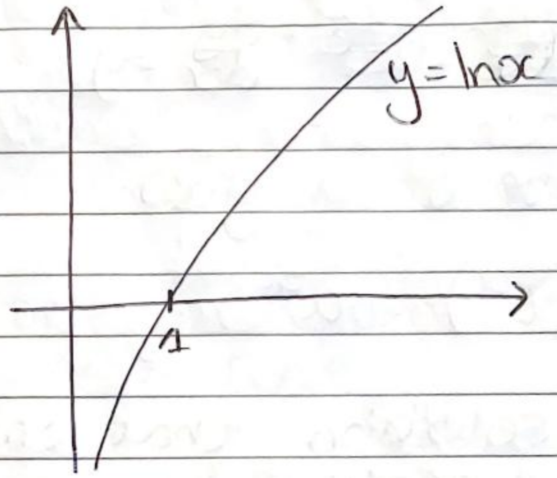
Functions 3

14/10/21

$*\ \ln\left(\frac{1}{x}\right) + \ln(x) = \ln\left(\frac{1}{x}\cdot x\right) = \ln 1 = 0$

$\ln(x^2) = \ln x + \ln x = 2\ln x$

$\Rightarrow \ln(x^n) = n\ln(x)$



tends to $+\infty$ with vanishing slope as $x \to \infty$.

The exponential function

Consider $x = \ln y$, what is the inverse of this function?

Let $x_1 = \ln y_1$ so that $y_1 = f(x_1)$
$x_2 = \ln y_2$ so that $y_2 = f(x_2)$

Then $x_1 + x_2 = \ln y_1 + \ln y_2 = \ln(y_1 y_2)$

$$y_1 y_2 = f(x_1 x_2)$$

and it must satisfy

$$f(x_1 + x_2) = f(x_1) f(x_2)$$

This implies that $f(x)$ to be of the form $f(x) = a^x$. $\left(a^{x_1 + x_2} = a^{x_1} a^{x_2}\right)$

We note also that

$$f(nx) = [f(x)]^n, \quad f(n) = [f(1)]^n, \quad f(0) = 1$$

If $x = \ln y$ then $\frac{dx}{dy} = \frac{1}{y}$

$$\Rightarrow \frac{dy}{dx} = y$$

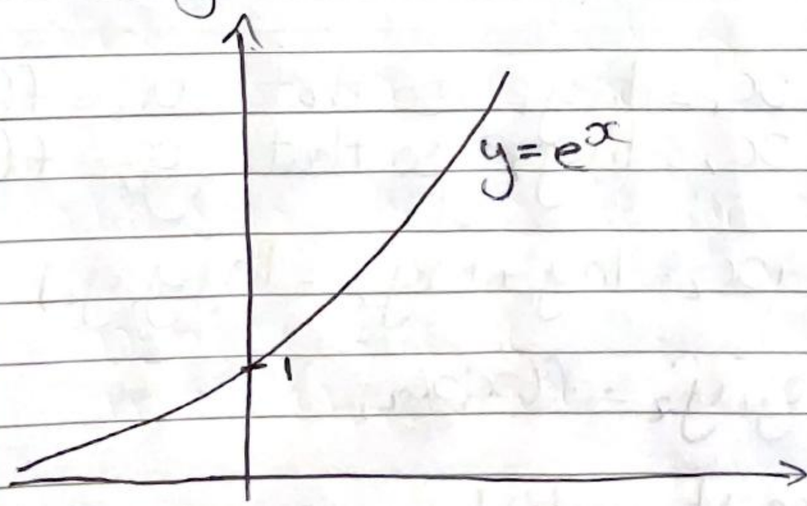
So what value of a gives

$$\frac{d}{dx}(a^x) = a^x$$

The UNIQUE solution that satisfies this is $e = 2.718281828 4\ldots$

$$e = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \ldots$$

$\therefore$ $y = e^x$ is the inverse of $y = \ln x$



How do log to other basis compare?

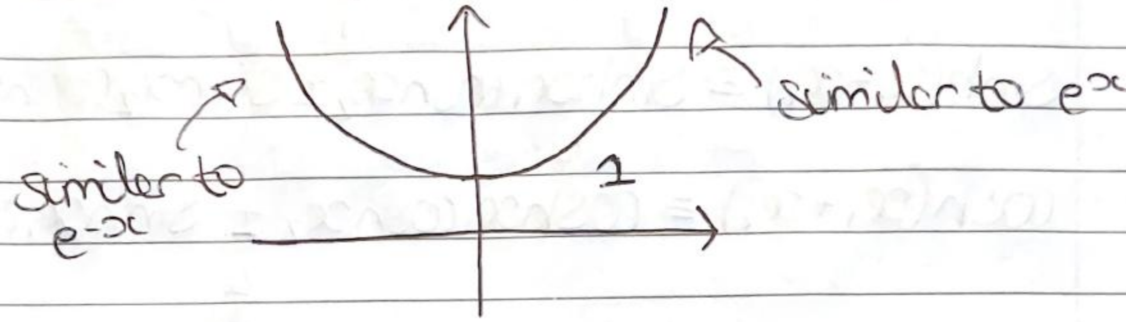
$$x = e^{\ln x} = 10^{\log_{10} x} = e^{\ln 10 \log_{10} x}$$

$$\ln x = (\ln 10)(\log_{10} x) \quad \boxed{\log_{10} x = \frac{\ln x}{\ln 10}}$$

