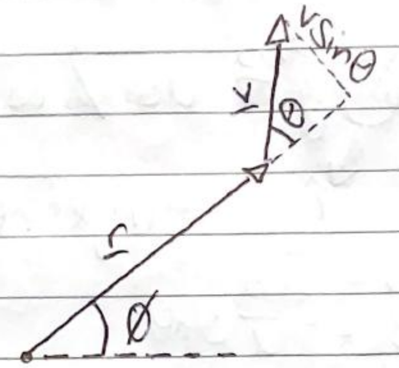
Classical Mechanics 1&4

Angular Momentum & Torque

$$\boxed{\underline{L} \triangleq \underline{r} \times \underline{p} = \underline{r} \times m\underline{v}}$$

angular momentum, position vector, momentum vector, velocity vector



☆ $|\underline{L}| = mrv\sin\theta = mrv_\phi$
v_ϕ is angular component of v.

☆ $mv_\phi r$ reduces to mvr for circular motion

☆ direction from r.h.r.

Rate of Change of Angular Momentum

$$\frac{d\underline{L}}{dt} = \frac{d}{dt}(\underline{r} \times \underline{p}) = \underbrace{\left(\frac{d\underline{r}}{dt} \times \underline{p}\right)}_{=0} + \left(\underline{r} \times \frac{d\underline{p}}{dt}\right) = \underline{r} \times \underline{F}$$

$=0$ as $\underline{v}$ parallel to $\underline{p}$

The torque (aka 'couple' or 'moment') $\underline{G}$ is defined by:

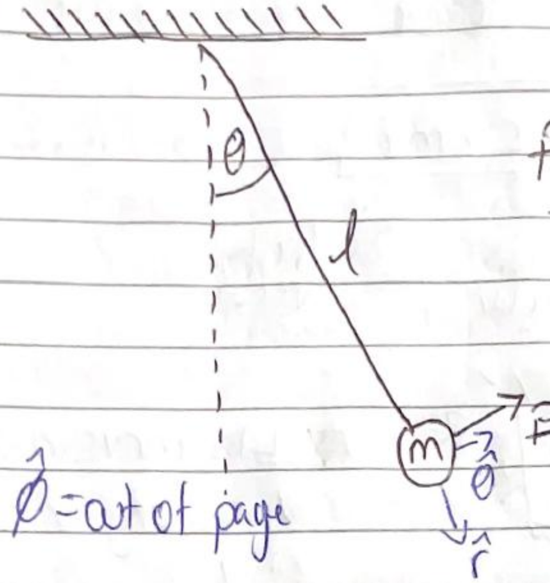
$$\boxed{G \triangleq r \times F}$$

$$\boxed{G = \frac{dL}{dt}}$$ analogy to force = rate of change of momentum.

Torque

The direction of $\underline{G}$ is along axis about which $\underline{F}$ is trying to cause rotation.

Example simple pendulum



$\hat{\phi}$ = out of page

$$\underline{p} = ml\dot{\theta}\hat{\theta}$$

$$\underline{L} = \underline{r} \times \underline{p} = ml^2\dot{\theta}\hat{\phi}$$

$$\underline{G} = \underline{r} \times \underline{F} = -mgl\sin\theta\,\hat{\phi}$$

as $\hat{\phi}$ is independent of t

$$-mgl\sin\theta\,\hat{\phi} = ml^2\ddot{\theta}\hat{\phi}$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Central Forces

central forces do not cause rotation and do not produce torque.

Angular Momentum is conserved in a central force field

Notes:

☆ $\underline{L} = \underline{r} \times m\underline{v}$ is perpendicular to the plane containing $\underline{r}$ and $\underline{v}$. If $\underline{L}$ is conserved, so is the orientation of that plane.

Motion in a central force field is always in a plane.

☆ A force field that is central about one origin is not central about other origins.

Angular Momentum for Multiple Particles

We cannot find a global origin for which $\underline{r}_j \times \underline{F}_{i on j} = 0$ for all inter-particle forces simultaneously, so how can angular momentum be conserved

Lets look at a 2-body example

$$\dot{\underline{p}}_1 = m_1 \ddot{\underline{r}}_1 = F_1^{ext} + F_{2on1}$$
$$\dot{\underline{p}}_2 = m_2 \ddot{\underline{r}}_2 = F_2^{ext} + F_{1on2}$$

where, $F_{1on2} = F_{12}(r_{12})\,\hat{r}_{12} = -F_{2on1}$

$$\underline{r}_{12} \triangleq \underline{r}_2 - \underline{r}_1$$

Total angular momentum is:

$$\underline{L} = (\underline{r}_1 \times \underline{p}_1) + (\underline{r}_2 \times \underline{p}_2)$$

differentiate. $$\frac{dL}{dt} = \cancel{(\dot{\underline{r}}_1 \times \underline{p}_1)} + (\underline{r}_1 \times \dot{\underline{p}}_1) + \cancel{(\dot{\underline{r}}_2 \times \underline{p}_2)} + (\underline{r}_2 \times \dot{\underline{p}}_2)$$

$$= \underline{r}_1 \times (F_1^{ext} + F_{2on1}) + \underline{r}_2 \times (F_2^{ext} + F_{2on1})$$
$$= (\underline{r}_1 \times \underline{F}_1^{ext}) + (\underline{r}_2 \times \underline{F}_2^{ext}) + (\underline{r}_1 \times F_{2on1}) + (\underline{r}_2 \times \underline{F}_{1on2})$$
$$= \underline{G}^{ext} + (r_2 - r_1) \times F_{1on2}$$

$F_{1on2} \parallel r_2 - r_1 \quad \therefore (r_2 - r_1) \times F_{1on2} = 0$

$$\boxed{\frac{dL}{dt} = G^{ext}}$$

$$\boxed{G^{ext} = \sum_i \underline{r}_i \times F_i^{ext}}$$