Vectors II

Basis Sets

Vectors as Matrices

$$A\underline{x} = \underline{b}$$

$\underline{a}$ and $\underline{b}$ are vectors but obey the rules of matrices.

Q. Can all vector operations be done by matrices?

☆ Addition and scalar multiplication can be.

☆ multiplication? $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

if we try $\underline{a}\,\underline{b}$, it is not possible. But we can do $\underline{a}^T\underline{b}$ which is just the dot product of $\underline{a}$ & $\underline{b}$.

What if we try $\underline{a}\,\underline{b}^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}(b_1\ b_2\ b_3)$

$$\underline{a}\,\underline{b}^T = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}$$ diag. is dot product terms.

It's quite hard to get the cross product terms from $\underline{a}\,\underline{b}^T$.

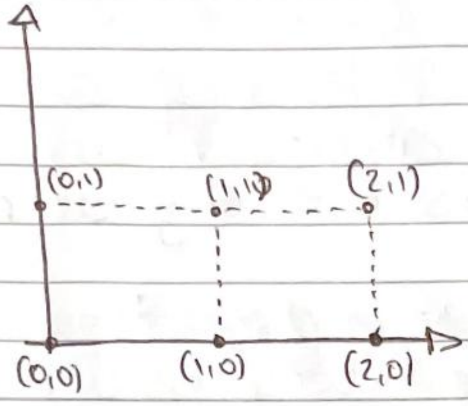
Basis Sets

$A\underline{x} = \underline{b}$ say we want to operate A on many x_i.

If we first form the matrix

$$X = (\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \ldots)$$

We can then do ~~A~~ $AX = B$



let $x_1^T = (0,0)$, $x_2^T = (1,0)$, $x_3^T = (0,1)$, $x_4^T = (1,1)$

then $X = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

if we let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$$AX = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\underline{A_1} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ $\underline{A_2} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$

We can write A as $A = (\underline{A_1}, \underline{A_2})$

So $A\underline{x} = \underline{b} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\Rightarrow \begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases} \Rightarrow \underline{A_1}x + \underline{A_2}y = \underline{b}$$

N.B. only a solution if $\det(A) \neq 0$.

if $\det(A) = 0$ then $\underline{A_1} = \lambda \underline{A_2}$, $\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \lambda \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$

Orthonormal Basis Sets

- non-zero vectors form an orthogonal set if they are orthogonal to each other. ie. their inner product is zero.
- In addition, if all vectors are of unit norm $\|\underline{v}\| = 1$, then it is called an <u>orthonormal</u> set.

We can use the gram-Schmidt process to ~~to~~ produce an orthonormal basis set.

Example if we start with a valid basis in $\mathbb{R}^2$ of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

N.B. $\underline{v}_1' = \underline{v}_1$

$$\hat{e}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}$$ ← $\hat{e}_1$ has same direction as $\underline{v}_1$, but of unit length.

$$\underline{v}_2' = \underline{v}_2 - \frac{\underline{v}_2 \cdot \underline{v}_1'}{\|\underline{v}_1\|}\,\underline{v}_1$$

← $\underline{v}_2 \cdot \underline{v}_1'$ = projection of $\underline{v}_2$ onto $\underline{v}_1$.

$\frac{\underline{v}_1}{\|\underline{v}_1\|}$ normalised length of $\underline{v}_1$ = 1, has direction.

aka. $\underline{v}_2' = \underline{v}_2$ - the projection of $\underline{v}_2$ onto $\underline{v}_1$ in the direction of $\underline{v}_1$

$$\hat{e}_2 = \frac{\underline{v}_2'}{\|\underline{v}_2'\|}$$ transform into a unit vector.

$$\underline{v}_3' = \underline{v}_3 - \underline{v}_3 \cdot \underline{v}_1' \frac{\underline{v}_1'}{\|\underline{v}_1'\|} - \underline{v}_3 \cdot \underline{v}_2' \frac{\underline{v}_2'}{\|\underline{v}_2'\|}$$

projection of $\underline{v}_3$ onto $\underline{v}_1$

projection of $\underline{v}_3$ onto $\underline{v}_2$

$$\hat{e}_3 = \frac{\underline{v}_3'}{\|\underline{v}_3'\|}$$ ← make unit length.

