Mechanics

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Chapter 1

Introduction

Mechanics was the field in which the ideas, approaches and techniques of modern science were invented and remains the platform on which most of physics is built. It was in mechanics that we first realised the mysterious link between mathematics — the manipulation of symbols written on pieces of paper — and the real world; and it was to understand mechanics that we first formulated the laws of nature mathematically. It was also in mechanics that we developed calculus as the mathematical language for describing things that change, that we became aware of the existence of conservation laws for energy, momentum and angular momentum, and that we first encountered the concept of (Galilean) relativity. All of these tools and ideas remain central to physics today. I am a theoretical and computational condensed matter physicist interested in the quantum mechanics of the vast numbers of interacting electrons in solids — a subject apparently unrelated to colliding billiard balls or masses sliding on inclined planes — and yet I use ideas from this course every day. Almost everything you learn this term will help you throughout your degree and the rest of your career in physics.

Although we all have intuition about mechanics, aspects of this intuition are often quite wrong. For example, it was not until the 16^{th} century that it was widely realised that objects did not naturally come to rest when nothing was pushing them, or that the planets in our solar system and the flight of a tennis ball are described by the same laws of motion. During this course we will derive some results that seem obvious, but also some that are surprising. It pays to be careful: do not assume that answers to mechanics problems are always obvious.

Limitations of Classical Mechanics

Classical mechanics is the theory of how bodies (humans, bricks, planets, molecules, ...) move in response to forces. Like most classical mechanics courses, this one ignores two important areas of physics. These have almost no direct impact on the motion of bodies as experienced in everyday life, but they matter at sub-atomic and galactic scales.

No quantum effects: This is a spectacularly good approximation for objects larger than atoms. At temperatures of 1 K or lower, it is sometimes possible to see individual atoms behaving quantum mechanically, but only just. At room temperature, although H and He atoms are very slightly affected by quantum mechanics, the motions of heavier atoms and larger objects are (almost) exactly as Newton described. Most chemical reactions, for example, can be described very well using Newton's laws. The forces felt by the atoms depend on chemical bonds made of electrons, which are very quantum mechanical, but Newton's second law tells you how those forces accelerate the atoms.

No special or general relativity: This is also a very good approximation everywhere on Earth for all objects moving at speeds v much slower than the speed of light c. Special relativity is essential if you want to understand how the fast-moving particles at CERN respond to forces. The core electrons of heavy atoms can have $v/c \approx 0.1$, so special relativity also matters there. General relativity (Einstein's theory of gravity) is important on galactic scales and near very massive objects such as black holes, but Newton's laws are more than good enough for use on Earth. Tiny general (and also special) relativistic effects have to be taken into account in global positioning systems, which require absurd accuracy, but it is hard to think of other terrestrial examples.

The fact that we neglect relativity and quantum theory may strike you as cheating, but all theories of physics, as far as I know, have limitations that restrict their range of validity. Most particle physicists now regard the standard model, once viewed as an essential part of the fundamental theory of everything, as an approximation valid only at long enough length scales and low enough energy scales. Just as quantum theory lurks below classical mechanics, something even more fundamental lurks below the standard model. For most aspects of the physics of everyday life, Newtonian mechanics is *the* best theory we have: an engineer trying to use quantum mechanics to design a bridge would be a fool.

Limitations of this Classical Mechanics Course

This course does not cover every aspect of classical mechanics. Some of the more difficult areas are left for later in your degree.

No fluid mechanics: The motion of (non-relativistic) fluids is governed by Newton's laws but is mathematically challenging. There is a fluid mechanics option in the final year of your degree.

No Lagrangian or Hamiltonian mechanics Although they look very different mathematically, Lagrangian and Hamiltonian mechanics are just reformulations of Newton's laws: they describe exactly the same physics and produce identical results. A few famous problems are easier to solve when reformulated, but the main use of Hamiltonian and Lagrangian mechanics is in quantum theory: Hamiltonian mechanics is central to non-relativistic quantum mechanics; and Lagrangian mechanics is at the heart of relativistic quantum field theory. Non-relativistic quantum mechanics is taught from the second year on; Lagrangian and Hamiltonian mechanics come in the third year; and quantum field theory is a final-year option.

Assumptions

The theory of classical mechanics rests on two assumptions so fundamental that they are almost philosophical.

Time is universal: All observers, no matter where they are or how fast they are moving, agree on the time that elapses between any two events.

Space is homogeneous, isotropic and Euclidean: A *homogeneous* space is one that looks the same wherever you are; an *isotropic* space looks the same in all directions; and a *Euclidean* space is the flat space familiar from school. Distances in a Euclidean space are given by Pythagoras's theorem: $r^2 = x^2 + y^2 + z^2$.

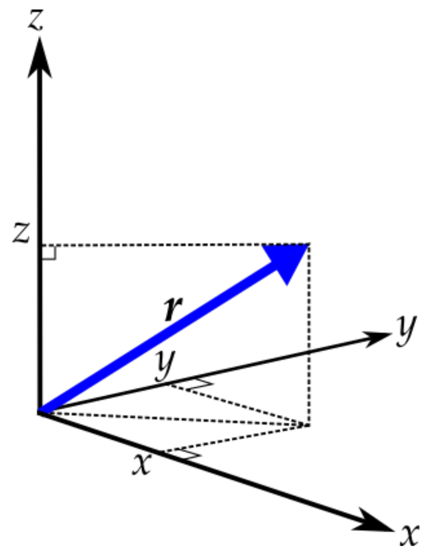
These assumptions seem so obviously true that it took thousands of years for people to realise that they were assumptions at all, and a few hundred more years to realise that they are wrong. Time is not universal in special relativity, and space-time is not homogeneous, isotropic and Euclidean in general relativity. Fortunately, although false, both assumptions are very accurate when $v \ll c$, length scales are less than galactic, and gravity is not enormously strong.

Definitions and Notation

A **particle** is a point-like object on which forces may act.

A **body** is a particle or assembly of particles bound together. When trying to work out how bodies respond to forces, they can often be treated as particles. Off-centre forces cause bodies to start rotating, which point-like particles cannot do, but the centre of mass still moves like a particle.

A **reference frame** is a set of coordinate axes used to measure positions.



Particle positions, velocities and accelerations are **vectors**. In these notes and most books, vectors are written in bold face. Unit vectors also have a hat (caret) on top. The unit vectors along the x, y and z axes are denoted $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. (Another common notation is $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.) In handwritten work, vectors are usually underlined, $\underline{r}$, or have an arrow on top, $\vec{r}$. Both are fine, although it is easier to add hats to underlined vectors.

Any vector $\boldsymbol{r}$ can be expressed as the sum of its components along the three coordinate axes:

$$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}.$$

Often, as a kind of shorthand, we write $\boldsymbol{r} = (x, y, z)$.

Position, Velocity and Acceleration

I am sure you already know that the velocity of a particle is the derivative of its position with respect to time.

- In one dimension, the **position** is a scalar: a positive or negative number. You can, if you want, think of the position in one dimension as a one-component vector and underline it, but this is not necessary and rarely useful. The velocity

$$v = \frac{dx}{dt}$$

  is also a scalar. If v is positive, the particle is moving in the $+x$ direction; if v is negative, it is moving in the $-x$ direction. The **speed**, $|v|$, is never negative. (The vertical bars, called modulus signs, mean the absolute value of a number: $|3| = |-3| = 3$.)

- In three dimensions, where the position $\boldsymbol{r}(t)$ is a vector, the **velocity**

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$$

  is also a vector. The absolute value of a three-dimensional vector such as $\boldsymbol{v}$ is (by definition) its length:

$$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Aside: How to Differentiate Vectors

Vectors are differentiated component by component:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \\ &= \frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\hat{\boldsymbol{k}}.\end{aligned}$$

(You cannot differentiate a vector with respect to another vector.)

