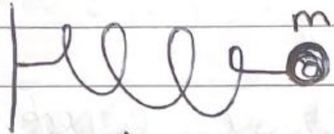
Forced Oscillations



← force applied to the mass.

↖ driver, move this end backwards and forwards.

$F_0\cos(\omega t)$

N.B. previously ω was part of the solution, now it is the applied driving frequency.

$$F = -kx - b\dot{x} + F_0\cos\omega t$$

restoring force

damping force (energy lost)

driving force (enegy supplied)

$$m\ddot{x} = -b\dot{x} - kx + F_0\cos\omega t$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

$$\gamma = b/m \quad \omega_0 = \sqrt{k/m}$$

This solution has two parts. I) <u>steady state</u> when the energy lost from damping equals the energy gained from the driver. System oscillates at driving frequency with const. amplitude.
II) <u>transient</u> the steady state is always present, but initally there is also a transient present which decays away leaving just the steady state.

Steady State

$$\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = \frac{F_0}{m}e^{i\omega t}$$

$$Re(RHS) = \frac{F_0}{m}\cos\omega t$$

We want to look for solutions where the energy from the driver compensates for energy lost via damping. The system will oscillate at some frequency ω^* with constant amplitude.

$$\tilde{x} = \tilde{A}e^{i\omega^* t} \quad \frac{d}{dt} = i\omega^*$$

$$-(\omega^*)^2\tilde{A}e^{i\omega^* t} + i\omega^*\gamma\tilde{A}e^{i\omega^* t} + \omega_0^2\tilde{A}e^{i\omega^* t} = \frac{F_0}{m}e^{i\omega t}$$

$$\tilde{A}e^{i\omega^* t}\left[-\omega^{*2}+i\omega^*\gamma+\omega_0^2\right] = \frac{F_0}{m}e^{i\omega t}$$

$$\tilde{A}e^{i(\omega^*-\omega)t}\left[-\omega^{*2}+i\omega^*\gamma+\omega_0^2\right] = \frac{F}{m}$$

therefore also independent of t.

independent of t.

$$\therefore \omega^* = \omega.$$

$$\tilde{A}\left[-\omega^2+i\omega\gamma+\omega_0^2\right] = \frac{F_0}{m}$$

$$\tilde{A} = Ae^{i\varphi} = \frac{F/m}{\omega_0^2-\omega^2-i\omega\gamma} = \frac{F_0/m\omega_0^2}{1-\frac{\omega^2}{\omega_0^2}-\frac{i\omega\gamma}{\omega_0^2}}$$

let's define:

$$A_0 = F_0/m\omega_0^2 = \frac{F_0}{m}\cdot\frac{m}{k} = \frac{F_0}{k}$$ ← displacement with const. force.

$$Q = \omega_0/\gamma$$

$$W = \omega/\omega_0$$

$$\tilde{A} = \frac{A_0}{1-W^2+i\frac{W}{Q}}$$

$$x = Re(\tilde{x}) = Re(\tilde{A}e^{i\omega t}) = Re(Ae^{i\varphi}e^{i\omega t}) = \cos(\omega t+\varphi)$$

N.B. both A and φ are I) dependent on ω, II) set by the system, not boundry conditions.

Resonance

$$RHS = \frac{A_0}{\left[(1-W^2)^2+W^2/Q^2\right]}\left[(1-\omega^2)-iW/Q\right]$$

Multiply top & bottom by complex conjugate.

$$A\cos\varphi = \frac{A_0(1-W^2)}{[(1-W^2)^2 + W^2/Q^2]}$$

$$A\sin\varphi = \frac{-A_0 W/Q}{[(1-W^2)^2 + W^2/Q^2]}$$

<u>Phase</u> - take $-\pi \leq \varphi \leq \pi$

~~This is from~~ the assumption $W > 0$, ~~which~~ means $\sin\varphi \leq 0$, which means that φ is actually $-\pi \leq \varphi \leq 0$.

