Standing Waves

Superpose two equal amplitude counterpropagating sinusoidal travelling waves.

$$\Psi = \Psi_0 \cos(kx - \omega t + \phi_f) + \Psi_0 \cos(kx + \omega t + \phi_b)$$

$$\Psi = 2\Psi_0 \cos\left(\frac{kx - \omega t + \phi_f + kx + \omega t + \phi_b}{2}\right)\cos\left(\frac{kx - \omega t + \phi_b - kx - \omega t - \phi_b}{2}\right)$$

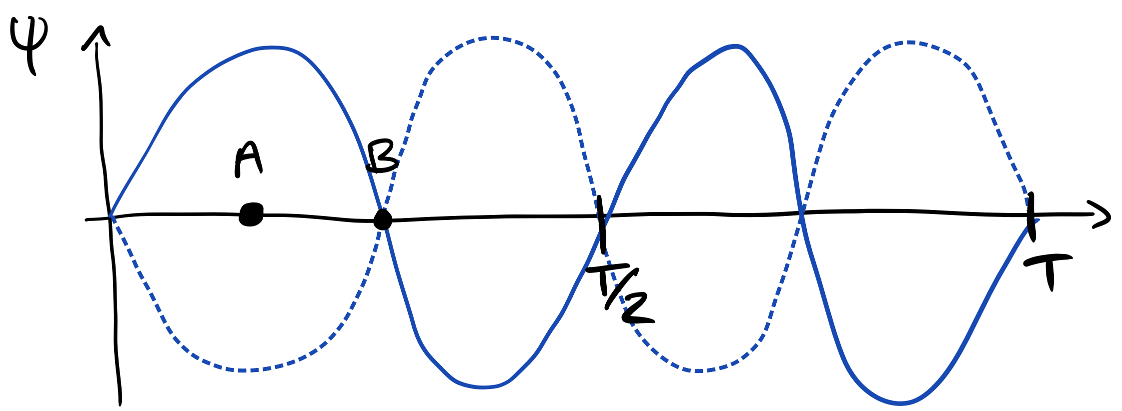
$$= 2\Psi_0 \cos\Big(kx + \underbrace{\frac{\phi_f + \phi_b}{2}}_{\phi_1}\Big)\cos\Big(-\omega t + \underbrace{\frac{\phi_f - \phi_b}{2}}_{\phi_2}\Big)$$

$$\Psi = 2\Psi_0 \cos(kx + \phi_1)\cos(\omega t + \phi_2)$$

This is the formula of a standing wave.

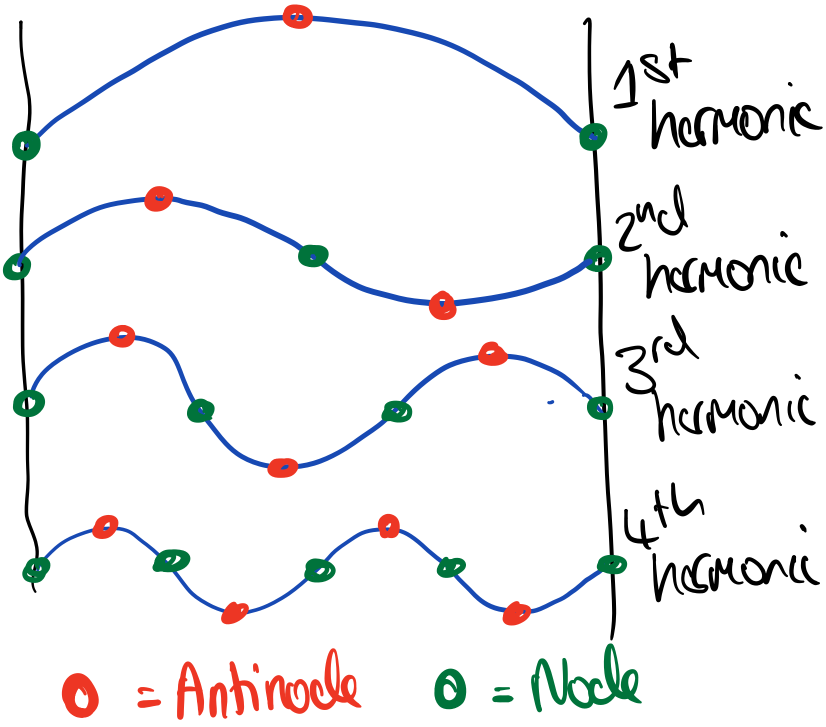
- disturbance does not travel
- does not transport energy
- are periodic

Ψ is not a function of $x - vt$, you can separate x & t.



