Functions 3+

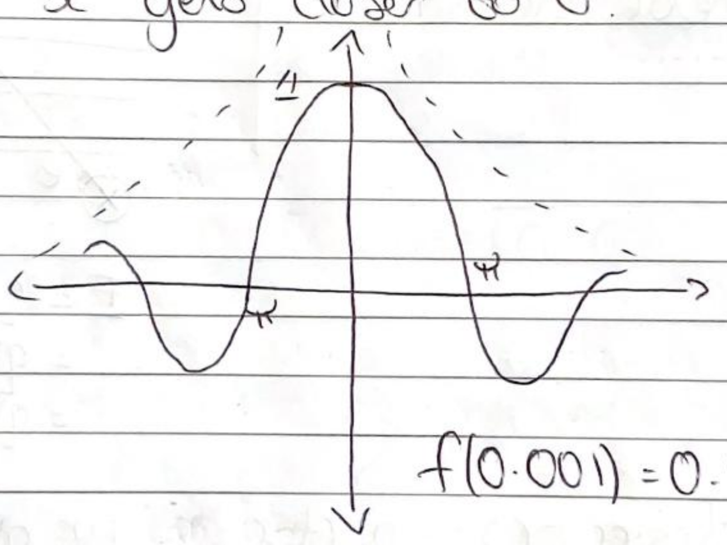
Limits of Functions

Example

$$f(x) = \frac{\sin x}{x} \qquad x \neq 0$$

$f(x)$ is not defined at $x = 0$ as $f(x)$ would be "$\frac{0}{0}$".

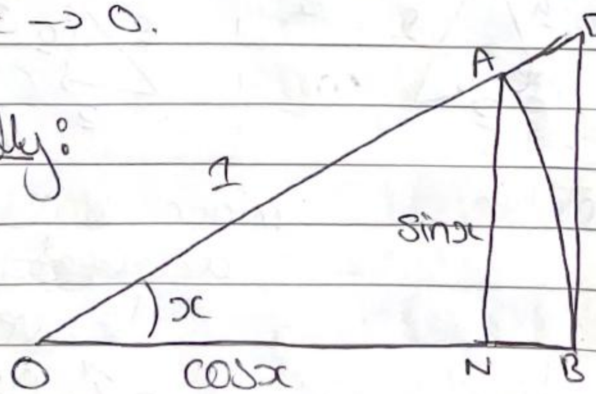
But plotting the function numerically shows that $f(x)$ get closer and closer to 1 as x gets closer to 0.



$f(0.001) = 0.99999983\ldots$

So can we confirm whats happens to $f(x)$ as $x \to 0$.

Geometrically:



Consider the sector of a unite circle

$OA = OB = 1$

$AN = \sin x$

$ON = \cos x$

$NB = 1 - \cos x$

$BD = \tan x$

Evidently,

$$\text{Area of } \Delta\, OAB < \text{Area of sector } OAB < \text{area of } \Delta\, ODB$$

$$\frac{1}{2}\sin x(1) < \frac{x}{2\pi}\pi(1)^2 < \frac{1}{2}\tan x(1)$$

$\div \frac{1}{2}\sin x$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$

As $x \to 0$, $\cos x \to 1$, $\therefore$ $\frac{\sin x}{x}$ is squessed to the value of 1.

Called "<u>The Squeeze Principle</u>"

<u>Notation</u>

$$\lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1.$$

Mathematically, a more formal process is needed to turn this 'proof' into a proof.

NB) There is no real issue if our function is 'well behaved' at x.

eg. $f(x) = x^2 + 4$ $\quad \lim_{x \to 4} f(x) = 20.$

<u>Simple Rules for Sums/Products</u>

If $f(x) \to F$ $\quad g(x) \to G$ $\quad$ as $x \to x_0$

then $af + bg \to aF + bG$ (sum)

$fg \to FG$ (product)

$f/g \to F/G$ (quotient)

Example

$$\lim_{x \to 2} \left[(x^2+2) \cos\left(\frac{\pi x}{2}\right) \right]$$

$$= \left[\lim_{x \to 2} (x^2+2) \right] \left[\lim_{x \to 2} \left(\cos\left(\frac{\pi x}{2}\right) \right) \right]$$

$$= (6)(\cos \pi)$$

$$= -6$$

Important first step

* check if we have a simple limit using rules we've just covered

Non-Trivial Limits

The concept of a limit is essential when we encounter combinations like:

