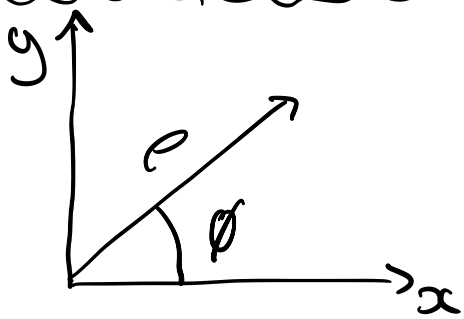
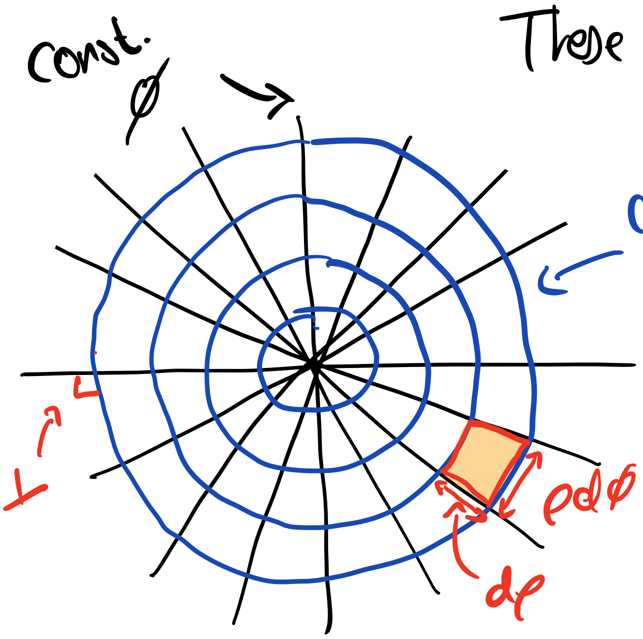
Plane-polar coordinates

Useful for problems with circular symmetry.



$$x(\rho, \phi) = \rho\cos\phi \qquad \rho(x,y) = \sqrt{x^2+y^2}$$
$$y(\rho, \phi) = \rho\sin\phi \qquad \phi(x,y) = \arctan(y/x)$$

These coordinates are orthogonal!



$A = dx\,dy = \rho\,d\rho\,d\phi$

this suggests that $J = \rho$

Analytic Proof

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix} = \rho(\cos^2\phi + \sin^2\phi) = \rho. \quad \square$$

Example

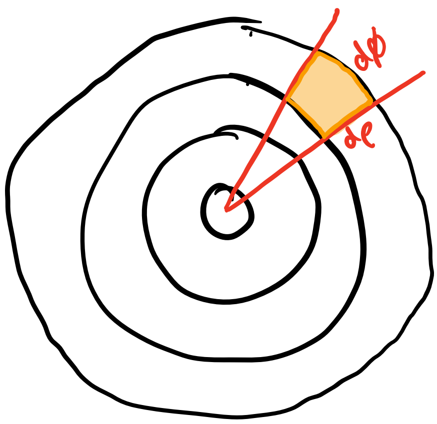
A disk of radius R has surface mass density $f(x,y) = B/\sqrt{x^2+y^2}$. What is the mass of the disk?

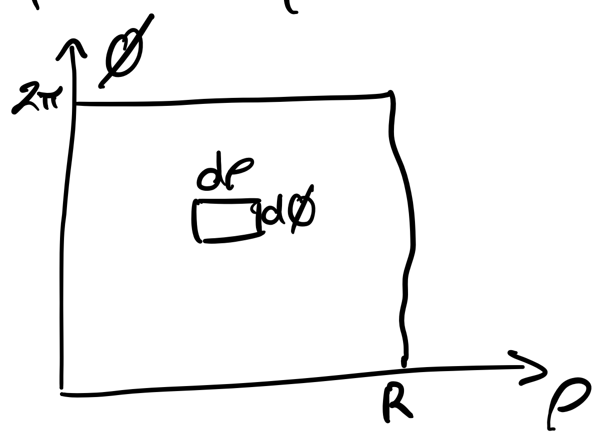
1) integrand
2) limits

$f(x,y) \to f(\rho, \theta) = B/\rho$

R_{xy} is a circle.

