Convolution

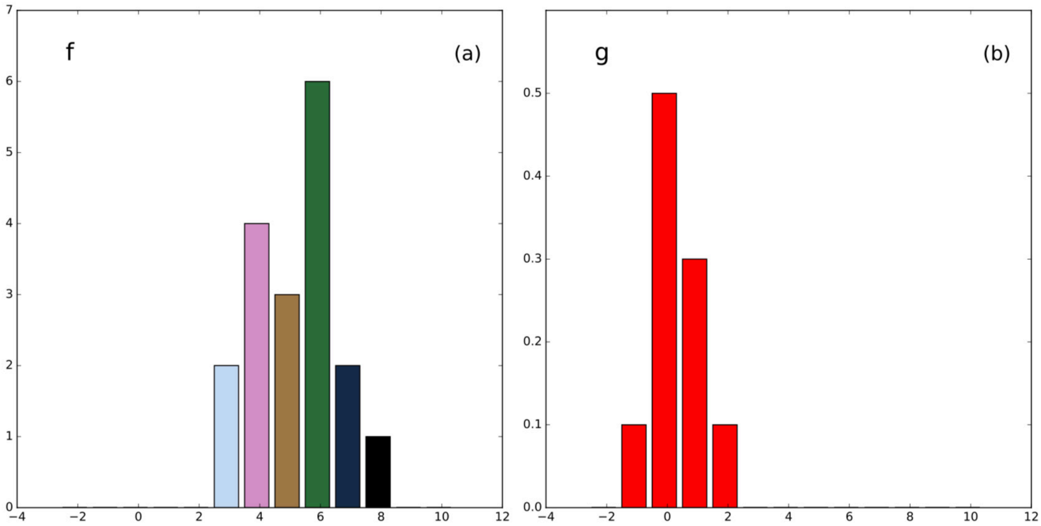
When making physical measurements we must account for the uncertainty in measurement.

For example using a 100mm ruler with precision of 0.5mm will result in a gaussian of mean 100mm and width 0.5mm.

Now imagine a diffraction pattern, there will be uncertainty in the measurement of each photon's position. How does this smear the diffraction pattern? In seismologi, can we understand how reflections occur after sending a pulse into the ground.

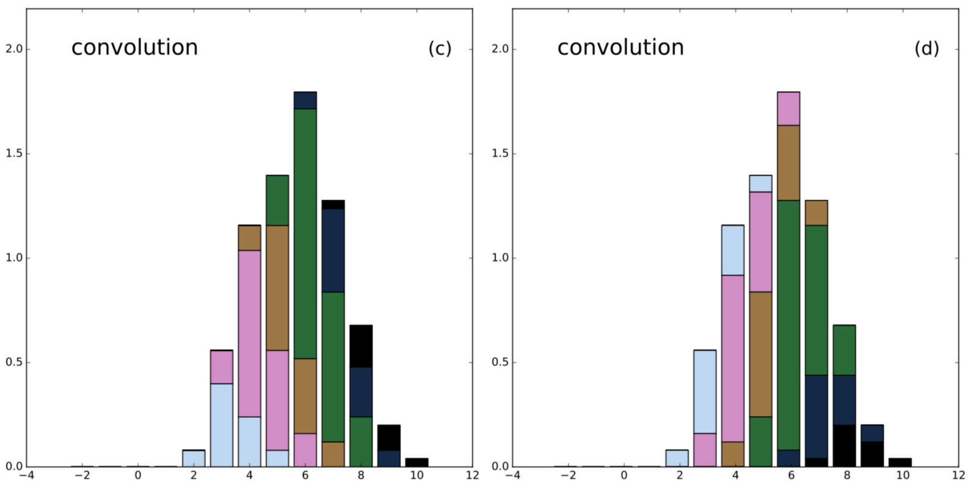
These ideas are related to convolution.

For an unbiased experiment, taking repeat expiments allows us to form a distribution. As the number of measurements goes to infinity we can find the mean to get our true value. However, given a limited number of measurements, then our resolution is limited.



The function f is our signal. Function g is our resolution function, note that's is antisymmetric.

To go from the true function to the measured function we take true function and apply the resolution function to each point. We can also do the reverse, we take our measured function and for each 'bin' ask how much of the constituent bins contribute to it.



c) shows this

d) convolution done in a different order but produce same function.

We calculate the convolution of f with g $f*g$ as

$$(f*g)(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau)\, d\tau$$

It can also be written as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$$

integrating

multiplying

shifting

If the functions are sampled using N datapoints, the operation will scale with N^2. This is a very 'brute-force' method.

Let's use fourier transforms to simplify this

$$F(\omega) = \mathcal{F}[f(t)]$$
$$G(\omega) = \mathcal{G}[g(t)]$$

Now lets look at the product

normalisation factor

$$\sqrt{2\pi}\, F(\omega) G(\omega)$$

Now lets take the inverse fourier transform

$$\mathcal{F}^{-1}[\sqrt{2\pi}\, F(\omega) G(\omega)] = \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau) e^{i\omega t} d\tau \right) \left(\int_{-\infty}^{\infty} g(s) e^{i\omega t} ds \right) e^{-i\omega t} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) g(s) \left(\int_{-\infty}^{\infty} e^{i\omega(\tau+s-t)} d\omega \right) ds\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) g(s) \delta(\tau+s-t)\, ds\, d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau \equiv (f * g)(t)$$

We've used the exponential form of the dirac delta function which forces the integral s to collapse to a single point at $s = t - \tau$.

We've shown that a complicated convolution is simply the product of two fourier transforms.

$$\mathcal{F}[f(t)g(t)](\omega) = \frac{1}{\sqrt{2\pi}} (F * G)(\omega)$$

Both the 'brute-force' method and the fourier transforms scale with $\sim N^2$. In reality we'll use a fast fourier transform (FFT) which scales $\sim N \log N$.

In convolution, order doesn't matter $(f*g) = (g*f)$

$$f*(g*h) = (f*g)*h = f*g*h$$

We can use convolution to shift a function. Consider

$$f(t) * \delta(t-d) = \int_{-\infty}^{\infty} f(\tau)\delta(t-d-\tau)\,d\tau$$

The delta function is even,

$$= \int_{-\infty}^{\infty} f(\tau)\delta(\tau-(t-d))\,d\tau$$

And now using the sifting properties of δ

$$= f(t-d)$$

<u>Example</u> Find the fourier transform of the same function shifted left & right by d.

$$\mathcal{F}[f(t-d)+f(t+d)] = \mathcal{F}[f(t)*\delta(t-d)+\delta(t+d)]$$
$$= \mathcal{F}[f(t)*\delta(t-d)] + \mathcal{F}[f(t)*\delta(t+d)]$$

applying the convolution theorem $F^{-1}[\sqrt{2\pi}\,F(\omega)G(\omega)] = f*g$

$$\sqrt{2\pi}\,F[f(t)]F[g(t)] = F[f*g]$$

$$= \sqrt{2\pi}\,F[f(t)]F[\delta(t-d)] + \sqrt{2\pi}\,F[f(t)]F[\delta(t+d)]$$
$$= \sqrt{2\pi}\,g(\omega)\left(e^{i\omega t}+e^{-i\omega t}\right)F[\delta(t)]$$
$$= \left(e^{i\omega t}+e^{-i\omega t}\right)g(\omega)$$
$$= 2\cos(\omega t)g(\omega)$$

A gaussian convoluted with another gaussian is itself a gaussian with variance and mean added.

$$g(\mu_1, \sigma_1) * g(\mu_2, \sigma_2) = g(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$$