Fourier Analysis

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Chapter 1

Introduction

Fourier transforms and Fourier series are a very general way to describe waves. With waves behind phenomena in almost every area of physics this short course will touch upon many of them. Examples will be given from areas as different as quantum mechanics, acoustics, particle physics, mechanics, solid state physics and cosmology. Fourier transforms are also essential for modern day life where mobile phones, televisions and cameras would not be possible without them.

Often differential equations can be formulated that describe the physical behaviour of a system. For boundary conditions that describe simple plane wave situations, the system can often be solved analytically. But in real life, boundary conditions are complicated and the solutions are no longer simple. This is where Fourier analysis comes into the picture. It allows for a problem to be broken down to the behaviour of the individual plane wave components and then, from using the principle of superposition, the full solution can be found.

1.1 Aims and objectives

1.1.1 Aims

To give students an understanding of Fourier series and Fourier transforms, and provide students with practice in their application and interpretation in a range of situations.

1.1.2 Objectives

At the end of the course, the student should:

- Be able to express a function as a sum of an even function and an odd function.
- Understand the concept of sets of orthogonal functions and appreciate that the Fourier series component functions are orthogonal.
- Understand the relationship between the complex exponential and trigonometric Fourier series.
- Know that periodic functions satisfying the Dirichlet conditions may be expressed as Fourier series.

- Be able to compute the Fourier coefficients of the complex exponential and the trigonometric Fourier series.
- Know and be able to apply Parseval's theorem and understand how it can be applied to physical situations.
- Know about the Gibbs phenomenon at discontinuities.
- Understand that if a Fourier series is constructed to represent an arbitrary function over a given range then the series represents that function periodically extended beyond that range.
- Know the defining properties for the Dirac delta function and be able to apply the sifting property of delta functions.
- Know and be able to apply expressions for the forwards and inverse Fourier transform to a range of non-periodic waveforms.
- Be able to calculate the Fourier transform and inverse Fourier transform of common functions including (but not limited to) top hat, Gaussian, delta, trigonometric, and exponential decays.
- Appreciate that Fourier transforming a Gaussian function produces another Gaussian.
- Know fundamental mathematical properties of the Fourier transform including linearity, shift, scaling, and convolution.
- Be able to express a convolution of smooth functions and the delta function mathematically and explain its function and relationship to measurement processes.
- Be able to give multiple examples of where Fourier series, Fourier transforms and convolutions are used in physics.

1.2 Lectures

The lectures will cover the following areas:

- Introduction, even and odd functions, orthogonality, integral of complex function of a real variable.
- Fourier series of periodic functions.
- Complex and trigonometric notation of Fourier series.
- Discontinuous functions, Diriclet conditions, Gibbs phenomenon, derivatives.
- Fourier transforms, Parseval's theorem.
- Evaluating the Fourier transform.
- Dirac delta function, Properties of the Fourier transform.
- Applications of Fourier transforms.

- Convolutions.
- Applications of convolutions.

1.3 Literature

While lecture notes are provided for this course, they should not be seen as a replacement for reading up independently in a text book. Three books are recommended for this course.

- Arfken, G. B., Weber, H.J., Harris, F. E., *Mathematical Methods for Physicists: a comprehensive guide*, 7$^{\text{th}}$ edition, Elsevier publishers.
- Boas, M. L., *Mathematical Methods in the Physical Sciences*, 3$^{\text{rd}}$ edition, Wiley publishers.
- K. B. Howell, *Principles of Fourier Analysis*, 2$^{\text{nd}}$ edition, CRC Press.

The Arfken book is fairly rigorous and is a very nice book for all undergraduate mathematics in Physics. I would recommend to purchase it. The Boas book is quite superficial in some places. The book by Howell, on the other hand, contains all possible detail you could ever think about.

All three books are available in the library as paper copies as well as available as e-books.

Concerning Fourier series, Fourier transforms and convolutions, there is a great variation in notation. I will make it clear, during the course, what notation I use and attempt to be consistent. In general you should always be aware of the exact notation used in a given situation. **The convention used across the whole degree coursess is not (and could never be!) consistent.**

Chapter 2

Mathematical prerequisites

2.1 Complex functions

Boas 2, AWH 1.8

A complex function is a function that takes a complex number as an argument and the value of the function is a complex number as well. We can divide a complex function into a real function and an imaginary function,

$$f(z) = u(z) + iv(z), \quad u(z) = \frac{f(z) + f^*(z)}{2}, \quad v(z) = \frac{f(z) - f^*(z)}{2i} , \tag{2.1}$$

where u and v are real valued functions that take a complex argument.

In this course, complex exponential functions will often be used as they provide for an easier notation than using trigonometric functions when representing wave phenomena. The translation between the two notations make use of Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta \; . \tag{2.2}$$

If we consider complex functions that only take a real argument, then integrating and differentiating them is exactly as for real functions. So

$$f'(x) = u'(x) + iv'(x) \tag{2.3}$$

and

$$\int f(x)dx = \int u(x)dx + i\int v(x)dx \; . \tag{2.4}$$

2.2 Even and Odd functions

Boas 7.9

To be able to spot if a function is even or odd, and indeed how it can be decomposed into an even and an odd part is of great advantage when it comes to considering periodic functions. If we consider a function of a real variable x that is defined on a symmetric interval around zero, we have

$$\begin{aligned} f(x) &= f(-x) \quad \text{if } f \text{ is even} \\ f(x) &= -f(-x) \quad \text{if } f \text{ is odd} \; . \end{aligned} \tag{2.5}$$

Just as a complex function can be decomposed into a real and imaginary part, an arbitrary function $f(x)$ defined on a symmetric interval around zero can be decomposed into a sum of an even part, $e(x)$, and an odd part, $o(x)$. We have

$$f(x) = e(x) + o(x) \;, \tag{2.6}$$

with

$$\begin{aligned} e(x) &= \frac{f(x)+f(-x)}{2} \\ o(x) &= \frac{f(x)-f(-x)}{2} \;, \end{aligned} \tag{2.7}$$

which is easy to show.

Example 2.1. *Consider the function $f(x) = e^x$ on the interval $(-\pi, \pi)$. Decomposing it into an even and an odd part we have*

$$\begin{aligned} e(x) &= \frac{e^x + e^{-x}}{2} = \cosh(x) \\ o(x) &= \frac{e^x - e^{-x}}{2} = \sinh(x) \;, \end{aligned} \tag{2.8}$$

There are several properties of even and odd functions that are useful in Fourier analysis:

- The product of two odd functions is an even function, $o_1(x)o_2(x) = e(x)$.
- The product of two even functions is an even function, $e_1(x)e_2(x) = e(x)$.
- The product of an even and an odd function is an odd function, $e_1(x)o_1(x) = o(x)$.
- The integral of an odd function on a symmetric interval is zero, $\int_{-a}^{a} o(x)dx = 0$.
- The integral of an even function on a symmetric interval is twice the integral of the postive part, $\int_{-a}^{a} e(x)dx = 2\int_0^a e(x)dx$.

The rules above can be combined with a coordinate transformation where the zero point is shifted.

Example 2.2. *Consider the function $f(x) = x - 3$ and let us evaluate $\int_0^6 f(x)dx$. If we let $f'(x') = f(x+3)$, we see that $f'(x')$ is odd. We can then use this to conclude that*

$$\int_{x'=-3}^{x'=3} f'(x')dx' = 0 \text{ or } \int_{x=0}^{x=6} f(x)dx = 0 \;. \tag{2.9}$$

2.3 Orthogonal and orthonormal functions

Boas 12.6, AWH 5.1

From vector analysis we know that two vectors **A** and **B** are orthogonal if their scalar product is zero, that is if

$$\mathbf{A} \cdot \mathbf{B} = \sum_i A_i B_i = 0 \;, \tag{2.10}$$

where the index i refer to the individual components in a Cartesian coordinate system. The concept of orthogonality can be extended to an arbitrary number of vectors $\mathbf{V}_n$. We say that the set of all vectors, $\{\mathbf{V}_n\}$ make an orthogonal set if

$$\mathbf{V}_n \cdot \mathbf{V}_m = 0 \, , \quad \text{for } m \neq n \, . \tag{2.11}$$

The set of vectors is called a *complete orthogonal set* if the size of the orthogonal set is the same as the number of dimensions in the space considered. The set is called *orthonormal* if the vectors are normalised. In that case we have

$$\mathbf{V}_n \cdot \mathbf{V}_m = \delta_{mn} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \, . \end{cases} \tag{2.12}$$

Notice how the *Kronecker delta* notation, δ_{mn} was used here.

If we have a complete orthonomal set of N vectors $\mathbf{V}_n$, then we can in a unique way write an arbitrary vector $\mathbf{A}$ as a *linear superposition* of the orthonormal vectors,

$$\mathbf{A} = \sum_{n=1}^{N} a_n \mathbf{V}_n = a_1 \mathbf{V}_1 + a_2 \mathbf{V}_2 + \cdots + a_N \mathbf{V}_N \, . \tag{2.13}$$

We are *constructing* the vector $\mathbf{A}$ using the $\mathbf{V}_n$ as building blocks. We call $\mathbf{V}_n$ an *orthonormal basis.*

Conversely, we can *decompose* a given vector $\mathbf{A}$ using its scalar product with the basis vectors $\mathbf{V}_n$ and using the definition (2.12), i.e. we have

$$a_n = \mathbf{A} \cdot \mathbf{V}_n \tag{2.14}$$

for the individual coordinates.

Example 2.3. *In 3-dimensional space the three unit vectors* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *form a complete orthonormal set. If we take a vector* $\mathbf{A} = (3, 7, 5)$ *and apply Eq. (2.14), we see that*

$$\mathbf{A} = 3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k} \, . \tag{2.15}$$

The definition of orthonormal vectors can be extended in several ways. If we consider the vectors to be complex, the definition of an orthonormal set is

$$\mathbf{V}_n \cdot \mathbf{V}_m^* = \delta_{mn} \, . \tag{2.16}$$

As the complex conjugate of a real function is the function itself, we can just consider this as the definition of an orthonormal set of vectors as it includes the real case.

We can also extend the definition of orthogonal or orthonormal to any kind of expression where we can formulate a scalar (or inner) product. In particular for complex functions, we define the inner product $\langle f, g \rangle$ between two (complex) functions f and g on an interval (a, b) as

$$\langle f, g \rangle = \int_a^b f(x) g^*(x) dx \, . \tag{2.17}$$

This leads us to that a set of functions, f_n, form an orthonormal set if

$$\langle f_n, f_m \rangle = \delta_{mn} \, . \tag{2.18}$$

Example 2.4. *Consider the set of complex exponentials*

$$\frac{1}{\sqrt{2\pi}}e^{in\theta} \text{ with } n \in \mathbb{Z}$$

defined on the interval $(-\pi,\pi)$. *The inner product is*

$$\langle e^{in\theta}, e^{im\theta}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}d\theta = \delta_{mn} \tag{2.19}$$

which shows that this set of functions forms an orthonormal set.

Without proof we state that the set of complex exponentials

$$\frac{1}{\sqrt{2\pi}}e^{in\theta} \text{ with } n \in \mathbb{Z} \tag{2.20}$$

form a complete set on the very large collection of *square integrable functions*, f, where

$$\langle f, f\rangle = \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \int_{-\pi}^{\pi} f(\theta)f^*(\theta)d\theta < \infty\,.$$

This set of functions (scaled to appropriate intervals) include just about any function you will ever consider in physics. Notice that the *function space* that is considered here has an infinite number of dimensions.

