Vectors 14

Eigenvalues

We've already met $A\underline{x} = \underline{b}$. Let's consider the special case when $\boxed{A\underline{x} = \lambda \underline{x}}$. This is called the eigenvalue problem. We want to find the values for λ and $\underline{x}$.

If A is a square matrix, then we can write

$$A\underline{x} = \lambda I \underline{x}$$
$$(A - \lambda I)\underline{x} = 0$$

This is a homogenous equation. When the determint is zero we get non-trivial solutions.

$$\det(A - \lambda I) = 0 = p(\lambda)$$

To find the eigenvalues λ_i, we solve for $p(\lambda) = 0$ and the corresponding eigenvector $\underline{x}_i$ can be found by substituting values of λ into homogenous eq^n.

Example $\quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$p(\lambda) = \left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 - 1 \quad = 4 - 4\lambda + \lambda^2 - 1 = \lambda^2 - 4\lambda + 3$$

$\lambda = 1, 3.$

N.B. up to 2 solutions for $n=2$ matrix.

To find eigenvectors, we sub our answers for λ.

for $\lambda = 1$ $\quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x + y = 0.$

set $x = 1$, then $y = -1$. $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

usual to quote normalised vector as the eigenvector.

$$\hat{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

for $\lambda = 3$ $\quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x - y = 0$

set $x = 1$ $\quad \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, eigenvector $\hat{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

<u>Invariants of Transformations</u>

$A_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad p(\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = -1 + \lambda^2 = 0$

$\lambda^2 = 1 \quad \lambda = \pm 1$

for $\lambda = 1$ $\quad \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies y = 0$

x can be anything.

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $\lambda = -1$ $\quad \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = 0$

$\underline{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ y can be anything.

eigenvectors are the y and x axis. can be scaled by any amount.