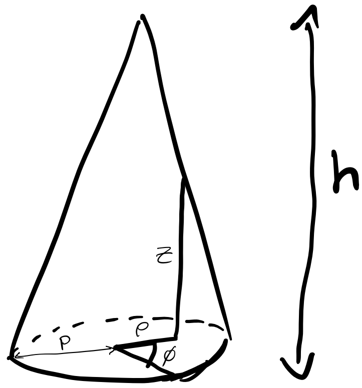
Example: Vertical CoM for solid cone



$$\bar{z} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\iiint z \cancel{\Delta}\, dV}{\iiint \cancel{\Delta}\, dV}$$

$$= \frac{1^{st} \text{ moment of vol.}}{0^{th} \text{ moment of vol.}}$$

Δ = density

$$\iiint dV = \frac{1}{3} \times \text{base} \times \text{height} = \frac{\pi P^2 h}{3}$$

at any point on the surface.

$$\rho = P\left(1 - \frac{z}{h}\right)$$

$$I = \int_{z=0}^{h} \int_{\rho=0}^{P(1-\frac{h}{z})} \int_{\phi=0}^{2\pi} z\rho\, d\phi\, d\rho\, dz$$

Jacobian

$$\int_0^{2\pi} z\rho\, d\phi = 2\pi z\rho$$

$$\int_{\rho=0}^{P(1-\frac{z}{h})} 2\pi z\rho = 2\pi \left[\frac{z\rho^2}{2}\right]_0^{P(1-\frac{z}{h})} = \pi z P^2 \left(1 - \frac{z}{h}\right)^2$$

$$= \pi z P^2 - 2\pi z P^2 \frac{z}{h} + \pi P^2 \frac{z^3}{h^2}$$

$$\int_{z=0}^{h} \pi z P^2 - 2\pi P^2 \frac{z^2}{h} + \pi P^2 \frac{z^3}{h^2}\, dz = \frac{1}{3}\pi h^2 P^2 - \frac{2}{3}\pi P^2 h^2 + \frac{1}{4}\pi P^2 h^2$$

$$= \frac{\pi P^2 h^2}{12}$$

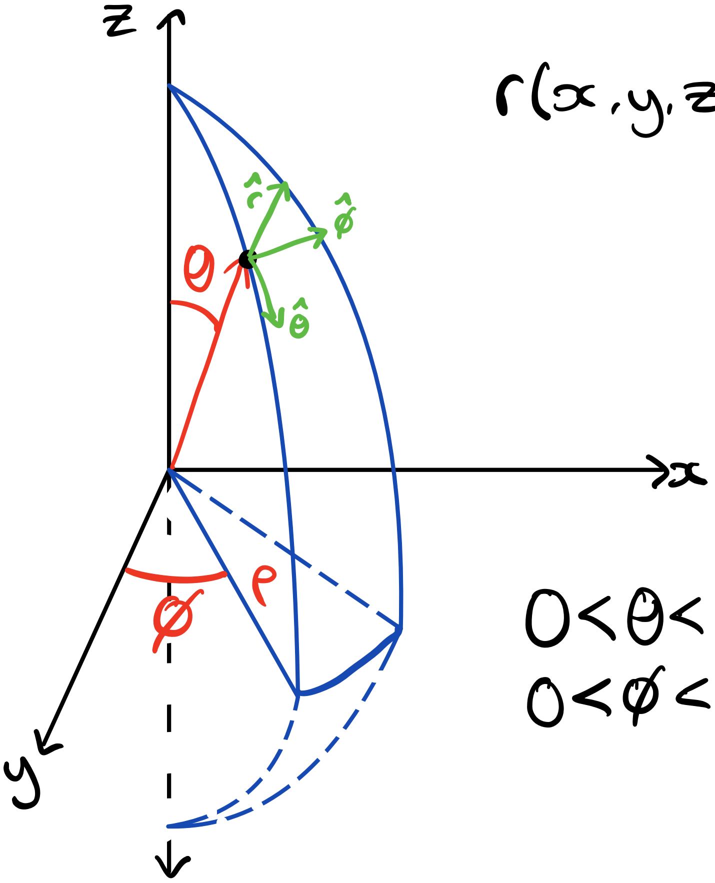
$$\bar{z} = \frac{\frac{\cancel{\pi P^2} h^{\cancel{2}}}{12}}{\frac{\cancel{\pi P^2 h}}{3}} = \frac{h}{4}$$

Spherical Polar Coordinates

(r, θ, ϕ)

latitude

longitude



$$r(x,y,z) = \bar{r}(r,\theta,\phi)$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$0<\theta<\pi$$
$$0<\phi<2\pi$$

$$\hat{r}(\theta,\phi) = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\hat{\theta}(\theta,\phi) = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,k$$
$$\hat{\phi}(\phi) = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,\hat{\phi} \qquad \frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$$

$$\frac{\partial\hat{r}}{\partial\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k} = \hat{\theta}$$

$$\frac{\partial\hat{r}}{\partial\phi} = -\sin\theta\sin\phi\,\hat{i} + \sin\theta\cos\phi\,\hat{j} = \sin\theta\,\hat{\phi}$$

$$d\underline{r} = \frac{\partial\underline{r}}{\partial r}dr + \frac{\partial\underline{r}}{\partial\theta}d\theta + \frac{\partial\underline{r}}{\partial\phi}d\phi \qquad \underline{r} = r\hat{r}$$

$$\frac{\partial\underline{r}}{\partial r} = \frac{\partial r}{\partial r}\hat{r} + \cancel{r\frac{\partial\hat{r}}{\partial r}} \qquad \frac{\partial\underline{r}}{\partial\theta} = \hat{\theta}r + \cancel{\frac{\partial r}{\partial\theta}}\hat{r} \qquad \frac{\partial\underline{r}}{\partial\phi} = r\sin\theta\,\hat{\phi} + \cancel{\frac{\partial r}{\partial\phi}\hat{r}}$$

$$d\underline{r} = \hat{r}\,dr + r\hat{\theta}\,d\theta + r\sin\theta\,\hat{\phi}$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi \qquad \boxed{J = r^2\sin\theta}$$