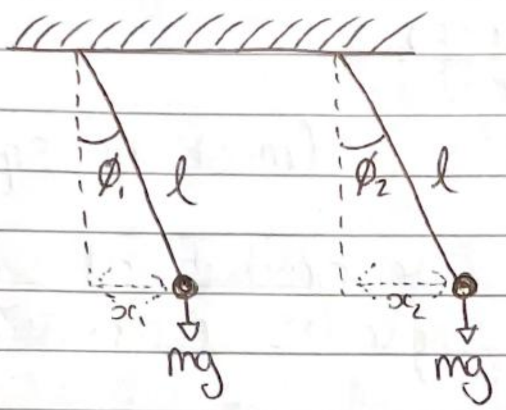
Vectors 16

Normal Modes

Linearisation



we know that in the limit ϕ_1, ϕ_2 are small

☆ $x_i \simeq l\phi_i$

☆ $F_x = -mg\sin\phi_i$

$\simeq -mg\phi_i$

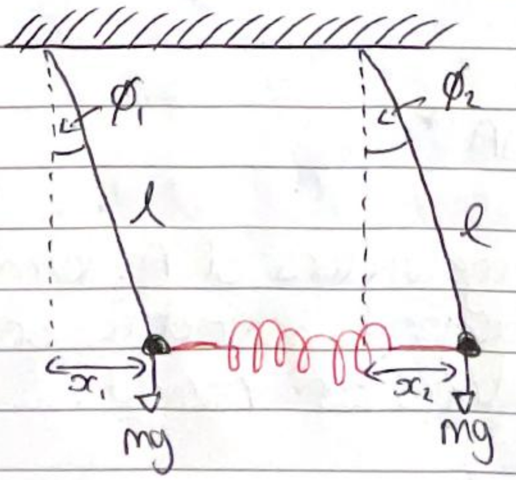
For small amplitudes, newton's law

$$m\frac{d^2x_i}{dt^2} = m\frac{d^2}{dt^2}(L\phi_i) = -mg\phi_i$$

$$\ddot{\phi}_i + \left(\frac{g}{l}\right)\phi_i = 0 \qquad \left(\text{SHM with } \omega = \sqrt{\frac{g}{l}}\right.$$

The process of taking only the first (linear) term of taylor series around equalibrium is called linearising.

Coupled Differential Equations



now we'll couple the masses together with a spring.

$$m\frac{d^2x_1}{dt^2} = m\frac{d^2}{dt^2}(l\phi_1) = -mg\phi_1 - k(x_1 - x_2)$$

$$\frac{d^2\phi_1}{dt^2} = -\left(\frac{g}{l}\right)\phi_1 - \left(\frac{k}{m}\right)(\phi_1 - \phi_2)$$

$$\frac{d^2\phi_2}{dt^2} = -\left(\frac{g}{l}\right)\phi_2 - \left(\frac{k}{m}\right)(\phi_2 - \phi_1)$$

$$\omega_l^2 = \frac{g}{l} \qquad \omega_k^2 = \frac{k}{m}$$

$$\frac{d^2}{dt^2}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}-\omega_l^2 - \omega_k^2 & \omega_k^2\\ \omega_k^2 & -\omega_l^2 - \omega_k^2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\frac{d^2}{dt^2}\underline{\phi} = M\underline{\phi}$$

dynamical matrix

dynamical matrix

Trail solution: $\underline{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \underline{n}e^{\nu t}$

$\underline{n}$ is called the normal mode.

$$\frac{d^2\phi}{dt^2} = \underline{n}\frac{d^2}{dt^2}(e^{\nu t}) = \underline{n}\nu^2 e^{\nu t} = M\underline{n}e^{\nu t}$$

$$M\underline{n} = \nu^2\underline{n} \qquad \text{(eigenvalue problem)}$$

$$(M - I\nu^2)\underline{n} = 0$$
$$\det(M - I\nu^2) = 0$$

$$\begin{vmatrix}-\omega_l^2 - \omega_k^2 - \nu^2 & \omega_k^2\\ \omega_k^2 & -\omega_l^2 - \omega_k^2 - \nu^2\end{vmatrix} = 0$$

$$(-\omega_l^2 - \omega_k^2 - \nu^2)^2 - (\omega_k^2)^2 = 0$$
$$-\omega_l^2 - \omega_k^2 - \nu^2 = \pm\omega_k^2$$
$$\nu^2 = \pm\omega_k^2 - \omega_l^2 - \omega_k^2$$

$$\mu^2 = \begin{cases} -\omega_\ell^2 \\ -\omega_\ell^2 - 2\omega_k^2 \end{cases}$$

$$\mu_1, \mu_2 = \begin{cases} \pm i\omega_\ell \\ \pm i\sqrt{\omega_\ell^2 + 2\omega_k^2} \end{cases}$$

This gives us two modes with four solutions.