One way to understand why differentiating component by component is the right thing to do is to start from the definition of the derivative as a limit:

$$\begin{aligned}\frac{d\boldsymbol{r}}{dt} &= \lim_{\delta t \to 0}\left(\frac{\boldsymbol{r}(t+\delta t) - \boldsymbol{r}(t)}{\delta t}\right) \\ &= \lim_{\delta t \to 0}\left(\frac{x(t+\delta t)\hat{\boldsymbol{\imath}} + y(t+\delta t)\hat{\boldsymbol{\jmath}} + z(t+\delta t))\hat{\boldsymbol{k}}}{\delta t}\right. \\ &\qquad\qquad \left. - \frac{x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}}{\delta t}\right) \\ &= \lim_{\delta t \to 0}\left(\frac{x(t+\delta t) - x(t)}{\delta t}\hat{\boldsymbol{\imath}} + \frac{y(t+\delta t) - y(t)}{\delta t}\hat{\boldsymbol{\jmath}}\right. \\ &\qquad\qquad \left. + \frac{z(t+\delta t) - z(t)}{\delta t}\hat{\boldsymbol{k}}\right) \\ &= \lim_{\delta t \to 0}\left(\frac{x(t+\delta t) - x(t)}{\delta t}\right)\hat{\boldsymbol{\imath}} + \lim_{\delta t \to 0}\left(\frac{y(t+\delta t) - y(t)}{\delta t}\right)\hat{\boldsymbol{\jmath}} \\ &\qquad\qquad + \lim_{\delta t \to 0}\left(\frac{z(t+\delta t) - z(t)}{\delta t}\right)\hat{\boldsymbol{k}} \\ &= \frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\hat{\boldsymbol{k}}.\end{aligned}$$

- If an object is at $\boldsymbol{r}_1$ at time t_1 and $\boldsymbol{r}_2$ at time t_2, its **average velocity** during that time interval is

  $$\bar{\boldsymbol{v}} = \frac{\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)}{t_2 - t_1} = \frac{\Delta\boldsymbol{r}}{\Delta t}.$$

  In the limit as $\Delta t \to 0$ the average velocity tends to the instantaneous velocity, but the two can be very different for longer time intervals.

Think, for example, about a particle orbiting in a circle. It velocity is never zero and its average speed is constant, but the average of its velocity around a complete orbit is exactly zero.

- **Acceleration** is the rate of change of velocity with respect to time.

$$\begin{aligned} a &= \frac{dv}{dt} = \frac{d^2x}{dt^2} && \text{in 1D,} \\ \boldsymbol{a} &= \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} && \text{in 3D.} \end{aligned}$$

- Newton denoted time derivatives by dots,

$$\dot{\boldsymbol{r}} \equiv \frac{d\boldsymbol{r}}{dt} \qquad \text{(the } \equiv \text{ sign means "is equivalent to")},$$

and space derivatives by primes,

$$V' \equiv \frac{dV}{dx}.$$

We often use Newton's notation in this course, mainly because it is more concise than Leibniz's $d\boldsymbol{r}/dt$ and dV/dx notation. Leibniz's approach is normally more powerful, but not always clearer. Suppose, for example, that you want to write down the time derivative of $\boldsymbol{r}(t)$ at some special time t_0:

$$\begin{aligned} &\text{Newton:} && \dot{\boldsymbol{r}}(t_0), \\ &\text{Leibniz:} && \left.\frac{d\boldsymbol{r}}{dt}\right|_{t=t_0} \text{ or perhaps } \left(\frac{d\boldsymbol{r}}{dt}\right)(t_0). \end{aligned}$$

The Newtonian version is clearer here.

The SUVAT Equations

The SUVAT equations,

$$\begin{aligned} v &= u + at, \\ s &= ut + \frac{1}{2}at^2, \end{aligned}$$

describe the motion of a body **accelerating at a constant rate**. The constant acceleration is a, the initial velocity is u, the time elapsed is t, the final velocity is v, and the distance travelled is s. The other SUVAT equations, such as $v^2 = u^2 + 2as$, are easily derived from the two above.

The SUVAT equations assume only that the acceleration is constant and stand independent of Newton's laws, but Newton's laws tell us when the SUVAT equations are likely to apply. According to Newton's second law, bodies accelerate at a constant rate if and only if the applied force is constant. When you take the mechanics exam after Christmas, I can almost guarantee that around half of you will forget this and use the SUVAT equations inappropriately.

Derivation of $v = u + at$

Start by stating that a is constant:

$$\frac{dv}{dt} = a = \text{const.}$$

Now integrate both sides with respect to time:

$$\cancel{\int_{t=0}^{t} \frac{dv(t)}{dt}\, dt = \int_{t=0}^{t} a\, dt.}$$

The idea of this is OK but the notation is confused: the same symbol t cannot be used for the upper limit of integration, which has a specific value, *and* for the variable of integration, which varies from 0 to the upper limit. The simplest solution is to change the name of the variable of integration and rewrite the equation above as

$$\int_{t'=0}^{t} \frac{dv(t')}{dt'}\, dt' = \int_{t'=0}^{t} a\, dt'.$$

(This is still a bit confusing because Newton used primes to indicate derivatives with respect to x. The symbol called t' here is just a new name for the variable of integration.)

Since a is a constant, the integrals are easy to do:

$$\begin{aligned} & [v(t')]_{t'=0}^{t} = [at']_{t'=0}^{t} \\ \Rightarrow \quad & v(t) - v(0) = at \\ \Rightarrow \quad & v(t) = u + at \qquad (\text{as } v(0) = u). \end{aligned}$$

Derivation of $s = ut + \frac{1}{2}at^2$

This time we start with $v = u + at$ and integrate both sides to get:

$$
\begin{aligned}
&& \int_{t'=0}^{t} \frac{dx(t')}{dt'}\,dt' &= \int_{t'=0}^{t} (u + at')\,dt' \\
\Rightarrow && \left[x(t')\right]_{t'=0}^{t} &= \left[ut' + \frac{1}{2}at'^2\right]_{t'=0}^{t} \\
\Rightarrow && x(t) - x(0) &= ut + \frac{1}{2}at^2 \\
\Rightarrow && s &= ut + \frac{1}{s}at^2.
\end{aligned}
$$

The SUVAT equations in three dimensions

The vector versions of these derivations are no harder (try them) and give the SUVAT equations in three dimensions:

$$
\begin{aligned}
\boldsymbol{v} &= \boldsymbol{u} + \boldsymbol{a}t, \\
\boldsymbol{s} &= \boldsymbol{u}t + \frac{1}{2}\boldsymbol{a}t^2,
\end{aligned}
$$

where $\boldsymbol{s}(t) = \boldsymbol{r}(t) - \boldsymbol{r}(0)$ is the vector displacement.

Chapter 2

Newton's Laws of Motion

2.1 Statement

Isaac Newton formulated his laws of mechanics in the *Philosophiæ Naturalis Principia Mathematica* published in 1687. From his point of view they were empirical laws, verified by comparison with experiment. Newton's laws can now be derived from quantum theory, but that is no less empirical in nature. At heart, all scientific theories are empirical.

Newton's First Law: A body on which no forces act remains at rest or moves at a constant velocity.

Newton's Second Law: The rate of change of momentum of a body is proportional to the force applied to it.

Newton's Third Law: When two bodies interact, they exert equal and opposite forces on each other.

2.2 Newton's First Law

Newton's first law (NI) can be viewed as a special case of Newton's second law (NII). If the applied force is zero, NII says that the momentum is constant. Since the momentum $\boldsymbol{p}$ is equal to $m\boldsymbol{v}$ in classical mechanics (this is a definition), it follows that the velocity is also constant. NI follows from NII.

Isaac Newton was perhaps the greatest physicist ever to have lived, so it is unlikely he missed this simple observation. Why, then, did he decide to state the first law separately? The main reason, I think, is historical. From the time of Aristotle (384–322 BC), the prevailing view had been that bodies slow down and come to rest unless a force is applied to keep them moving. Newton wanted to make sure his readers understood that this was wrong.

Inertial Frames

A reinterpreted version of NI remains useful today for another reason, which Newton himself may or may not have understood. The first thing to realise is that NI does not always hold. If, for example, you draw your coordinate axes on the floor of an accelerating train carriage and drop a ball, you will see the ball accelerate backwards, just as if pushed by a force. We understand that this force is fictitious — its apparent existence is a side effect of the acceleration of the coordinate axes — but in order to make that deduction we have to know that the frame of reference is accelerating. How can we tell? We could look at the landscape passing the train window, but perhaps the rest of the world is accelerating backwards and the train is stationary? Suppose the train is in outer space, far from any visible landmarks? How can we work out whether it is accelerating then? We could check whether objects accelerate when none of the real forces we know about are acting on them, but perhaps there are forces we have not yet discovered?

Frames of reference in which NI holds are called **inertial frames**, and NI is nowadays often reformulated as a statement about their existence.

Reformulation of Newton's First Law: There exist frames of reference, known as inertial frames, in which bodies on which no forces act remain at rest or move at a constant velocity.

Although the Earth's surface is not an inertial frame, the rotation of the coordinate system is slow and the corresponding acceleration is small. It is usually small enough to ignore.