A is an antinode. B is a node.

All parts of the string oscillate at the same frequency. We defined a normal mode when the whole system oscillated at the same frequency. $\therefore$ Standing waves are normal modes.



Harmonics: $\lambda = 2L$

$\lambda = L$

$\lambda = \frac{2}{3}L$

$\lambda = \frac{1}{2}L$

the n^{th} harmonic has a wavelength of:

$$\lambda_n = \frac{2L}{n}$$

$$k_n = \frac{2\pi}{\lambda_n} = \frac{\pi n}{L}$$

$$v = \frac{\omega}{k} = \sqrt{\frac{T}{\rho}}$$ wave velocity

Energy of Standing Waves

Given a standing wave of the form:

$$\varphi = A\sin(kx)\cos(\omega t)$$

it's derivatives are:

$$\frac{\partial \varphi}{\partial t} = -\omega A \sin(kx)\sin(\omega t)$$

$$\frac{\partial \varphi}{\partial x} = kA\cos(kx)\cos(\omega t)$$

in Lecture 10, we found equations for PE & KE for a travelling wave:

$$KE = \frac{1}{2}\rho_0\left(\frac{\partial\psi}{\partial t}\right)^2 \qquad PE = \frac{1}{2}\rho_0 v^2\left(\frac{\partial\psi}{\partial x}\right)^2$$

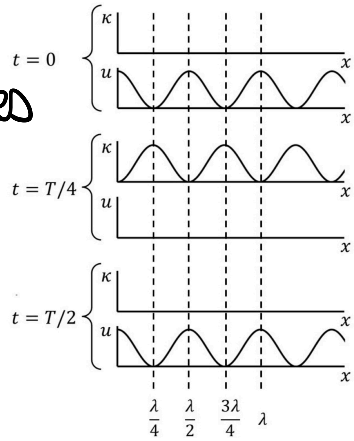
$$KE = \frac{1}{2}\rho_0\left[-\omega A\sin(kx)\sin(\omega t)\right]^2 \qquad PE = \frac{1}{2}\rho_0 v^2\left[kA\cos(kx)\cos(\omega t)\right]^2$$

$$= \frac{1}{2}\rho_0\,\omega^2 A^2\sin^2(kx)\sin^2(\omega t) \qquad = \frac{1}{2}\rho_0 v^2 k^2 A^2\cos^2(kx)\cos^2(\omega t)$$

$$= \frac{1}{2}\rho_0\,\omega^2 A^2\cos^2(kx)\cos^2(\omega t)$$

In general, for a standing wave $KE \neq PE$. This is different to a travelling wave where $PE = KE$. L10

In a standing wave, the energy changes forms from potential to kinetic energy.



Eigenvalue Equation

Given the wave equation $\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$,

we want to find all solutions which are normal modes: All parts of the system will oscillate at the same frequency.

separable, not a func. of $x-vt$

$$\psi(x,t) = \Phi e^{-i\omega t}$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\Phi e^{-i\omega t}$$

$v = \sqrt{\frac{T}{\rho}}$

$$-\omega^2\Phi e^{-i\omega t} = \frac{T}{\rho}e^{-i\omega t}\frac{d^2\Phi}{dx^2}$$

$$-\omega^2 \Phi = \frac{T}{\rho}\frac{d^2\Phi}{dx^2}$$

$$-\frac{T}{\rho}\frac{d^2\Phi}{dx^2} = \omega^2 \Phi$$

This is the eigenvalue equation, it takes the generic form:

$$F(\psi) = \Lambda\psi$$

differentiable operator → F; ψ = eigenfunction; Λ = eigenvalue

Given the boundry conditions of $\Phi(x=0)=0$ $\Phi(x=L)=0$, lets try a trail solution of $\Phi = A\sin(kx)$

$$\frac{\partial^2\Phi}{\partial x^2} = -k^2 A\sin(kx) = -k^2\Phi$$

∴ does satisfy the eigenvalue equation, but what is k?

$$-\omega^2 \cancel{A\sin(kx)} = -\frac{T}{\rho}k^2\cancel{A\sin(kx)}$$

$$\omega^2 = \frac{T}{\rho}k^2$$

$$\omega = vk$$

$A\sin(kx)$ is a solution if $\omega = vk$ is true.

Applying the boundry conditions,

$$A\sin(kL) = 0 \implies kL = n\pi \implies k_n = \frac{n\pi}{L}$$

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$$