$R_{\phi\rho}$ is a square



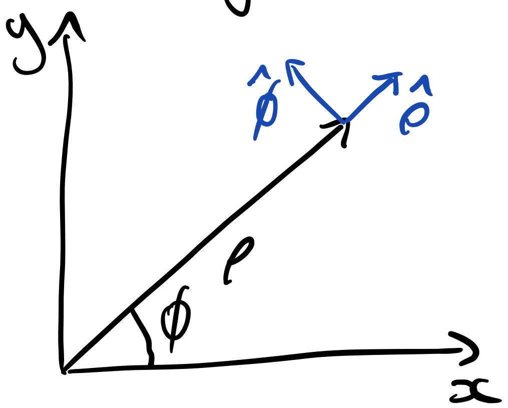


3) $J = \rho$

$$I = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{R} \frac{B}{\rho}\,\rho\, d\rho\, d\phi = \int_{\phi=0}^{2\pi} [B\rho]_0^R\, d\phi = \int_{\phi=0}^{2\pi} BR\, d\phi = 2\pi BR$$

Unit Vectors in Plane-Polar Coordinates

Differentiating vector fields in plane polar coordinates is more difficult. We define radial $\hat{\rho}$ and tangential $\hat{\phi}$ directions (depend on ϕ).



$$\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\boxed{\begin{array}{ll} \dfrac{\partial\hat{\rho}}{\partial\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} = \hat{\phi} & \dfrac{\partial\hat{\rho}}{\partial\rho} = 0 \\ \dfrac{\partial\hat{\phi}}{\partial\phi} = -\cos\phi\,\hat{i} - \sin\phi\,\hat{j} = -\hat{\rho} & \dfrac{\partial\hat{\phi}}{\partial\rho} = 0 \end{array}}$$

LEARN THESE EQUATIONS!

Differentiating Vector Fields in Plane-Polar Coordinates

Generally, we write a vector field as

$$\underline{A} = A_\rho(\rho,\phi)\,\hat{\rho}(\phi) + A_\phi(\rho,\phi)\,\hat{\phi}(\phi)$$

$$\frac{\partial \underline{A}}{\partial\phi} = \frac{\partial}{\partial\phi}(A_\rho\hat{\rho}) + \frac{\partial}{\partial\phi}(A_\phi\hat{\phi})$$

$$= \frac{\partial A_\rho}{\partial\phi}\hat{\rho} + A_\rho\underline{\underline{\frac{\partial\hat{\rho}}{\partial\phi}}} + \frac{\partial A_\phi}{\partial\phi}\hat{\phi} + A_\phi\underline{\underline{\frac{\partial\hat{\phi}}{\partial\phi}}}$$

Given the position vector $\underline{r} = \rho\hat{\rho}$ (scalar: ρ), we can derive the Jacobian for plane polar.

$$d\underline{r} = \underbrace{\frac{\partial \underline{r}}{\partial \rho}}\, d\rho + \underbrace{\frac{\partial \underline{r}}{\partial \phi}}\, d\phi$$

Jacobian is cross product of these vectors

$$J = \frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi}$$

$$\frac{\partial \underline{r}}{\partial \rho} = \frac{\partial}{\partial \rho}(\rho\hat{\rho}) = \underbrace{\frac{\partial \rho}{\partial \rho}}_{=1}\hat{\rho} + \underbrace{\frac{\partial \hat{\rho}}{\partial \rho}}_{=0}\rho = \hat{\rho}$$

$$\frac{\partial \underline{r}}{\partial \phi} = \frac{\partial}{\partial \phi}(\rho\hat{\rho}) = \underbrace{\frac{\partial \rho}{\partial \phi}}_{=0}\hat{\rho} + \underbrace{\frac{\partial \hat{\rho}}{\partial \phi}}_{=\hat{\phi}}\rho = \rho\hat{\phi}$$

$$\frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{k} \\ 1 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix}$$

Orthogonal so $J = \rho$.

$$= \rho\hat{k} \qquad \therefore J = \rho$$