Galilean Relativity

If you can find one inertial frame, any other frame moving at constant velocity with respect to the first is also inertial. The Galilean version of the principle of relativity states that any two such frames are completely equivalent. Neither has the right to declare itself "special" and the laws of physics must be the same in both. This does not mean that everything you can measure (velocity, position, ...) has to be the same in both frames, but that the laws of physics relating the measured quantities to each other must have the same form.

It is interesting to check whether Newton's second law obeys this principle. Suppose the coordinate axes of inertial frame 2 move at constant velocity $\boldsymbol{u}$ relative to the coordinate axes of inertial frame 1. A body with velocity $\boldsymbol{v}_1(t)$ as measured in frame 1 has velocity $\boldsymbol{v}_2(t) = \boldsymbol{v}_1(t) - \boldsymbol{u}$ as measured in frame 2. If a force $\boldsymbol{F}(t)$ acts on the body, NII in frame 1 reads:

$$m\frac{d\boldsymbol{v}_1}{dt} = \boldsymbol{F}(t).$$

Since $\boldsymbol{u}$ is constant, $d\boldsymbol{u}/dt = \mathbf{0}$. (Note that a zero vector is still a vector, $\mathbf{0} = (0, 0, 0)$, and should still be written in bold face.) This means that

$$m\frac{d\boldsymbol{v}_2}{dt} = m\frac{d\boldsymbol{v}_1}{dt} - m\frac{d\boldsymbol{u}}{dt} = m\frac{d\boldsymbol{v}_1}{dt} = \boldsymbol{F}(t),$$

showing that NII has exactly the same form in both frames. NII does obey the Galilean principle of relativity.

2.3 Newton's Second Law

Why did we write NII as $\boldsymbol{F} = d\boldsymbol{p}/dt$ instead of $\boldsymbol{F} = md\boldsymbol{v}/dt = m\boldsymbol{a}$? In classical mechanics, where the momentum $\boldsymbol{p}$ is just another name for $m\boldsymbol{v}$, the two equations are equivalent and the choice is irrelevant. In special relativity, however, the equation $\boldsymbol{F} = d\boldsymbol{p}/dt$ still holds but the momentum is defined differently:

$$\boldsymbol{p} \triangleq \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}. \qquad (\triangleq \text{ means "is defined to be equal to"})$$

Because of the more complicated $\boldsymbol{v}$ dependence of the relativistic momentum, the $\boldsymbol{F} = md\boldsymbol{v}/dt = m\boldsymbol{a}$ version of NII no longer holds.

[The m appearing in the special relativistic definition of momentum is the rest mass of the particle and does not depend on how fast it is moving. Many older books call $m/\sqrt{1 - v^2/c^2}$ the "relativistic mass" and say that the mass of the particle increases as its speed increases. This is just a difference of perspective.]

The essence of NII in classical physics is the statement that the acceleration of an object is proportional to the force applied. If you double the force, the acceleration also doubles. There is a bit more to it than that, though. Although Newton did not state this explicitly, NII also assumes that the mass is a property of the body itself; the value of F/a does not depend on where the body is located, what sort of force is applied, or whether it happens to be tea time.

Measuring Mass

One can use NII to establish a mass scale. First choose an object (until very recently this was a platinum cylinder in Paris) and say that it has mass $m_1 = 1$. This defines the unit of mass. To measure the mass m_2 of any other object, all you have to do is apply the same force to m_1 and m_2 and measure the accelerations $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Since

$$\boldsymbol{F} = m_1\boldsymbol{a}_1 = m_2\boldsymbol{a}_2$$

and $m_1 = 1$,

$$m_2 = \frac{m_1|\boldsymbol{a}_1|}{|\boldsymbol{a}_2|} = \frac{|\boldsymbol{a}_1|}{|\boldsymbol{a}_2|}.$$

If you can measure accelerations, you can also measure masses.

How can you be sure that you have applied exactly the same force to both masses? One way would be to use the same spring. That might work, but a neater way is to allow the two masses to collide and exert forces on each other. By Newton's third law (NIII), the force exerted on mass 1 by mass 2 is equal and opposite to the force exerted on mass 2 by mass 1. Equal and opposite is not the same as equal, of course, but because $m_2 = |\boldsymbol{a}_1|/|\boldsymbol{a}_2|$ only depends on the absolute values of the two accelerations it is good enough.

Groups of Particles

NII holds (bar small quantum corrections) for each of the more than 10^{27} atoms in my body. Why does it also work for my body as a whole? The mystery is that, although every atom exerts forces on every other atom, only the *externally* applied forces appear in NII for my body as a whole. What happened to all the internal forces?

Start by thinking about a body made of only two particles and write down NII for each:

$$\frac{d\boldsymbol{p}_1}{dt} = \boldsymbol{f}_1^{\text{ext}} + \boldsymbol{f}_{2\,\text{on}\,1}, \qquad \frac{d\boldsymbol{p}_2}{dt} = \boldsymbol{f}_2^{\text{ext}} + \boldsymbol{f}_{1\,\text{on}\,2}.$$

Adding these two equations gives

$$\frac{d\boldsymbol{p}_1}{dt} + \frac{d\boldsymbol{p}_2}{dt} = \boldsymbol{f}_1^{\text{ext}} + \boldsymbol{f}_2^{\text{ext}} + \boldsymbol{f}_{2\,\text{on}\,1} + \boldsymbol{f}_{1\,\text{on}\,2} = \boldsymbol{f}_1^{\text{ext}} + \boldsymbol{f}_2^{\text{ext}},$$

where the last step made use of NIII. Since $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ is the total momentum of the body and $\boldsymbol{F}^{\text{ext}} = \boldsymbol{f}_1^{\text{ext}} + \boldsymbol{f}_2^{\text{ext}}$ is the total applied force, this is just NII for the body as a whole:

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F}^{\text{ext}}.$$

The argument is easy to extend to bodies made of arbitrarily large numbers of particles. (Try it.)

Notice that we would *not* be able to apply NII to bodies consisting of many particles if NIII did not hold. Without NIII, NII would be more or less useless.

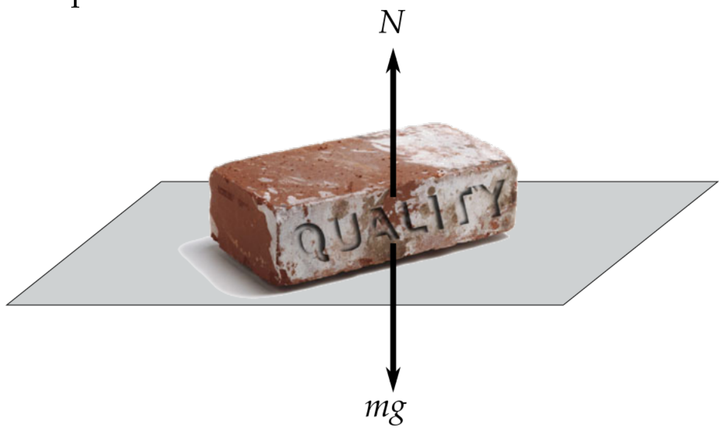
We have just shown that the total momentum of an isolated system of interacting particles is always conserved, no matter how complicated the internal interactions. Nothing is truly isolated because gravitational and electromagnetic forces are so long ranged, but the external forces on many objects — the solar system, an astronaut in free fall, a free-wheeling bicycle — are often small enough to be ignored.

It is important to understand that, although the total momentum of an isolated system is conserved, its total kinetic energy need not be. The total momentum of the two astronauts playing tug of war in problem sheet 2 remains constant before, during and after the collision, but their kinetic energy increases as they pull in on the rope and decreases when they run into each other. As they pull on the rope, potential energy stored in chemical bonds in their muscles is turned into kinetic energy. As their bodies crunch together, that kinetic energy becomes a mixture of elastic energy and heat energy. Sadly for the astronauts, people are not very springy, so the energy stored elastically soon turns to heat as well. Some of the kinetic energy might also be stored as potential energy associated with broken chemical bonds in fractured bones!

One can argue that most of the heat energy left after the collision exists as vibrations of atoms and that this is also a kind of kinetic energy (or, more accurately, a rapidly interchanging mixture of kinetic energy and potential energy, as in a vibrating mass on a spring). This is true, but the kinetic energy associated with disordered atomic motion is not the sort of kinetic energy we talk about in classical mechanics. Only the kinetic energy associated with macroscopic ("human-scale") motions of the body as a whole counts.

2.4 Newton's Third Law

Although it looks simple, Newton's third law is often misunderstood. The first thing to realise is that the action and reaction forces in NIII always affect *different* bodies. Two forces acting on the same body are *never* an action-reaction pair.



