Lecture 12

Case 2: $\alpha_1 = \alpha_2$ both $\mathbb{R}$

$a\alpha^2 + b\alpha + c = 0$: $b^2 - 4ac = 0$

General Solution: $y = Ay_1(x) + By_2(x)$

$$y_1(x) = e^{\alpha_1 x} \quad y_2(x) = ???$$

How can we calculate y_2? We can take limits

$$\lim_{\alpha_1 \to \alpha_2} (\text{Case } 1) = ?$$

$$y = \frac{\alpha_2 y_0}{\alpha_2 - \alpha_1} e^{\alpha_1 x} + \frac{\alpha_1 y_0}{\alpha_1 - \alpha_2} e^{\alpha_2 x} - \frac{y_0'}{\alpha_2 - \alpha_1} e^{\alpha_1 x} - \frac{y_0'}{\alpha_1 - \alpha_2} e^{\alpha_2 x}$$

$$= y_0 \left[\frac{\alpha_2 e^{\alpha_1 x} - \alpha_1 e^{\alpha_2 x}}{\alpha_2 - \alpha_1} \right] + y_0' \left[\frac{e^{\alpha_1 x} - e^{\alpha_2 x}}{\alpha_1 - \alpha_2} \right]$$

$\therefore$ We can use l'Hôpital as we have a "$\frac{0}{0}$" fraction. We take the derivate with respect to α_1.

$$\lim_{\alpha_1 \to \alpha_2} y(x) = y_0 \left[\lim_{\alpha_1 \to \alpha_2} \frac{(\alpha_2 e^{\alpha_1 x} - \alpha_1 e^{\alpha_2 x})'_{\alpha_1}}{(\alpha_2 - \alpha_1)'_{\alpha_1}} \right] +$$
$$y_0' \left[\lim_{\alpha_1 \to \alpha_2} \frac{(e^{\alpha_1 x} - e^{\alpha_2 x})'_{\alpha_1}}{(\alpha_1 - \alpha_2)'_{\alpha_1}} \right]$$

$$= y_0 \left[\lim_{\alpha_1 \to \alpha_2} \frac{\alpha_2 x e^{\alpha_1 x} - e^{\alpha_2 x}}{-1} \right] +$$
$$y_0' \left[\lim_{\alpha_1 \to \alpha_2} \frac{x e^{\alpha_1 x}}{1} \right]$$

$$= y_0 e^{\alpha_2 x} + (y_0' - \alpha_2 y_0) x e^{\alpha_2 x}$$

$$= Ae^{\alpha x} + Bxe^{\alpha x}$$

$$A = y_0 \qquad B = y_0' - \alpha y_0$$

$$\boxed{y(x) = y_0 e^{\alpha x} + (y_0' - \alpha y_0) x e^{\alpha x}}$$

Alternative Approach:

$$ay'' + by' + \frac{b^2}{4a}y = 0$$

$$y_1(x) = e^{-\frac{b}{2a}x} \quad \text{(from characteristic equation)}$$

$$y_2(x) = U(x) \cdot y_1(x)$$

$$\to a(U''y_1 + 2U'y_1' + Uy_1'') + b(U'y + Uy') + \frac{b^2}{4a}(Uy) = 0$$

$$(ay)U'' + \cancel{(2ay' + by)U'} + \underbrace{\cancel{(ay'' + by' + \tfrac{b^2}{4a}y)}}_{=0}U = 0$$

$$y = e^{-\frac{b}{2a}x} \qquad y' = -\frac{b}{2a}e^{-\frac{b}{2a}x} \qquad 2ay' + by = 0$$

$$\neq 0 \qquad (ay)U'' = 0 \qquad = 0$$

$$\therefore U(x) = C_1 x + C_2$$

$$\therefore y_2(x) = (C_1 x + C_2) y_1(x) \qquad \left(\begin{array}{l} C_2 y_1(x) \text{ is} \\ \text{linearly dependent} \\ \therefore \text{ignore} \end{array}\right)$$

$$y_2(x) = C_1 x y_1(x)$$

Case 3:

