Sinusoidal Waves

The simplist periodic wave is $\psi(x,t) = A\cos\phi$.

$$\phi = \text{phase} = k(x-vt)$$

dimensionless   wavenumber

$$\psi(x,t) = A\cos(kx - \underbrace{kvt}_{=\omega})$$

$$\omega = \frac{2\pi}{T} \qquad k = \frac{2\pi}{\lambda}$$

"temporal frequency"   "spacial frequency"

$$\psi(x,t) = A\cos(kx - \omega t + \phi) = Re\{\tilde{\psi}(x,t)\}$$

where $\tilde{\psi}(x,t) = \tilde{A}e^{i(kx-\omega t)} = Ae^{i\phi}e^{i(kx-\omega t)}$

if $k>0$, wave travels in +ve x direction.

Superposition of Waves

Given that $f(x-vt)$ and $g(x+vt)$ are solutions to the wave equation. Show that the superposition is also a solution.

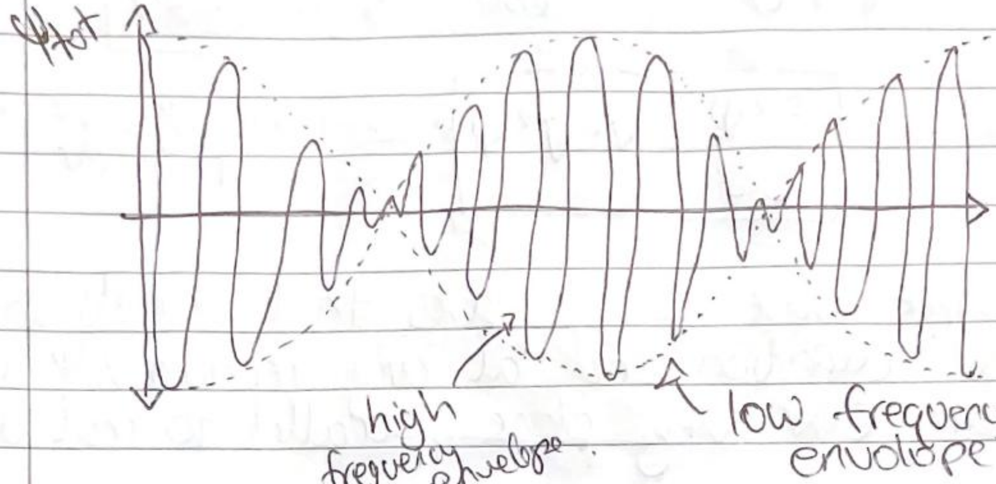
$$\psi_{tot} = A\cos(k_1x - \omega_1 t) + A\cos(k_2x - \omega_2 t)$$

$$= 2A\cos\left(\frac{k_1x - \omega_1 t + k_2x - \omega_2 t}{2}\right)\cos\left(\frac{k_1x - \omega_1 t - k_2x + \omega_2 t}{2}\right)$$

lets define $k_h = \frac{1}{2}(k_1+k_2)$, $k_l = \frac{1}{2}(k_1-k_2)$

$$\omega_h = \frac{1}{2}(\omega_1+\omega_2); \quad \omega_l = \frac{1}{2}(\omega_1-\omega_2)$$

$$\Psi_{tot} = 2A\cos(k_n x - \omega_n t)\cos(k_\Delta x - \omega_\Delta t)$$



If this was a sound wave, a person would hear a sound at ~~frequency~~ the average frequency of the two.

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\left[\frac{1}{2}(\omega_1 + \omega_2)\right] = \frac{1}{2}\left[f_1 + f_2\right]$$

The amplitude would be modulated by the low frequency envelope. Phenomenon known as beats.

The beat period is found by

$$f_b = \frac{1}{2\pi} \times 2 \times \frac{1}{2}(\omega_1 - \omega_2) = f_1 - f_2.$$

The frequency of the beats is the average of the input frequencies.

