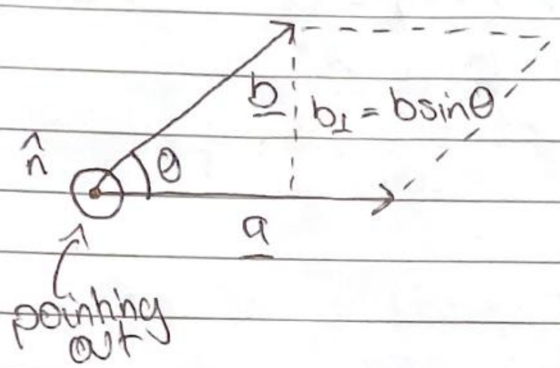
Vectors 4

Cross Product

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\theta \, \hat{n}}$$

normal vector to $\underline{a}$ & $\underline{b}$

'right hand rule'.



$|\underline{a}||\underline{b}|\sin\theta$ is equal to area of the parrale..ogram traced by $\underline{a}$ and $\underline{b}$.

$$\underline{a} \times \underline{b} = A\hat{n} = \underline{A}$$

$$\hat{n} = \frac{(\underline{a} \times \underline{b})}{|a \times b|}$$

Rule 1

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$ ← anticommutative

Rule 2

$$\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$$ ← distributive.

Rule 3

$\hat{i} \times \hat{i} = 0 \quad \hat{j} \times \hat{j} = 0 \quad \hat{k} \times \hat{k} = 0 \qquad (\underline{a} \times \underline{a} = 0)$

$\hat{i} \times \hat{j} = k \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k} \quad k \times \hat{j} = -\hat{i} \quad \hat{i} \times \hat{k} = -\hat{j}$

consider $\underline{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \quad \underline{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\underline{a} \times \underline{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$$

$$= \cancel{a_x b_x (\hat{i} \times \hat{i})} + a_x b_y (\hat{i} \times \hat{j}) + a_x b_z (\hat{i} \times \hat{k}) +$$
$$a_y b_x (\hat{j} \times \hat{i}) + \cancel{a_y b_y (\hat{j} \times \hat{j})} + a_y b_z (\hat{j} \times \hat{k}) +$$
$$a_z b_x (\hat{k} \times \hat{i}) + a_z b_y (\hat{k} \times \hat{j}) + \cancel{a_z b_z (\hat{k} \times \hat{k})}$$

$$= (a_y b_z - a_z b_y)(\hat{j} \times \hat{k}) + (a_z b_x - a_x b_z)(\hat{k} \times \hat{i}) + (a_x b_y - a_y b_x)(\hat{i} \times \hat{j})$$
$$= (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

$$= \hat{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} + \hat{j} \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

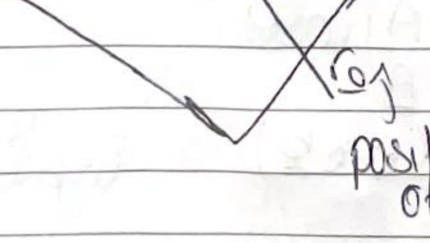
Planes

In $\mathbb{R}^3$, a plane can be defined by 3 points, or by one point and 2 vectors.

~~(0,0)~~

any point on the plane take the following equation:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r_0 + \lambda(r_1 - r_0) + \mu(r_2 - r_0)$$



position of plane

There is an easier way to define a plane.

vector, n, is normal to the plane.

$$n = (a, b, c)$$

$$\underline{r} \cdot \underline{n} = \underline{r_0} \cdot \underline{n} + \cancel{\lambda(r_1 - r_0) \cdot n} + \cancel{\mu(r_2 - r_0) \cdot n} \quad \perp$$

$$\underline{r} \cdot \underline{n} = \underline{r_0} \cdot \underline{n} \qquad \text{eq}^n \text{ of plane}$$

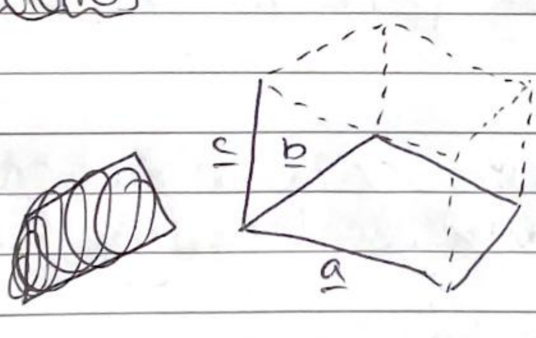
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \text{const} \Leftrightarrow ax + by + cz = \text{const}$$

coefficients of normal vector.

can find n by:

$$n = (r_2 - r_0) \times (r_1 - r_0)$$

Volumes



volume = height x area

height = $c \cos\theta$

angle between vertical & c.

area = $|\underline{A}|$

$$\begin{aligned} \text{volume} &= \underline{c}\,|\underline{A}| \cos\theta \\ &= \underline{c} \cdot \underline{A} \\ &= \underline{c} \cdot (\underline{a} \times \underline{b}) \end{aligned}$$

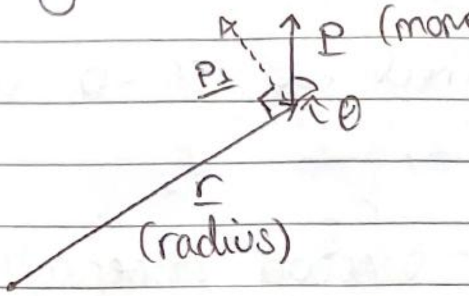
scalar triple product.

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$$ vector triple product

Physics Applications

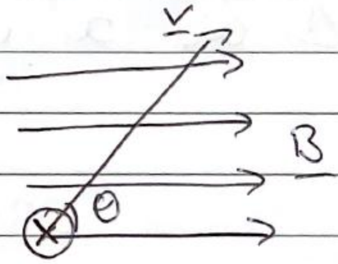
Angular momentum (L)



$$|L| = p_\perp r = p \sin\theta \, r$$

$$\underline{L} = \underline{r} \times \underline{p}$$

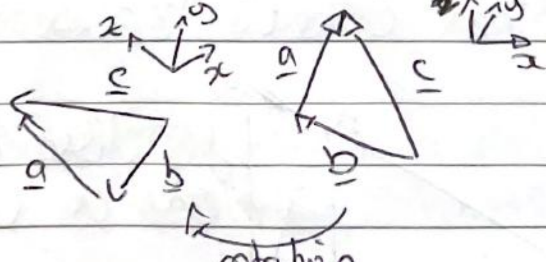
Magnetic Field

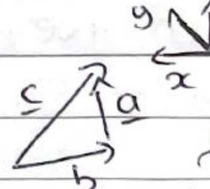


$$\underline{F} = q V_\perp B \hat{n}$$
$$= qVB\sin\theta \hat{n}$$
$$= q\underline{V} \times \underline{B}$$

Pseudovectors

vector $\underline{c}$ is made from $\underline{a}$ & $\underline{b}$





when a new vector is formed via addition, we can mirror translate/rotate the axis ~~vector~~ without changing the vector. This is NOT true for cross product.

$\therefore$ pseudovector.