$$a\alpha^2 + b\alpha + c = 0$$

$$b^2 - 4ac < 0$$

$$\alpha_1 = \alpha_2^*$$

$$\alpha_{1,2} = \frac{-b \pm i\sqrt{b^2-4ac}}{2a}$$

$$y(x) = Ae^{\alpha x} + Be^{\alpha^* x}$$

$$\alpha = u + iv \qquad \alpha^* = u - iv$$

$$y_1(x) = e^{(u+iv)x} = e^{ux}\left(\cos(vx) + i\sin(vx)\right)$$

$$y_2(x) = e^{(u-iv)x} = e^{ux}\left(\cos(vx) - i\sin(vx)\right)$$

$$y(x) = Ay_1(x) + By_2(x) = (A+B)e^{ux}\cos(vx) + i(A-B)e^{ux}\sin(vx)$$

$$A+B = C \qquad i(A-B) = D$$

$$y(x) = e^{ux}\left(C\cos(vx) + D\sin(vx)\right)$$

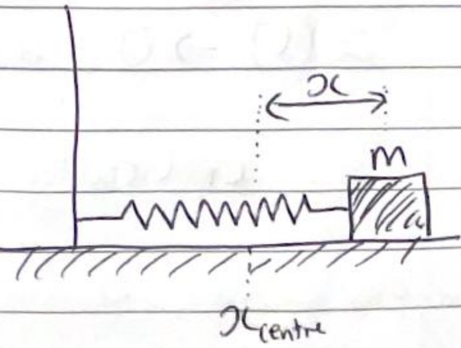
$$\boxed{y(x) = A_0 e^{ux}\left(\cos(vx+\phi)\right)}$$

$$C = A_0\cos\phi \qquad D = -A_0\sin\phi$$

Applications: Damped Harmonic Oscillator

$$m\ddot{x} = -kx - b\dot{x}$$

$$\left.\begin{aligned} \gamma &= \frac{b}{m} \\ \omega^2 &= \frac{k}{m} \end{aligned}\right\} \text{constants}$$



$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

$a = 1 \qquad b = \gamma \qquad c = \omega^2$

Case 1: $\quad \gamma^2 - 4\omega_0^2 > 0$

$$x(t) = e^{\gamma t/2}\left(Ae^{\rho t} + Be^{-\rho t}\right)$$

$$\alpha_{1,2} = -\frac{\gamma}{2} \pm \rho \qquad \rho = \sqrt{\frac{\gamma^2}{4} - \omega_0^2} < \frac{\gamma}{2}$$

$x(t) \to 0$ exponentially as $t \to \infty$

"overdampted" solution

Case 2: $\quad \gamma^2 - 4\omega_0^2 = 0$

$$\alpha = -\frac{b}{2a} = -\frac{\gamma}{2}$$

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

I.C. of $\quad x(0) = x_0 \quad \dot{x}(0) = 0$

$A = x_0 \qquad B - \frac{1}{2}\gamma A = 0 \qquad B = \frac{1}{2}\gamma A = \frac{1}{2}\gamma x_0$

$$x(t) = x_0\left(1 + \frac{\gamma t}{2}\right)e^{-\frac{\gamma t}{2}}$$

$x(t) \to 0$ as $t \to \infty$.

"critically dampeel".

Case 3: $\gamma^2 - 4\omega_0^2 < 0$

$$x(t) = e^{-\frac{\gamma t}{2}}\left(Ae^{i\omega t} + Be^{-i\omega t}\right)$$

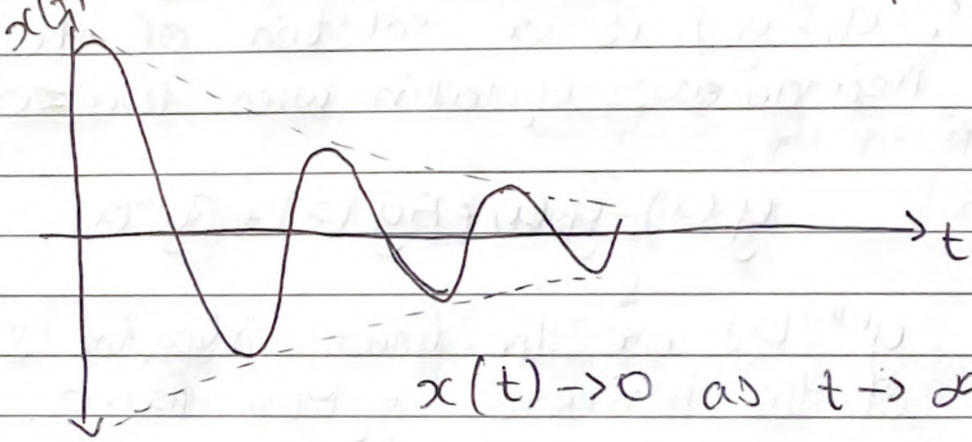
$$\alpha_{1,2} = -\frac{\gamma}{2} \pm i\omega \qquad \omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$x(t) = x_0 e^{-\frac{\gamma t}{2}}\left(\cos(\omega t) + \frac{\gamma}{2\omega}\sin(\omega t)\right)$$

$$= x_0 \frac{\omega_0}{\omega} \underbrace{e^{-\frac{\gamma t}{2}}\cos(\omega t + \phi)}$$

← easy to plot

$$\tan\phi = -\frac{\gamma}{2\omega}$$



$x(t) \to 0$ as $t \to \infty$.