Chain Rule

given the formula $f(x,y)$, there are 3 cases to consi.

Case 1: $y = y(x)$
Case 2: $x = x(t)$, $y = y(t)$
Case 3: $x = x(u,v)$, $y = y(u,v)$

Case 1: $y = y(x)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

But $dy = \frac{dy}{dx}dx$!

$$\Rightarrow df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{dy}{dx}dx$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y} + \frac{dy}{dx}\frac{dx}{dx}$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Case 2: $x = x(t)$, $y = y(t)$

$$dx = \frac{dx}{dt}dt \qquad dy = \frac{dy}{dt}dt$$

$$df = \frac{\partial f}{\partial y}\frac{dy}{dt}dt + \frac{\partial f}{\partial x}\frac{dx}{dt}dt$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\frac{dt}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Case 3: $x = x(u,v)$ $y = y(u,v)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

if we keep v constant then

$$\Rightarrow df = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}du + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}du$$

$$\frac{df}{du} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{df}{dv} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$