~~Slide Notes~~

Hyperbolic Function

$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$ **EVEN**

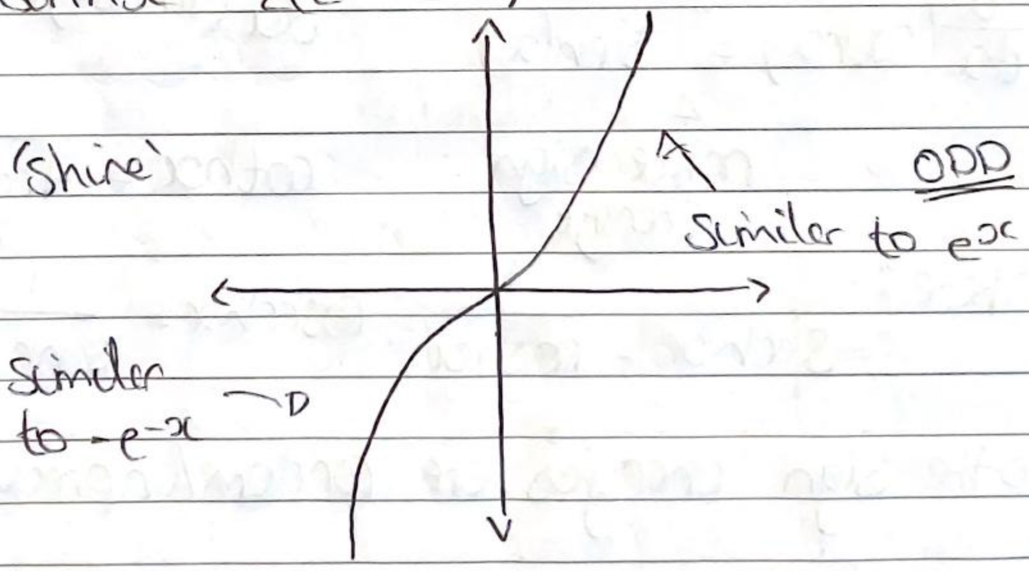


Shape a chain makes when resting is cosh curve.

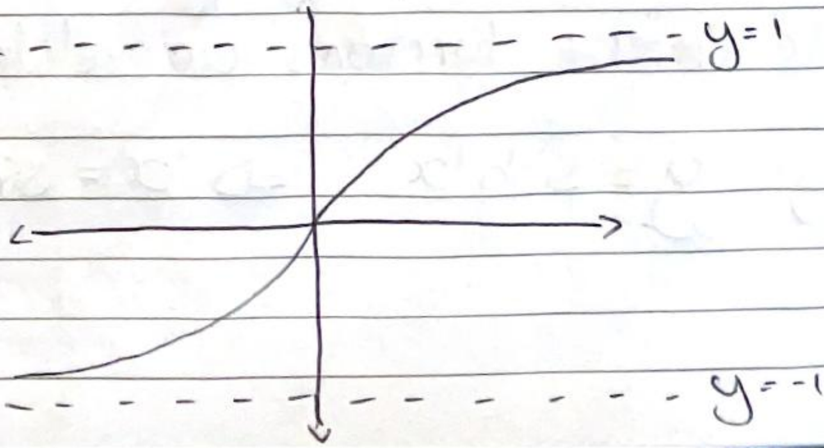
$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

'shine'

ODD



$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$



Hyperbolic functions are similar to trig functions, but not periodic.

$$\cosh^2 x + \sinh^2 x \equiv 1$$

$$\sinh(x_1 + x_2) \equiv \sinh x_1 \cosh x_2 \pm \sinh x_2 \cosh x_1$$

$$\cosh(x_1 + x_2) \equiv \cosh x_1 \cosh x_2 \pm \sinh x_1 \sinh x_2$$

notice sign change

$$\frac{d}{dx}(\sinh x) = \cosh(x)$$

$$\frac{d}{dx}(\cosh x) = \sinh(x)$$

notice sign change

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\coth x = \frac{1}{\tanh x}$$

Where:

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

→ the sign changes are random/unpredictible.

$\cosh(ix) = \cos x$
$\sinh(ix) = i \sin x$ → See later chapters.

The inverse functions are related to logs.

eg $y = \sinh^{-1} x \Rightarrow x = \sinh y$

$$x = \tfrac{1}{2}(e^y - e^{-y})$$
$$2x = e^y - e^{-y}$$
$$2xe^y = e^{2y} - 1$$
$$e^{2y} - 2xe^y - 1 = 0$$
$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}$$

just drop -ve for now

$$y = \ln\left[x \pm \sqrt{x^2+1}\right]$$

$$\sinh^{-1}x = \ln\left[x + \sqrt{x^2+1}\right]$$

similary

$$\cosh^{-1}x = \ln\left[x \pm (x^2-1)^{\frac{1}{2}}\right] \quad \text{for } \underline{\underline{x \geq 1}}$$
$$= \pm\ln\left[x + \sqrt{x^2-1}\right]$$

why? we note that

$$\left[x + \sqrt{x^2-1}\right]\left[x - \sqrt{x^2-1}\right] = 1$$

$$\ln\left[x - \sqrt{x^2-1}\right] = -\ln\left[x + \sqrt{x^2-1}\right]$$

<u>Note</u>