The force of gravity, mg, pulling down on the brick in the figure is equal and opposite to the force N exerted on the brick by the table, but the two are not an action-reaction pair because both are acting on the brick. The reaction corresponding to N is the force the brick exerts on the table top. The reaction corresponding to mg is the upward gravitational force the brick exerts on the Earth.

The second thing to realise about NIII is that it admits no ifs or buts. If two objects are interacting, the force exerted by the first on the second is *always* equal and opposite to the force exerted by the second on the first. It does not make any difference whether the objects are moving at constant velocity or accelerating, and it does not matter whether you regard one object as pushing the other. In reality, they are always pushing each other.

2.5 Inertial and Gravitational Mass

The mass in $\boldsymbol{F} = m\boldsymbol{a}$ measures the inertia of a body: it tells you how much force is required to make that body accelerate at $1\,\mathrm{m{\cdot}s^{-2}}$. The mass in Newton's law of gravitation, $F = GMm/r^2$, tells you how hard gravity pulls on a body: it is a gravitational "charge", analogous to the electrical charge in $F = q_1 q_2 / 4\pi\varepsilon_0 r^2$. Why should the inertial and gravitational masses, m_{I} and m_{G}, be the same? The two concepts appear unrelated. The electrical charge of a body is not proportional to its inertia. Why should gravity be different?

The equivalence of inertial and gravitational mass was first investigated by Galileo in his famous (and probably apocryphal) Leaning Tower of Pisa experiment showing that different masses drop at the same rate. It has since been verified experimentally to about 1 part in 10^{12}, but that does not make the reason for the equivalence any more obvious.

It is surprising that we had to wait for Albert Einstein to point out the strangeness of this equivalence and investigate its consequences. He noticed that the fictitious forces seen by observers living in accelerated frames of reference are proportional to the inertial mass and wondered whether gravity might also be a kind of fictitious force. Perhaps some of the frames we think of as inertial are not inertial after all? Einstein reasoned that the frame of reference attached to a free-falling lift, in which the unfortunate riders experience zero gravity, is (locally) inertial, but that the frame associated with the Earth's surface into which the lift plummets is not. This was the first step of the brilliant chain of reasoning that led him to the General Theory of Relativity.

Chapter 3

One-Dimensional Motion

3.1 Motion due to Forces

The force acting on an object may depend on its position (perhaps it is moving in a non-uniform electric or gravitational field), its velocity (if it is experiencing drag, friction or magnetic $q\boldsymbol{v} \times \boldsymbol{B}$ forces), and time:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t).$$

This vector differential equation is shorthand for three coupled differential equations,

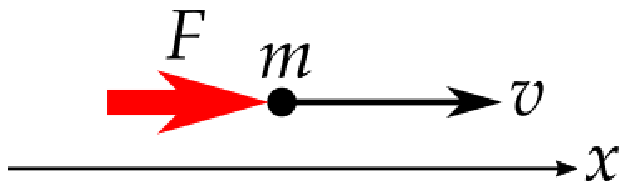
$$\begin{aligned} m\frac{d^2x}{dt^2} &= F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \\ m\frac{d^2y}{dt^2} &= F_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \\ m\frac{d^2z}{dt^2} &= F_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \end{aligned}$$

which already look forbidding enough. If there are lots of particles (the 6×10^{23} C atoms in a mole of diamond, for example), the force on one particle depends on the coordinates of all of the others and the problem gets *really* scary.

Newton's equations of motion can only be solved analytically (that is, by using a pencil and paper to find a formula for $\boldsymbol{r}(t)$) in a few very simple cases. Even solving them on a computer is often challenging. Many dynamical systems are *chaotic*, which means that tiny differences in the initial positions and velocities of the particles or bodies grow exponentially with time. To get accurate numerical results at long times, the initial conditions have to be specified to an exponentially large number of digits. Unfortunately, standard computers (using 8-byte real numbers) keep only about 14 significant figures. Predicting the future accurately using Newton's laws is impossible in practice in chaotic systems. When trying to work out the weather a week from now, the best that one can do is run simulations starting from lots of slightly different initial conditions and use the set of results to estimate probabilities.

Because of these difficulties, the best way to learn about classical mechanics is to study simple examples motivated by the real world. The knowledge we gain will help us derive general principles, especially conservation laws, which are often more useful than Newton's laws themselves.

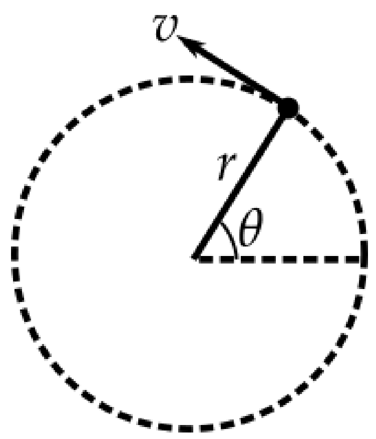
3.2 Motion in One Dimension



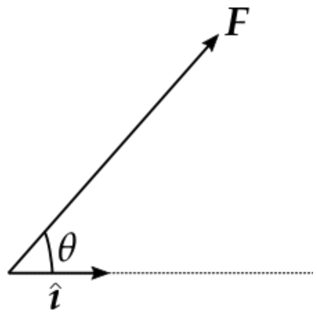
The position x, velocity v, and acceleration a of an object moving in one dimension are one-dimensional vectors: the only directions they can have are $+$ (right) or $-$ (left). There is usually no need to write one-dimensional vectors in bold or underline them or put arrows above them, but you can if you want to and it sometimes helps to distinguish, say, between speed (which is always positive) and velocity (which can be positive or negative). One-dimensional vector spaces are valid mathematical objects.

The point of working in one dimension is to keep the mathematics simple, but from now on we move beyond the A-level world in which almost all forces are constant, independent of time, position and velocity. The SUVAT equations, which only apply when the acceleration is constant, will only rarely be useful.

There are more examples of one-dimensional motion than you might imagine. Objects falling under gravity move along lines, as do trains on tracks, particles in head-on collisions, and bodies in circular motion. The circumference of the circle is admittedly not straight, but positions on it can be specified using a single variable, the angle of rotation θ, so the coordinate space is still one dimensional.



Another reason for studying one-dimensional motion is that any three-dimensional motion can be resolved along any axis.



The x-component of the force is easily obtained using a dot product,

$$F_x = |\boldsymbol{F}| \cos\theta = |\boldsymbol{F}|\,|\hat{\boldsymbol{\imath}}|\,\cos\theta = \boldsymbol{F} \cdot \hat{\boldsymbol{\imath}},$$

and only the x component causes acceleration in the x direction:

$$m\ddot{x} = F_x.$$

This means that the x-component of the motion is governed by the familiar one-dimensional version of Newton's second law, the only proviso being that F_x may now depend on y and z (and perhaps also v_y and v_z).

3.3 Impulse

You probably already know that impulse = force × time. This section shows how to work out the impulse when the force depends on time, as it does, for example, during a collision between two objects.

You might reasonably ask whether assuming that the force depends on time is general enough? What about a spaceship moving through the solar system? The gravitational forces it feels depend on its position, not time. Fortunately, as the location of the spaceship $\boldsymbol{r}(t)$ is a function of time, we can think of the force it experiences as a function of time too. We just have to understand that the force is to be measured along the track of the spaceship. If $\boldsymbol{F}(\boldsymbol{r})$ is the force as a function of position, the corresponding time-dependent force $\tilde{\boldsymbol{F}}(t)$ is given by:

$$\tilde{\boldsymbol{F}}(t) \triangleq \boldsymbol{F}(\boldsymbol{r}(t)).$$

The mathematical forms of $\tilde{\boldsymbol{F}}(t)$ and $\boldsymbol{F}(\boldsymbol{r})$ are of course very different: $\tilde{\boldsymbol{F}}$ maps a single variable, t, to a force, whereas $\boldsymbol{F}(\boldsymbol{r})$ maps a three-dimensional vector, $\boldsymbol{r}$, to a force. This explains why I have used different symbols, $\tilde{\boldsymbol{F}}$ and $\boldsymbol{F}$, for the two functions. To a mathematician this is the right thing to do. From the point of view of a physicist, however, both functions give the same result — the force on the spaceship — and assigning them different names is a bit fussy. From now on I will call both functions by the same name: $\boldsymbol{F}(\boldsymbol{r})$ and $\boldsymbol{F}(t)$.