Returning to Eqs. (2.13) and (2.14) and applying them to functions, we see that if we have a complete set of orthonormal functions g_n, we can expand an arbitrary function $f(x)$ on a closed interval as

$$f(x) = \sum_{n\in\mathbb{Z}} a_n g_n(x)\;, \tag{2.21}$$

with

$$a_n = \langle f, g_n\rangle = \int_a^b f(x)g_n^*(x)dx \tag{2.22}$$

for the individual coordinates. Using complex exponentials as the complete orthonormal set is the essential part of a Fourier series expansion.

You might wonder what the definition is of an inner product. It is a mapping, $V\times V \to S$, from a pair of generalised vectors (like the functions above) to a real or a complex number. It has to satisfy the following criteria;

- conjugate symmetry: $\langle x,y\rangle = \langle y,x\rangle^*$,
- linearity: $\langle ax,y\rangle = a\langle x,y\rangle$,
- additivity: $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$,
- positive-definiteness: $\langle x,x\rangle \geq 0$,
- $\langle x,x\rangle = 0 \iff x = 0$.

It is an easy exercise to prove that the inner products defined above are satisfying these conditions.

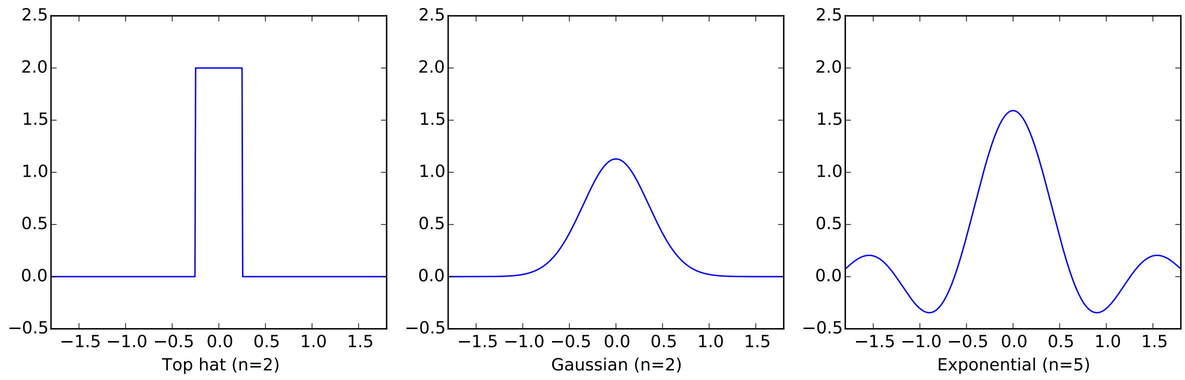

Figure 2.1: The top hat, Gaussian and complex exponential functions that each in the limit $n \to \infty$ approaches the delta function.

2.4 Dirac delta function

Boas 8.11, AWH 1.11

Consider a real function $\delta(x)$, called the *(Dirac) delta function*, that is defined by the criteria

$$\delta(x) = 0 \text{ if } x \neq 0, \text{ and} \tag{2.23}$$

$$\int_a^b f(x)\delta(x)dx = f(0) \text{for } a < 0 \text{ and } b > 0, \tag{2.24}$$

where $f(x)$ is an arbitrary real function. Trying to use the definition with $f(x) = 1$ we see that we have the trivial result

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \; . \tag{2.25}$$

The delta function must be infinitely high at the origin, but somehow still adjusted such that its integral takes on the unit value. In fact it is not a function at all but rather has to be thought of as the limit of a sequence of functions. In this way the delta function itself only makes sense when considered under an integral sign. We can use the delta function to express the value of a function $f(x)$ as an integral,

$$f(x) = \int_{-\infty}^{\infty} f(t)\delta(t-x)dt \; . \tag{2.26}$$

This is called the *sifting* property of the of the δ function.

There are many different sequences that define the Dirac delta function and which one to use will depend on the situation. Here the top hat function, a complex exponential and a Gaussian are considered. In each case we consider

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) \; , \tag{2.27}$$

where $\delta_n(x)$ in the three cases is defined as either of

$$\delta_n(x) = \begin{cases} n & \text{if } |x| \leq \frac{1}{2n} \\ 0 & \text{if } |x| > \frac{1}{2n} \end{cases} , \tag{2.28}$$

$$\delta_n(x) = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt}dt \; , \tag{2.29}$$

or

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \ . \tag{2.30}$$

All three functions are shown in Fig. 2.1. It is fairly easy to see that the tophat and Gaussian series in the limit satisfy the definiton of the delta function, Eq. (2.23), while for the complex exponential form, it helps to rewrite it as

$$\delta_n(x) = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt = \frac{\sin nx}{\pi x} \ . \tag{2.31}$$

For the complex exponential form we can substitute x with $t - x$ to reach

$$\delta(t - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega \ , \tag{2.32}$$

which is the form that we will meet in Fourier integrals.

Chapter 3

Fourier Series

3.1 Definition

Boas 7.1, 7.5, 7.7, AWH 19.1.

The complex exponential Fourier series of a periodic function $f(x)$ with period 2π is defined as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \,, \tag{3.1}$$

in which

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \,. \tag{3.2}$$

The set of functions $\{e^{inx}\}$ form a complete orthogonal set (see Sec. 2.3. However, the set is not an orthonormal set due to the way that the normalisation is defined. Be aware that the normalisation differs between text books.

If we rewrite the sum in Eq. (3.1) to have pairs with n and $-n$, we get

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + c_{-n}e^{-inx} \,. \tag{3.3}$$

The individual terms in the sum can be rewritten as

$$\begin{aligned}
&\frac{c_n + c_{-n}}{2}(e^{inx} + e^{-inx}) + \frac{c_n - c_{-n}}{2}(e^{inx} - e^{-inx}) \\
=&\frac{c_n + c_{-n}}{2}2\cos(nx) + \frac{c_n - c_{-n}}{2}2i\sin(nx) \\
=&a_n\cos(nx) + b_n\sin(nx) \,,
\end{aligned} \tag{3.4}$$

where we in the last line have defined

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \,. \tag{3.5}$$

Plugging the definition of c_n into this we have

$$\begin{aligned}
a_n =&\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\
=&\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{e^{inx} + e^{-inx}}{2}dx \\
=&\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
b_n =& \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\
=& -\frac{i}{\pi}\int_{-\pi}^{\pi} f(x)\frac{e^{inx}-e^{-inx}}{2}dx \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{e^{inx}-e^{-inx}}{2i}dx \\
=& \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx
\end{aligned} \tag{3.7}$$

With this information in hand we are now ready to write down the Fourier series using trigonometric functions as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right), \tag{3.8}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{3.9}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\,. \tag{3.10}$$

These expressions deserve a few comments. While harder to remember than the expression of the Fourier series with complex exponentials, it is easier to interpret in terms of different wave components adding up to the full function. The division by 2 in the first term of the sum is simply a convenience to make the expression for a_n work for the term a_0 as well. You can confirm that this is correct by looking at the expression for c_0 in the exponential notation. However, you have to remember this factor when you write down the sum!

Example 3.1. *We can use the first terms of a Fourier series to approximate a function. If we consider the square wave defined as*

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 < x < \pi\,, \end{cases} \tag{3.11}$$

we have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} dx = 1 \tag{3.12}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos(nx)dx = 0\,, n \ge 1 \tag{3.13}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin(nx)dx = \begin{cases} \frac{2}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases} \tag{3.14}$$

and

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \cdots\right). \tag{3.15}$$

In Fig. 3.1 the Fourier series with an increasing number of terms can be seen. The overshoot close to the discontinuity that persist even with a very large number of terms will be considered later.

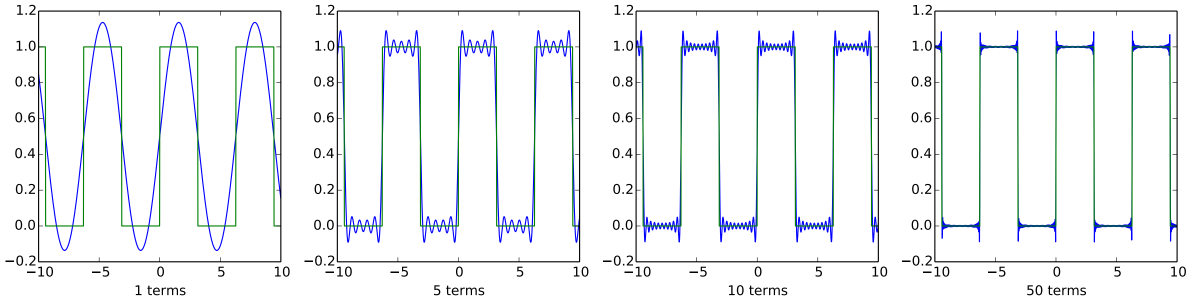

Figure 3.1: The approximation of the square wave with an increasing number of terms from the Fourier series. Notice the small spikes present at the discontinuities even with 50 terms used. This will be returned to later.

3.2 Reality Condition

What happens when we use the Fourier series to construct a real-valued function $f(x) \in \mathbb{R}$? In this case the function is equal to its complex conjugate $f(x) = f^*(x)$ but the coefficients c_n are still complex, the Fourier basis function in the scalar product ensures.

This should seem strange. A real valued function is represented by a single real numbers at each point x but we are constructing it using two (ral and imaginary) numbers in each of the coefficients c_n. The two should be equivalent representations but one seems to use redundant information.

We can resolve this by looking at the complex conjugate of the coefficients

$$c_n^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(x)e^{inx}dx\,. \tag{3.16}$$

but since $f(x) = f^*(x)$ we have

$$c_n^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \equiv c_{-n}\,. \tag{3.17}$$

This tells us that to construct a real function we only need to specify complex coefficients c_n for $n \geq 0$. The c_n for $n < 0$ are not independent quantities as they are all determined by the complex conjugates of the $n > 0$ coefficients. Note also that c_0 is the only coefficients that is real since $c_0^* = c_0$ for real $f(x)$.

This is known as the reality condition and is important in signal analysis which usually involved real functions (e.g. audio signal as a function of time). We will see how tuncated series are used to store the information of discreetly sampled functions and the reality conditions ensures the amount of information is equivalent in the equivalent representations.

3.3 Dirichlet conditions of convergence

Boas 7.6, AWH 19.1
Having defined the Fourier series, it is of interst to examine what is actually meant by the equal sign in Eq. (3.1). For this we can use the **Dirichlet conditions** that state that for a function $f(x)$, if

1. $f(x)$ is periodic of period 2π,

2. it is single valued,
3. has a finite number of maxima and minima values in the 2π interval,
4. has a finite number of discontinuities in the 2π interval,
5. and $\int_{-\pi}^{\pi} |f(x)|dx$ is finite,

then

- the Fourier series converges[1] to $f(x)$ at all points where $f(x)$ is continuous,
- and the series converges to the midpoint between the values of $f(x)$ from the left and the right at points of discontinuity.

The Dirichlet conditions are sufficient for convergence but not necessary so there will be (pathological) functions where the Fourier series converge despite that they fail the Diriclet conditions. As we will see later, the Dirichlet conditions can be defined for any other interval than 2π.

