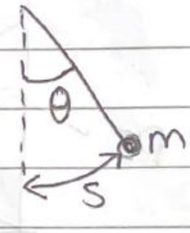
Damped Harmonic Motion

We'll assume a damping force of $F = -bv$,
This isn't generally true, but for slow velocities its works reasonabley well.



$$m\frac{d^2s}{dt^2} = -\frac{mg}{l}s - b\frac{ds}{dt}$$

$$\Rightarrow \ddot{s} = -\frac{g}{l}s - \frac{b}{m}\dot{s}$$

$$\ddot{s} + \frac{b}{m}\dot{s} + \frac{g}{l}s = 0$$

let $\gamma = \frac{b}{m}$ $\qquad \omega_0^2 = \frac{g}{l}$

$$\ddot{s} + \gamma\dot{s} + \omega_0^2 s = 0$$

$$-\omega^2 s + i\omega\gamma s + \omega_0^2 = 0$$

$$-\omega^2 + i\omega\gamma + \omega_0^2 = 0$$

$$\omega = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

Now lets define $Q = \omega_0/\gamma$. This is called the quality factor. It measures the strength of damping.

ω is complex if $Q > 1/2$
ω is imaginary if $Q < 1/2$

~~NB~~

Light & Heavy Damping

Light Damping

This occurs when $\gamma/2 < \omega_0$ aka $Q > 1/2$.

$$\omega = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$ ← this part is real (called ω_d)

$$i\omega t = -\frac{\gamma}{2}t \pm i\omega_d t$$

$$\tilde{\psi}(t) = \tilde{B}e^{-\frac{\gamma t}{2}}\underbrace{e^{i\omega_d t}} + \tilde{C}e^{-\frac{\gamma t}{2}}\underbrace{e^{-i\omega_d t}}$$

$$e^{i\omega_d t} = \cos(\omega_d t) + i\sin(\omega_d t)$$
$$e^{-i\omega_d t} = \cos(-\omega_d t) + i\sin(-\omega_d t) = \cos(\omega_d t) - i\sin(\omega_d t)$$

$$\tilde{\psi} = e^{-\frac{\gamma t}{2}}\left[(\tilde{B}+\tilde{C})\cos\omega_d t + i(\tilde{B}-\tilde{C})\sin\omega_d t\right]$$

$$\psi = Re(\tilde{\psi}) = e^{-\frac{\gamma t}{2}}\left[Re(\tilde{B}+\tilde{C})\cos\omega_d t - Im(\tilde{B}-\tilde{C})\sin\omega_d t\right]$$

$$= e^{-\frac{\gamma t}{2}}\left[A\cos(\omega_d t + \varphi)\right]$$

It will oscillate at angular frequency of ω_d ($<\omega_0$) with an amplitude which decreases exponentially.

Heavy Damping

This occurs when $\frac{\gamma}{2} > \omega_0 \rightarrow Q < \frac{1}{2}$

Let's write $\gamma^* = \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}$ this is real.

$$\omega = i\frac{\gamma}{2} \pm i\gamma^*$$

$$i\omega t = -\left(\frac{\gamma}{2} \pm \gamma^*\right)t$$

$$\tilde{\psi}(t) = \tilde{B}e^{-\left(\frac{\gamma}{2}+\gamma^*\right)t} + \tilde{C}e^{-\left(\frac{\gamma}{2}-\gamma^*\right)t}$$

both terms decay with no oscillations, all exponentials are real.

Critical Damping

This happens when $Q = 0.5$. Then $\omega = \frac{i\gamma}{2}$.

$$\tilde{\psi} = Ae^{i\varphi}e^{-\gamma t/2}$$

$$\psi = Re(\tilde{\psi}) = A\cos(\varphi)e^{-\frac{\gamma t}{2}}.$$

$$\dot{\psi} = -\frac{\gamma}{2}A\cos(\varphi)e^{-\frac{\gamma t}{2}} = -\frac{\gamma}{2}\psi.$$

We cannot use this as our final solution as we cannot specify $\psi(t=0)$ and $\dot{\psi}(t=0)$ separatley.

$$\psi = (A+Bt)e^{-\frac{\gamma t}{2}}$$

Consider a system with a given ω_0 and a variable Q. τ = time for amplitude to fall by a factor of e. Then plot $\omega_0\tau$ vs. Q.

light – $\text{amp} = \alpha e^{-\gamma t/2}$ $\therefore \tau = \frac{2}{\gamma}$

$$\omega_0\tau = \frac{2\omega_0}{\gamma} = 2Q.$$

Heavy – $\text{amp} = Be^{-\gamma t/2} + Ce^{-\gamma t/2}$ $\bar{\tau} = \frac{1}{2}(\tau_1 + \tau_2)$

$$\omega_0\bar{\tau} = 1/2Q$$