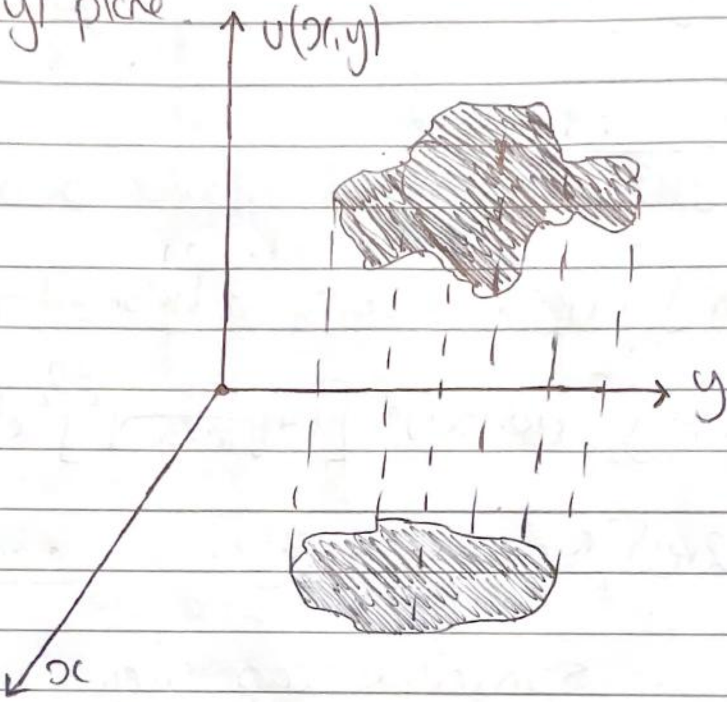
Functions 10

Partial Differentiation

Consider a function $v = v(x,y)$ of 2 independent variables (x,y). We can think of a surface over a (x,y) plane

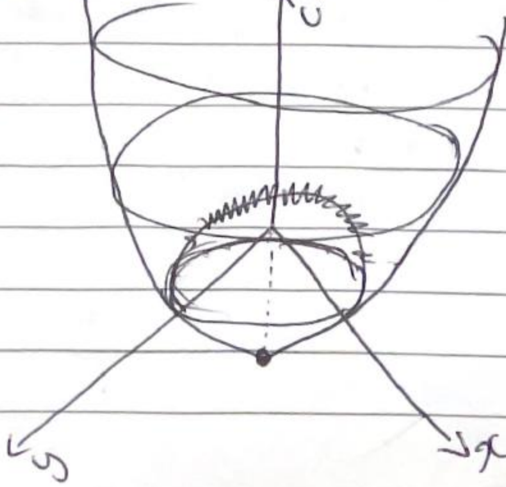


It is often helpful to visualise the surface using contour lines.

$$v(x,y) = c \quad (\text{const.})$$

for various values of c.

Example $\quad v = x^2 + y^2 - 5 \quad$ (circular contours)



Physically, we could represent a geometrical object or temperature or pressure or... . We now look at (spatial) rates of change.

We start at $P(x,y)$ and move a small distance in the x direction (δx) to $Q(x+\delta x, y)$. We define this limit as:

$$\frac{\partial u}{\partial x} = \lim_{h \to 0}\left(\frac{u(x+h,y)-u(x,y)}{h}\right)$$

Similarly,

$$\frac{\partial u}{\partial y} = \lim_{h \to 0}\left(\frac{u(x,y+h)-u(x,y)}{h}\right)$$

<u>Example</u> $\quad u = x^2\sin y + y^3$

$$\frac{\partial u}{\partial x} = 2x\sin y \qquad \frac{\partial u}{\partial y} = x^2\cos y + 3y^2$$

We can, of course, consider higher derivatives:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

note the order.

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

For the example above we get the following:

$$\frac{\partial^2 u}{\partial x^2} = 2\sin y \qquad \frac{\partial^2 u}{\partial y^2} = -x^2\sin y + 6y$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2x\cos y \qquad \frac{\partial^2 u}{\partial y \partial x} = 2x\cos y$$

We note that in our case:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial u}$$

This is a <u>general result</u>.

<u>Example</u> $\quad u(x, t) = a\sin(x-ct)$

$$\frac{\partial u}{\partial x} = a\cos(x-ct) \qquad \frac{\partial u}{\partial t} = -a\overset{c}{\vee}\cos(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = -a\sin(x-ct) \qquad \frac{\partial^2 u}{\partial t^2} = -ac^2\sin(x-ct)$$

$\Rightarrow u(x,t)$ satisfies

The 1D wave equation $\rightarrow$ $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$ $\leftarrow$ 2^{nd} order linear partial differential equation.

Any reasonable function of the form $u = a\sin(x-ct)$ satisfies this equation. It represents a wave form moving (with speed c) to the right.