Lecture 8

We've explored when z^a for both $a=n$ and $a=\frac{1}{n}$. We've also look at irrational number eg. $\sqrt{2}$.

But what happens if

$$z^{\omega} \quad \text{when} \quad \omega = x + iy.$$

for simplicity, lets take $|z|$ for $z = e^{i\theta}$

$$(e^{i\theta})^{\omega} = e^{i(\theta+2\pi k)\cdot(a+ib)}$$

$$= \underbrace{e^{-b\theta - 2\pi k b}}_{\text{real part, equivelnt to } |z|} \cdot \underbrace{e^{i(a\theta + 2\pi k a)}}_{\text{complex part.}}$$

$$= (e^{-b\theta - 2\pi k b})\cdot\left(\cos(a\theta + 2\pi k a) + i\sin(a\theta + 2\pi k a)\right)$$

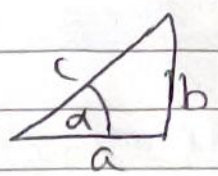
notice - there are an infinate number of solutions.

Now, what would happen if we were looking at: $f(z) = e^z$ $\quad f(z) = \sin(z)$ $\quad f(z) = \ln(z)$?

To answer this, we first must answer a more basic question. What is a function?

For example:

$\sin(\alpha)$:



$\sin(\alpha) = \frac{b}{c}$

this is an awful way to define $\sin(\alpha)$, for example when $\alpha > \frac{\pi}{2}$, there is no definition.

Secondly, we cannot use this to find non-trivial angles using a computer using the triangle method.

How to define a function?

There are 3 ways to define a function.

1) Power series (taylor)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

the right-hand side can be computed using a machine $(+, -, \times, \div)$.

2) Intergral Representation.

$$y = f(x) = \int_1^x \frac{dt}{t} = \ln x - \ln 1 = \ln x$$

this is a definite integral, we gives a numerical value.

t is a dummy variable.

3) Differential Equation

same as above, but not included as found later in course.

Q: given the power series, can we recover/ prove the properties of $f(z)$.

A: We take one of the three standard ways of defining function sub in complex variable Analysis the output.

Issue of convergence

$$g(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$a_n = \frac{1}{n!} \frac{d^n g}{dx^n} \Big|_{x=0}$$

is $g(x) = \sum_{n=0}^{\infty} a_n x^n$ finite? ← does it reach a finite value.

Convergence Ratio Test

$$\rho = \lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right|$$

if $\rho < 1$, we have convergence
if $\rho > 1$, we have divergence
if $\rho = 1$, undefined (ratio test doesn't work)

Proof for $\rho < 1$ convergence

Suppose that $\lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = \rho < 1$

define $R = \frac{\rho + 1}{2}$ $\quad (\rho < R < 1)$

$|A_{n+1}| < R |A_n|$

when n is large enough:

$$|A_{n+i}| < R^i |A_n|$$

for each $n > N$, $i > 0$.

$$\Rightarrow \sum_{i=N+1}^{\infty} |A_i| = \sum_{i=1}^{\infty} |A_{N+i}| < \sum_{i=1}^{N} R^i |A_N|$$

$$< |A_N| \sum_{i=1}^{N} R^i$$

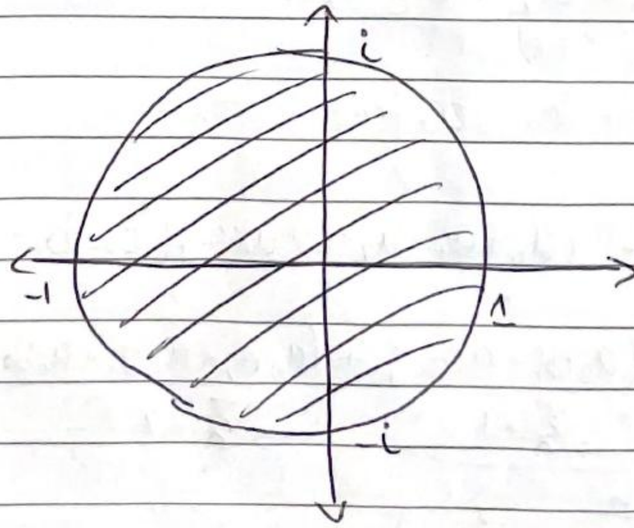
$$< |A_N| \frac{R}{1-R} < \infty.$$

Proofs convergence!

Example

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

$$\rho = \lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|$$



convergence when $|z| < 1$, ie, inside the unit circle

outside the unit circle gives divergence

Example

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

constant.

$$\rho = \lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \left| \frac{z}{n+1} \right| = 0$$

goes to infinity

e^z converges absolutly for ALL finite values of z.

Power Series for e^z

convergence $\Rightarrow$ use power series to derive properties of e^z.

Example

$e^{z_1+z_2} = e^{z_1}e^{z_2}$ - yet to be proved for $\mathbb{C}$

Step 1 Cauchy Product ($\neq$ Cauchy Search)

$$\sum_{n=0}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

$$\sum_{n=0}^{\infty} b_n = b_1 + b_2 + b_3 + \dots b_n + \dots$$

lets compute their product.

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = (a_0 + a_1 + a_2 + \dots + a_n + \dots)(b_0 + b_1 + b_2 + \dots b_n + \dots)$$

$$= a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots$$

$\Sigma_n = 0$ $\qquad \Sigma_n = 1$ $\qquad \Sigma_n = 2$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right)$$

sum of indices