Waves in 3D

In general $\Psi(x, y, z, t)$. The wave equation becomes:

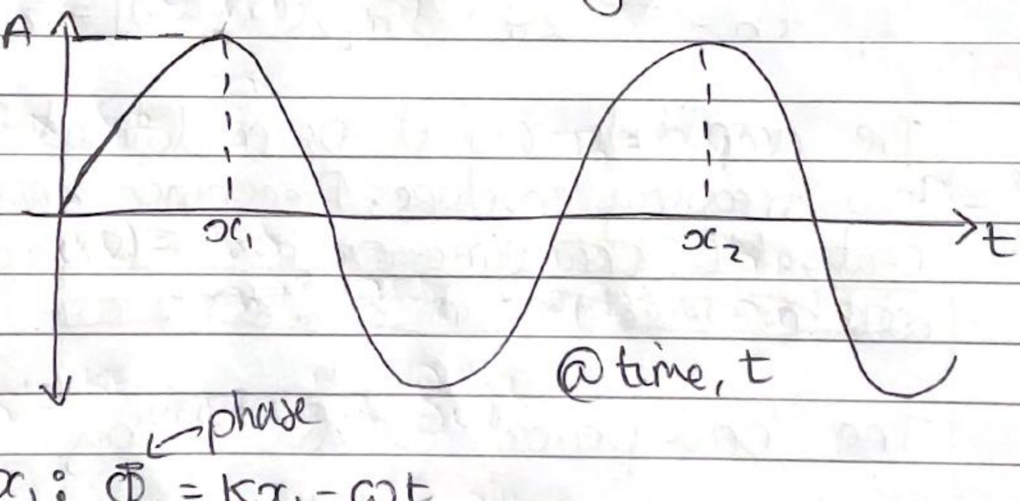
$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = v^2\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right)}$$

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi}$$ ← see vector fields module.

A plane wave is a waves that travels in a fixed direction and at any instant, ψ is uniform over any plane parallel to that direction.

$$\tilde{\psi}(x,y,z,t) = \tilde{A} e^{i(kx - \omega t)}$$

∴, the wave propogates in the x direction and is independent of y & z.



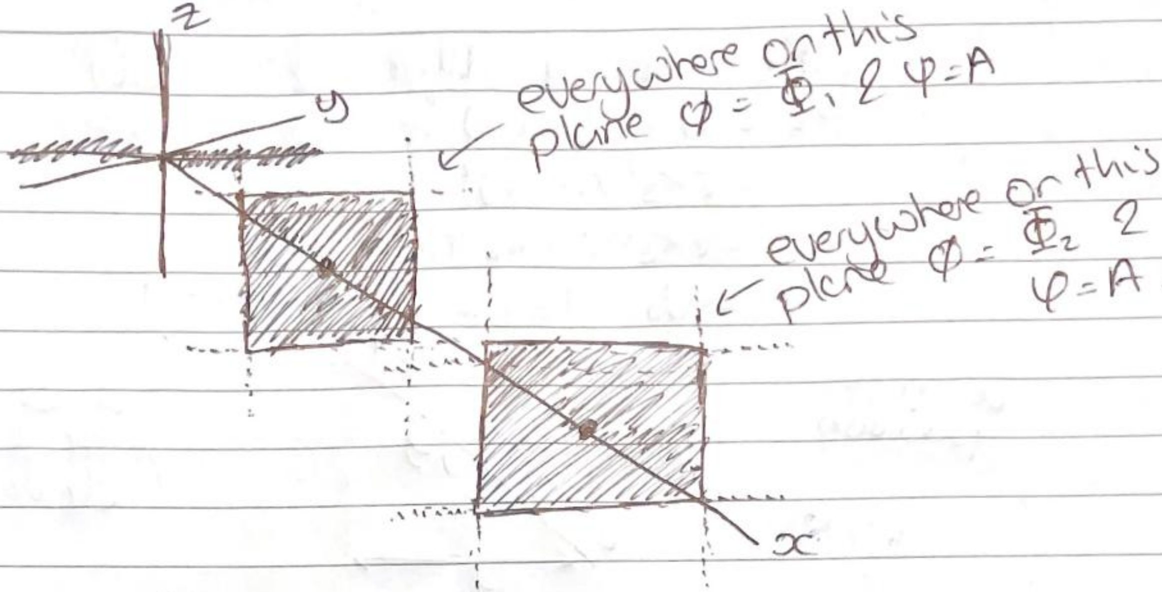
phase

@ x_1: $\Phi_1 = kx_1 - \omega t$

@ x_2: $\Phi_2 = kx_2 - \omega t = \Phi_1 + 2\pi$

Planes of constant phase move in the x direction, known as wavefronts, with speed v.

For a general plane wave, we use the equation



$$\tilde{\psi}(\underline{r},t) = \tilde{A}e^{i(\underline{k}\cdot\underline{r}-\omega t)}$$

$\underline{k}$ is the wavevector.

- ✡ direction of $\underline{k}$ = direction of propogation
- ✡ $|\underline{k}| = 2\pi/\lambda$
- ✡ $\underline{k}\cdot\underline{r} = k_x x + k_y y + k_z z$