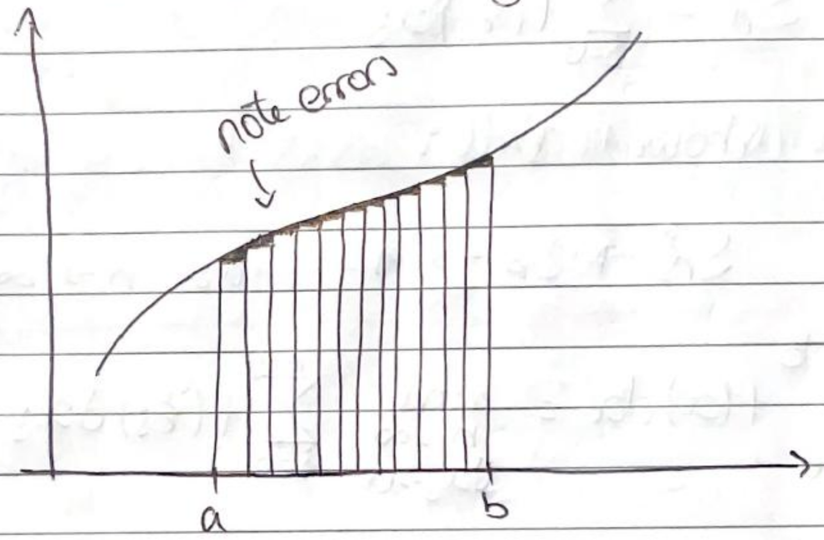
Functions 7

Intergration

Riemann's Definition

Intergration arose from intuitive ideas about area and volume in geometry.

Consider the area A under a curve $f(x)$ between coordinates at $x = a$, $y = b$.



To calculate the area A we imagine a large number of rectangular strips locateded at $x_1, x_2, \ldots x_n$ with $x_0 = a$, $x_n = b$

Area of Strips $= S_n = \sum_{j=0}^{n-1} f(x_j)\,\delta x_j$ $\quad$ ($\delta x_j = x_{j+1} - x_j$

strip height $\quad$ strip width

Intuition leads us to expect $S_n \to A$ as the number of strips $n \to \infty$, we expect that the errors vanish at this limit.

Reimann generalised this to show this intuition is correct.

- Reimann used upper and lower limits which have the same limit as $n \to \infty$.
- He used the height of the strip $f(\xi_j)$ where ξ_j is any point on $x_j \leq \xi_j \leq x_{j+1}$

So Reimann's definition is the limit of the sum over strips

$$S_n^* = \sum_{j=0}^{n-1} f(\xi_j)\delta x_j$$

and he showed that:

$$S_n^* \to S_n \to A \qquad \text{as } n \to \infty.$$

$$\int_a^b f(x)\,dx = \lim_{\substack{n \to \infty \\ \delta x_j \to 0}} \sum_{j=0}^{n-1} f(\xi_j)\,\delta x_j$$

is the integral of $f(x)$ between a & b.

Notes

$f(x)$ = 'intergrand'

x = 'dummy variable' - any symbol will do.

$$\int_a^b f(s)\,ds = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

We define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Hence,

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Valid for any b.

The 'Fundamental Theorem of Calculus'

Integration is the inverse of differentiation.

That is to say:

$$\text{If} \quad F(x) = \int_{\underset{\text{fixed const}}{a}}^{\overset{\text{variable}}{x}} f(u)\,du$$

$$\text{then} \quad \frac{dF(x)}{dx} = f(x)$$

Proof:

$$\frac{dF(x)}{dx} = \lim_{\delta x \to 0}\left[\frac{F(x+\delta x) - F(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\int_a^{x+\delta x} f(u)\,du - \int_a^x f(u)\,du}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{\int_x^{x+\delta x} f(u)\,du}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{f(x)\,\delta x}{\delta x}\right]$$

$$= f(x)$$

Remarks

I) In the above definition of $F(x)$ the lower limit a is arbitrary, an arbitrary constant can be added to $F(x)$: hence 'indeffinite integral)

II) The definite integral

$$\int_a^b f(u)\,du = F(b) - F(a)$$

~~Indefinite~~ finite & Improper Integrals

Infinite integrals have a $+\infty$ (or $-\infty$) in the upper (or lower) limit. What is the meaning of

$$\int_0^\infty f(x)\,dx = ???$$

To decide if this is indeed meaningful we write

$$I(N) = \int_a^N f(x)\,dx$$

If this has a finite limit as $N \to \infty$ then the infinite intergral exists.

Example

$$\int_a^\infty e^{-x}\,dx = \lim_{N\to\infty} \int_a^N e^{-x}\,dx$$

$$= \lim_{N\to\infty} \left(e^{-a} - e^{-N}\right)$$

$$= e^{-a}$$

Example

$$\int_a^\infty \frac{dx}{x} = \lim_{N\to\infty} \int_a^N \frac{dx}{x}$$

$$= \lim_{N\to\infty} (\ln N - \ln a)$$

$$= \infty$$ $\therefore$ intergral does not exist.

In a similar fashion, improper integrals involve a singularity of the integrand on the range of integration. Of course we might need to spot whether this might be at an end point or within the range.

Example $\int_0^1 x^{-\frac{1}{2}} dx$

This might be a problem here because $\frac{1}{\sqrt{x}}$ is infinite at the end point $x=0$.

To resolve this issue we can integrate from ϵ to 1, where $0<\epsilon<<1$, then take $\epsilon \to 0$.

$$I(\epsilon) = \int_\epsilon^1 \frac{1}{\sqrt{x}} dx = \left[2x^{\frac{1}{2}}\right]_\epsilon^1 = 2-2\sqrt{\epsilon}$$

$$\cancel{\int_0^1 x^{-\frac{1}{2}} dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-\frac{1}{2}} dx}$$

As $\epsilon \to 0$, $2-2\sqrt{\epsilon} \to 2$. ✓ Fine

FINITE AREA

Example

$$\int_0^1 \frac{1}{x^2} dx = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{x^2}$$

$$= \lim_{\epsilon \to 0} \left[\frac{-1}{x}\right]_\epsilon^1$$

$$= \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} - 1\right)$$

$$\Rightarrow \infty .$$

This integral does not exist.

INFINITE AREA

Example $\int_{-1}^{1} \frac{dx}{x^2} = \left[\frac{-1}{x}\right]_{-1}^{1} = -2$ what!

But integrand is surley posabve (and assymtote at $x=0$).

The problem is at, (and near) $x=0$. The intergrand is singuler there.

Write: $\int_{-1}^{+1} \frac{dx}{x^2} = \lim_{\epsilon \to 0} \int_{-1}^{-\epsilon} \frac{dx}{x^2} + \lim_{\epsilon \to 0} \int_{\epsilon}^{1} \frac{dx}{x^2}$

$= \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} - 1\right) + \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} - 1\right)$

$= +\infty$.

Useful Intergration Techniques

Partail Fractions

$$\int \frac{dx}{x(x+1)} = \int \frac{1}{x} - \frac{1}{x+1} \, dx$$

$$= \ln(x) - \ln(x+1) + c$$

$$= \ln\left(\frac{x}{x+1}\right) + c$$

more complicated: $\int \frac{(\text{polynomial})}{(\text{polynomial})} dx$