<u>Normal Modes</u>

μ_1, $(\mu^2 = -\omega_\ell^2)$ $\begin{pmatrix} -\omega_k^2 & \omega_k^2 \\ \omega_k^2 & -\omega_k^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

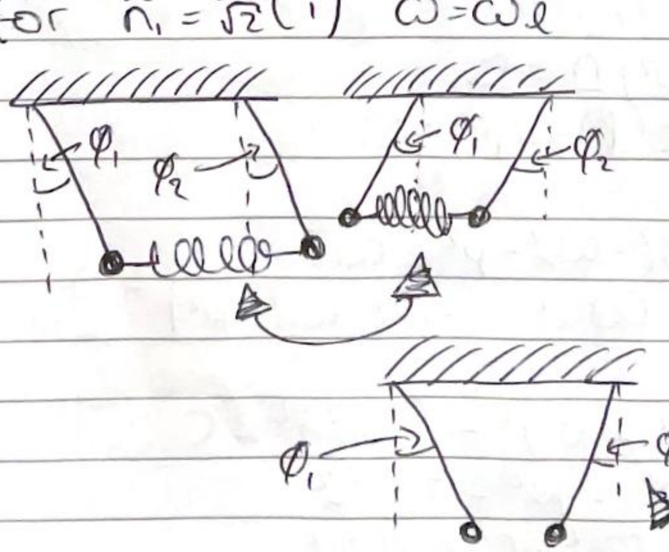
sol. $x = y$. $\hat{n}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

μ_2, $\mu^2 = (-\omega_\ell^2 - 2\omega_k^2)$ $\begin{pmatrix} \omega_k^2 & \omega_k^2 \\ \omega_k^2 & \omega_k^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

sol. $x = -y$ $\hat{n}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

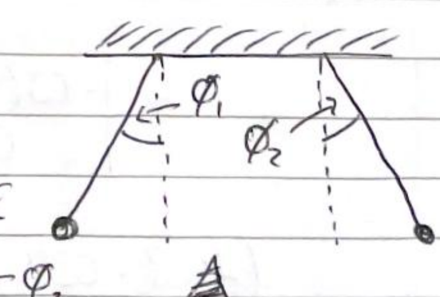
N.B. $\hat{n}_1 \cdot \hat{n}_2 = 0$ ie. modes are orthogonal.

for $\hat{n}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\omega = \omega_\ell$

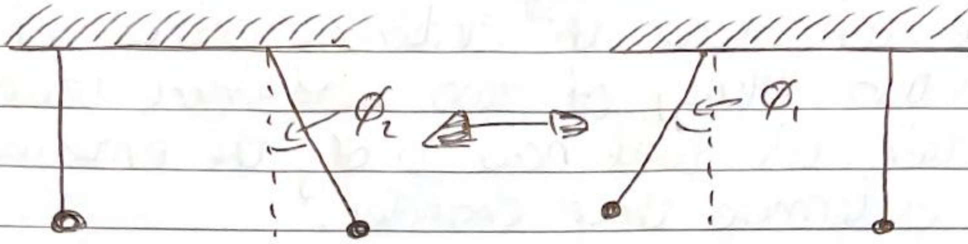


for $\hat{n}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\omega = \sqrt{\omega_\ell^2 - \omega_k^2}$



Every displacement of the system can be written as a combination of these models.



<u>Example</u> the 'R' number infection rate

asymtomatic $\dfrac{dI_1}{dt} = \underbrace{\beta_1 I + \beta_2 I_2}_{} - \eta I_1 - \gamma_1 I_1$

symtomatic $\dfrac{dI_2}{dt} = \eta I_1 - \gamma I_2$ recovery rate

we can write this with matrix notation

$$\frac{dI}{dt} = A\underline{I}$$

$$A = \begin{pmatrix} \beta_1 - \eta - \gamma_1 & \beta_2 \\ \eta & -\gamma_2 \end{pmatrix} \qquad \underline{I} = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

we could look at this day-by-day.

1st day: $I(1) = AI(0)$ 2nd day: $A(2) = A\underline{I}(1) = A^2 I(0)$

after n days: $I(n) = A^n I(0)$

$$I(n) = A^n I(0) = S\Lambda^n S^{-1} I(0) = S\begin{pmatrix} R_0^n & 0 \\ 0 & R_1^n \end{pmatrix} S^{-1} I(0)$$

the largest eigenvalue becomes the dominant solution, which is the growth rate 'R'.