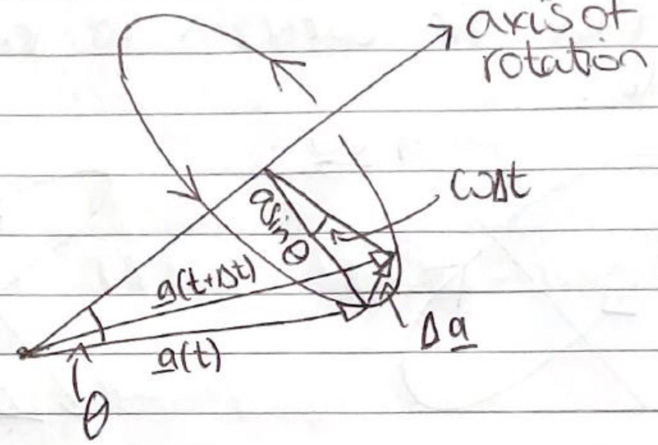
Classical Mechanics 13

Time-dependent vectors

Suppose $\underline{a}(t)$ is a time-dependent vector of fixed length:

$$|\underline{a}(t)| = a = \text{const.}$$



The radius vector of length $a\sin\theta$ sweeps around the circle at rate ω rad·s^{-1}.

☆ as $\Delta t \to 0$:

$$|\Delta\underline{a}| \to \text{arc length} = (a\sin\theta)(\omega\Delta t)$$

☆ the direction of $\Delta\underline{a}$ becomes tangent to the circle. Let's call this the $\hat{\phi}$ direction.

this gives

$$\frac{d\underline{a}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\underline{a}}{\Delta t} = \lim_{\Delta t \to 0}\left(\frac{(a\sin\theta)(\omega\Delta t)\hat{\phi}}{\Delta t}\right)$$

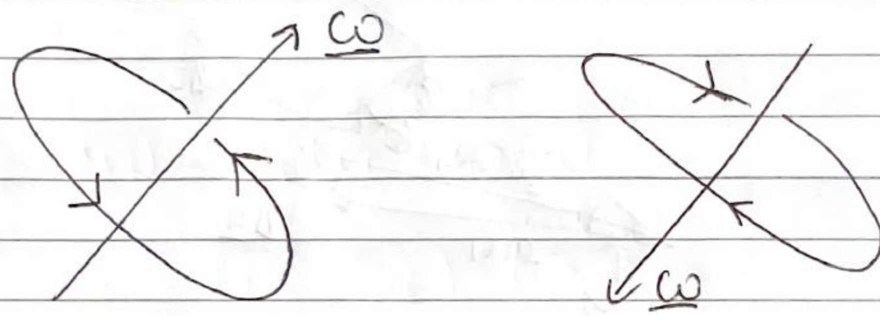
$$\frac{d\underline{a}}{dt} = a\omega\sin\theta\,\hat{\phi}$$

describing $\hat{\phi}$ in words is awkward, there is a better way.

We begin with the angular velocity:

$$\underline{\omega} = \omega \, \hat{\underline{\omega}}$$

where $\hat{\underline{\omega}}$ is the unit vector along the rotation axis, where the direction is found via the right hand rule.



Cross Product

$$\underline{\omega} \times \underline{a} = |\underline{\omega}||\underline{a}| \sin\theta \, \hat{\phi} = a\omega \sin\theta \, \hat{\phi}$$

$$\boxed{\frac{d\underline{a}}{dt} = \underline{\omega} \times \underline{a}}$$

Derivative of a General Time-Dependent Vector

A general time-dependent vector has a time-dependent direction and a time-dependent vector.

$$\underline{a}(t) = \underbrace{a(t)}_{\substack{\text{scalar} \\ =\text{length}}} \underbrace{\hat{a}(t)}_{\substack{\text{unit vector} \\ =\text{direction}}}$$

Product Rule:

$$\frac{d\underline{a}(t)}{dt} = \frac{da}{dt}\hat{a} + a\frac{d\hat{a}}{dt}$$

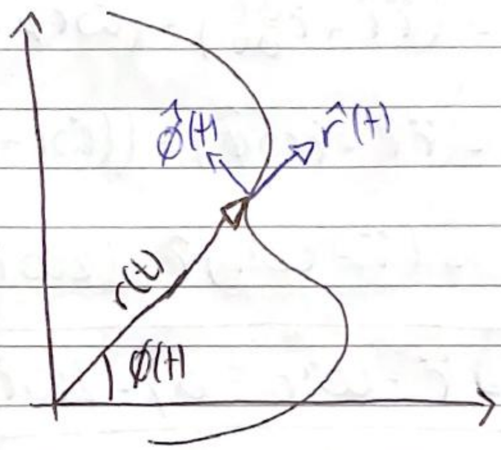
$$\boxed{\frac{d\underline{a}(t)}{dt} = \dot{a}\hat{a} + a\underline{\omega} \times \hat{a}}$$

This equation splits the rate of change of any vector in radial and angular contributions. The radial and angular parts are perpendicular to each other: $\hat{a} \cdot (\underline{\omega} \times \hat{a}) = 0$.

N.B. The angular velocity $\underline{\omega}$ is itself a time-dependent vector.

Plane Polar Coordinates

Consider a mass moving in 2D described by plane polar coordinates.



The angular velocity $\underline{\omega}(t) = \dot{\phi}(t)\hat{k}$ is always perpendicular to the plane of the paper.

Because $\underline{r}(t)$ depends on t, so do the directions of the plane polar unit vectors:

$$\frac{d\hat{r}}{dt} = \underline{\omega} \times \hat{r} = \omega\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = \underline{\omega} \times \hat{\phi} = -\omega\hat{r}$$

Velocity in Plane Polar

$$\underline{r}(t) = r(t)\hat{r}(t)$$

$$\underline{v}(t) = \frac{d\underline{r}}{dt}$$

$$= \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \quad \text{(product rule)}$$

$$= \dot{r}\hat{r} + r\omega\hat{\phi}$$

$$\boxed{v_r = \dot{r} \qquad v_\phi = \omega r}$$

<u>Acceleration in Plane Polar</u>

$$\underline{a}(t) = \frac{d}{dt}\left(\dot{r}\hat{r} + \omega r\hat{\phi}\right)$$

$$= \left(\ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt}\right) + \left(\dot{\omega}r\hat{\phi} + \omega\dot{r}\hat{\phi} + \omega r\frac{d\hat{\phi}}{dt}\right)$$

$$= \left(\ddot{r}\hat{r} + \dot{r}\omega\hat{\phi}\right) + \left(\left(\dot{\omega}r + \omega\dot{r}\right)\hat{\phi} - \omega^2 r\hat{r}\right)$$

$$= \left(\ddot{r} - \omega^2 r\right)\hat{r} + \left(2\omega\dot{r} + \dot{\omega}r\right)\hat{\phi}$$

$$\boxed{a_r = \ddot{r} - \omega^2 r \qquad a_\phi = 2\omega\dot{r} + \dot{\omega}r}$$

think centripetal acceleration

$$a = \frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r$$

$$a_\phi r = 2\omega r\dot{r} + \dot{\omega}r^2$$

$$= \underbrace{\frac{d}{dt}(\omega r^2)}_{\text{this links to angular momentum}}$$