Example 3.2. *If we take*

$$f(x) = \frac{1}{1+\exp(\sin(x))} \quad , -\pi \le x < \pi \tag{3.18}$$

it is easy to see that the first four parts of the Dirichlet conditions are fulfilled but the last one is a bit tricky. However, we notice that $f(x)$ is bounded such that $|f(x)| < 1$ for all x and thus have

$$\int_{-\pi}^{\pi} |f(x)|dx < \int_{-\pi}^{\pi} 1dx = 2\pi \tag{3.19}$$

which is finite. Thus the last condition is fulfilled as well and the Fourier series converges.

Example 3.3. *If we let $f(x) = 1/x$ we have that*

$$\int_{-\pi}^{\pi} |f(x)|dx = \int_{-\pi}^{\pi} \left|\frac{1}{x}\right| dx = 2\ln x|_0^\pi = \infty \tag{3.20}$$

and part five of the Dirichlet conditions is not satisfied. On the other hand if we take $f(x) = 1/\sqrt{|x|}$, we get

$$\int_{-\pi}^{\pi} |f(x)|dx = 2\int_{0}^{\pi} \left|\frac{1}{\sqrt{x}}\right| dx = 4\sqrt{x}|_0^\pi = 4\sqrt{\pi} \tag{3.21}$$

and as the other conditions are trivially satisfied, the series converges.

3.4 Properties of Fourier series

Boas 7.3, 7.8 AWH 19.1, 19.2

[1] At each point x, the series will converge to $f(x)$ but as Gibb's phenomena later illustrates, the convergence is not guaranteed to be uniform.

3.4.1 Different intervals

The basic idea of expanding a periodic function as a sum of complex exponentials or a sum of trigonometric functions does not depend on using the interval $(-\pi,\pi)$. For an arbitrary interval of length $2l$, we indeed have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} , \tag{3.22}$$

with

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}dx , \tag{3.23}$$

and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l)\right) , \tag{3.24}$$

with

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(n\pi x/l)dx \tag{3.25}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(n\pi x/l)dx . \tag{3.26}$$

The Diriclet conditions are the same as for the *standard* interval of $(-\pi,\pi)$ apart from the the function is required to have a period of $2l$.

3.4.2 Power spectrum

The Fourier series can, in addition to the complex exponential form and the trigonometric form with cos and sin functions that we have already seen, be written on the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n) . \tag{3.27}$$

It can be shown (see Problem Sheet) that this is an equivalent representation with

$$a_n = \alpha_n \cos\theta_n, \qquad b_n = \alpha_n \sin\theta_n \tag{3.28}$$

$$\alpha_n^2 = a_n^2 + b_n^2, \qquad \tan\theta_n = b_n/a_n . \tag{3.29}$$

The form here is interesting as it allows for an easier interpretation of the data. The α_n terms represents the strength of each frequency term and the θ_n terms represents the phase shift of the terms with respect to each other. In many applications there is an overall phase which has no physical meaning and we can see here that if we shift the wave by an overall phase, ϕ, the consequence is that all θ_n are shifted by ϕ but the α_n terms are unchanged. The set represented by α_n^2 is called the power spectrum of a function. An example of the use of this can be seen in Fig. 3.2.

3.4.3 Parseval's identity

Let us take a (complex) function and represent it in terms of the complex exponential Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} , \tag{3.30}$$

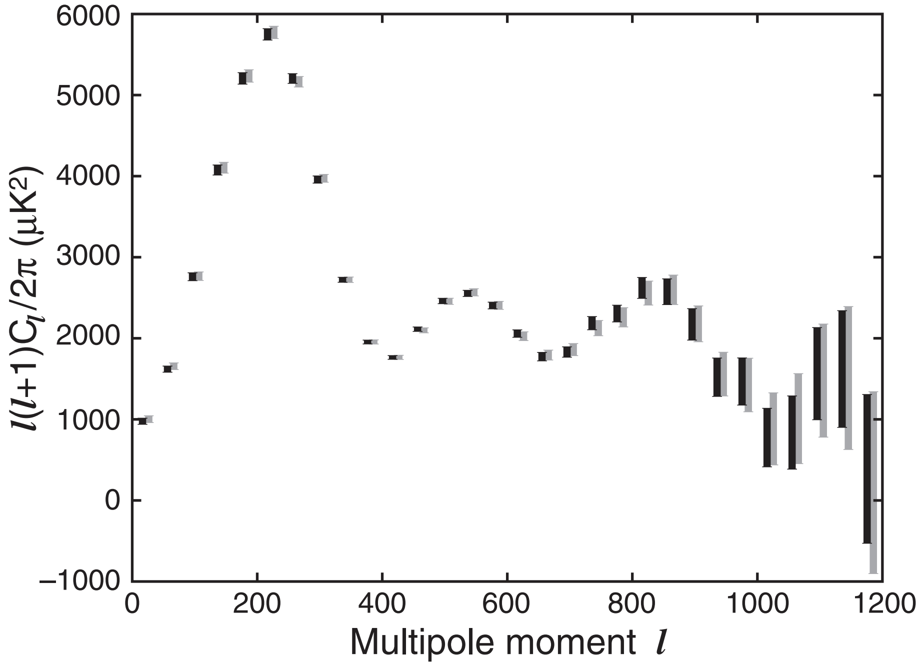

Figure 3.2: The power spectrum of the variation of in the cosmic microwave background as derived from a Fourier Series expansion (on a sphere). Source arXiv:1212.5225.

and try to calculate the integral of the magnitude squared of $f(x)$ on each side. We get

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} c_n e^{inx} \right|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \,. \tag{3.31}$$

Make sure to take a bit of time to consider the last equality above (where did all the cross terms go). Rearranging we have

$$\frac{\int_{-\pi}^{\pi} |f(x)|^2 dx}{2\pi} = \sum_{n=-\infty}^{\infty} |c_n|^2 \,. \tag{3.32}$$

which is called **Parseval's identity** and express that the average of the magnitude squared of the function is equal to the sum of the magnitude squared of the Fourier coefficients. As all the terms in the sum are positive, this also leads to **Parseval's inequality** for the truncated Fourier series, where for any positive $N < \infty$, we have

$$\sum_{n=-N}^{N} |c_n|^2 \leq \frac{\int_{-\pi}^{\pi} |f(x)|^2 dx}{2\pi} \,. \tag{3.33}$$

The equivalent form of Parseval's identity for the trigonometric Fourier series is

$$\frac{\int_{-\pi}^{\pi} |f(x)|^2 dx}{2\pi} = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \,. \tag{3.34}$$

Parseval's identity is very useful for considering how good an approximation to the function, a Fourier series with a given number of terms is. Be aware that it only talks about the quality of the approximation in the mean and not about how good the approximation is at a given point.

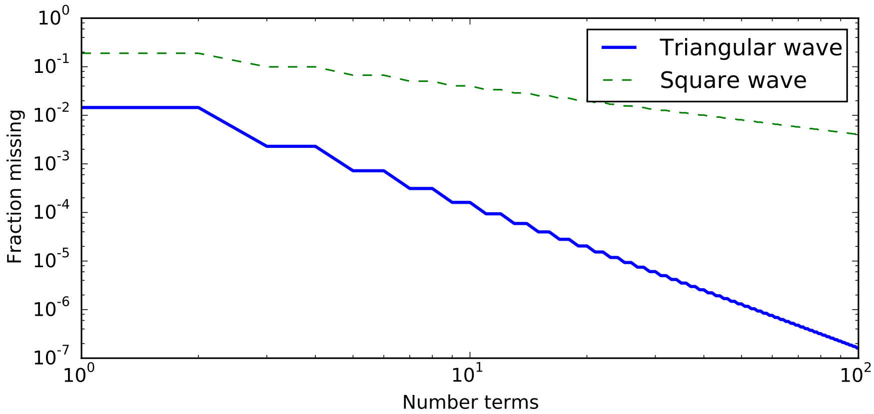

Figure 3.3: The fraction of the total that is missing when only taking the first n terms in the sum of Parseval's identity. It can be seen how the triangular wave with no discontinuities is a much better approximation than the square wave (with discontinuities) for a given truncation.

Example 3.4. *Consider the square wave and the triangular wave again. We can see that the sum in Parceval's identity will have* $1/n^2$ *terms for the square wave and* $1/n^4$ *terms for the triangular wave. The sum in the triangular wave will thus approach the average of the magnitude squared much faster than the sum for the square wave, as seen in Fig. 3.3. We learn from this that the more smooth a function is, the better the approximation of the Fourier series is for a truncation at a given* n*.*

3.4.4 Gibb's phenomena

When a truncated Fourier series is used to approximate a function with discontinuities in, we get a set of artifacts showing up as an overshoot (undershoot) with respect to the function on each side of the discontinuity. The effect is very clearly seen in Fig. 3.1 for the square wave. The overshoot is not reducing in size as we add more terms to our truncated series but rather just moves closer to the discontinuity. The effect is present for all functions with a discontinuity and is called Gibb's phenomena.

So why does Gibb's phenomena occur? It is an effect of that a discontinuity represents a change taking place in zero time and thus includes infinite high frequencies. When we then approximate this by only finite frequencies as in the truncated Fourier series, we get a *ringing* effect around the transition where the first *ring* is the largest overshoot.

Gibb's phenomena is not just a theoretical oddity but has real implications. In electronics the ringing can give rise to *after pulses* that may themselves be detected as signal events, in compressed images there might be shadowing effects (see Fig. 3.4) and in compressed audio, it can give rise to what is called *pre echo*, a faint echo that occurs *before* a sharp sound like a drum. In the MP3 algorithm, the effect of this is minimised by having an algorithm where the frequency cutoff can adapt to the audio that is compressed.

Figure 3.4: An illustration of the shadowing that occurs from an image of a sharp edged ellipse in the JPEG compression algorithm due to Gibb's phenomena. The leftmost picture is compressed with the PNG algorithm and shows no shadowing, while the middle picture is using a high quality setting for the JPEG algorithm. The rightmost picture uses a low quality JPEG setting (so high compression) but suffers from shadowing effects.

3.5 Applications

Fourier series have a huge number of applications. As it is often easy to understand the behaviour of a system in response to a simple harmonic force, we can use the Fourier series decomposition to represent any function as a set of such responses. However, this relies completely on that the problem we want to resolve is linear. There is a large number of applications related to the Fast Fourier Transform (FFT) which is the computational implementation of Fourier transforms and Fourier series.

3.5.1 Low and high pass filters

Modern electronics relies on the fact that we can transfer digital signals around. A digital signal takes the form of a series of binary values, 0 and 1, transferred down a line. These values are (in a simplified form) presented by different voltages and thus the signal will look like a square wave with different step lengths. If the attenuation of the wire is known (see Fig. 3.5) the behaviour of the line can be completely modelled. In this example it behaves as a low pass filter and the measured copper cable is quite similar to standard phone wiring in the UK. Examples of how frequency dependent attenuation can affect a signal are shown in Fig. 3.6.

3.5.2 Ordinary differential equations

Note: This section is not examinable. It was included in previous versions of the course given to second years. They would have been familiar with solving ODE systems. I have left it in the notes as additional material. It is a great introduction for how Fourier analysis is useful in analytical work.

