Total Differential

for scalar fields

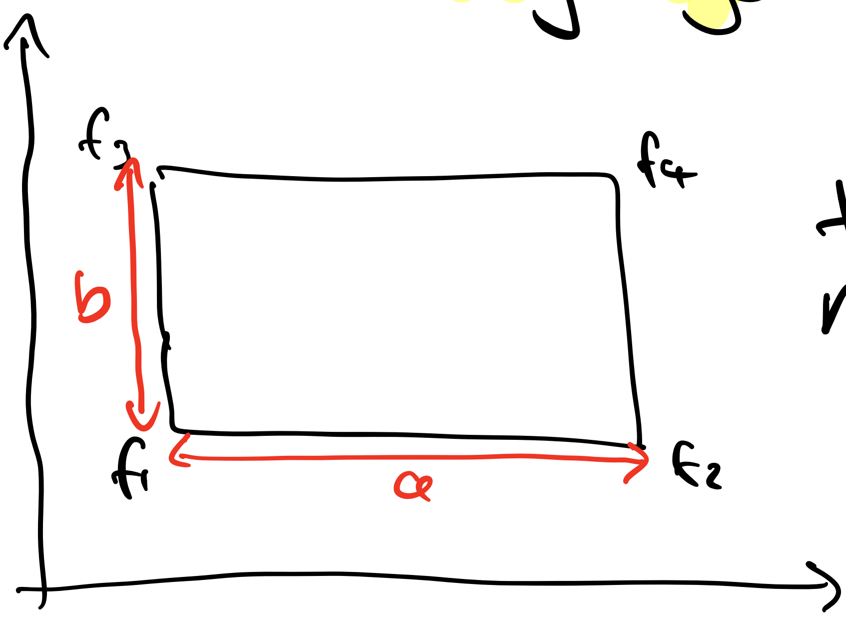
The total differential is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

it's the 'tangent-plane' approximation to Δf. It can also be viewed as the change in f for an infintesimal change dx, dy. We don't require x & y to be orthogonal, only independent.

Clairault's Theorem

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$



this is a non-rigourous proof

$$\frac{\partial f}{\partial x} \simeq \frac{f_2 - f_1}{a} \qquad \frac{\partial f}{\partial y} \simeq \frac{f_3 - f_1}{b}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\frac{(f_4 - f_3)}{a} - \frac{(f_2 - f_1)}{a}}{b} = \frac{f_1 + f_4 - (f_2 + f_3)}{ab}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\frac{(f_4 - f_2)}{a} - \frac{(f_3 - f_1)}{a}}{b} = \frac{f_1 + f_4 - (f_2 + f_3)}{ab}$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Partial Diff. Vector Fields

A 2D vector field defines a vector at every x & y coordinate. Eg. velocity.

$$\underline{A}(x,y) = \underbrace{A_x(x,y)}_{\text{Scaler field}}\hat{\imath} + \underbrace{A_y(x,y)}_{\text{Scaler field}}\hat{\jmath}$$

Example $\underline{A} = (xy)\hat{\imath} + (x^2y^2)\hat{\jmath}$

this is a 2D vector field

$$\frac{\partial \underline{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\underline{A}(x+\Delta x, y) - \underline{A}(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{A_x(x+\Delta x, y)\hat{i} + A_y(x+\Delta x, y)\hat{j} - \underline{A}(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[A_x(x+\Delta x, y) - A_x(x,y)]\hat{i} + [A_y(x+\Delta x, y) - A_y(x,y)]\hat{j}}{\Delta x}$$

$$= \boxed{\frac{\partial A_x}{\partial x}\hat{i} + \frac{\partial A_y}{\partial x}\hat{j}}$$

The partial derivative of a vector field is the partial derivative of each component.

The differential is defined as:

$$d\underline{A} = \frac{\partial \underline{A}}{\partial x} dx + \frac{\partial \underline{A}}{\partial y} dy$$

The Exact Differential

A particular differential

$$A(x,y)dx + B(x,y)dy$$

is a total differential of some parent function, f, if:

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \qquad \left(\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \right)$$

This is a sufficient condition. We call this differential <u>exact</u>.