" $\frac{0}{0}$ " " $\frac{\infty}{\infty}$ " " $\infty \times 0$ " " $\infty - \infty$ "

Type " $\frac{0}{0}$ "

For this we can use L'Hôpitals rule (1696)

Formal proof normally requires $f(x), g(x)$ to have taylor expansion at $x = x_0$

For $\lim_{x \to x_0} \left[\frac{f(x)}{g(x)}\right]$ ← where $f(x_0) = 0 = g(x_0)$

then $\lim_{x \to x_0} \left[\frac{f(x)}{g(x)}\right] = \lim_{x \to x_0} \left[\frac{f'(x)}{g'(x)}\right]$

If $f'(x)$ and $g'(x)$ are still <u>BOTH</u> zero, then differentiate again

$$\lim_{x \to x_0} \left[\frac{f(x)}{g(x)}\right] = \left[\frac{f''(x)}{g''(x)}\right] \lim_{x \to x_0}.$$

<u>Example</u>

$$\lim_{x \to 0} \left[\frac{\sin x}{x}\right] = \lim_{x \to 0} \left[\frac{\cos x}{1}\right] = 1 \checkmark$$

<u>Example</u> $\lim_{x \to 1} \left(\frac{x^3 - x^2 + 2x - 2}{x^3 + x^2 - 2}\right)$ is of "$\frac{0}{0}$" type.

L'Hôpital ⇒ $\lim_{x \to 1} \left(\frac{3x^2 - 2x + 2}{3x^2 + 2x}\right) = \frac{3}{5} \checkmark$

<u>Alternative</u> ⇒ equally valid
we can use the expansion $x = h + 1$ and then allow $h \to 0$.

$$\lim_{x \to 1} \left(\frac{x^3 - x^2 + 2x - 2}{x^3 + x^2 - 2}\right) = \lim_{x \to 0} \left(\frac{(h+1)^3 - (h+1)^2 + 2(h+1) - 2}{(h+1)^3 + (h+1)^2 - 2}\right)$$

$$= \lim_{h \to 0} \left(\frac{(1 + 3h + 3h^2 + h^3) - (1 + 2h + h^2) + 2h + 2 - 2}{(1 + 3h + 3h^2 + h^3) + (1 + 2h + h^2) - 2}\right)$$

$$= \lim_{h\to 0} \left(\frac{3h + 2h^2 + h^3}{5h + 4h^2 + h^3} \right)$$

$$= \lim_{h\to 0} \left(\frac{3 + 2h + h^2}{5 + 4h + h^2} \right) = \frac{3}{5} \checkmark$$

Type "$\frac{\infty}{\infty}$"

No L'Hôpital!

cannot expand taylor series around ∞.

Example $\lim_{x\to\infty} \left(\frac{2x^5 + 2x^2 - 1}{x^5 - x^3 + 1} \right)$

$$= \lim_{x\to\infty} \left(\frac{x^5(2 + 2\frac{1}{x^3} - \frac{1}{x^5})}{x^5(1 - \frac{1}{x^2} + \frac{1}{x^5})} \right)$$

$$= \lim_{x\to\infty} \left(\frac{2 + \frac{2}{x^3} - \frac{1}{x^5}}{1 - \frac{1}{x^2} + \frac{1}{x^5}} \right)$$

$$= 2$$

We should expect a finite answer as the dominant power in the numerator and denominator are the same.

Type "1^∞"

$$\lim_{x\to 0} (1-x)^{\frac{1}{x}} = ?$$

Take logs and consider

$$\lim_{x\to 0} \left[\frac{1}{x} \ln(1-x) \right] \quad \text{"}\frac{0}{0}\text{"}$$

now we can use L'Hôpital

$$\lim_{x \to 0} \left[\frac{-\frac{1}{1-x}}{1} \right] = -1$$

Hence, $$\lim_{x \to 0} \left[(1-x)^{\frac{1}{x}} \right] = e^{-1}$$

In (slight) disguise this is:

$$\lim_{n \to \infty} \left[\left(1 - \frac{1}{n}\right)^n \right] = e^{-1}$$

An alternative definition of the exponential function is found by

$$\lim_{n \to \infty} \left[\left(1 + \frac{x}{n}\right)^n \right] = e^x$$