Fourier Theorems

☆ Shift theorem: $\mathcal{F}[f(x-x_0)] = \mathcal{F}[f(x)]e^{-ik_x x_0}$

☆ Fourier Pairs: $\mathcal{F}\left[\delta\left(x+\frac{d}{2}\right)+\delta\left(x-\frac{d}{2}\right)\right] = 2\cos\left(\frac{k_x d}{2}\right)$

☆ Convolution: $(f * g)(x) = \mathcal{F}[f(x)\cdot g(x)]$

Single Narrow Slit

When the aperture function is a delta function $A(x) = \delta(x)$.
The fourier transform of a delta function is a constant.

$$E(k_x) = \int_{-\infty}^{\infty} \delta(x) e^{-ik_x x}\, dx = 1$$

Two Extended Slits

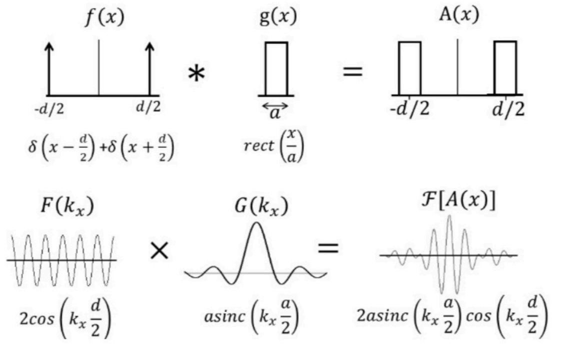
We can express this aperture function by

$$A(x) = \left[\delta\left(x-\frac{d}{2}\right)+\delta\left(x+\frac{d}{2}\right)\right] * \mathrm{rect}\left(\frac{x}{a}\right)$$

Using the convolution theorem to get

$$\mathcal{F}[A(x)] = \mathcal{F}\left[\delta\left(x-\frac{d}{2}\right)+\delta\left(x+\frac{d}{2}\right)\right]\cdot\mathcal{F}\left[\mathrm{rect}\left(\frac{x}{a}\right)\right]$$

$$E_p(\theta) = \mathcal{F}[A(x)] = 2a\,\mathrm{sinc}\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_x d}{2}\right)$$



N-infinitely-narrowed slits

Formed from N equally spaced delta functions.

$$A(x) = \sum_{n=0}^{N-1} \delta(x-nd)$$

We can now use the shift theorem $\delta(x-nd) = \delta(x)e^{-ik_x nd}$

$$\mathcal{F}[A(x)] = \sum_{n=0}^{N-1} \mathcal{F}[\delta(x)]e^{-ik_x nd} = \sum_{n=0}^{N-1} e^{-ik_x nd}$$

To solve this we use properties of geometric series.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\mathcal{F}[A(x)] = \sum_{n=0}^{N-1} e^{-ik_x nd} = \frac{1-e^{-iNk_x d}}{1-e^{-ik_x d}} = e^{\frac{-i(N-1)k_x d}{2}} \cdot \frac{\sin\left(\frac{Nk_x d}{2}\right)}{\sin\left(\frac{k_x d}{2}\right)}$$

Minima occur when $\frac{k_x d}{2} = m\pi \Rightarrow d\sin\theta = m\lambda$. The central peak scales linearly with N.

N extended slits

We simply convolve our N delta functions with our $\text{rect}(\frac{x}{a})$ function. The N narrow slit diffraction pattern is modulated by a single slit pattern.

Diffraction Gratings

When there are infinite number of slits $(N \to \infty)$, we call this a diffraction grating. We call this an infinite comb of $\delta(x)$ is another infinite comb of δ-functions in k_x space.

$$\frac{2\pi}{d}\sum_{m=-\infty}^{\infty}\delta\left(k_x - m\frac{2\pi}{d}\right)$$

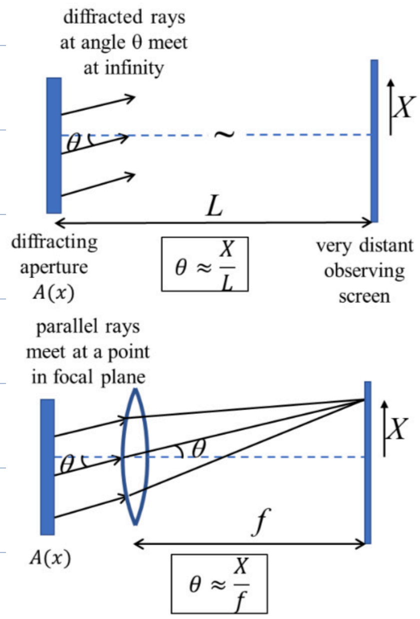
with <u>discrete</u> diffraction orders $k_{md} = m\frac{2\pi}{d}$. In terms of angles, we get the equation

$$d\sin\theta = m\lambda$$

Diffraction orders must be limited to $\theta < 90°$. We often use diffraction gratings to determine wavelength.

Lens as a Fourier Transform

All diffracted rays from the aperture $A(x)$ going at angle θ will be bought to the focal point of the lens at position $X = f\theta$ (small angles).



The lens fourier transform has units $k_x = k\sin\theta \approx k\theta = \frac{kX}{f}$ which gives the integral

$$E(X) = \int_{-\infty}^{\infty} A(x)\, e^{-i\frac{kXx}{f}}\, dx$$