Return now to working in one dimension and suppose that a time-dependent force $F(t)$ acts from time t_i to time t_f $(> t_i)$ on a mass m. Newton's second law tells us that

$$\begin{aligned} m\frac{dv}{dt} &= F(t) \\ \Rightarrow \qquad m\int_{t_i}^{t_f} \frac{dv(t)}{dt}dt &= \int_{t_i}^{t_f} F(t)dt. \end{aligned}$$

We cannot do anything more with the right-hand side without knowing more about the form of $F(t)$, which we are trying to keep general, but we can integrate the left-hand side to get:

$$m\left[v(t)\right]_{t_i}^{t_f} = \int_{t_i}^{t_f} F(t)dt,$$

and so

$$\boxed{\int_{t_i}^{t_f} F(t)dt = mv_f - mv_i = \text{change in momentum}}$$

The integral $\int_{t_i}^{t_f} F(t)dt$ is called the **impulse**, often denoted J, and this equation tells us that the impulse is equal to the change in momentum. The units of $\int_{t_i}^{t_f} F(t)dt$ are N·s. Since $F = ma$, 1 Newton-second = 1 kg·m·s^{-2}·s = 1 kg·m·s^{-1}. This is also the unit of momentum, mv.

We have shown that the equation "impulse = change in momentum" works even when the applied force is *not* constant. This is no surprise but can be useful, especially in collisions. The forces acting during a collision are complicated and hard to work out, but their time integral, the impulse, is just the difference between the momentum before and after the collision, which is easy to measure. In "inelastic" collisions such as car crashes, some of the kinetic energy turns into heat and kinetic energy is not conserved, but Newton's second law always holds and the change in momentum is still equal to the impulse.

You already know how to differentiate a time-dependent vector component by component. Integrating vectors works the same way:

$$\begin{aligned}\int \boldsymbol{F}(t)dt &\triangleq \left(\int F_x(t)dt\right)\hat{\boldsymbol{\imath}} + \left(\int F_y(t)dt\right)\hat{\boldsymbol{\jmath}} + \left(\int F_z(t)dt\right)\hat{\boldsymbol{k}} \\ &= \left(\int F_x(t)dt, \int F_y(t)dt, \int F_z(t)dt\right).\end{aligned}$$

Once you know this, it is easy to derive the three-dimensional version of the impulse equation:

$$\boxed{\int_{t_i}^{t_f} \boldsymbol{F}(t)dt = m\boldsymbol{v}_f - m\boldsymbol{v}_i = \text{change in momentum}}$$

Please try the derivation yourself.

3.4 One-Dimensional Motion with a Velocity-Dependent Force

Our aim here is to find the trajectory $x(t)$ of an object moving in response to an externally applied force that depends on velocity. The SUVAT equations do not hold, but we can still obtain analytic solutions in simple cases. The particle velocity $v(t)$ depends on time, so a velocity-dependent force is also a time-dependent force in the sense discussed in Sec. 3.3. This means that impulse is still a useful concept (although we are not going to use it).

Objects moving through fluids (gases or liquids) feel velocity-dependent drag forces of the form:

$$\boldsymbol{F} = -\alpha(v)\boldsymbol{v}.$$

The minus sign indicates that the force opposes the velocity. The speed-dependent function $\alpha(v)$ is complicated and depends on the type of fluid and the shape of the moving object, but is more or less constant at low speed, when the flow is *laminar*, and more or less proportional to v at higher speed, when the flow is *turbulent*.

The simplest model of friction says that the forces between solid objects in relative motion are independent of their relative speed, although the direction of the force depends on the direction of the relative velocity. If N is the normal force acting between the two objects, μ_k is the kinetic friction coefficient, and $\boldsymbol{v}$ is the relative velocity of the two objects, the frictional force is:

$$\boldsymbol{F} = -\mu_k N\hat{\boldsymbol{v}}.$$

Note that $\hat{\boldsymbol{v}}$ is a *unit* vector in the $\boldsymbol{v}$ direction.

Example: Fluid Drag Proportional to Speed

At low speeds it is often a good approximation to assume that $F = -\alpha v$, with α a constant.

Suppose the wind pushing the sailing boat pictured on the next page drops at time $t = 0$, after which the boat gradually drifts to a stop. New-

The velocity-dependent drag force on a sailing boat.

ton's second law reads:

$$
\begin{aligned}
& m\frac{dv}{dt} = -\alpha v \\
\Rightarrow \quad & \frac{1}{v}\frac{dv}{dt} = -\frac{\alpha}{m} \\
\Rightarrow \quad & \frac{dv}{v} = -\frac{\alpha}{m}dt \qquad \text{("physicist's" notation)} \\
\Rightarrow \quad & \int_{v_0}^{v} \frac{dv'}{v'} = -\frac{\alpha}{m}\int_0^t dt' \\
\Rightarrow \quad & [\ln v']_{v_0}^{v} = -\frac{\alpha}{m}[t']_0^t \\
\Rightarrow \quad & \ln v - \ln v_0 = -\frac{\alpha}{m}t \\
\Rightarrow \quad & \ln\left(\frac{v}{v_0}\right) = -\frac{\alpha}{m}t \\
\Rightarrow \quad & v = v_0 e^{-\alpha t/m} \\
\Rightarrow \quad & v = v_0 e^{-t/\tau}, \qquad \text{where } \tau \triangleq \frac{m}{\alpha} = \begin{cases} \text{characteristic time} \\ 1/e \text{ time} \end{cases}
\end{aligned}
$$

The speed drops exponentially with time, decreasing by a factor of e every $\tau = m/\alpha$ seconds. A graph of speed against time can be found near the end of this section.

Aside: Physicist's Notation

At school I was told that the "physicist's" notation (my name, not standard terminology) used on page 27 was illegal and would have been expected to write something like this instead:

$$\begin{aligned} \frac{1}{v}\frac{dv}{dt} &= -\frac{\alpha}{m} \\ \Rightarrow \qquad \int_0^t \frac{1}{v(t')}\frac{dv(t')}{dt'}dt' &= -\frac{\alpha}{m}\int_0^t dt'. \end{aligned}$$

Changing the variable of integration, setting $u = v(t')$, then gives

$$\int_{v_0}^{v} \frac{1}{u}du = -\frac{\alpha}{m}\int_0^t dt',$$

which is the same as line 4 of the version on page 27 (although with v' renamed as u).

Despite its supposed illegality, almost all working physicists and mathematicians are happy with physicist's notation and use it all the time. If your teachers warned you against it, now is the time to start ignoring them.

The same notation is used when changing variables in integrals. Asked to work out

$$\int_0^t \frac{1}{v(t')}\frac{dv(t')}{dt'}dt',$$

we say:

Let $u = v(t')$. Then $du = \frac{dv(t')}{dt'}dt'$ and

$$\int_0^t \frac{1}{v}\frac{dv}{dt'}dt' = \int_{v(0)}^{v(t)} \frac{1}{u}du.$$

What is the range of the boat? To find that we need to integrate again:

$$
\begin{aligned}
& \frac{dx}{dt} = v_0 e^{-t/\tau} \\
\Rightarrow \qquad & dx = v_0 e^{-t/\tau} dt \\
\Rightarrow \qquad & \int_{x_0}^{x} dx' = \int_0^t v_0 e^{-t'/\tau} dt' \\
\Rightarrow \qquad & x - x_0 = \left[-v_0 \tau e^{-t'/\tau} \right]_0^t \\
\Rightarrow \qquad & x - x_0 = -v_0 \tau e^{-t/\tau} + v_0 \tau \\
\Rightarrow \qquad & x - x_0 = (1 - e^{-t/\tau}) v_0 \tau.
\end{aligned}
$$

It is worth investigating how the position x depends on time when $t \ll \tau$ and $t \gg \tau$:

- When $t \ll \tau$,

$$
\begin{aligned}
e^{-t/\tau} &= 1 - \frac{t}{\tau} + \frac{1}{2!}\left(\frac{t}{\tau}\right)^2 - \ldots \\
&\approx 1 - \frac{t}{\tau}
\end{aligned}
$$

