Line Integrals

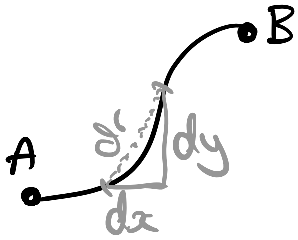
Integral along a path, a curve C, in space. There are different forms.

scaler　　　　　　　most important case

$$\int_C \Omega \, d\underline{r} \qquad \int_C \underline{F} \cdot d\underline{r} \qquad \int_C \underline{F} \times d\underline{r}$$

C ← curve

How To Calculate:



We have a curve $y = y(x)$ and a vector field $\underline{F}(x,y) = F_x(x,y)\hat{i} + F_y(x,y)\hat{j}$

$$d\underline{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\int_A^B \underline{F} \cdot d\underline{r} = \underbrace{\sum_i F_{x_i}(x_i, y_i)\,dx_i}_{①} + \underbrace{\sum_i F_{y_i}(x_i, y_i)\,dy_i}_{②}$$

Using the curve, we can write each term entirely using only x or only y. Same is true in 3D.

We can express ① as:

$$\int_{x_A}^{x_B} F_x(x, y(x))\,dx \quad \text{or} \quad \int_{y_A}^{y_B} F_x(x(y), y)\,\frac{dx}{dy}\,dy$$

We can express ② as:

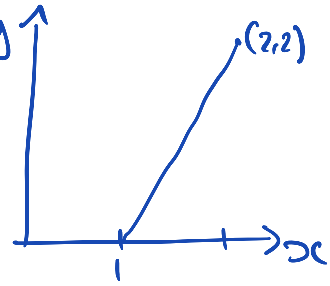
$$\int_{y_A}^{y_B} F_y(x(y), y)\,dy \quad \text{or} \quad \int_{x_A}^{x_B} F_y(x, y(x))\,\frac{dy}{dx}\,dx$$

Four Examples

We will express each integral using only parameter and work out the limits for that parameter.

Example 1: the vector field $\underline{F} = 2xy\hat{i} + x^2\hat{j}$ along the path $y = 2x - 2$ from $(1,0)$ to $(2,2)$.

$$\int_A^B \underline{F}\cdot d\underline{r} = \int_{A_x}^{B_x} F_x\,dx + \int_{A_y}^{B_y} F_y\,dy \qquad F_x = 2xy \quad F_y = x^2$$



$$y(x) = 2x - 2 \qquad x(y) = \frac{y}{2} + 1$$

$$= \int_1^2 2xy\,dx + \int_0^2 x^2\,dy$$

$$= \int_1^2 2x(2x-2)\,dx + \int_0^2 \left(\frac{y}{2}+1\right)^2 dy$$

$$= \int_1^2 4x^2 - 4x\,dx + \int_0^2 \frac{y^2}{4} + y + 1\,dy$$

$$= \left[\frac{4}{3}x^3 - 2x^2\right]_1^2 + \left[\frac{1}{12}y^3 + \frac{1}{2}y^2 + y\right]_0^2$$

$$= \left(\frac{4}{3}(8) - 2(4)\right) - \left(\frac{4}{3} - 2\right) + \left(\frac{1}{12}(8) + 2 + 2\right) - 0$$

$$= \frac{32}{3} - 8 - \frac{4}{3} + 2 + \frac{8}{12} + 4 = \frac{4}{3} + \frac{2}{3} + 6 = 8$$

Example 2: the line may be expressed parametrically, when we cannot write $y = f(x)$.

eg. $\left.\begin{aligned} x(t) &= 2t+1 \\ y(t) &= 4t \end{aligned}\right\}$ from $t=0$ to $t=\frac{1}{2}$

$$dx = 2\,dt \qquad dy = 4\,dt$$

$$\int_C \underline{F}\cdot d\underline{r} = \int_C 2xy\,dx + \int_C x^2\,dy$$

$$= \int_{t=0}^{1/2} 2(2t+1)(4t)\,2\,dt + \int_{t=0}^{1/2} (2t+1)^2\,4\,dt$$

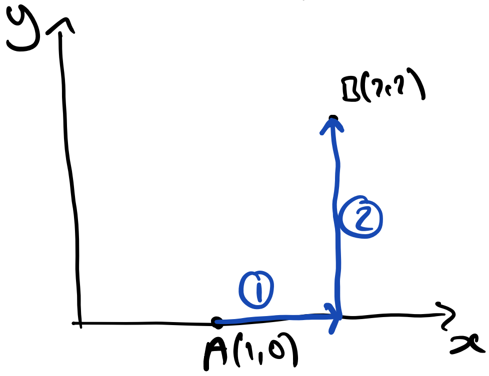
$$= \int_{t=0}^{1/2} 32t^2 + 16t\;dt + \int_{t=0}^{1/2} 16t^2 + 16t + 4\;dt$$

$$= \left[\frac{32}{3}t^3 + 8t^2\right]_0^{1/2} + \left[\frac{16}{3}t^3 + 8t^2 + 4t\right]_0^{1/2}$$

$$= \frac{32}{3}\cdot\frac{1}{8} + \frac{8}{4} + \frac{16}{3}\cdot\frac{1}{8} + \frac{8}{4} + 2$$

$$= \frac{4}{3} + 2 + \frac{2}{3} + 2 + 2 = 8$$

Example 3: same $\underline{F}$, same end points, different path



along 1: $\displaystyle\int \underline{F}\cdot d\underline{r} = \underbrace{\int_C 2xy\,dx}_{=0,\ \text{since } y=0} + \underbrace{\int_C x^2\,dy}_{=0,\ \text{since } dy=0.}$

along 2: $\displaystyle\int \underline{F}\cdot d\underline{r} = \underbrace{\int_C 2xy\,dx}_{=0,\ \text{since } dx=0} + \underbrace{\int_C x^2\,dy}_{\neq 0\,!}$

$$\int_{y=0}^{2} x^2 \, dy = \int_{y=0}^{2} 4 \, dy = 8 \quad \text{(same as before).}$$