Vectors 6

Determinant

The determinant is an $n \times n$ array evaluated to a scalar.

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \leftarrow \text{minor}$$

or $$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \leftarrow \text{minor}$$

A_{ij} is the minor - array left when excluding the corresponding row and column of that element.

eg. $$A_{22} = \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

The co-factor is $(-1)^{i+j} A_{ij}$ the minor times the $(-1)^{i+j}$. If i & j are both odd or even, $(-1)^{i+j} = 1$. If only one of $i+j$ is odd: $(-1)^{i+j} = -1$.

Reduce determinant until all minors are 2×2.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

called the laplace method.

Example

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -2 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1(8) - 2(4) + 2$$

$$= \underline{\underline{2}}$$

Properties

makes negative

I) swapping two rows (or columns) 'negates' the determinant.

eg. $D_3 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ $\quad D_3^* = \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$

$$D_3^* = b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + c\begin{vmatrix} e & d \\ h & g \end{vmatrix}$$

$$= -a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - c\begin{vmatrix} d & e \\ h & g \end{vmatrix}$$

$$= -D_3$$

II) transpose determinant leaves value unchanged.

$$D_3^T = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - d\begin{vmatrix} b & h \\ c & i \end{vmatrix} + g\begin{vmatrix} b & e \\ c & f \end{vmatrix}$$

$$= a(ei - fh) - d(bi - ch) + g(bf - ce)$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= D_3$$

III) combination - any row or column can be split or combined.

e.g. $\begin{vmatrix} a+j & b+k & c+l \\ d & e & f \\ g & h & i \end{vmatrix} = (a+j)\begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b+k)\begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c+l)\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} + j\begin{vmatrix} e & f \\ h & i \end{vmatrix} - k\begin{vmatrix} d & f \\ g & i \end{vmatrix} + l\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} j & k & l \\ d & e & f \\ g & h & i \end{vmatrix}$

IIII) a scaler multiplying a determinant multiples just one row or column.

$\lambda D_3 = \lambda\left[a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}\right]$

$= \begin{vmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{vmatrix}$

V) if any row or column hass all elements equal to zero, then the determinant is zero.

VI) if any two rows/columns are equivalent then the determinant is zero.

$D_3 = \begin{vmatrix} a & b & c \\ d & e & f \\ d & e & f \end{vmatrix} = a\begin{vmatrix} e & f \\ e & f \end{vmatrix} - b\begin{vmatrix} d & f \\ d & f \end{vmatrix} + c\begin{vmatrix} d & e \\ d & e \end{vmatrix}$

$= 0$

N.B. you can add/subtract multiples of any row or colown from a determinant without changing its value.

$$\begin{vmatrix} a+\lambda b & b & c \\ d+\lambda e & e & f \\ g+\lambda h & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} \lambda b & b & c \\ \lambda e & e & f \\ \lambda h & h & i \end{vmatrix}$$

$$= \underline{\underline{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}} + \cancel{\lambda \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}}$$

VII) triangular determinant equates to a product of diagonal terms.

$$\begin{vmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & 0 \\ h & i \end{vmatrix} = ahi$$