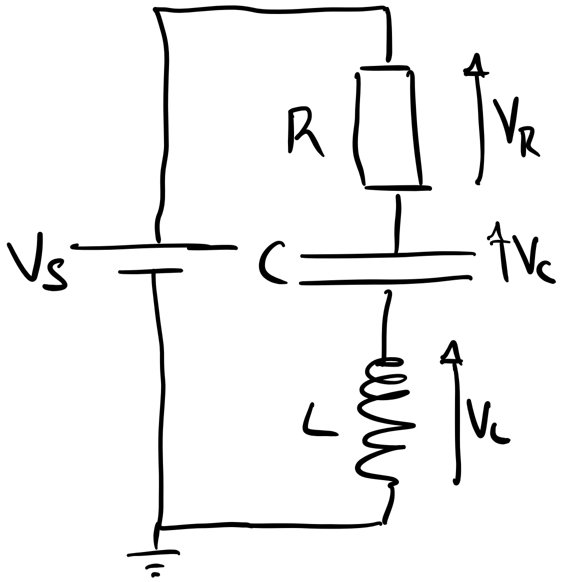
Damped Harmonic Oscillator

$$V_S = \begin{cases} \mathcal{E}, & t<0 \\ 0, & t \geq 0 \end{cases}$$



for $t \geq 0$:

$$0 = V_R + V_C + V_L$$
$$= iR + q/C + L\frac{di}{dt}$$
$$= q/C + R\frac{dq}{dt} + L\frac{d^2q}{dt^2}$$

$$0 = \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q \qquad \gamma = \frac{R}{L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$
$$= \ddot{q} + \gamma\dot{q} + \omega_0^2 q$$

Our initial conditions are: $V_R = V_L = 0$, $V_C = \mathcal{E}$.
The form of our solution depends upon the sign of $\gamma^2 - 4\omega_0^2$.

$\omega_0 > \frac{\gamma}{2}$	$\omega_0 = \frac{\gamma}{2}$	$\omega_0 < \frac{\gamma}{2}$
Δ underdamped	Δ critically damped	Δ overdamped
Δ exponentially decaying oscillation	Δ no oscillation	Δ exponential

We'll mostly explore the underdamped solution but also the critically damped one too.

Under-Damped Solution $\left(\omega_0 > \frac{\gamma}{2}\right)$

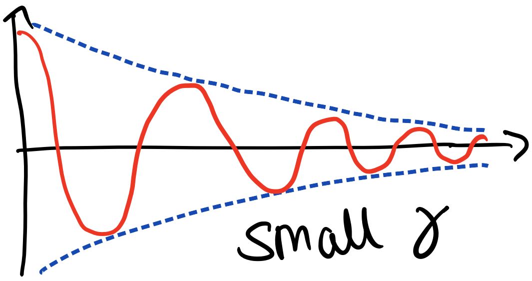
From complex analysis, we get the solution:

$$q = \frac{q_0 \omega_0}{\omega_d} e^{-\frac{\gamma t}{2}} \cos(\omega_d t + \phi)$$

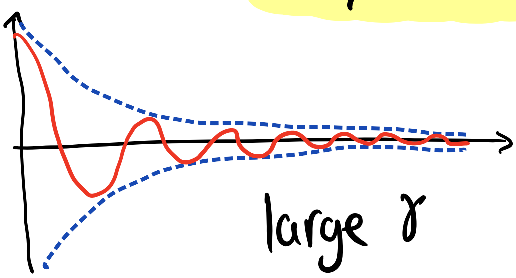
q_0 = initial charge on capacitor

ω_d = damping frequency

$$q_0 = C\mathcal{E}$$
$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}$$
$$\tan\phi = -\gamma/2\omega_d$$



small γ



large γ

power
$$\begin{cases} P_R = Vi = \dot{q}Ri = \dot{q}^2 R \\ P_L = Vi = L\ddot{q}i = L\dot{q}\ddot{q} \\ P_C = Vi = q/C\, i = \frac{q\dot{q}}{C} \end{cases}$$

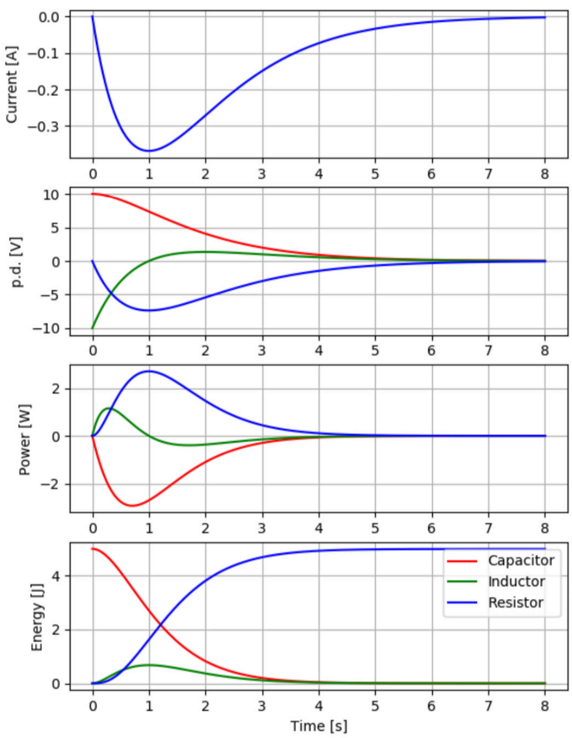
$(P = I^2 R)$

$\left(P_L = \frac{d}{dt}\left(\frac{LI^2}{2}\right)\right)$

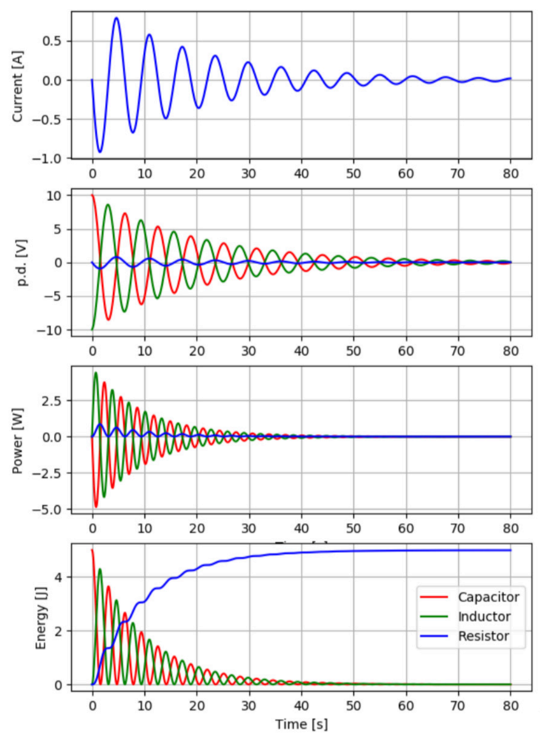
$\left(P_C = \frac{d}{dt}\left(\frac{Q^2}{2C}\right)\right)$ $\left(\frac{CV^2}{2} = \frac{Q^2}{2C}\right)$

Critically Damped Solution

$$q = q_0\left(1 + \frac{\gamma t}{2}\right)e^{-\frac{\gamma t}{2}}$$



under damped



critically damped