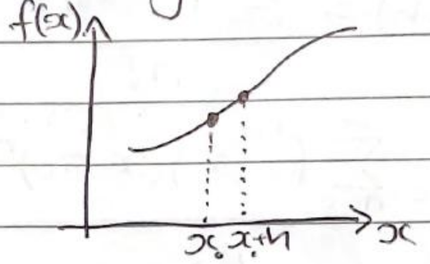
Functions 14

Taylor Series

Consider a function $f(x)$ of a single independent variable where the values of all of its derivates are known at a point x_0.

Try to evaluate the function at a nearby point x_0+h using a power series.



h is 'small'

$$f(x_0+h) = \alpha_0 + \alpha_1 h + \alpha_2 h^2 + \dots$$
$$= \sum_{n=0}^{\infty} \alpha_n h^n$$

the coefficents α_n are to be determined.

Putting $h=0 \Rightarrow f(x_0) = \alpha_0$ *

Then, if we differentiate with respect to h. $\Rightarrow$

$$f'(x_0+h) = \alpha_1 + 2\alpha_2 h + 3\alpha_3 h^2 + \dots$$

Let $h=0 \Rightarrow f'(x_0) = \alpha_1$ *

Continuing in this manner:

$$f''(x_0+h) = 2\alpha^2 + 6\alpha_3 h + \dots$$
$$\Rightarrow \frac{f''(x_0)}{2} = \alpha_2.$$ *

$$f'''(x_0+h) = 6\alpha_3 + 24\alpha_4 h + \dots$$
$$\Rightarrow \frac{f'''(x_0)}{6} = \alpha_3$$ *

The general result can be found by

$$\alpha_n = \frac{f^{(n)}(x_0)}{n!}$$

We can obtain the taylor series of $f(x)$ about x_0 in the form:

$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)\, h^n}{n!} \qquad h = (x - x_0)$$

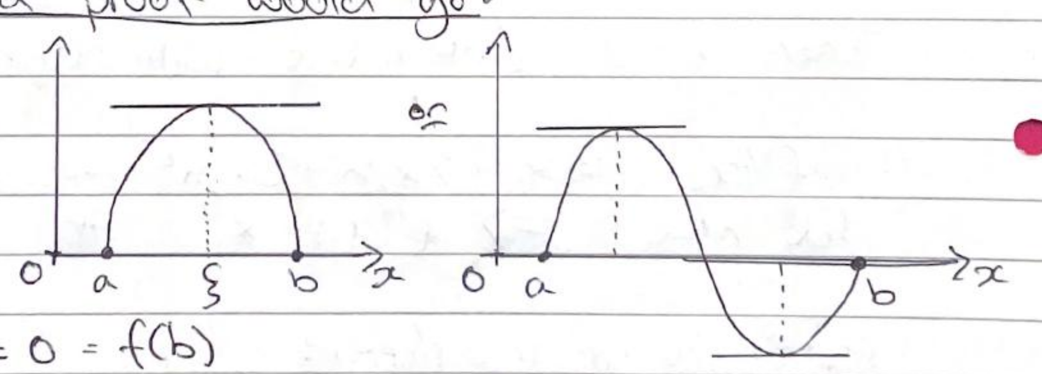
alternative expression:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

The above is not a formal proof, since it does rely on various assumptions.

<u>How a proof would go:</u>

Firstly:



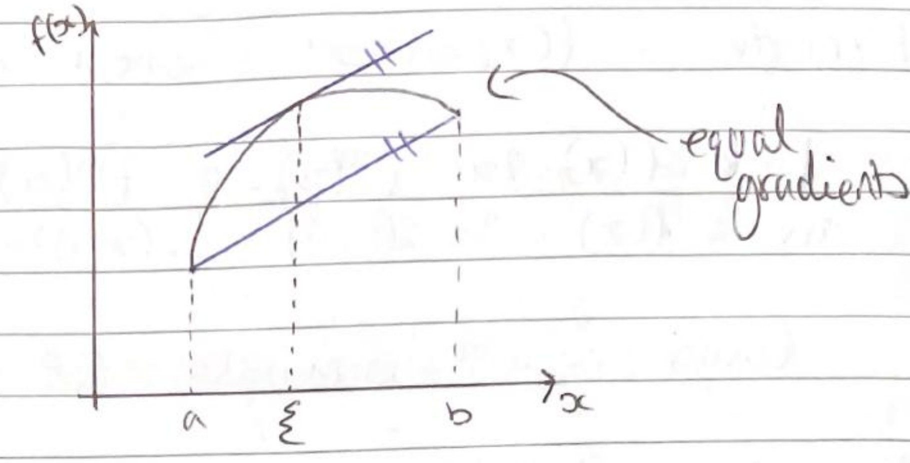
$f(a) = 0 = f(b)$

$f(x)$ continous & differentiable on (a,b)

$\Rightarrow$ there exists $a < \xi < b$ such that $f'(\xi) = 0$.

Known as <u>ROLLE'S THEOREM</u>

Then,



there must be ξ on (a,b) such that

$$f'(\xi) = \frac{f(b) - f(a)}{b-a}$$

i.e. $f(b) = f(a) + f'(\xi)(b-a)$

This is known the (first) mean value theorem.

Now putting $b = x$ and $a = x_0$ (ie. a = const.)

$$f(x) = f(x_0) + f'(\xi)(x - x_0)$$

We can extend the mean value theorem to find: (by induction)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$$
$$+ f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + R_n \longleftarrow \text{remainder term.}$$

where $R_n = \frac{(x-x_0)^n}{n!} f^{(n)}(\xi_n)$ with $x_0 < \xi_n < x$.

The utility of this result depends upon:
→ $f(x)$ having the requisite no. of derivatives
→ the behaviour of the remainder term as n increases.

Example $f(x) = 1 + x^2$ about $x_0 = 1$

Here, $f'(x) = 2x$ $f''(x) = 2$ $f'''(x) = 0$
and $f(x) = 2 + 2(x-1) + \frac{2}{2!}(x-1)^2$

taylor series terminates (of course).

Example $f(x) = \sin x$ about $x_0 = \pi/4$

$$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x$$

So. $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) - \frac{1}{\sqrt{2}}\left(\frac{(x-\pi/4)^2}{2!}\right) + \dots$

This series does not terminate but works well for small $|x - \pi/4|$ values. This should be equivalent to

$$\sin\left[\frac{\pi}{4} + (x - \frac{\pi}{4})\right] \equiv \sin\frac{\pi}{4}\cos(x - \frac{\pi}{4}) + \cos\frac{\pi}{4}\sin(x - \frac{\pi}{4})$$

Maclaurin Series
Simply put, the maclaurin series of a function $f(x)$ is the taylor series about the origin.

$$f(x) = \frac{\sum_{n=0}^{\infty} f^{(n)}(0)\, x^n}{n!}$$

independent derivation

Example $y = e^x$

$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example $\quad f(x) = \sin x$

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x$

$f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = -1$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Example $\quad f(x) = \cos x$

$f'(x) = -\sin x \quad f''(x) = -\cos x \quad f'''(x) = \sin x$

$f'(0) = 0 \quad f''(0) = -1 \quad f'''(0) = 0$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$