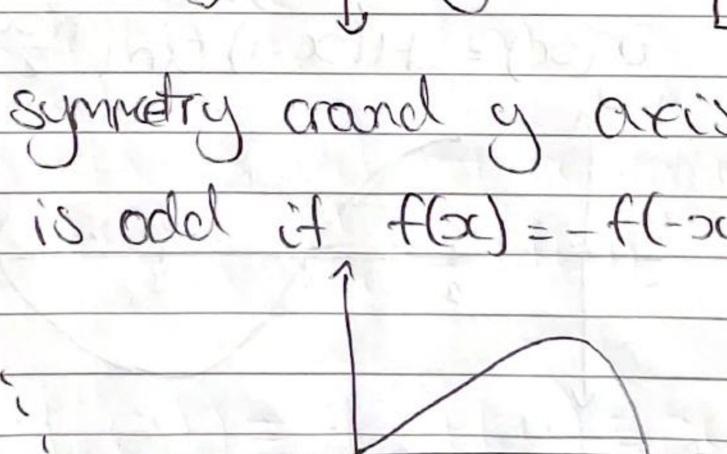
$f(x)$ is even if $f(x) = f(-x)$



$\boxed{\text{eg. } \cos x}$

Note symmetry around y axis.

$f(x)$ is odd if $f(x) = -f(-x)$



$\boxed{\begin{array}{l}\text{eg. } \sin x \\ \quad \tan x\end{array}}$

Note symmetry around origin.

Eg $y = \sin x^5$ $\quad \sin(-x)^5 = -\sin(x)^5$

$\therefore y(x)$ is odd.

Not all functions are odd or even.

eg. $f(x) = x + x^2$

(even f)(even f) = even f
(odd f)(odd f) = even f
(odd f)(even f) = odd f

For a general function, it can always be expressed as the sum of odd and even functions.

$$g(x) \equiv \underbrace{\tfrac{1}{2}\left[g(x) + g(-x)\right]}_{\text{even}} + \underbrace{\tfrac{1}{2}\left[g(x) - g(-x)\right]}_{\text{odd}}$$

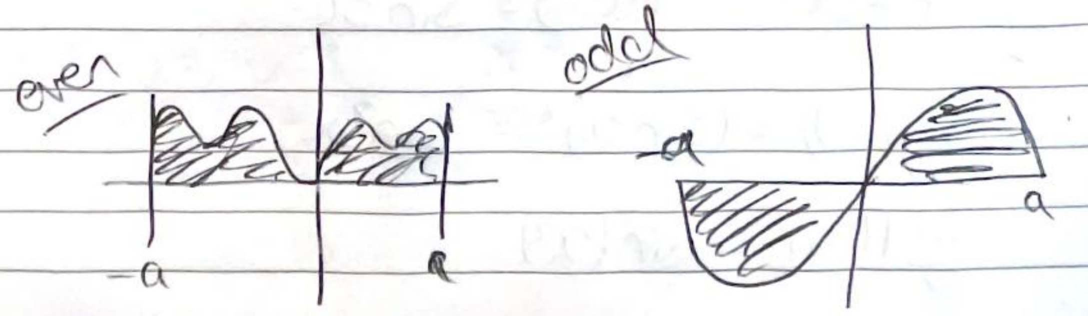
Later important applications (fourier)

For an even function, we have

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

For an odd function, we have

$$\int_{-a}^{a} f(x)\,dx = 0$$



Inverse functions

A function $y = f(x)$ can sometimes be inverted to get x in terms of y.

$$x = g(y)$$

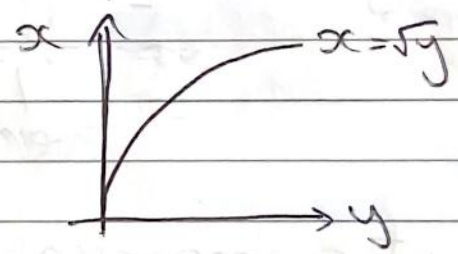
g is inverse of f

Eg

$$y = x^2 \quad \Rightarrow x = \pm\sqrt{y}$$

We will only take the positive values

$\therefore \; x = \sqrt{y}$



The inverse function $g(x)$ of $f(x)$ is often written $f^{-1}(x)$.

