Electric Flux

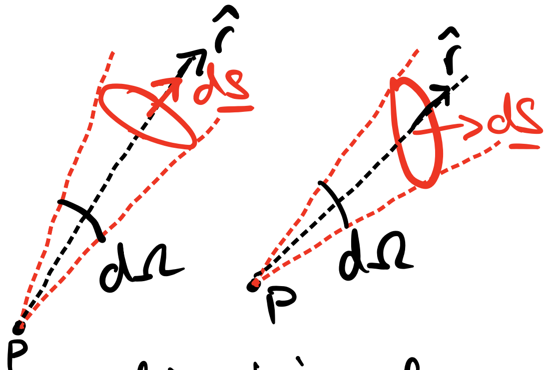
scalar $\rightarrow$ $$\Phi_E = \iint_R \underline{E} \cdot d\underline{S}$$ $\leftarrow$ points outwards

Can be though of as the number of electric field lines flowing through a surface.

Solid Angles

Consider a surface element $d\underline{S}$ which is a distance r from point P. The surface element is defined to subtend a solid angle $d\Omega$ by:

$$d\Omega = \frac{d\underline{S} \cdot \hat{r}}{r^2}$$



$\hat{r}$ is a unit vector along the direction from P to the surface element. The the surface element $d\underline{S}$ is parallel to $\hat{r}$, we can write,

$$d\Omega = \frac{dS}{r^2}$$

Solid angles have the unit of steradians. There are 4π steradians covering the surface of the sphere.

Flux Density from Point Charge

The electric field $\underline{E}$ due to a point charge Q is given by

$$\underline{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r}$$

The electric flux, Φ, through a spherical surface S of radius r is given by:

surface area of sphere

E is radial & uniform

$$\Phi = 4\pi r^2 E_1 = \frac{Q}{\varepsilon_0}$$

Now if we consider the flux of a small portion of the sphere ds, it's given by

$$d\underline{\Phi}_s = \underline{E}_1 \cdot d\underline{s} = \frac{\underline{\Phi}_s}{4\pi r^2} ds$$

$$\frac{\underline{\Phi}_s}{4\pi r^2} dS = \frac{\underline{\Phi}_s}{4\pi} d\Omega = \frac{Q}{4\pi\varepsilon_0}$$

$$\therefore \quad d\underline{\Phi}_s = \frac{Q}{4\pi\varepsilon_0} d\Omega$$

Now lets define a new surface R which encloses S. The flux through R will be

$$d\underline{\Phi}_R = \underline{E} \cdot d\underline{R} = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r} \cdot d\underline{S} = \frac{Q}{4\pi\varepsilon_0} d\Omega = d\underline{\Phi}_s$$

The flux through the two surface elements are the same, even though the direction of $d\underline{R}$ is arbitrary.

The flux through any closed surface is always Q/ϵ_0.

Gauss's Law

To extend this idea to multiple charges, we use the superposition principle.

FYI - surface integrals

$$\Phi = \oint_S \underline{E_1} \cdot d\underline{S} + \oint_S \underline{E_2} \cdot d\underline{S} + \oint_S \underline{E_3} \cdot d\underline{S} + \dots + \oint_S \underline{E_N} \cdot d\underline{S}$$

$$\Phi = \frac{Q_1}{\epsilon_0} + \frac{Q_2}{\epsilon_0} + \frac{Q_3}{\epsilon_0} + \dots + \frac{Q_N}{\epsilon_0}$$

Since the enclosed charge is just $Q_{enc} = Q_1 + Q_2 + Q_3 + \dots + Q_N$

$$\oiint_S \underline{E} \cdot d\underline{S} = \frac{Q_{enc}}{\epsilon_0}$$

This is gauss's law in integral form. Observations:

I) all charges contribute to $\underline{E}$ but not all charges contribute to Q_{enc}.

II) only charges inside the surface matter.

III) we'll mainly look at symmetric situations to simplify our calculations.