Classical Mechanics 4

Work-Energy Theorem

A body starts at x_i at time t_i and moves to x_f at time t_f under the influence of the general force F.

We normally think of the applied force F and the velocity, v, of the body as a function of t, but we can equally well think of them as functions of the position of the body, which is itself a function of time.

$$\left.\begin{aligned} v &= v(x) \\ F &= F(x) \end{aligned}\right\} \text{ where } x = x(t)$$

By the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{1}{2}\frac{d}{dx}(v^2)$$

So Newton's second law becomes:

$$F(x) = \frac{1}{2}m\frac{d}{dx}(v^2)$$

$$\int_{x_i}^{x_f} \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = \int_{x_i}^{x_f} F(x)\,dx$$

$$\left[\frac{1}{2}mv^2\right]_{x_i}^{x_f} = \int_{x_i}^{x_f} F(x)\,dx$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W_{if}$$ ← Work done. This is a definition.

$$KE_f - KE_i = W_{if}$$

W_{if} is the work done on the body, by the applied forces. It is measured in J.

Note what occurs when force is constant.

$$\int_{x_i}^{x_f} F\,dx = F\int_{x_i}^{x_f} dx = F(x_f - x_i) = \text{force} \times \text{distance}$$

Example

A spring like restoring force $F = -3x$ N pulls a mass from position $x = 3$m to $x = 2$m.

$$W_{if} = \int_3^2 -3x\,dx = \left[-\frac{3}{2}x^2\right]_3^2 = -\frac{3}{2}(4) + \frac{3}{2}(9) = \underline{\underline{7.5\text{J}}}$$

3D version (preview)

$$W_{if} = \int_{\underline{r}_i}^{\underline{r}_f} \underline{F(r)} \cdot d\underline{r} \quad \} \text{ dot product}$$

We will cover path integrals later!

KE vs momentum

KE	momentum
* $\propto m$	* $\propto m$
* ~~ve~~ scaler ($\propto v^2$)	* vector ($\propto v$)
* exchanged via forces	* exchanged via forces
* conserved in a closed system/cannot be converted to sntg new.	* total KE not conserved, can be converted, ΣE is conserved.

Power

Power = work done by external force per second.

Suppose a body moves a tiny distance δx in time δt.

The work done, δW, by force F is given (almost exactly) by.

$$\delta W = F\delta x$$

How can we keep F constant? By making the time interval as small as possible so the force during ~~consecutive~~ time intervals is ~~very similar~~ constant - i.e. $\Delta F \approx 0$. The smaller the δt, the better.

By shrinking δt until F is constant during δt, we can use the simpler equation. $W =$ Force $\times$ distance.

This trick is called a linear approximation and is used all the time in physics. (first-order).

$$\frac{\delta W}{\delta t} = F\frac{\delta x}{\delta t} \qquad (\div \delta t)$$

$$\frac{dW}{dt} = F\frac{dx}{dt} = Fv \qquad (\lim \delta t \to 0)$$

<u>3D</u> $\quad \delta W = F_{\parallel}\delta r = \underline{F}\cdot\underline{\delta r} = \underline{F}\cdot\underline{v}\,\delta t$

$$\text{Power} = \lim_{\delta t \to 0} \frac{\delta W}{\delta t} = \underline{F}\cdot\underline{v}$$

<u>Example</u>
A mass m hanging from a spring of spring constant s undergoes SHM at angular fequeny $\omega = \sqrt{\frac{s}{m}}$ with amplitude A. Whats the power.

We know that:

$$x(t) = A\sin(\omega t) \quad v(t) = \omega A\cos(\omega t) \quad F(t) = -Sx(t)$$

$$\text{Power} = F(t)v(t) = -Sx(t)\omega A\cos(\omega t) = -SA\sin(\omega t)\omega A\cos(\omega t$$

$$= -S\omega A^2\sin(\omega t)\cos(\omega t) = -\frac{1}{2}S\omega A^2\sin(2\omega t)$$