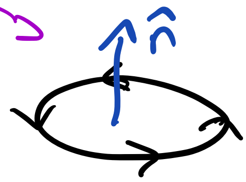
Curl $\quad \nabla \times \underline{B}$

curl is a vector that quantifies the 'circulation surface density'.

right-hand rule



$$\nabla \times \underline{B} \cdot \hat{n} = \lim_{A \to 0} \left(\frac{1}{A} \oint \underline{B} \cdot \underline{dr} \right)$$

In cartesian coordinates, to derive (eg. $\hat{k}$ component) consider loop of finite size & shrink.

$$\frac{1}{A} \oint \underline{B} \cdot \underline{dr} = \frac{1}{A} \oint B_x dx + B_y dy = \frac{1}{A} \iint_R \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \, dx dy$$

now shrink (take limit)

$$\nabla \times \underline{B} \cdot \hat{k} = \lim_{dxdy \to 0} \frac{\iint_R \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \, dxdy}{\iint_R dxdy} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$

and similarly for $\hat{i}$ & $\hat{j}$:

$$(\nabla \times \underline{B}) \cdot \hat{i} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \qquad (\nabla \times \underline{B}) \cdot \hat{j} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}$$

we can write this mathematically as:

$$\nabla \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix}$$

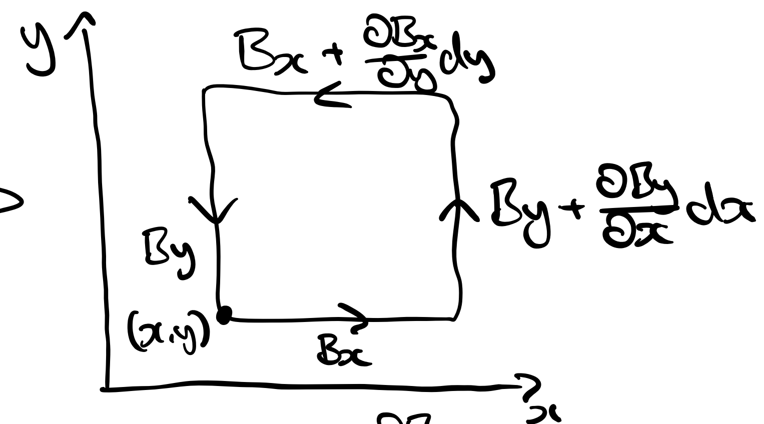
$$\nabla = \hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z$$

$$\underline{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$\nabla \times \underline{B}$ looks like the cross product of two vectors!

To understand 'circulation surface density', let's look again at the $\hat{k}$ component and consider an infinitesimal area dxdy.

linear approximation



$$\oint \frac{\underline{B}\cdot d\underline{r}}{A} = \frac{B_x dx + (B_y + \frac{\partial B_y}{\partial x}dx)dy - (B_x + \frac{\partial B_x}{\partial y}dy)dx - B_y dy}{dydx}$$

$$= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}$$ ← these cause rotation

A conservative field has zero curl!

$$\nabla \times (\nabla \Omega) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \Omega}{\partial x} & \frac{\partial \Omega}{\partial y} & \frac{\partial \Omega}{\partial z} \end{vmatrix} = \hat{i}\underbrace{\left(\frac{\partial^2 \Omega}{\partial y \partial z} - \frac{\partial^2 \Omega}{\partial z \partial y}\right)}_{=0} + \hat{j}(\ldots$$

$\nabla \times \underline{B} = 0$ implies $\underline{B}$ is a conservative field or 'irrotational' field.

Example 1 $\quad \underline{B} = 0\hat{\imath} + x^2\hat{\jmath} + 0\hat{k}$

$$\nabla \times \underline{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^2 & 0 \end{vmatrix} = 2x\hat{k}$$

Example 2 $\quad B = -y\hat{\imath} + x\hat{\jmath} + 0\hat{k}$

$$\nabla \times B = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{k}$$

Other Coordinate Systems:

Cyclindrical Polar:

$$\nabla = \hat{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\underline{B} = B_\rho \hat{\rho} + B_\phi \hat{\phi} + B_z \hat{z}$$

you can get the correct form by 'differentiate first, cross product second'.

$$\nabla \times \underline{B} = \left(\hat{\rho} \times \frac{\partial \underline{B}}{\partial \rho}\right) + \left(\frac{1}{\rho}\hat{\phi} \times \frac{\partial \underline{B}}{\partial \phi}\right) + \left(\hat{z} \times \frac{\partial \underline{B}}{\partial z}\right)$$

$$\nabla \times \underline{B} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ B_\rho & B_\phi & B_z \end{vmatrix}$$

Spherical:

$$\nabla \times \underline{B} = \frac{1}{\rho^2 \sin\theta} \begin{vmatrix} \hat{\rho} & r\sin\theta\,\hat{\phi} & r\hat{\theta} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial\theta \\ B_\rho & \rho\sin\theta\, B_\phi & rB_\theta \end{vmatrix}$$