Fermat's Principle

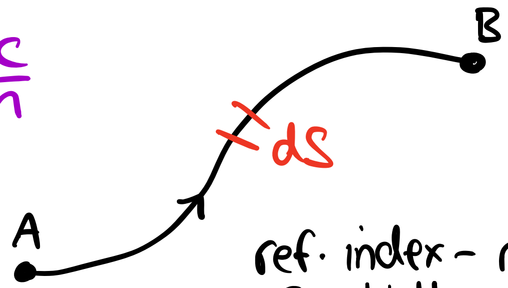
The original 1662 edition: 'The path taken by a ray between 2 points is the one traversed in least time'.

The modern version: 'The path taken by a ray between 2 points is <u>stationary</u> with respect to small variations of the path'.

Ray-time Path Integral

$v = \frac{c}{n}$

$$\frac{ds}{dt} = \frac{c}{n(s)} \Rightarrow dt = \frac{n(s)\,ds}{c}$$



ref. index - $n(s)$ - spatially varying

$$T = \int_A^B dt = \int_A^B \frac{n(s)\,ds}{c}$$

$$T = \frac{1}{c}\int_A^B n(s)\,ds$$

The modern version uses the calculus of variations,

$$\frac{dT}{ds} = 0$$

(the change in T with a small change in path is zero.)

Optical Path Length

$$OPL = \int_A^B n(s)\,ds$$

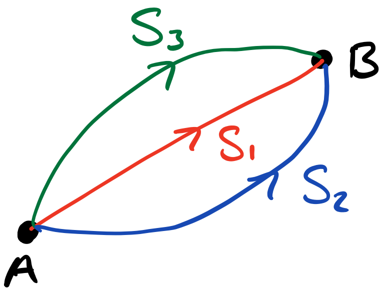
$$\therefore \quad T = \frac{OPL}{c}$$

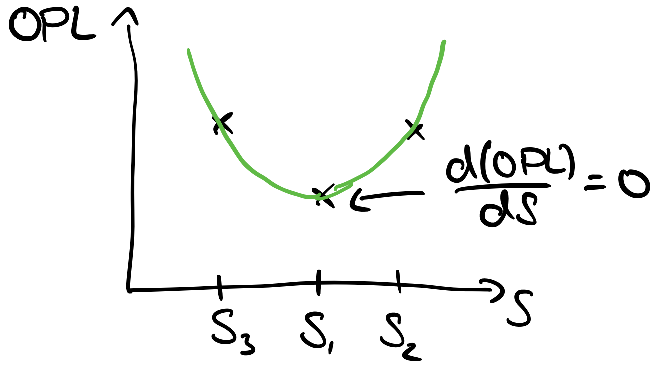
We can state FP as

$$\frac{d(OPL)}{ds} = 0$$

Homogeneous Medium

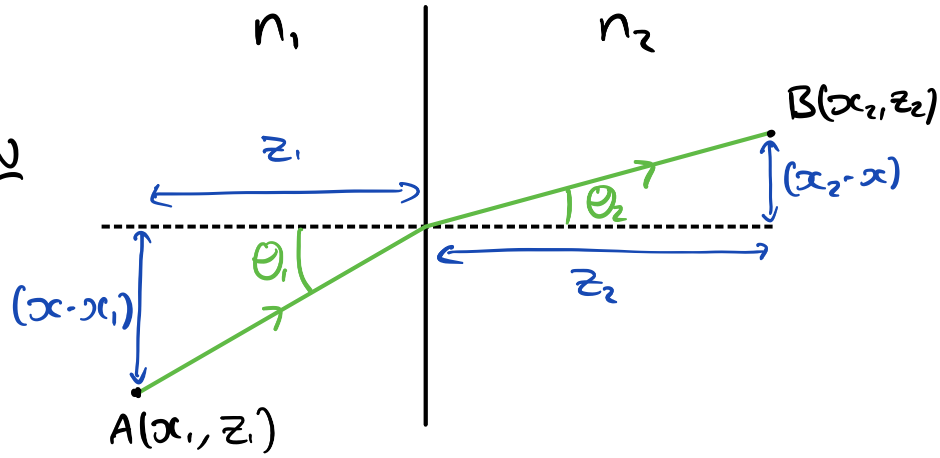
In a homogeneous equation $n(s) = \text{const.} = n$, $\therefore OPL = n \cdot L$.





In a homogeneous medium, the light travels in a straight line.

Deriving Snell's Law



$$OPL = n_1 \sqrt{z_1^2 + (x - x_1)^2} + n_2 \sqrt{z_2^2 + (x_2 - x)}$$

$$\frac{d(OPL)}{dx} = n_1 \frac{(x - x_1)}{\sqrt{z_1^2 + (x - x_1)^2}} - n_2 \frac{(x_2 - x)}{\sqrt{z_2^2 + (x_2 - x)^2}}$$

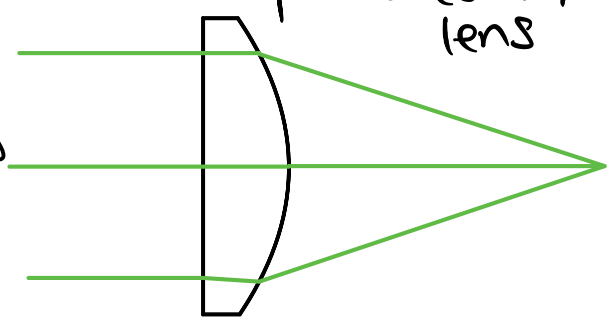
$$= n_1 \sin\theta_1 - n_2 \sin\theta_2 = 0$$