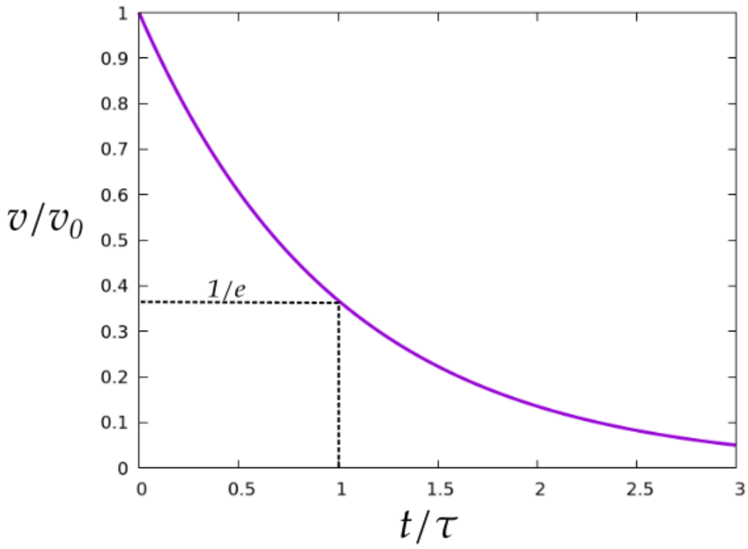
  and so

$$
x(\tau) \approx x_0 + v_0 t.
$$

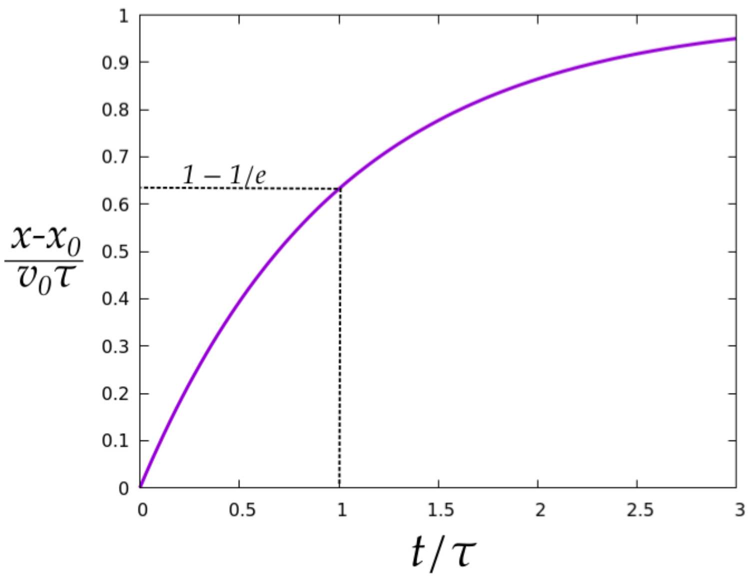
  The boat starts at x_0 and is initially moving at speed v_0, exactly as expected.

- When $t \gg \tau$, $e^{-t/\tau} \approx 0$ and $1 - e^{-t/\tau} \approx 1$, so $x - x_0 \approx v_0 \tau$, which is constant. The range is $v_0 \tau$.

A graph of position versus time can be found on the next page.



Boat speed versus time.



Boat position versus time.

3.5 The Work-Energy Theorem

We normally think of the velocity of a body and the force it experiences as functions of t. We might ask, for example: "how fast and in what direction was the car moving at 3pm?" If we choose to, however, we can equally well think of the velocity and applied force as functions of the position of the body. We could instead ask: "how fast and in what direction was the car moving when it passed the front door of the Blackett Laboratory?" This viewpoint is useful in the derivation of the work energy theorem.

A body starts at position x_i at time t_i and moves to x_f at time t_f under the influence of an applied force F that may depend on position, velocity and time. We are going to write the velocity and applied force as functions of the position x of the body, but this does not mean that they may not be different on different journeys from x_i to x_f. Once all the details of the journey (starting time, velocity profile) have been specified, we can work out the force and velocity as functions of the position x of the body, but the results may be different on different journeys.

Although we are going to treat the velocity and applied force as functions of the position of the body, the position depends on t:

$$\left.\begin{array}{rcl} v & = & v(x) \\ F & = & F(x) \end{array}\right\} \quad \text{where } x = x(t).$$

This means that we can use the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = \frac{1}{2}\frac{d}{dx}\left(v^2\right).$$

Substituting into Newton's second law then gives

$$\frac{1}{2}m\frac{d}{dx}\left(v^2\right) = F(x),$$

which integrates to

$$\begin{aligned} & \int_{x_i}^{x_f} \frac{d}{dx}\left(\frac{1}{2}mv^2\right) dx = \int_{x_i}^{x_f} F(x)dx \\ \Rightarrow \quad & \left[\frac{1}{2}mv^2\right]_{x_i}^{x_f} = \int_{x_i}^{x_f} F(x)dx \\ \Rightarrow \quad & \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int_{x_i}^{x_f} F(x)dx. \end{aligned}$$

We have derived:

The Work-Energy Theorem

The change in kinetic energy of a body moving under the influence of an externally applied force is equal to the work done:

$$K_f - K_i = W_{if}$$

where $K_f \triangleq \frac{1}{2}mv_f^2$ is the final kinetic energy of the body, $K_i \triangleq \frac{1}{2}mv_i^2$ is the initial kinetic energy of the body, and

$$W_{if} \triangleq \int_{x_i}^{x_f} F(x)dx$$

is the work done on the body.

Notes

- W_{if} is the work done **on** the body by the external apparatus that applies the forces. By Newton's third law, the work done **by** the body on the external apparatus is $-W_{if}$.
- The work done is measured in Joules:

  $$1\,\mathrm{J} = 1\,\mathrm{N{\cdot}m} = 1\,\mathrm{kg{\cdot}m^2{\cdot}s^{-2}} = 1\,\mathrm{W{\cdot}s}.$$

- If $F = \text{const}$,

  $$W_{if} = \int_{x_i}^{x_f} F dx = F \int_{x_i}^{x_f} dx = F \times (x_f - x_i) = \text{force } \times \text{ distance},$$

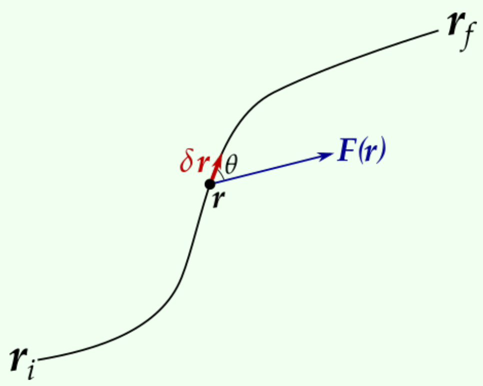
  as at school.
- The three-dimensional version of the work-energy theorem reads:

  $$K_f - K_i = \int_{\boldsymbol{r}_i}^{\boldsymbol{r}_f} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}.$$

  The left-hand side is exactly as in one dimension, but the right-hand side is more interesting. The integral is now along the path $\boldsymbol{r}(t)$ taken by the body, which need not be a straight line. You will not need to use or understand path integrals for this course, but you will meet them again in your "Functions" mathematics course and use them in electromagnetism next term.

Aside: Path Integrals (not examinable)

To understand the path integral that gives the work done in three dimensions, imagine dividing the path $\boldsymbol{r}(t)$ into tiny segments, $\delta\boldsymbol{r}$, and working out the scalar product, $\boldsymbol{F}(\boldsymbol{r}) \cdot \delta\boldsymbol{r}$, for each segment. Because the segments are so small, the force can be approximated as a constant within each one, although it varies from segment to segment along the path. The value of the path integral is the limit as the segment length tends to zero of the sum of the tiny scalar products, $\boldsymbol{F}(\boldsymbol{r}) \cdot \delta\boldsymbol{r}$, from all of the segments along the path.



The work δW associated with the segment from $\boldsymbol{r}$ to $\boldsymbol{r} + \delta\boldsymbol{r}$ is a dot product, $\delta W = \boldsymbol{F} \cdot \delta\boldsymbol{r}$, because the force need not be parallel to $\delta\boldsymbol{r}$. Only the component of $\boldsymbol{F}$ parallel to $\delta\boldsymbol{r}$ does any work:

$$\delta W = \boldsymbol{F} \cdot \delta\boldsymbol{r} = |\boldsymbol{F}||\delta\boldsymbol{r}| \cos\theta = F_{||}|\delta\boldsymbol{r}|.$$

Example

A spring-like restoring force $F = -3x\,\mathrm{N}$ pulls a mss from $x = 3\,\mathrm{m}$ to $x = 2\,\mathrm{m}$. How much work does it do?

$$W_{if} = \int_3^2 (-3x)dx = \left[-\frac{3}{2}x^2\right]_3^2 = -\frac{3}{2}2^2 + \frac{3}{2}3^2 = \frac{15}{2} = 7.5\,\mathrm{J}.$$

The final velocity v_f is measured at $x = 2\,\mathrm{m}$; the initial velocity v_i is measured at $x = 3\,\mathrm{m}$. The work done by the external force, 7.5 J, is equal to the change in kinetic energy of the mass.

