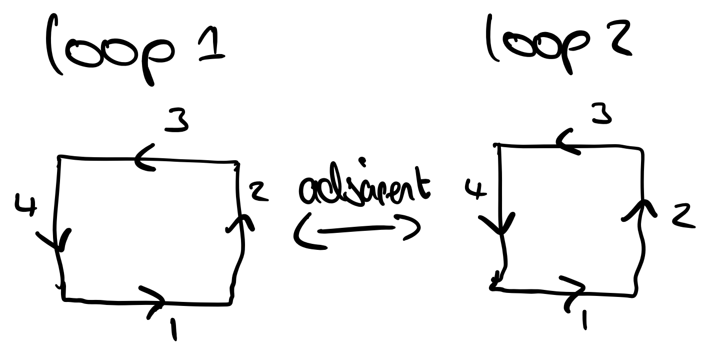
Stoke's Theorem

consider two adjacent infinitesimal square loops, not necessarily in the same plane

$$(\nabla \times \underline{B})\cdot\hat{n} = \lim_{A\to 0} \frac{\oint \underline{B}\cdot d\underline{r}}{A} \qquad (d\underline{s} = A\hat{n})$$

$$\therefore (\nabla\times\underline{B})\cdot d\underline{s} = \oint \underline{B}\cdot d\underline{r}$$ for an infinitesimal loop



$$\sum_{i=1}^{4} \underline{B}\cdot d\underline{r} = \sum_{i=1}^{4}\omega_i^1 = (\nabla\times\underline{B}_1)\cdot d\underline{s} \qquad \sum_{i=1}^{4} \underline{B}\cdot d\underline{r} = \sum_{i=1}^{4}\omega_i^2 = (\nabla\times\underline{B}_1)\cdot d\underline{s}$$

now lets join the two loops together

$$\sum_{1+2} \underline{B}\cdot d\underline{r} = \sum_{i=1}^{4}\omega_i^1 + \sum_{i=1}^{4}\omega_i^2 \cancel{-\omega_2^1 - \omega_4^2}^{\,0}$$

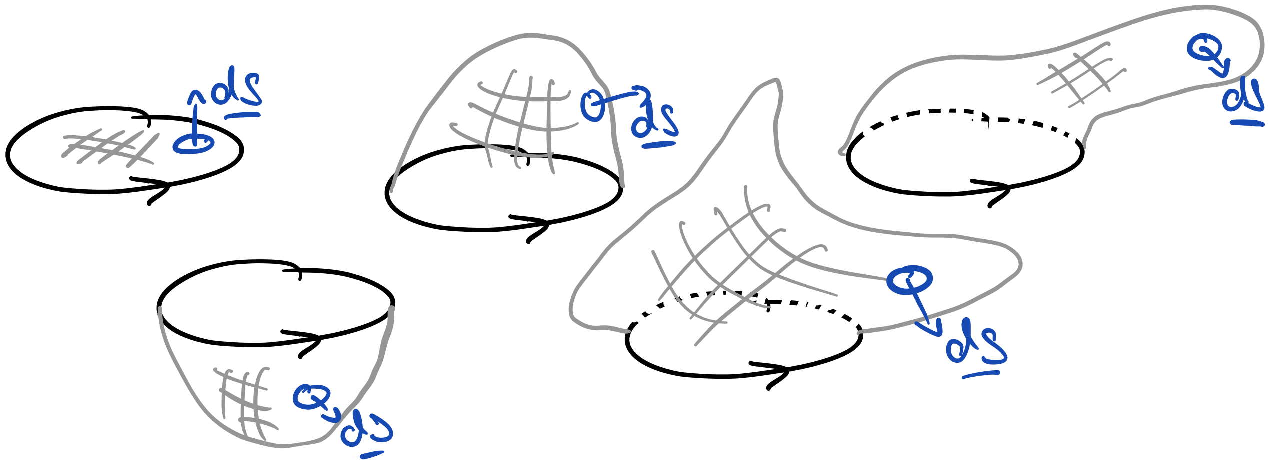
But, $\omega_2^1 = -\omega_4^2$ $\quad\therefore \sum_{1+2}\underline{B}\cdot d\underline{r} = (\nabla\times\underline{B}_1)\cdot d\underline{s} + (\nabla\times\underline{B}_2)\cdot d\underline{s}$

$$= \sum_{j=1}^{2}(\nabla\times\underline{B}_j)\cdot d\underline{s}$$

now, lets keep adding loops to form a macroscopic surface attached to a large loop.

closed $\longrightarrow$ $$\oint \underline{B}\cdot d\underline{r} = \iint_S (\nabla\times\underline{B})\cdot d\underline{s}$$ $\longleftarrow$ open

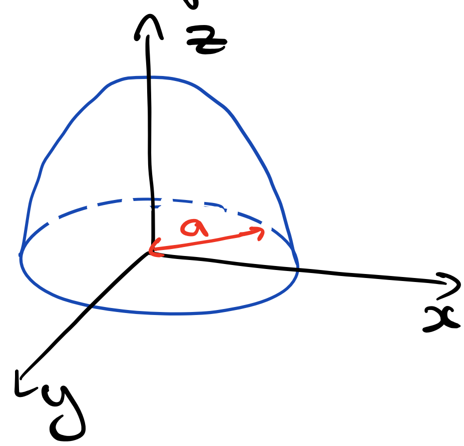
This applies to <u>any</u> surface attached to the loop.