$f(x) = x^2 \quad f^{-1}(x) = \sqrt{x}$

The notation should not be confused with a reciprocal. $\frac{1}{f(x)} = (f(x))^{-1}$

$$f(f^{-1}(x)) \equiv f^{-1}(f(x)) \equiv x$$

Function of a function

eg $f = x^2 \quad g(x) = \sin x$

$$f(g(x)) = (\sin x)^2 = \sin^2 x$$

$$g(f(x)) = \sin(x^2)$$

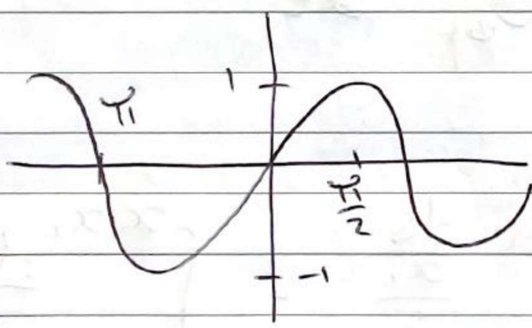
$fg(x) = \sin^2 x$

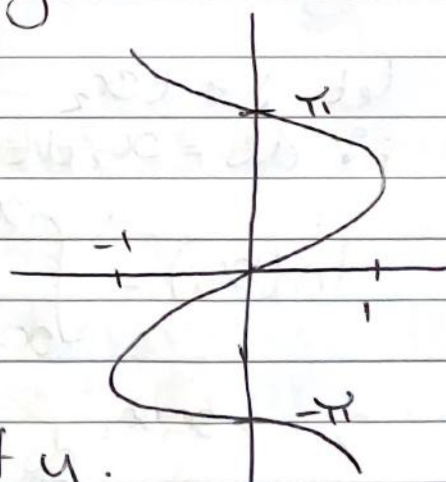
$gf(x) = \sin(x^2)$

Evidently not the same function. We say that this composition <u>does not commute</u>.

Many valued functions ....

Consider $y = \sin x$ and the inverse $y = \sin^{-1} x = \arcsin x$





For each value of x there are many values of y.

To clarify, we define the <u>principle</u> value from

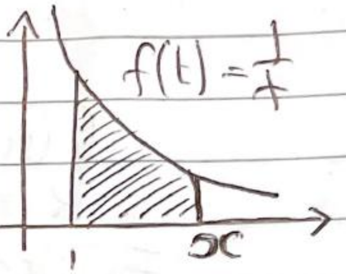
$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$$

for $\cos^{-1} x$: $\quad 0 \leq \cos^{-1} x \leq 1$

Logs, Exponential, Hyperbolic

Natural logarithm $= \ln(x) = \log_e(x)$

defined by $\ln(x) = \int_1^x \frac{dt}{t}$



It follows that:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \ln(1) = 0$$

$$\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$$

s & t are dummy variables

Why? $\ln(x_1) = \int_1^{x_1} \frac{dt}{t}$

let $s = t x_2$

$\therefore ds = x_2 \, dt$

$$\ln(x_1) = \int_{x_2}^{x_1 x_2} \frac{\frac{ds}{x_2}}{\frac{s}{x_2}} = \int_{x_2}^{x_1 x_2} \frac{ds}{s}$$

$$= \int_1^{x_1 x_2} \frac{ds}{s} - \int_1^{x_2} \frac{ds}{s}$$

difference in area

$$\ln(x_1) = \ln(x_1 x_2) - \ln(x_2) \quad \blacksquare$$