3.6 Power

Power $\triangleq$ work done by external force per second

Suppose that a body moves a distance δx in time δt. An externally applied force F, which may depend on time, position and velocity, acts during the displacement. According to Sec. 3.5, the work done during the infinitesimal move is

$$\delta W = F\delta x,$$

where F is the force currently acting on the body. The time interval δt is so short that the applied force hardly changes from beginning to end.

Dividing both sides by δt,

$$\frac{\delta W}{\delta t} = F\frac{\delta x}{\delta t},$$

and taking the limit as $\delta t \to 0$ (so that δx also tends to 0) gives

$$\frac{dW}{dt} = F\frac{dx}{dt} = Fv.$$

The power supplied by the external apparatus that applies the force is given by the formula:

$$\boxed{\text{Power supplied} = Fv}$$

- Given the power dW/dt as a function of time, the total work done between any time t_i and any later time t_f $(> t_i)$ can be found by integration:

$$W_{if} = \int_{t_i}^{t_f} \frac{dW}{dt}dt = \int_{t_i}^{t_f} F\frac{dx}{dt}dt = \int_{x_i}^{x_f} F dx.$$

- Section 3.5 explained that only the component of the force parallel to $\delta \boldsymbol{r}$ does work in three dimensions. The displacement of a body moving at velocity $\boldsymbol{v}$ for time δt is $\delta \boldsymbol{r} = \boldsymbol{v}\delta t$, so the three-dimensional formula for the power is:

$$\text{Power supplied} = \boldsymbol{F} \cdot \boldsymbol{v}.$$

Example

A mass m attached to a spring of spring constant s undergoes simple harmonic motion of amplitude A at angular frequency $\omega = \sqrt{s/m}$:

$$x(t) = A\sin(\omega t).$$

Find an expression for the power supplied by the spring.

The force applied by the spring is given by Hooke's law:

$$F(t) = -sx(t) = -sA\sin(\omega t).$$

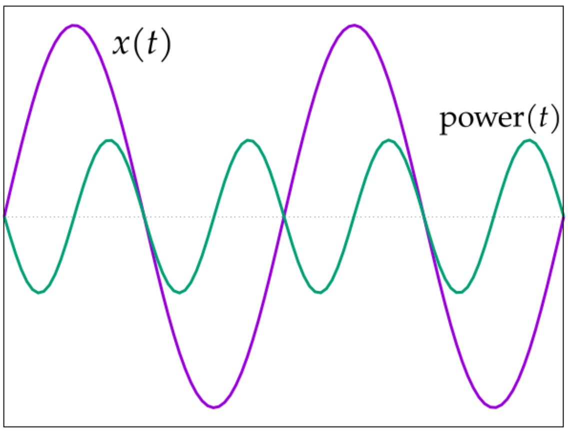
Differentiating the expression for $x(t)$ gives the velocity:

$$v(t) = A\omega\cos(\omega t).$$

These are the ingredients we need to work out the power supplied by the spring:

$$\begin{aligned} \text{Power}(t) = F(t)v(t) &= -s\omega A^2 \sin(\omega t)\cos(\omega t) \\ &= -\frac{1}{2}s\omega A^2 \sin(2\omega t). \end{aligned}$$

Power flows from the spring, increasing the kinetic energy of the mass, when the mass is moving towards the origin, and back into the spring, decreasing the kinetic energy of the mass, when the mass is moving away from the origin. No work is done on average because the mass is oscillating backwards and forwards.



The displacement $x(t)$ and power supplied to an oscillating mass hanging from a spring.

Aside: Momentum versus Kinetic Energy

Kinetic energy and momentum are both properties of bodies in motion, but there are important differences between them:

Momentum	Kinetic Energy
• Proportional to m.	• Proportional to m.
• A vector proportional to $\boldsymbol{v}$.	• A positive scalar proportional to v^2.
• Can be exchanged with other bodies via forces.	• Can be exchanged with other bodies via forces.
• Cannot convert to anything else. Always conserved.	• Can convert to other forms of energy (potential energy, heat, ...). Although total energy is conserved, the kinetic energy alone need not be.

3.7 One-Dimensional Motion with a Position-Dependent Force

This section concerns forces that depend on the position of a body but not on its velocity or on time. Gravitational forces, for which $F = GMm/r^2$, and electrostatic forces, for which $F = q_1 q_2 / 4\pi\varepsilon_0 r^2$, are obvious examples. Almost all of the forces we experience in everyday life are gravitational or electrostatic in origin, so this is a very important special case. It is gravity that accelerates a sled down a snowy hill, holds the atmosphere to the Earth, and keeps the Earth orbiting the Sun. It is electrostatic forces you feel when the Earth pushes up on your feet or you bump into something. Electrostatic forces also bind electrons to nuclei and atoms to each other, and so determine the chemical properties of matter. The only commonly experienced forces that are not functions of position alone are drag and friction (where the force depends on v or the sign of v) and Lorentz magnetic forces given by $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.

This section introduces the idea of a potential energy function, $U(x)$, which provides an alternative description of any position-dependent force field

$F(x)$. (To a physicist, a "force field" is just a function $F(x)$ that maps position to force. This is consistent with the use of the term in Star Trek, I suppose, but rather less exciting.) Switching from a force-based description to a potential-based description brings limited gains in one dimension, but is very useful in three dimensions. This is because the three-dimensional potential function, $U(\boldsymbol{r})$, is a position-dependent scalar and easier to deal with than the three-dimensional force field, $\boldsymbol{F}(\boldsymbol{r})$, which is a position-dependent vector.

Work Done by a Position-Dependent Force

If the applied force only depends on position, the work done in moving a body from x_i to x_f is always the same:

$$W_{if} = \int_{x_i}^{x_f} F(x)dx.$$

The form of the function $F(x)$ is given, so it does not matter when the body left x_i, how fast it travelled, or whether it went past x_f and back again. The work done is just the area under the curve $F(x)$ between $x = x_i$ and $x = x_f$. Contrast this with the case of a force that depends on t and/or v. The force experienced at position x may then be different on different journeys, so W_{if} will be journey-dependent too.

The Potential Energy Function

The potential energy function $U(x)$ is defined by the equation

$$\boxed{F(x) = -\frac{dU(x)}{dx}} \qquad \textbf{(IMPORTANT)}$$

- Beware the minus sign!
- Only the value of $-dU/dx$ matters (it's the force), so we are free to add any constant to $U(x)$. We say that $U(x)$ is defined "to within an arbitrary constant."

It is usual to pick the arbitrary constant by:

(a) Choosing a convenient point x_0.

(b) Setting $U(x_0) = 0$.

Integrating $F = -dU/dx$ from x_0 to x then gives

$$\int_{x_0}^{x} \frac{dU(x')}{dx'} dx' = -\int_{x_0}^{x} F(x')dx'$$
$$\Rightarrow \qquad U(x) - \cancel{U(x_0)} = -\int_{x_0}^{x} F(x')dx',$$

where the $U(x_0)$ term vanished because we chose to set it to zero. We can now write $U(x)$ as an integral:

$$\boxed{U(x) = -\int_{x_0}^{x} F(x')dx'} \qquad \textbf{(IMPORTANT)}$$

In words, the potential function at x is minus the work done in moving from the reference point x_0 to x.

If the force is a function of time or velocity, the work done may depend on when the body left x_0 or the speed at which it moved from x_0 to x. The integral $-\int_{x_0}^{x} F(x')dx'$ from the reference point x_0 to the final point x can still be evaluated, but the result depends on details of the journey from x_0 to x and different journeys give different answers. Since the value of the integral cannot be expressed as a function of x alone, it is impossible to define a potential energy function $U(x)$. There is no equivalent of potential energy for friction and drag forces.

Conservation of Energy

Since $F = -dU/dx$, the work done in moving from x_i to x_f can be expressed in terms of U:

$$W_{if} = \int_{x_i}^{x_f} F(x)dx = \int_{x_i}^{x_f} \left(-\frac{dU(x)}{dx}\right) dx = -U(x_f) + U(x_i).$$

Plugging this into the work-energy theorem,

$$K_f - K_i = W_{if},$$

gives $K_f - K_i = U(x_i) - U(x_f)$, which rearranges into a statement of the conservation of energy:

Conservation of Energy

$$K_f + U(x_f) = K_i + U(x_i)$$

Using only Newton's laws, and assuming that the force $F(x)$ is a function of position alone, we have shown that the sum of the kinetic and potential energies remains constant as the body moves around.