Consider the second order inhomogeneous linear ordinary differential equation

$$y'' + 2\gamma\omega_0 y' + \omega_0^2 y = f(t) \,, \tag{3.35}$$

where $f(t)$ is a function satisfying the Diriclet conditions with period T. We know that the way to find the general solution to this ODE is to find a single solution to the inhomogenenous

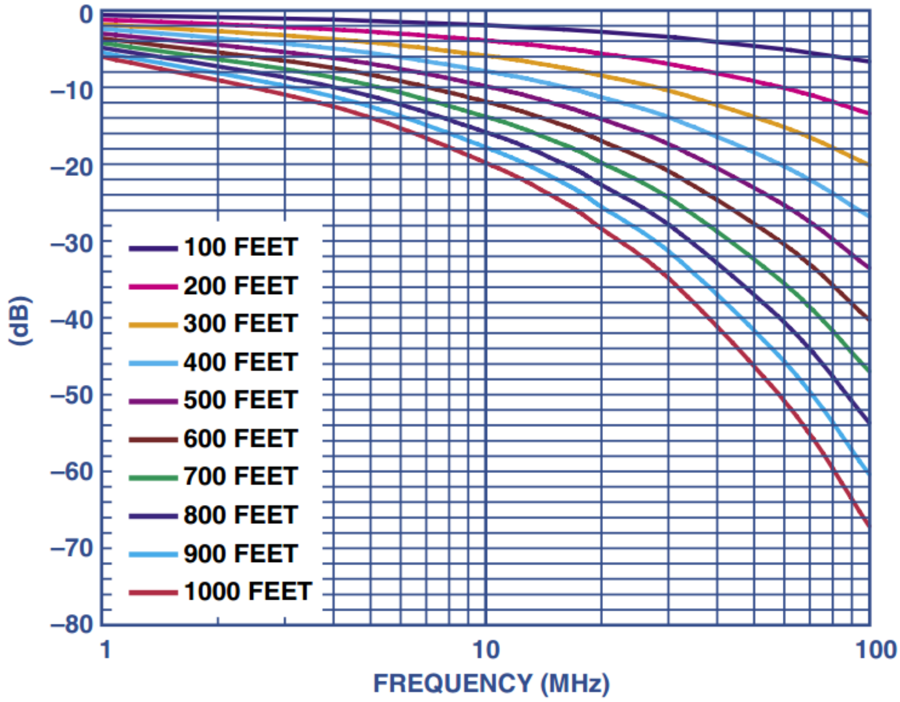

Figure 3.5: The attenuation as a function of frequency for different lengths of twisted pair copper cable. For an ADSL2+ connection (up to 24 Mb/s), very limited attenuation is required up to 2 MHz. From *Analog Dialogue 38-07, July (2004)*.

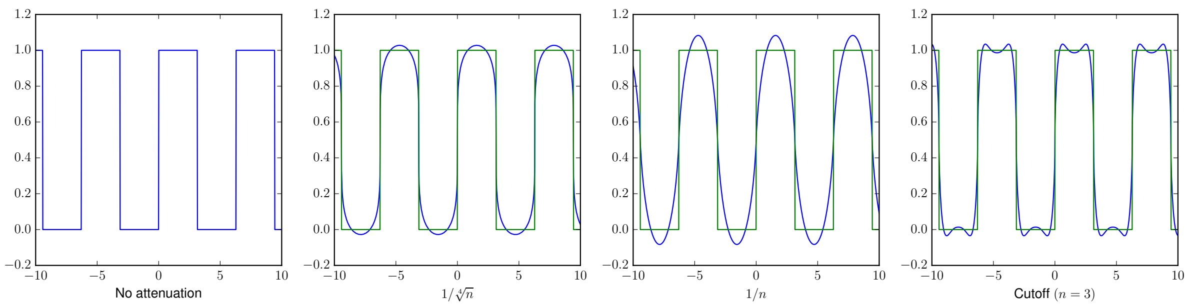

Figure 3.6: Example of how attenuation with different forms of attenuation as a function of the n^{th} term is applied.

equation and then just add it to the well known general solution of the homogeneous ODE. We also know that, if we can write f as a sum $f(x) = f_1(t) + f_2(t)$, then we can find separate single solutions to the inhomogeneous ODE involving just f_1 and just f_2 and then add them up. This works because the ODE is linear.

We can now extend the argument, and instead of expressing f as a sum of two functions, we express it as a Fourier series. We thus instead have to solve

$$y'' + 2\gamma\omega_0 y' + \omega_0^2 y = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\omega n t) + b_n \sin(\omega n t)\right) , \tag{3.36}$$

where $\omega = 2\pi/T$. This involves finding specific solutions to ODE's of the type

$$y'' + 2\gamma\omega_0 y' + \omega_0^2 y = \frac{F_0}{m} \sin(\omega n t) . \tag{3.37}$$

which are well known and the steady state solutions are of the form

$$y(t) = \frac{F_0}{mZ\omega n} \sin(\omega n t + \phi) , \tag{3.38}$$

with

$$Z = \sqrt{(2\omega_0\gamma)^2 + \frac{1}{\omega^2 n^2}(\omega_0^2 - \omega^2 n^2)^2} \quad , \quad \phi = \arctan\left(\frac{2\omega n \omega_0 \gamma}{\omega^2 n^2 - \omega_0^2}\right) . \tag{3.39}$$

These solutions can then be summed up. Thus we can see how we can solve the forced harmonic oscillator for any function where we can find the Fourier series. How many terms to include in the Fourier series before a good approximation is achieved will depend on the resonance frequency of the oscillator. A good rule of thumb might be to include terms up to N, where $N = 10\omega_0/\omega$.

Chapter 4

Fourier transforms

4.1 Definition

AWH 20.2, *Boas 7.12*

So far we have only considered creating a Fourier series for a function which is periodic. Using the scaling laws derived in the previous chapter, the period of the function can be extended to an arbitrary length. If, in this extension, the Fourier series is kept to have the same highest frequency, then the sum will become longer and longer. The limit of this, when the period is expanded to infinity is that the sum is changing to an integral. Using this description, the connection can be seen between the Fourier series in the previous example and the Fourier transform presented here.

We define the **Fourier transform** of a real or complex valued function $f(t)$, $t \in \mathbb{R}$ as

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{4.1}$$

and the **inverse Fourier transform** as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega \,. \tag{4.2}$$

Several things should be noted here. The definition of the Fourier transform is not unique. The notation followed here is the one from Arfken (and thus *not* the one from Boas). In general we regard f to be a function of time and g to be a function of angular frequency.

The way to understand the Fourier transform and its inverse is to look at the inverse transfom first. We can see from Eq. (4.2) that a non-periodic function f can be represented by a superposition of waves $e^{-i\omega t}$, each with amplitude $g(\omega)$. The Fourier transform, Eq. (4.1) shows how to calculate the amplitudes.

Using the exponential form of the delta function (see Appendix 2.4), we have

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} f(t)\delta(t-x) dt \\
&= \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\delta_n(t-x) dt \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-n}^{n} e^{i\omega(t-x)} d\omega dt \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} \int_{-\infty}^{\infty} f(t) e^{i\omega(t-x)} dt d\omega;,
\end{aligned} \tag{4.3}$$

where in the last part, the order of integration has interchanged. Further taking the limit $n \to \infty$ gives

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\omega(t-x)}dt d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} g(\omega) d\omega \;, \end{aligned} \tag{4.4}$$

with the definition of $g(\omega)$ from the Fourier transform, Eq. (4.1), inserted. We have thus proven[1] that the inverse Fourier transform, Eq. (4.2) brings us back to the original function.

A few points on notation. If for a function $f(x)$ the function $g(\omega)$ is the Fourier transform, we will write $\mathscr{F}(f) = g$, and will often also identify the Fourier transfer of a function by a capital letter, so in this case $F = g$. The inverse Fourier transfer, we write as $\mathscr{F}^{-1}(g) = f$.

4.2 Specific Fourier transforms

Let us now consider a few specific Fourier transforms.

4.2.1 The Gaussian

Consider the Gaussian function $f(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}$. The Fourier transform is

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}e^{i\omega t}dt \tag{4.5}$$

which we will evaluate by transferring it into an ordinary differential equation where we know the solution. So first we differentiate the expression above with respect to ω to give

$$\frac{dg}{d\omega} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{t^2}{2\sigma^2}}ite^{i\omega t}dt \;. \tag{4.6}$$

We then do an integration by parts,

$$\int_{-\infty}^{\infty} u dv = [uv]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v du \tag{4.7}$$

where we identify

$$u = e^{i\omega t} \qquad du = i\omega e^{i\omega t}dt \tag{4.8}$$

$$dv = ite^{-\frac{t^2}{2\sigma^2}}dt \qquad v = -i\sigma^2 e^{-\frac{t^2}{2\sigma^2}} \tag{4.9}$$

we get

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}ite^{i\omega t}dt &= \left[-i\sigma^2 e^{i\omega t}e^{-\frac{t^2}{2\sigma^2}}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -i\sigma^2 e^{-\frac{t^2}{2\sigma^2}}i\omega e^{i\omega t}dt \\ &= 0 - \sigma^2\omega\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}e^{i\omega t}dt \;. \end{aligned} \tag{4.10}$$

[1] As in many other parts of the course, we are relaxed about in which order integrals and limits are taken. If you want to see the formal proofs, look in the Howell book.

Combining Eqs. (4.6) and (4.10) gives

$$\frac{dg}{d\omega} = -\sigma^2 \omega g(\omega) \tag{4.11}$$

which is an ordinary differential equation with the solution

$$g(\omega) = g(0)e^{-\frac{\sigma^2\omega^2}{2}}\ , \tag{4.12}$$

where we still need to identify the integration constant. Inserting $\omega = 0$ in the original definition of the Fourier transform, Eq. (4.5), we get

$$g(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = \frac{1}{\sqrt{2\pi}} \tag{4.13}$$

and thus finally the Fourier transform of our Gaussian as

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2\omega^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2\rho^2}}\ , \tag{4.14}$$

with $\rho = 1/\sigma$. We see from this long derivation that the Fourier transform of a Gaussian is in itself a Gaussian and the widths of the Gaussian and the transformed Gaussian are reciprocal. Thus when f becomes narrower, g becomes wider and the other way around and the product of the two widths is always 1.

4.2.2 The delta function

In physics, we often operate with the notion of an impulse force. This is a force that is applied in such a short time that the time evolution is not relevant within that time. As an example we can think of a cricket bat hitting the ball. So what frequencies are involved in such an impulse force? To answer the question we model the impulse force as a delta function, $\delta(t)$, and find the Fourier transform.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt \tag{4.15}$$

Considering the delta function as the limit of a Gaussian (see Eq. (2.30) it is obvious that the corresponding limit of the Fourier transforms will be a series of Gaussians that are turning wider and wider.

Finding the limit is in fact easier the other way around. If we take the identity function $g(\omega) = 1/\sqrt{2\pi}$, we see that the inverse transform of g is

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} d\omega && (4.16)\\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega && (4.17)\\ &= \delta(-t) = \delta(t)\ . && (4.18) \end{aligned}$$

Now taking advantage of that the inverse Fourier transform brings us back to the original function, we can see that

$$\mathscr{F}(\delta(t)) = g(\omega) = \frac{1}{\sqrt{2\pi}} \tag{4.19}$$

so the Fourier transform of the delta function is a constant function. This means that all frequencies contribute at the same level to the delta function.

4.2.3 The Top-Hat Function

The "top-hat" or rectangular function is a very important one in signal analysis because it describes the case where the coordinate domain (t, x, etc.) is truncated.