$\hat{e}_1, \hat{e}_2, \hat{e}_3$ are an orthonormal basis set.
$\underline{v}_1', \underline{v}_2', \underline{v}_3'$ are an orthogonal basis set.

Homogeneous Equations
consider

$$A\underline{x} = \underline{0}$$ ← vector

this is called a homogeneous equation.

$$\begin{vmatrix} a & 0 & c \\ d & 0 & f \\ g & 0 & i \end{vmatrix} = 0$$

from cramer's rule, $x_i = \frac{\Delta_i}{\Delta} = \frac{0}{\Delta}$

only when $\Delta = 0$ do we get non-trivial solutions.

We know we can write a matrix as a combination of vectors.

$$\underline{A} = \underline{A_1} + \underline{A_2} + \underline{A_3}$$
$$\underline{b} = x_1 \underline{A_1} + x_2 \underline{A_2} + x_3 \underline{A_3}$$

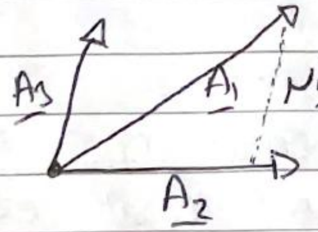
but now
$$\underline{0} = x_1 \underline{A_1} + x_2 \underline{A_2} + \dots$$
$$\underline{A_1} = \frac{-1}{x_1}\left[x_2 \underline{A_2} + x_3 \underline{A_3} + \dots\right]$$

So if $x_1 \neq 0$, then $\underline{A_1}$ is linearly dependent upon $\underline{A_2}, \underline{A_3}, \dots$

<u>Example</u> 3D vector space.

$$\underline{0} = x_1 \underline{A_1} + x_2 \underline{A_2} + x_3 \underline{A_3}$$
$$A_1 = \lambda A_2 + \mu A_3$$

$$\det(\underline{A}) = |\underline{A_1}\ \underline{A_2}\ \underline{A_3}| = |\lambda \underline{A_2} + \mu \underline{A_3}\ \ A_2\ \ A_3| = 0$$



← all co-planar

← vol. parallelipiped $= 0$ (triple product)

<u>Example</u> $A\underline{x} = \underline{b}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

to solve for $\underline{x}$, we will use gaussian elimination.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

R2 - R1   R3 - R1

We cannot procede any further using gaussian elimination. lets make z arbitary

$$z = \lambda.$$

$$\begin{aligned} x + z &= 0 \\ y - z &= 1 \end{aligned} \Rightarrow \begin{aligned} x &= -\lambda \\ y &= \lambda \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{x} = \underline{x_0} + \lambda \underline{x_1}$$

particular solution

$$A\underline{x_0} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \underline{b}$$

$$A\underline{x_1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$\therefore$ $\underline{x_1}$ is general solution to the homogeneous solution.

<u>Example</u>

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ -2 & 4 & -2 & -4 \\ 3 & -6 & 3 & 6 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

R2/-2, R3/3
R2-R1, R3-R1

let $y = \lambda$ $z = \mu$

$$x - 2\lambda + \mu = 2$$
$$x = 2 + 2\lambda - \mu$$

$$\underline{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\underline{x} = \underline{x_0} + \lambda \underline{x_1} + \mu \underline{x_2}$$

$$A\underline{x_0} = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \underline{b}$$ particular solution

$$A\underline{x_1} = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A\underline{x_2} = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 3 & -6 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In general, n linearly dependant vectors can span a ~~space~~ $\mathbb{R}^{m-n}$ in $\mathbb{R}^m$.