Functions 15

L'Hopital Revisited

Earlier we considered the limits of:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \quad \text{with } f(x_0) = 0 \ \& \ g(x_0) = 0$$

Using L'Höpital's rule

What is our justification?

If we can assume that $f(x)$ and $g(x)$ each have a Taylor series expansion in the neighbourhood of x_0, we can write the above as...

VITAL

$$\lim_{h \to 0} \frac{f(x_0+h)}{g(x_0+h)}$$

$$\equiv \lim_{h \to 0} \left[\frac{\cancel{f(x_0)} + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots}{\cancel{g(x_0)} + hg'(x_0) + \frac{h^2}{2!}g''(x_0) + \dots} \right]$$

if $f(x_0) = 0 = g(x_0)$

$$= \lim_{h \to 0} \left[\frac{f'(x_0) + \frac{h}{2!}f''(x_0) + \dots}{g'(x_0) + \frac{h}{2!}g''(x_0) + \dots} \right]$$

$$= \frac{f'(x_0)}{g'(x_0)}$$ if at least one of the numerator or denominator is non-zero.

If $f'(x_0) = 0 = g'(x_0)$, then we go to the next terms.

Evidently if $f(x_0)/g(x_0)$ is of the form '∞/∞', this justification fails.

L'Hôpital Warnings

There are occasions L'Hôpital fails - see 15.3.

Double Taylor Series

We now consider a function $u(x,y)$ of two variables (independent) x,y in the neighbourhood of (x_0,y_0). That is we seek an expansion in powers of $h=x-x_0$, $k=y-y_0$.

$$u(x,y) = u(x_0+h, y_0+k)$$

x first.

$$= u(x_0,y_0+k) + h\frac{\partial u}{\partial x}(x_0,y_0+k) + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2}(x_0,y_0+k) + \dots$$

now y

$$u(x_0,y_0) + k\frac{\partial u}{\partial y}(x_0,y_0) + \frac{k^2}{2!}\frac{\partial^2 u}{\partial y^2}(x_0,y_0)$$

$$\frac{h^2}{2!}\left[\frac{\partial^2 u}{\partial x^2}(x_0,y_0) + k\frac{\partial^3 u}{\partial y \partial x^2}(x_0,y_0) + \dots\right]$$

$$h\left[\frac{\partial u}{\partial x}(x_0,y_0) + k\frac{\partial^2 u}{\partial y \partial x}(x_0,y_0) + \dots\right]$$

u_0 = around the point x_0,y_0.

$$= \underbrace{u_0}_{\text{zero order}} + \underbrace{\left[h\left(\frac{\partial u}{\partial x}\right)_0 + k\left(\frac{\partial u}{\partial y}\right)_0\right]}_{\text{first order}} + \underbrace{\frac{1}{2!}\left[h^2\left(\frac{\partial^2 u}{\partial x^2}\right)_0 + 2hk\left(\frac{\partial^2 u}{\partial x \partial y}\right)_0 + k^2\left(\frac{\partial^2 u}{\partial y^2}\right)_0\right]}_{\text{second order}}$$

There is a straight forward pattern to these terms.

We can write $D \equiv h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$, where D is a differential opperator.

$$\Rightarrow U(x_0+h, y_0+k) = U_0 + DU_0 + \frac{D^2}{2!}U_0 + \frac{D^3}{3!}U_0 + \dots$$

<u>Example</u> $U(x,y) = e^{2x-y}$ $\quad x_0 = 0 = y_0$

$U_0 = U(0,0) = 1$

$$\frac{\partial U}{\partial x} = 2e^{2x-y} \qquad \frac{\partial U}{\partial y} = -e^{2x-y}$$

$$\frac{\partial^2 U}{\partial x^2} = 4e^{2x-y} \quad \frac{\partial^2 U}{\partial y^2} = e^{2x-y} \quad \frac{\partial^2 U}{\partial y \partial x} = -2e^{2x-y}$$

$$\left(\frac{\partial U}{\partial x}\right)_0 = 2 \quad \left(\frac{\partial U}{\partial y}\right)_0 = -1 \quad \left(\frac{\partial^2 U}{\partial x^2}\right) = 4 \quad \left(\frac{\partial^2 U}{\partial y^2}\right) = 1 \quad \left(\frac{\partial^2 U}{\partial y \partial x}\right)_0 = -2$$

$$e^{2x-y} = e^{2h-k} = 1 + [2h-k] + \frac{1}{2!}[4h^2 - 4hk + k^2] + \dots$$