To be clear on orientation of $\underline{ds}$, "collapse" surface onto the loop.

Why? Sometimes we want a surface integral but the line integral is easier. Or, you choose a different surface on the same loop.

<u>Example</u> $\underline{B} = z\hat{i} - y\hat{j} - x\hat{k}$, Shape: hemisphere $z>0$



$$\underline{ds} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{\rho}$$

$$\oint_C \underline{B} \cdot \underline{dr} = \iint_R \nabla \times \underline{B} \cdot \underline{ds}$$

$$\nabla \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & -y & 0 \end{vmatrix} = z\hat{j}$$

along loop $z=0, dz=0$

$$\oint \underline{B} \cdot \underline{dr} = \oint B_x dx + B_y dy + \cancel{B_z dz}^{\,0} = \cancel{z dx}^{\,0} - y dy$$

We now have to convert into cyclindrical so that our start & end limits are different.

$$x = a\cos\phi \qquad dx = -a\sin\phi\, d\phi$$
$$y = a\sin\phi \qquad dy = a\cos\phi\, d\phi$$

$$-y\,dy = -a^2\sin\phi\cos\phi\, d\phi$$

$$\oint \underline{B}\cdot d\underline{r} = \int_0^{2\pi} -a^2\sin\phi\cos\phi\, d\phi = -a^2\left[\tfrac{1}{2}\sin^2\phi\right]_0^{2\pi} = 0$$

now let's calculate using curl. $d\underline{s} = a^2\sin\theta\, d\phi\, d\theta\, \hat{r}$

$$\iint_R \nabla\times\underline{B}\cdot d\underline{s} = \iint_R 2\hat{j}\cdot(a^2\sin\theta\, d\theta\, d\phi)\hat{r}$$

$\hat{}$ see earlier notes

$$\hat{r}(\theta,\phi) = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} 2a^2\sin^2\phi\sin\theta\, d\theta\, d\phi = 0$$

even quicker: choose a surface in the x-y plane.

$$d\underline{S} = dS\,\hat{k}$$
$$(\nabla\times B)\cdot d\underline{s} = 2\hat{j}\cdot ds\,\hat{k} = 0$$

Vector Identities

There are many vector identities. 3 are particually important!

I) $\nabla \times (\nabla \Omega) = 0$

if $\underline{B} = \nabla \Omega$, the $\underline{B}$ is conservative/irrotational

II) $\nabla \cdot (\nabla \times \underline{V}) = 0$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$= 0$

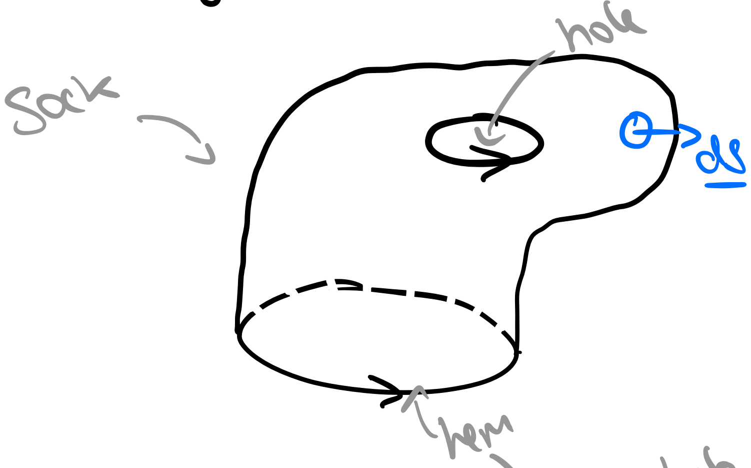
if $\underline{B} = \nabla \times \underline{V}$, $\underline{B}$ is 'solenoidal'. $\underline{V}$ is called the 'vector potential'.

III) $\nabla \times (\nabla \times \underline{B}) = \nabla(\nabla \cdot \underline{B}) - \nabla^2 \underline{B}$

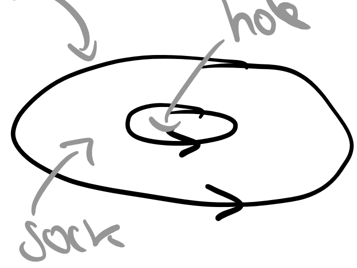
Using in E&M to derive wave equation.

$$\nabla^2 \underline{B} = \nabla^2 B_x \hat{\imath} + \nabla^2 B_y \hat{\jmath} + \nabla^2 B_z \hat{k}$$

Final Point



$$\iint_{\text{sock only}} (\nabla \times \underline{B}) \cdot d\underline{s} = ?$$



We instead think of the problem as

$$\oint_{\text{hem}} \underline{B} \cdot d\underline{r} = \iint_{\text{hole}} (\nabla \times \underline{B}) \cdot d\underline{s} + \iint_{\text{sock}} (\nabla \times \underline{B}) \cdot d\underline{s}$$

2020 Exam

$$\iint_{\text{sock}} (\nabla \times \underline{B}) \cdot d\underline{s} = \oint_{\text{hem}} \underline{B} \cdot d\underline{r} - \oint_{\text{hole}} \underline{B} \cdot d\underline{r}$$