$$\therefore n_1 \sin\theta_1 = n_2 \sin\theta_2$$

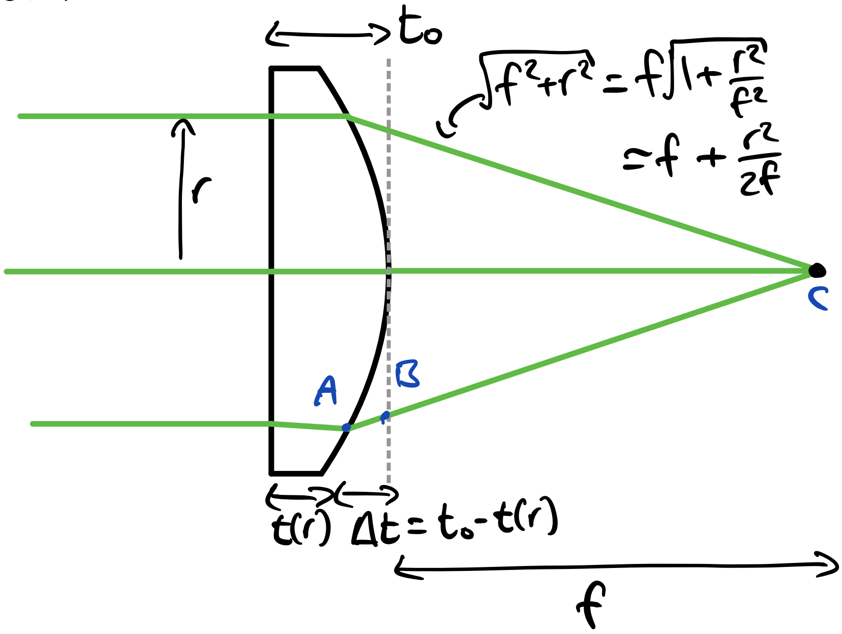
Lens Imaging

Imaging means bringing all the ray paths from the object to an image point. But, FP states ray path between 2 points is one with shortest time! How!

$\frac{d(OPL)_n}{ds} = 0$ can be satisfied if $(OPL)_n = \text{const}$. $\therefore$ we make all paths have the same OPL to get an image to form.



the shortest direct path has the most glass whereas the least direct has less glass.



Derivation of Thin Lens

Assumptions:
- thin lens ($\Delta t \ll f$)
- Paraxial ($r \ll f$)
- Axial object
- $AC \approx BC$

Now we can use fermat's theorem and state that

$$OPL(0) = OPL(r)$$

$$nt(0) + f = nt(r) + [t(0) - t(r)] + f + f^2/2r$$

$$n[t(0) - t(r)] - [t(0) - t(r)] = f^2/2r$$

$$(n-1)[t(0) - t(r)] = f^2/2r$$

$$t(r) = t(0) - \frac{r^2}{2(n-1)f}$$

$$t(r) = t(0) - Cr^2$$

I) the lens thickness is parabolic (r^2)

II) the curvature $C = 1/2(n-1)f$ depends upon f & n, more bulged for small f & n.

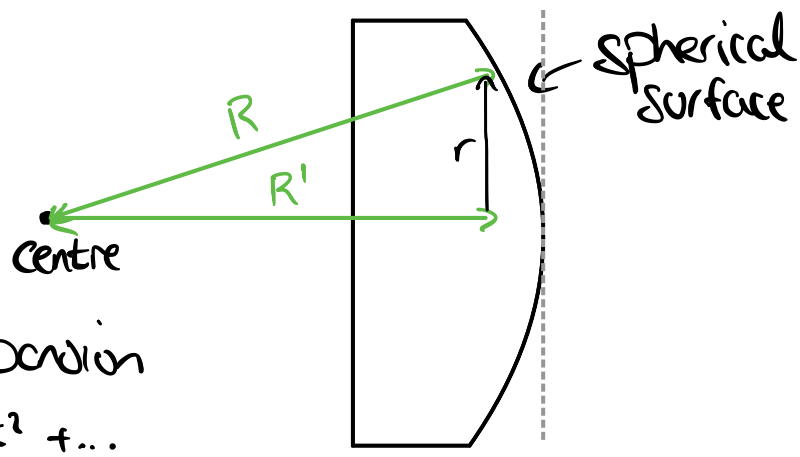
The lens shape is given by $\Delta t = t(0) - t(r)$

$$\Delta t(r) = t(0) - t(r) = \frac{r^2}{2(n-1)f}$$

Spherical Lens

It is often cheaper & quicker to produce spherical lenses than parabolic lenses.

$$R' = \sqrt{R^2 - r^2} = R\sqrt{1 - \left(\frac{r}{R}\right)^2}$$



now we use the taylor expansion

$$(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \dots$$

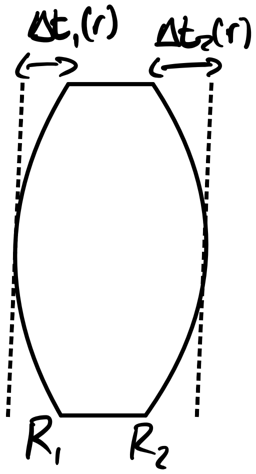
$$R' = R - \frac{r^2}{2R} - \frac{r^4}{8R^3} + \dots$$

$$\Delta t(r) = R - R' = \underbrace{\frac{r^2}{2R}}_{\text{paraxial parabolic}} + \underbrace{\frac{r^4}{8R^3} + \dots}_{\text{higher order 'spherical abberations'}}$$

The focal length for a spherical lens is

$$f = \frac{R}{(n-1)}$$

For a generalised spherical lens, Δt is



$$\Delta t(r) = \Delta t_1(r) + \Delta t_2(r) = \frac{r^2}{2R_1} + \frac{r^2}{2R_2} = \frac{r^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Lens Maker's Formula

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

By convention when R is positive its convex, when negative its concave.

Len's Shapes