Consider the function

$$f(t) = \begin{cases} \frac{\sqrt{2\pi}}{2T} & \text{if } t \leq |T| \\ 0 & \text{if } t > |T| \,. \end{cases} \tag{4.20}$$

The normalisation is chosen for convenience. The Fourier transform is

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2T} \int_{-T}^{T} e^{i\omega t}\, dt \,. \tag{4.21}$$

Notice the intgration domain is now bounded. For finite T this integral is definite (the limit $T \to \infty$ is the Dirac delta function).

Working out the integral we have

$$g(\omega) = \frac{1}{2T\, i\omega} \left[e^{i\omega t} \right]_{-T}^{T} , \tag{4.22}$$

$$= \frac{\sin(\omega T)}{\omega T} \equiv \operatorname{sinc}(\omega T) \,. \tag{4.23}$$

The Fourier transform of a top-hat is a sinc function whose width is reciprocal to the width of the top-hat i.e. $T \leftrightarrow 1/T$. There is a one-to-one correspondence with the limiting functions we used to describe the Dirac delta function. The top-hat (2.28) and exponential (2.29) limiting function for the Dirac delta are related by a Fourier Transform.

For example, as $T \to \infty$ then $f(t)$ tends to a constant and $g(\omega) \to \delta(\omega)$. The reciprocal is also true, as $T \to 0$ then $f(t) \to \delta(t)$ and $g(\omega)$ tends to a constant.

4.3 Properties

AWH 20.3

When calculating the Fourier transform, there are a number of properties that can be used to transform the function in question into a function with an already known transform. The proof of the individual forms will be left for the problems. In each of the cases below we use the notation that $g(\omega) = \mathscr{F}(f(t))$ is the Fourier transform of the original function $f(t)$.

Linearity The Fourier transform is linear, so the Fourier transform of a linear combination is the linear combination of the individual transforms

$$\mathscr{F}(\alpha f_1(t) + \beta f_2(t)) = \alpha \mathscr{F}(f_1(t)) + \beta \mathscr{F}(f_2(t)) \tag{4.24}$$

Change of sign Inverting the function results in inverting the Fourier transform

$$\mathscr{F}(f(-t)) = g(-\omega) \tag{4.25}$$

Translation Shifting a function by a certain amount results in a phase shift

$$\mathscr{F}(f(t - t_0)) = e^{i\omega t_0} g(\omega) \tag{4.26}$$

Scaling Stretching a function by a factor α results in the Fourier tranform to be compressed by the same factor

$$\mathscr{F}(f(\alpha t)) = \frac{1}{|\alpha|} g(\omega/\alpha) \tag{4.27}$$

Conjugation The transform of the complex conjugate is the complex conjugate of the transform with the sign of the frequency changed.

$$\mathscr{F}(f^*(t)) = g^*(-\omega) \tag{4.28}$$

A consequence of this is the **Reality Condition**. If the function $f(t)$ is real (i.e. $f(t) = f^*(t)$) then the negative frequencies are redundant

$$g(-\omega) = g^*(\omega) \text{ if } f(t) = f^*(t) \tag{4.29}$$

Derivative Finding the transform of a derivative, simply translates into multiplication

$$\mathscr{F}(f'(t)) = -i\omega g(\omega) \tag{4.30}$$

Parseval's identity Just as for the Fourier series, we have Parseval's identity for Fourier transforms. It states that

$$\int_{-\infty}^{\infty} f(t) g^*(t)\, dt = \int_{-\infty}^{\infty} F(\omega) G^*(\omega)\, d\omega \tag{4.31}$$

or as a special case

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega\,. \tag{4.32}$$

This demonstrates conservation of energy in diffraction (light going in is the same as light going out) and conservation of probability in quantum mechanics (whether looking at the wave equation in position or momentum space).

4.4 Applications

4.4.1 Partial differential equations

AWH 20.3

A Fourier transform can be used to change a partial differential equation (PDE) into an oridinary differential equation (ODE). As ODEs are in general easier to solve than PDEs this sounds great but it comes at the price of having to make first a Fourier transform and subsequently an inverse Fourier transform. The method will be outlined with an example of the heat flow equation. It was in fact to solve this problem that Fourier transforms were invented.

Let us consider the heat flow equation in 1 dimension,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}\,, \tag{4.33}$$

where $u(x,t)$ represents the temperature at the point x and time t. Let us Fourier transfer the x dependence of this equation and let y be the transformed variable of x. We get

$$\frac{\partial U}{\partial t} = -a^2 y^2 U\,, \tag{4.34}$$

where $U(y,t)$ is the Fourier transform of $u(x,t)$. For the second derivative with respect to x, we have used Eq. (4.30) twice.

The great thing about Eq. (4.34) is that it only has derivatives with respect to t so it is an ODE. Think carefully about how the Fourier transform achieved that trick. The ODE can be solved by separation of variables and we get

$$U = C(y)e^{-a^2y^2t} . \tag{4.35}$$

Here $C(y)$ is the integration constant. By inserting $t = 0$ in Eq. (4.35), it can be seen that $C(y)$ is the Fourier transform of the temperature distribution at $t = 0$,

$$C(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)e^{ixy} dx . \tag{4.36}$$

To get the solution in terms of x and t we need to perform an inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(y)e^{-a^2y^2t}e^{-iyx} dy . \tag{4.37}$$

4.4.2 Fraunhofer diffraction

When light passes through a small aperture it creates a diffraction pattern. If the pattern is viewed at a distance that is much greater than the size of the aperture, it can be described through Huygens principle that each point inside the aperture emits a plane wave.

The diffraction pattern can be understood by considering the path length difference taken by each light ray incident at a point on a screen in the far-field. The path differences cause phase differences in the incident radiation. It is easiest to derive the resulting pattern for a coherent source (radiation is emitted with the same phase) which is monochromatic (single wavelength λ).

In this limit we can use Fourier transforms to calculate the resulting pattern. Consider a slit of width d along the vertical x. The slit extends from $x = -d/2$ to $x = d/2$. The plane wave ariving from the infinitesimal interval dx centred at x will give a contribution to the amplitude radiation field

$$d\psi(\theta) = a(x) \exp\left[i\frac{2\pi}{\lambda}(R + x\sin\theta)\right] dx . \tag{4.38}$$

Here $a(x)$ is the relative amplitude of the of the radiation arriving at the aperture from the source, R is the path length of the radiation arriving at the screen from point x along the slit. $x\sin\theta$ is the path difference with respect to radiation arriving from the middle of the slit.

Note that we have assumed that the overall attenuation of the field does not depend ont the (small) path differences and that phase differences give the dominant contribution, this is fine in the limit where $\lambda \ll d$ and $R \gg d$.

We then have

$$d\psi(\theta) \sim a(x) \exp\left[i\frac{2\pi}{\lambda}x\sin\theta\right] dx , \tag{4.39}$$

where we have dropped the common phase - we are only interested in the pattern on the screen.

Assuming the small angle limit we then obtain

$$d\psi(\ell) \sim a(x) \exp\left[i\frac{2\pi}{\lambda}\ell x\right] dx , \tag{4.40}$$

where $\ell \sim \sin\theta$. The total contribution to the field at ℓ on the screen will then be given by

$$\psi(\ell) = A\int_{-d/2}^{d/2} a(x)e^{i\frac{2\pi}{\lambda}\ell x}\,dx\,, \tag{4.41}$$

where A is an overall normalisation. If we introduce $k \equiv 2\pi\ell/\lambda$ we can see that this is just the Fourier transform of the aperture function $a(x)$.

In the idealised limit a slit can be represented by a top-hat aperture function $a(x)$ of width d. The equivalence with the Fourier transform with infinite integration limits is then exact.

Having derived this result we can immediately generalise it to yield a number of interesting results without having to replicate the ray phase calculation.

- A very narrow slit (so much narrower than the wavelength) is represented by the delta function. Hence, the diffraction pattern is a smooth spread of light in all directions.

- If on the other hand the slit is very wide, the light will pass straight through as the Fourier transform of a very wide function is the delta function.

- To calculate the light from two narrow slits we can use the Fourier transform of a single slit and then add it with the Fourier transform of the same slit shifted (see Eq. (4.26). See the problem sheets for an example of this.

The interesting point here is actually the other way around. By analysing the diffraction pattern we can through the inverse Fourier transform say something about the shape of the aperture. This is also how X-ray crystallography works. The pattern arising from Bragg scattering is the Fourier transform of the scattering centres in the crystal and can thus through the inverse Fourier transform be used to determine these.

4.4.3 Resonance shape

The electron in the Hydrogen atom can through the absorption of a photon be excited into a higher energy level if the energy of the photon is matching the difference in energy between the ground state and the excited state. This is what is observed as spectral lines in a star spectra as illustrated in Fig. 4.2.

When the electron is in the excited state, it will decay into the ground state with a probability to decay after a certain time t following an exponentially falling distribution with lifetime τ,

$$P(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-t/\tau} & \text{if } t > 0\,. \end{cases} \tag{4.42}$$

In Quantum Mechanics, the wave function corresponding to this is

$$\psi(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-i\omega_0 t}e^{-t/2\tau} & \text{if } t > 0\,, \end{cases} \tag{4.43}$$

where $\omega_0 = E/\hbar$ with E the energy of the excited level. To get the distribution of angular frequencies (and thus energies) of the photons emitted, we have to perform the Fourier transform of the wave function. A bit of manipulation gives

$$\phi(\omega) = \mathscr{F}(\psi(t)) = \frac{1}{\sqrt{2\pi}}\frac{i}{\omega - \omega_0 + i\frac{\Gamma}{2}}\,, \tag{4.44}$$

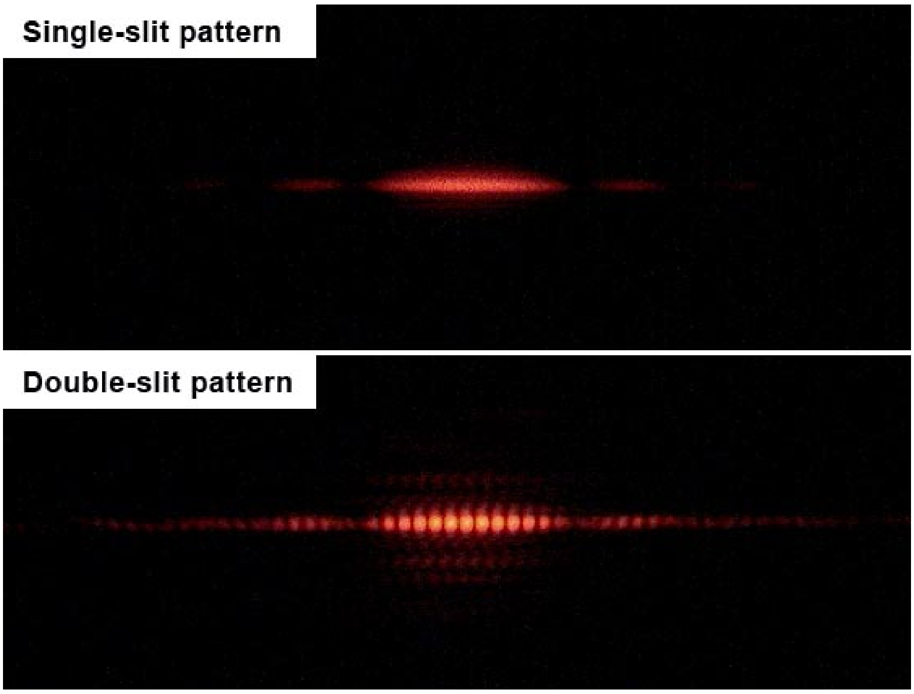

Figure 4.1: A single slit and double slit pattern made with the same aperatus. In the single slit image the diffraction pattern is seen (the Fourier transform of a top hat). In the double slit experiment this is overlaid with the narrow interference pattern (resulting as the Fourier transform of two top hat functions). © User:Jordgette / Wikimedia Commons / CC-BY-SA-3.0

