Current Density

The current density is defined as

$$\underline{j} = \sum_{i=1}^{N} n_i q_i \underline{v}_i$$

$\underline{v}_i$ - velocity

n_i - number density

q_i - charge

The current is defined as

$$I = \iint_S \underline{j} \cdot d\underline{S}$$

We must be careful taking current as a scaler, often it is too inflexible in most situations.

Conservation of Charge

If volume V has a charge Q, then I is the current out of the volume

$$-\frac{dQ}{dt} = I$$

We can also say that

$$I = \oiint_{\partial V} \underline{j} \cdot d\underline{S}$$

which leads to

$$\oiint_{\partial V} \underline{j} \cdot d\underline{S} = -\frac{dQ}{dt}$$

This leads to the conservation of charge. Applying the divergence theorem

$$I = \oiint_S \underline{j} \cdot d\underline{S} = \iiint_V \nabla \cdot \underline{j} \, dV$$

Now if we assume that charge is evenly distributed,

$$Q = \iiint_V \rho \, dV$$

ρ - charge density

Then we can say that

$$\frac{\partial}{\partial t} \iiint_V \rho \, dV + \iiint_V \nabla \cdot \underline{j} \, dV = 0$$

$$\frac{\partial \rho}{\partial t} dV + \nabla \cdot \underline{j} \, dV = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = 0$$

This is conservation of charge in differential form.

Ohm's Law

Consider a conducting wire of length l and cross-sectional area A to which we apply an electric field $\underline{E}$. Assuming a steady current density $\underline{j}$ across the wire, we get that

$$\underline{E} = \eta \underline{j}$$ η - resistivity

We define conductivity σ is defined as $\frac{1}{\eta}$, thus we can also write

$$\underline{j} = \sigma \underline{E}$$

This is <u>Ohm's Law</u>. If we now integrate along the length of the wire $d\ell$.

$$\int \underline{E} \cdot d\ell = \eta \int \underline{j} \cdot d\ell$$

The LHS is the potential difference, for the RHS we note that $I = jA$ so

$$V = \frac{I \eta \ell}{A}$$

We now define resistance $R = \frac{\eta \ell}{A}$ which gives

$$V = IR$$

Joule Heating

In a metallic wire we can treat the electrons as moving & the ions at rest. The electrons are accelerated by the electric field $\underline{E}$.

Collisions with the ions cause the electrons to lose energy. $\therefore$ they move at an average velocity of $\underline{v_e}$. The work done will be $\underline{F} \cdot \underline{v_e}$, $\therefore$ the power per unit volume:

$$\underline{F} \cdot \underline{v_e} = n(-e\underline{E})\underline{v_e} = \underline{j} \cdot \underline{E} = \eta \underline{j}^2$$

resistivity

where n is the no. of electrons per unit volume (aka the number density). This gives the heating rate per unit volume (Wm^{-3}). For a conductor of length l and area A we get total heating rate

$$P = (\eta j^2) l A = \eta \left(\frac{I}{A}\right)^2 A l = \frac{\eta l}{A} I^2 = RI^2$$