$$\tan\varphi = \frac{\sin\varphi}{\cos\varphi} = \frac{-W/Q}{1-W^2}$$

<u>Amplitude</u> -

$$|Ae^{i\varphi}|^2 = \frac{A_0^2}{(1-W^2)^2 + W^2/Q^2}$$

$$A = \frac{A_0}{\sqrt{(1-W^2)^2 + W^2/Q^2}}$$

$\rightarrow W = 0$: $\sin\varphi = 0 \Rightarrow \varphi = 0$

$A = A_0$

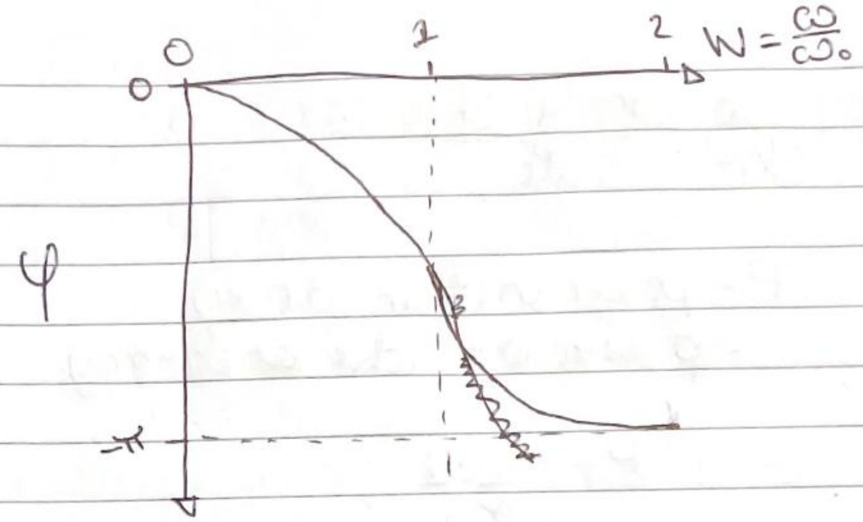
$\rightarrow W = 1$: $\cos\varphi = 0 \Rightarrow \varphi = -\pi/2$

$A = QA_0$

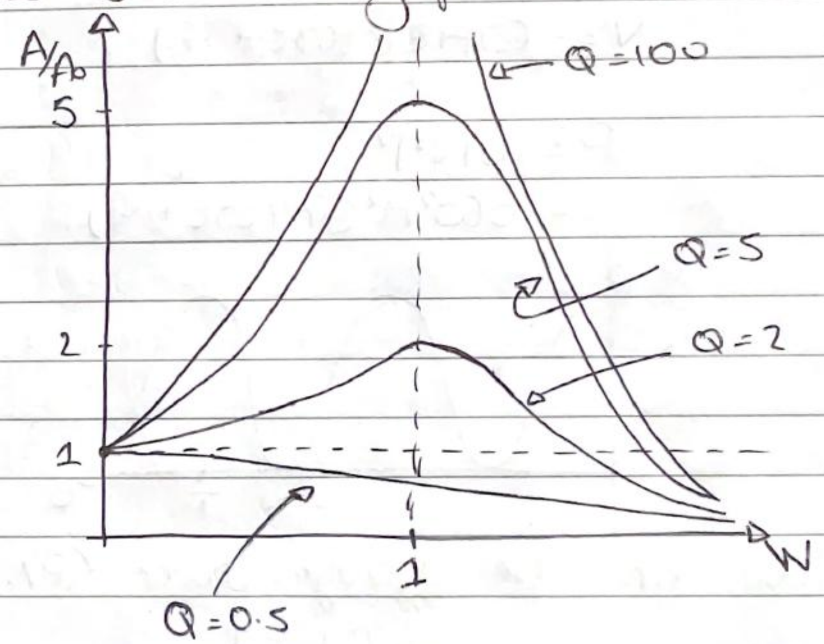
$\rightarrow W = \infty$: $\tan\varphi = 0 \Rightarrow 1/QW = 0 \Rightarrow \varphi = -\pi$

$A \rightarrow 0$.

if we now sketch a graph of W against φ...



Now lets sketch the graph of A/A_0 vs. W.



for very lightly damped, peaks at $W \simeq 1$.

$$A_{MAX} = A(W \simeq 1) = QA_0$$

Driving a system close to its natural frequency, thus producing a large amplitude is called resonance.

Resonant frequency is when

$$W = \sqrt{1 - \frac{1}{2Q^2}}$$