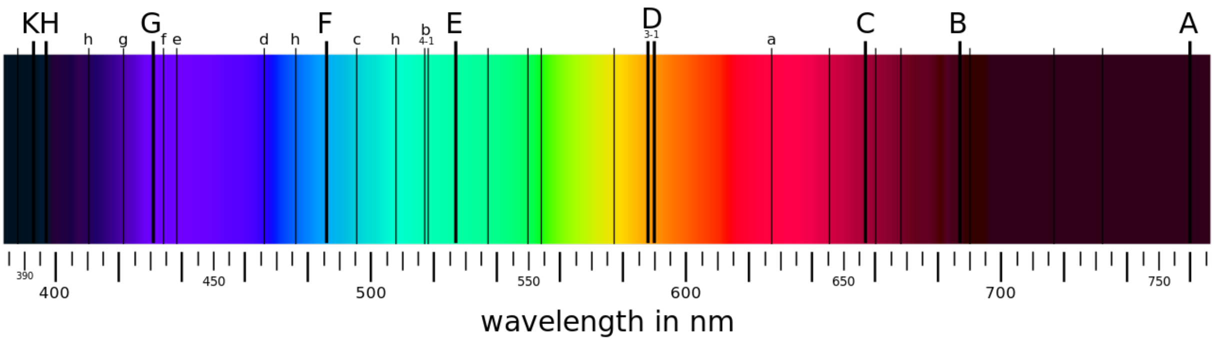

Figure 4.2: An illustration of absorption lines in a spectra.

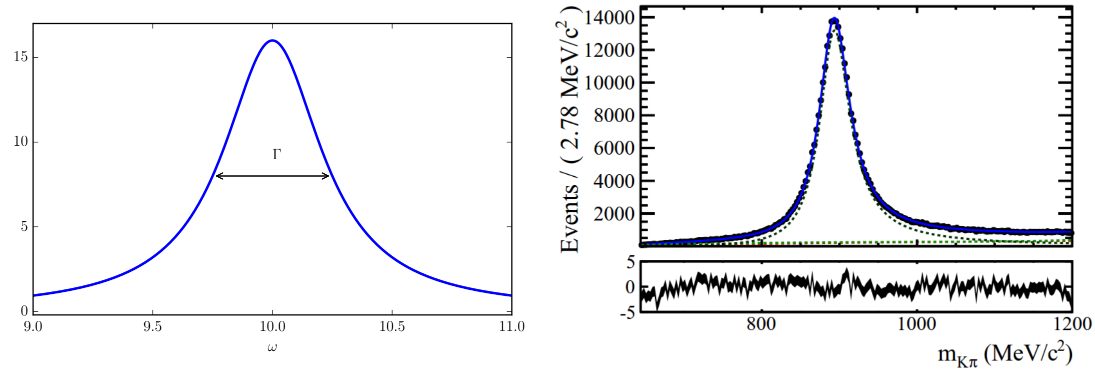

Figure 4.3: To the left a plot of the Breit-Wigner function. To the right a plot of the invariant mass distribution of kaons and pions coming from the decay $K^* \to K^+\pi^-$. The short lifetime of the K^* gives a clear Breit-Wigner shape to the resonance with a mass of $896\,\mathrm{MeV/c^2}$ and a width of $47\,\mathrm{MeV/c^2}$. Source S. Cunliffe, LHCb collaboration.

where $\Gamma = 1/\tau$. The intensity distribution is then

$$I(\omega) = |\phi(\omega)|^2 = \frac{1}{2\pi}\frac{1}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}}\,. \tag{4.45}$$

This distribution is called the Breit-Wigner distribution and is of paramount importance in atomic physics, nuclear physics and particle physics. It shows that the emitted (or absorbed) energies follow a distribution, given by $I(\omega)$ above, where Γ corresponds to the *Full Width Half Maximum* of the distribution. Thus the shorter the lifetime of an excited state, the less well defined its energy. The effect is clearly illustrated in Fig. 4.3.

4.5 Multi-Dimensional Transforms

The Fourier transform can be generalised to the case of mutiple coordinate dimensions. The application where this occurs often is image analysis. An image is spanned by two coordinates e.g. cartesian coordinates x and y.

Consider a function $f(x,y)$. A multi-dimensional Fourier transform is simply a transform acting on each of the variables (x and y in this case). We can write this as

$$g(k_x,k_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x,y)\, e^{ik_x x} e^{ik_y y}\,, \tag{4.46}$$

where we have introduced two wavenumbers k_x and k_y relating to directions x and y respectively. Think of k_x and k_y representing the frequencies of cosine and sine waves that are aligned in the x and y directions respectively.

We can generalise this further by making use of vectors. We package the coordinates into a vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and also introduce a wavevector $\mathbf{k} = k_x\hat{\mathbf{i}} + k_y\hat{\mathbf{j}}$. The Fourier transform now becomes

$$g(\mathbf{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^2\mathbf{r}\,. \tag{4.47}$$

The basis functions have become waves with general vector direction $\hat{\mathbf{k}}$ with respect to the cartesian coordinates. The frequency (or wavelength) of the waves is set by the wavenumber which is now $|\mathbf{k}|$.

The reciprocal transformation is defined as using the complex conjugate of the basis functions as usual

$$f(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^2\mathbf{k}. \tag{4.48}$$

The transform generalises for any number of dimensions

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty} f(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^n\mathbf{r}. \tag{4.49}$$

4.5.1 Inhomogeneous Ordinary Differential Equations

Note: This section is not examinable. It is included to show how Fourier transforms can be used to solve ODEs from first principles. The equation used here is of the same kind you will have seen in the OScillations component (forced harmonic oscillator) and Basic Electronics (LRC circuits). You will have solved the system by making some ansatz of the solution and verifying it is consistent with the equation. Here we use Fourier transforms to obtain a solution from first principles (for a suitably simple forcing function).

In 4.4.1 we saw how Fourier transforms change derivatives to algebraic factors. This can can be used to solve ODEs. We will use the general second order equation

$$\frac{d^2u(t)}{dt^2} + \gamma\frac{du(t)}{dt} + \omega_0^2 u(t) = f(t)\,, \tag{4.50}$$

where $u(t)$ is the solution describing the state of an oscillator with natural frequency ω_0 and damping coefficient γ. $f(t)$ is a forcing function.

It is useful to comare the equation to that of a damped harmonic oscillator or an LRC circuit to identify the various components.

We will assume a simple forcing function in this example, a pure cosine,

$$f(t) = 2|A|\cos(\Omega t + \phi)\,, \tag{4.51}$$

where Ω is the forcing frequency, ϕ is a phase offset, and $|A|$ is a real amplitude. Note that this focring term is real. [2] We can re-write this as

$$f(t) = Ae^{i\Omega t} + A^* e^{-i\Omega t}\,, \tag{4.52}$$

where $A = |A|e^{i\phi}$. Check that this is consistent and that $f(t)$ is still real - can you see how the reality condition discussed previously applies?

We now "Fourier transform" the equation as we did in 4.4.1 but we do it in a single step

$$-\omega^2 g(\omega) - i\omega\gamma g(\omega) + \omega_0^2 g(\omega) = F(\omega)\,, \tag{4.53}$$

where $g(\omega) \equiv \mathscr{F}[u(t)]$ and $F(\omega) \equiv \mathscr{F}[f(t)]$.

We can work out $F(\omega)$ by taking the Fourier transform of $f(t)$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt\,, \tag{4.54}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[Ae^{i\Omega t} + A^* e^{-i\Omega t}\right] e^{i\omega t}\, dt\,, \tag{4.55}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left[Ae^{i(\Omega+\omega)t} + A^* e^{-i(\Omega-\omega)t}\right] dt\,. \tag{4.56}$$

[2] You may have used a complex forcing term to solve this system in the past to solve the system, then just take the real part of the final solution. We won't need to do that here

Comparing to the exponential form of the limiting function for the Dirac delta we see that

$$F(\omega) = \sqrt{2\pi}\left[A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)\right]. \tag{4.57}$$

Notice how the single frequency in the time domain is equivalent to two delta functions at $\omega = \pm\Omega$ in the frequency domain as expected. The amplitudes also satisfy the reality condition explicitly.

Inserting this back into (4.53) we have the spectral solution

$$g(\omega) = \sqrt{2\pi}\frac{\left[A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)\right]}{\omega_0^2 - \omega^2 - i\omega\gamma}. \tag{4.58}$$

This is the solution in the frequency domain. We can use an inverse Fourier transfrom toobtain the time domain solution $u(t)$

$$\begin{aligned} u(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}\,d\omega, && (4.59)\\ &= \int_{-\infty}^{\infty}\frac{\left[A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)\right]}{\omega_0^2 - \omega^2 - i\omega\gamma}e^{-i\omega t}\,d\omega. && (4.60)\\ & && (4.61) \end{aligned}$$

The two delta functions simply collapse the integrals at values $\omega = \pm\Omega$ so this simplifies to

$$u(t) = \frac{\left[Ae^{i\Omega t}\right]}{\omega_0^2 - \Omega^2 + i\Omega\gamma} + \frac{\left[A^*e^{-i\Omega t}\right]}{\omega_0^2 - \Omega^2 - i\Omega\gamma}. \tag{4.62}$$

It may not be immediately obvious that this solution is real, as expceted, but notice how the two terms are complex conjugates. We can re-write this in a more useful form by introducing

$$B = |B|e^{i\alpha} \equiv \omega_0^2 - \Omega^2 + \imath\Omega\gamma, \tag{4.63}$$

to get

$$\begin{aligned} u(t) &= \frac{A}{B}e^{i\Omega t} + \frac{A^*}{B^*}e^{-i\Omega t}, && (4.64)\\ &= \frac{|A|}{|B|}\cos(\Omega t + \phi - \alpha). && (4.65) \end{aligned}$$

This tells us that the solution is a cosine at the forcing frequency Ω but offset by a phase α from the forcing function. The phase and amplitude of the solution are set by a combination of Ω, ω_0, and γ. It is interesting to note what happens in the limit where ther eis no damping, with $\gamma = 0$. In this case, when the system is being forced at the natural frequency i.e. $\Omega = \pm\omega$, $|B| \to 0$ and $u(t) \to \infty$ - resonance. The presence of a non-zero damping term will always counteract this tendency.

Chapter 5

Convolution

Boas 2, *AWH 1.8*

When we make a measurement of physical phenomena, we always have to bear in mind the precision of the instruments we use for making the measurement. First imagine that we with a ruler measure the length of a box that has a true length of 100.0 mm. If the individual measurements we make have a resolution of 0.5 mm, we know that we from repeated measurements will get a Gaussian distribution with a mean of 100.0 mm and a width of 0.5 mm. More complicated examples are not so easy to comprehend though. Imagine that we look at the diffraction pattern that arises from a double slit experiment. The intensity of the beam coming through is so low that photons are counted individually as they interact with the detection plane. We know that our detector has a certain uncertainty in detecting the position of each photon so we will somehow see a smeared version of the true distribution. But how will it be smeared and is it possible to recover the true distribution? In a different area, is it possible in seismologi to understand the signals that are produced when a sound pulse is sent into the ground and multiple reflections occur.

Shared for all of these phenomena is that they can be understood through a process called *convolution* of the original signal. Starting from a very simple example, we will then afterwards introduce convolutions in the formal way. In Fig. 5.1, we have in (a) a true distribution and in (b) the resolution of our measurements. It is on purpose that this resolution is asymmetric (i.e. we are more likely to measure a too large value than to measure a too small value). To get the measured distribution, we can then for each true value distribute it in the measured distribution according to the resolution function. This is illustrated in (c). We achieved this by going through the true distribution and then redistribute the signal according to the resolution function. But we could also chosse to start with the expected signal and then for each bin ask which bins in the true signal contribute to it. To do this we need to overlay the resolution function at each bin and **reverse** the distribution. The concept is illustrated in (d) where the different parts of the true function now arrive in each bin in a different order. Notice that the distributions are identical; the colours within the bars simply represent how the calculation was carried out.

If we let f be the signal and g the resolution function, then the generalisation of the above example leads us to define the *convolution of f with g*, $f * g$, as

$$(f * g)(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau)\, d\tau\ . \tag{5.1}$$

As the definiton is identical (use a simple substitution to verify), the convolution is in many places defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau\ . \tag{5.2}$$

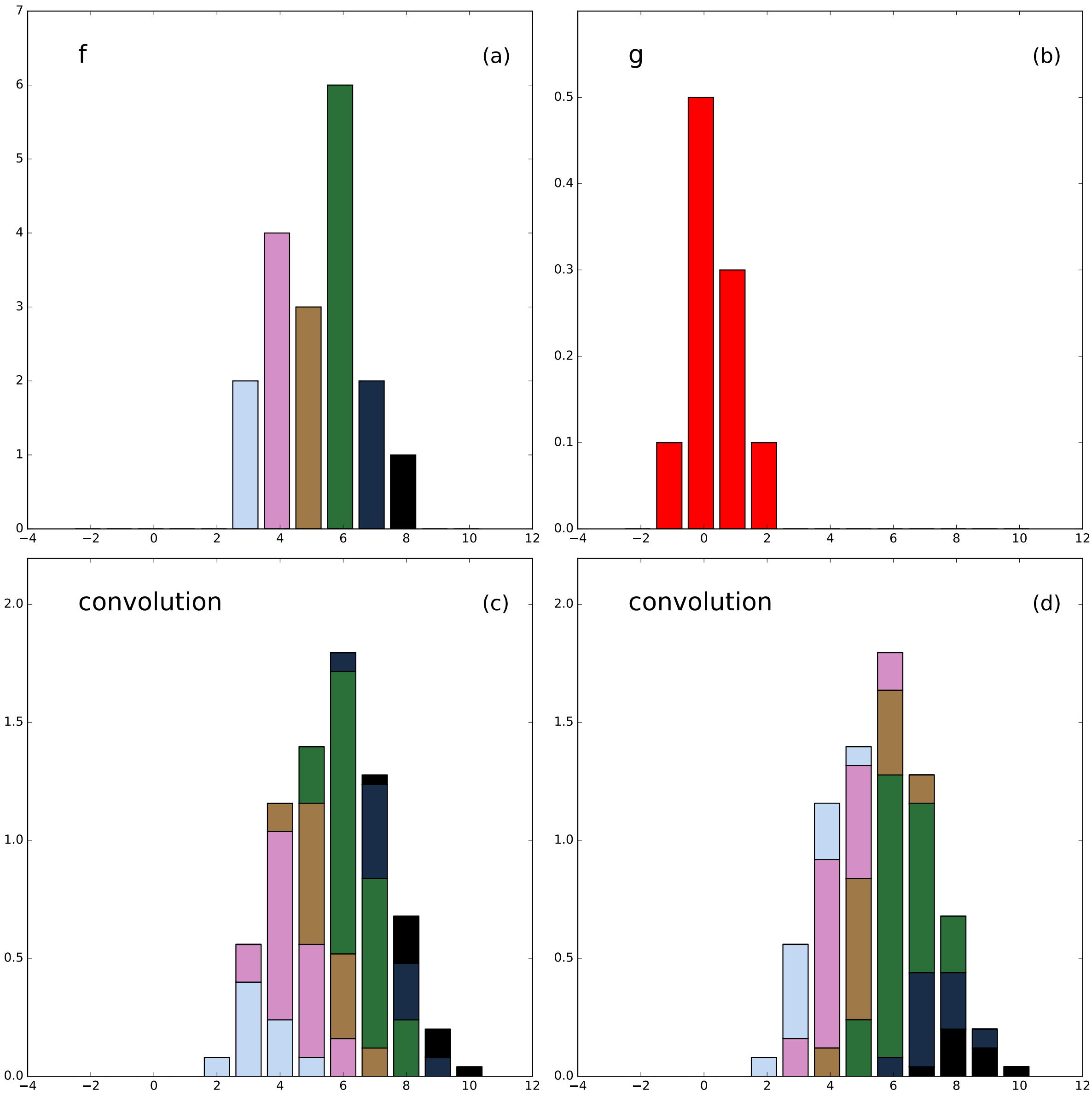

Figure 5.1: The convolution of f with g on a very simple discrete example. The two bottom parts of the figure illustrate how the convolution is built up following two different ways of obtaining it. See main text for details.

It can be seen that Eq. 5.1 exactly replicates the method of having the inverted resolution function sliding across f as part of the integral. Often the same dependent variable is used in both $f(t)$ and $(f*g)(t)$. From a physics point of view this makes sense as the smearing of a distribution (lets say a time) is still a time and is still measured in the same units. However, from a mathematical point of view this can cause confusion.

It is worth noting the number of oprations we needed to calculate a convolution using the integral (5.2). If the functions are sampled discretely using N separate points the opertation will sclae as N^2. To see this consider that to evaluate the sum (integral) will require N summations and at each step (shift) you are multiplying N things together.

To understand how Fourier transforms can be used as an alternative to the "brute-force" evaluation of the integral (5.2) $F(\omega)$ and $G(\omega)$ denote the Fourier tranform of $f(t)$ and $g(t)$, respectively. The proof is easiest to understand in reverse - by considering the product

$$\sqrt{2\pi}F(\omega)G(\omega)\,. \tag{5.3}$$

Taking the inverse Fourier transform we have

$$\mathscr{F}^{-1}\left[\sqrt{2\pi}F(\omega)G(\omega)\right](t) = \int_{-\infty}^{\infty} F(\omega)G(\omega)\,e^{-i\omega t}\,d\omega\,. \tag{5.4}$$

Exoand the individual functions in the integrand using the Fourier transform, note that we introduce two new integration variables when inserting the two new integrals

$$\mathscr{F}^{-1}\left[\sqrt{2\pi}F(\omega)G(\omega)\right](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)\,e^{i\omega\tau}\,d\tau\right)\left(\int_{-\infty}^{\infty} g(s)\,e^{i\omega s}\,ds\right)e^{-i\omega t}\,d\omega\,. \tag{5.5}$$

Rearranging the order of integrations we have

$$\begin{aligned}
\mathscr{F}^{-1}\left[\sqrt{2\pi}F(\omega)G(\omega)\right](t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\,g(s)\left(\int_{-\infty}^{\infty} e^{i\omega(\tau+s-t)}\,d\omega\right)ds\,d\tau\,, && (5.6)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\,g(s)\delta(\tau+s-t)\,ds\,d\tau\,, && (5.7)\\
&= \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau \equiv (f*g)(t)\,, && (5.8)
\end{aligned}$$

where we used the exponential form of the Dirac delta function which foces the integral in s to collapse to a single value at $s = t-\tau$. We have the amazing result that the complicated convolution of two funtions after a Fourier transform simply turns into multiplication. This result is called the convolution theorem.

As the Fourier transform and the inverse Fourier transform are almost identical, we can in the same way look at the Fourier transform of a product. The result is

$$\mathscr{F}([f(t)g(t)]\,(\omega) = \frac{1}{\sqrt{2\pi}}(F*G)(\omega)\,, \tag{5.9}$$

which is easy to prove in the same way as done above.

It may seem that we have just substituted the "brute-force" integration method with two Fourier transforms and a single product to do the same thing. Both these operations scale a $\sim N^2$. However, in practice the Frourier Transforms can be calculated using Fast Fourier Transforms (FFTs). These scale as $\sim N\log(N)$ which is *much* fatser than $\sim N^2$ for large N.

We have already, when we introduced the convolution, seen that the order doesn't matter, such that $f * g = g * f$. When the convolution is regarded as a smearing the two functions take on very different roles though. We can also have repeated convolutions of the type $f*(g*h)$, where we have the result that

$$f*(g*h) = (f*g)*h = f*g*h\,, \tag{5.10}$$

where the last notation simply indicates that the order doesn't matter. See problem sheets for a proof of this.

The convolution can be used to shift a function. Consider first using Eq. (5.2) for the convolution of a function f with a shifted δ function:

$$f(t)*\delta(t-d) = \int_{-\infty}^{\infty} f(\tau)\delta(t-d-\tau)d\tau$$

now use that the δ function is even

$$= \int_{-\infty}^{\infty} f(\tau)\delta(\tau-(t-d))d\tau$$

and the sifting property from Eq. (2.26)

$$= f(t-d) \tag{5.11}$$

This might seems a very complicated way of writing down the shift of a function and is only really useful when it is combined with the convolution theorem as shown in the following example.

Example 5.1. *Let us consider a situation where we want to find the Fourier transform of the same function shifted left and right by a distance d. We first use the shifting property of a convolution with a δ function as shown in Eq. (5.11)*

$$\begin{aligned}\mathscr{F}(f(t-d)+f(t+d)) &= \mathscr{F}(f(t)*(\delta(t-d)+\delta(t+d)))\\ &= \mathscr{F}(f(t)*\delta(t-d)) + \mathscr{F}(f(t)*\delta(t+d))\end{aligned}$$

and then the convolution theorem,

$$\begin{aligned}&= \sqrt{2\pi}\mathscr{F}(f(t))\mathscr{F}(\delta(t-d)) + \mathscr{F}(f(t))\mathscr{F}(\delta(t+d))\\ &= \sqrt{2\pi}g(\omega)(\mathscr{F}(\delta(t-d)) + \mathscr{F}(\delta(t+d)))\end{aligned}$$

followed by the translation property of a Fourier transform, Eq. (4.26)

$$= \sqrt{2\pi}g(\omega)\left(e^{i\omega d}+e^{-i\omega d}\right)\mathscr{F}(\delta(t))$$

and finally the Fourier transform of the δ function, Eq. (4.16)

$$\begin{aligned}&= \left(e^{i\omega d}+e^{-i\omega d}\right)g(\omega)\\ &= 2\cos(\omega d)g(\omega)\,. \end{aligned}\tag{5.12}$$

This shows that as long as we know the Fourier transform of our basic function $\mathscr{F}(x) = g(\omega)$, then we can trivially obtain the Fourier transform of the same function summed up with shifts along the axis. This will prove fundamental for the understanding of diffraction in optics.

A Gaussian function convoluted with another Gaussian function is itself a Gaussian function with the variance and the means added,

$$g(\mu_1,\sigma_1)*g(\mu_2,\sigma_2) = g(\mu_1+\mu_2,\sqrt{\sigma_1^2+\sigma_2^2})\,. \tag{5.13}$$

As part of the problems, this will be proved.

5.1 Applications

5.1.1 Measurement of a resonance

A quark and anti-quark pair can form a bound resonance state which decays through the strong force. An example is the ϕ meson that decays to a pair of opposite sign kaons. The lifetime of the ϕ is so short that it can't be meaasured directly. However, from measuring the distribution in invariant mass of many ϕ decays, we can indirectly measure the lifetime through a measurement of the width of the resonance as discussed in section 4.4.3. However if the width is just a few MeV and the resolution in the individual measurements maybe 10 MeV, care has to be taken. The observed mass distribution will be a Breit-Wigner distribution convoluted with a Gaussian resolution function. The convoluted function itself look quite a bit like a wider version of the original Breit-Wigner distribution, meaning that the resolution function should be very well understood to avoid making a biased measurement.

5.1.2 Measurement of oscillation amplitude

Imagine an experiment where we trap cosmic ray muons in an aluminium bar inside a magnetic field. When the muon is trapped within the magnetic field its spin will undergo Larmor precession and the direction of the electron from the eventual decay depends on the spin direction at that time. If the initial spin direction is not random (i.e. the muons are polarised), then the number of electrons observed in a given direction will on top of the general exponential decay time show an oscillation. The period of the oscillation is given by the magnetic field and is not in itself interesting, but the amplitude of the oscillation tells us the level of the initial polarisation.

If the time resolution for the decay time of the individual muon decays measured is not perfect, the expected amplitude of the oscillation will be smeared out giving the wrong result if not taken into account. The effect is illustrated in Fig. 5.2. Here the true net polarisation was $p = 0.50$ and a smearing of the lifetime of 0.2 was made. As can be seen this leads to an apparent polarisation of around 0.38 instead if not corrected for. With the correct knowledge about the time resolution function that convoluted the true signal, we can avoid making a biased measurement.

5.1.3 Weighting

When waiting in the passport queue to get into the UK, you might be interested in an estimate for how long it takes before you reach the front. For this we need an estimate of the average time it takes to treat a person in the queue. Several ways can be used to be used to reach this estimate like:

- the average time taken per person over the past month;
- the average time taken in the past hour;
- some kind of weighted average where recent measurements are given more weight than historical ones.

All of these could be implemented in a fully predictable system through a convolution. The first two would take the hourly measurements and convolute them with a box function of width a month or an hour, respectively. For the last option we could use a falling exponential for positive

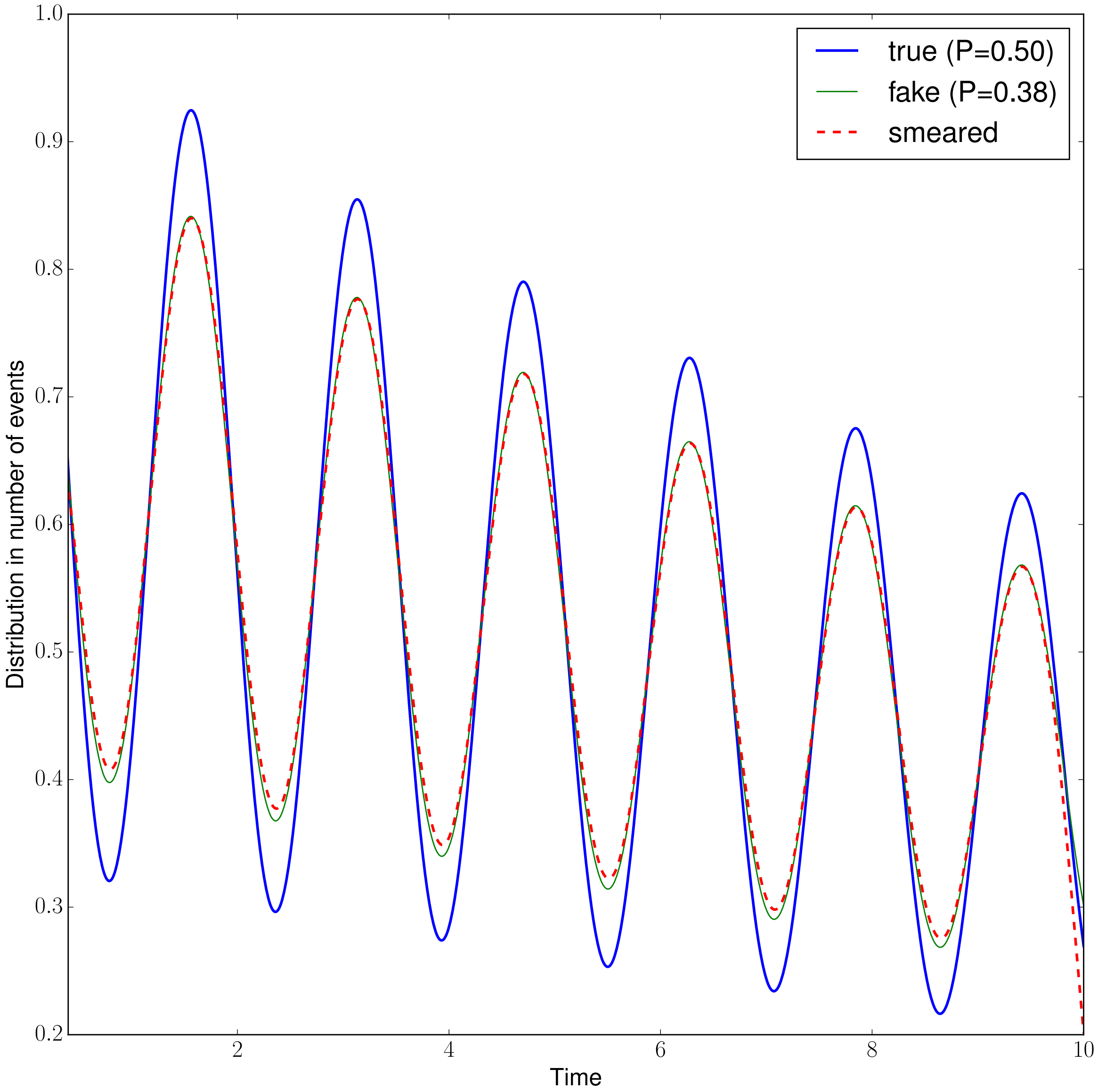

Figure 5.2: An example of how a smearing can lead to an incorrect result if not taken into account. A simulation is made of a polarisation measurement and it is illustrated how a biased measurement can be made if the time resolution of measurements is not taken into account correctly.

values. To understand the frequencies in the variation of rate that the prediction is sensitive to, we can just calculate the Fourier Transform of our convolution function.

Similar types of weighting to provide a prediction for the current status based on the past can be used in signal processing or indeed for giving a status message on your computer about how busy the CPU or the network interface is.

5.1.4 Filters

When we in section 3.5.1 discussed low and high pass filters, we applied them as a mutiplicative factor to the individual frequency components[1]. The filter efficiency thus takes the role of G in Eq. (5.6). Using the equation backwards then means that we can interpret a filter as applying a convolution to the signal with the inverse Fourier transform of the filter.

We know that the Fourier transform of a Gaussian is another Gaussian with the two width σ and ρ related by $\rho = 1/\sigma$. We thus have that a smearing with a Gaussian of width ρ, corresponds to applying a low pass filter with an efficiency shaped like a Gaussian with width ρ. This corresponds quite well to our intuition that a Gaussian smearing *wash out* the high frequency components. It is also clear from this that a convolution leads to information loss. If the Fourier transform of the smearing function is zero (or very small) for some frequencies, then any component of the signal with those frequencies can never be recovered.

5.1.5 Deconcolution - Sharpening of images

A photo will often, under magnification, show artefacts like diffraction rings from the limitations of the optics. The convolution that this represents is called the Point Spread Function (PSF). Images of stars are particulary good at illustrating this[2]. Much of the information about the true image can be recovered through what is called deconvolution. It works the best if the convolution causing the image artefacts is well known. If our true image is $f(x)$ and it is distorted by a convolution with $p(x)$ to give the recorded image $r(x)$, we have in terms of their corresponding Fourier transforms that $R(k) = \sqrt{2\pi}F(k)P(k)$ and thus $F(k) = \frac{R(k)}{\sqrt{2\pi}P(k)}$. We can apply the inverse Fourier transform to this and get

$$f(\mathbf{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{R(\mathbf{k})}{P(\mathbf{k})} e^{-i\mathbf{k}\cdot\mathbf{x}} d^2\mathbf{k}, \tag{5.14}$$

where we have generalise the procedure to a 2d-image.

While this in principle fully recovers the original image it is obvious that this only works if $P(k)$ is non-zero everywhere. As this is in general not the case, some adaptions of the algorithm are required in order to make sure that results do not diverge. An example of sharpening in terms of a deconvolution can be seen in Fig. 5.3. Most often the resolution function is well understood only at small k and $R(k) \to 0$ at larger k the bias in the division (deconvolution) grows. This is the reason why deconvolved images look sharper but noisier.

[1] The discussion in that section was on Fourier series but it might just as well be applied to Fourier transforms.

[2] The star represents a delta function and as the delta function convoluted with a function g is g itself, we get a direct recording of the convolution function.

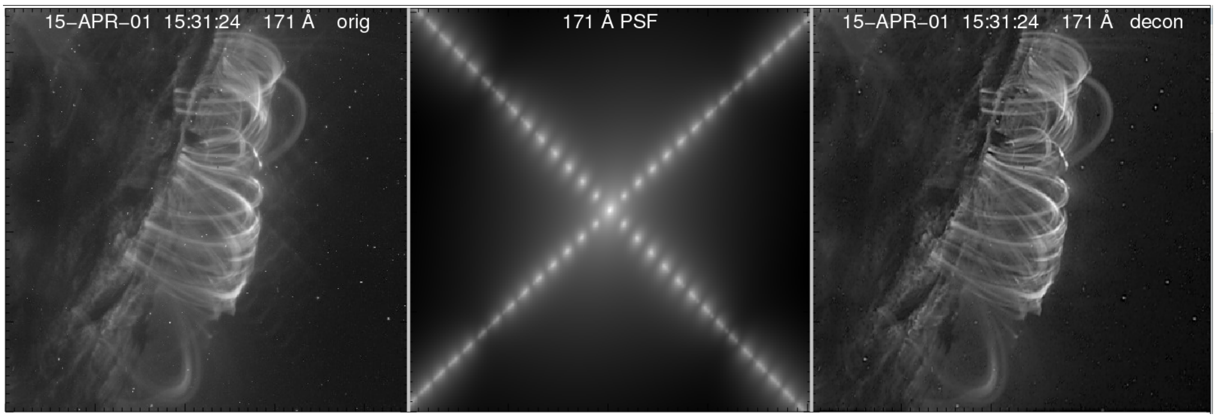

Figure 5.3: An illustration of the process of image sharpening on an image of solar corona. The left image illustres the originally recorded image, the middle one is the point spread function and the right image is after image sharpening has been applied. Note that the central image is hugely magnified compared to the two outermost ones. © 2007. Source J. Sylwester and B. Sylwester, Cent. Eur. Astrophys. Bull. **31** 2007 1, 229–43.