The **potential energy** $U(x)$ is stored in the external system (springs, petrol, gravitational field, electric field) that applies the forces. As the body moves around, the external system may do work on the body, converting some of its stored potential energy into kinetic energy, or the body may do work on the external system, converting some of its kinetic energy into stored potential energy. The **total energy** $K + U(x)$ is always conserved.

Notes

- Since $K + U(x) = \text{const.}$, K is also a function of x. Whenever the body passes position x, regardless of which way it is moving, it always has the same kinetic energy.
- Any force that depends only on x is called **conservative**.
  - Any conservative force field $F(x)$ has a corresponding potential function $U(x)$.
  - Motion in any conservative force field conserves energy.
- Beware:
  - In classical mechanics, $U(x)$ is the potential energy of the body at x.
  - In electromagnetism and electronics, $U(x)$ is the potential energy per unit charge at x. The potential energy of the body is $qU(x)$, where q is the body's charge.
- Like all energies in classical mechanics, $U(x)$ is a (positive or negative) scalar. Its gradient $F(x) = -dU/dx$ is of course a vector.
- Over time, we have identified other forms of energy. These include heat energy (the random jiggling of atoms), field energy (stored in

electric, magnetic, and gravitational fields), and mass energy (Einstein's famous $E = mc^2$ equivalence). It is an empirical observation that total energy is conserved once you have taken all stores of energy into account.

Physicist's believe in energy conservation so strongly that any evidence to the contrary immediately leads them to start looking for new types of energy to balance the books. So far, this has always worked out. A good example was the discovery of the neutrino. After experiments showed an apparent failure of energy (and momentum and angular momentum) conservation in beta decay, Wolfgang Pauli proposed that a hitherto unseen particle must be responsible. Experiments proved him right.

Aside: Conservative Forces in 3D (not examinable)

You might expect conservative force fields to be very rare in three dimensions. Even if the force depends on position only, there are so many possible paths from the reference point $\boldsymbol{r}_0$ to $\boldsymbol{r}$ that it seems hard to imagine that

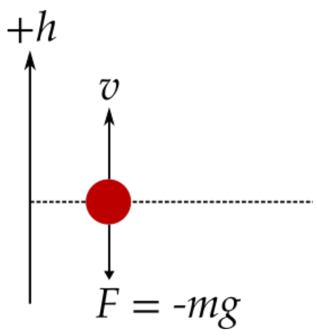
$$-\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}'$$

could be independent of the path taken. If we cannot express the result of the integral as a simple function of the end point $\boldsymbol{r}$, we cannot define a potential energy function $U(\boldsymbol{r})$.

This reasoning is correct: if you invent a three-dimensional force field at random, it is very unlikely to correspond to a potential function. The three-dimensional force fields that occur in nature are not chosen at random, however, and many of them *are* conservative. Why this should be is not obvious, but perhaps it is because energy conservation is a fundamental aspect of physical reality. The gravitational and electrostatic forces that govern the everyday aspects of the world around us are both conservative.

Example: Gravity near the Earth

Imagine throwing a ball of mass m upwards.



The force acting on it is

$$F(h) = -\frac{dU}{dh} = -mg,$$

so the potential function $U(h)$ is given by

$$U(h) = -\int_{h_0}^{h} F(h')dh' = \int_{h_0}^{h} mg\,dh' = mgh - mgh_0.$$

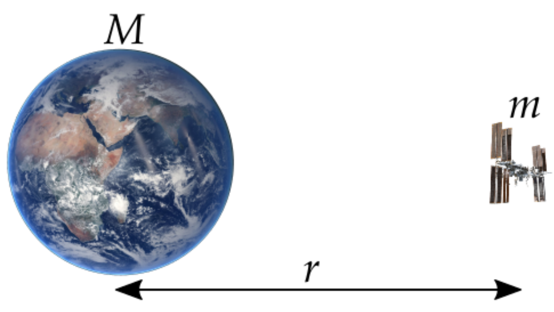
If we choose $h_0 = 0$, we get $U(h) = mgh$.

The conservation of energy in a uniform magnetic field reads:

$$\frac{1}{2}mv^2 + mgh = \text{constant} = E.$$

Example: Gravity far from the Earth

What is the gravitational potential energy far from the Earth?



The force field is

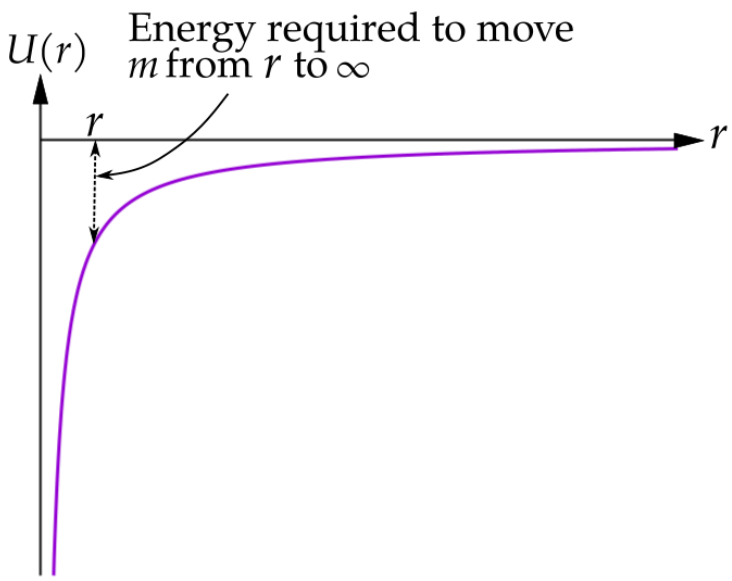
$$F(r) = -\frac{GMm}{r^2} \qquad \text{(negative because force is in } -r \text{ direction)}$$

so

$$U(r) = -\int_{r_0}^{r} F(r')dr' = \int_{r_0}^{r} \frac{GMm}{r'^2}dr' = -\frac{GMm}{r} + \frac{GMm}{r_0}.$$

We now have to pick the position r_0 at which we set $U(r)$ to zero. Choosing $r_0 = 0$ is no good because $GMm/r_0 \to \infty$ as $r_0 \to 0$. The simplest and standard choice is to set r_0 to infinity, in which case

$$U(r) = -\frac{GMm}{r}. \qquad \left(\text{Check: } F(r) = -\frac{dU}{dr} = -\frac{GMm}{r^2} \checkmark\right).$$



- The positive number

$$-U(r) = \int_{\infty}^{r} F(r')dr' \qquad \text{(notice the minus sign)}$$

  is the work done **on** m by the gravitational field as m moves in from ∞ to r. It is positive because $F(r') < 0$ and $dr' < 0$.

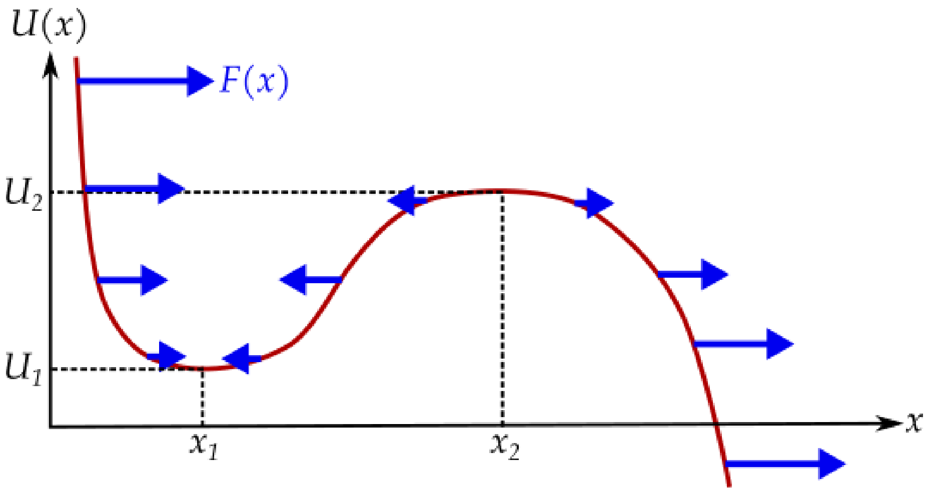
- The same positive number, $-U(r)$, is the work done **by** m on the gravitational field as the particle moves out from r to ∞.

3.8 Potential Functions and Equilibrium

The force $F(x)$ corresponding to the potential function $U(x)$ is given by

$$F(x) = -\frac{dU(x)}{dx} = -\text{ slope of } U(x).$$

To illustrate this idea, here is a diagram of a complicated potential function and the corresponding forces.



The force arrows show that any minimum of $U(x)$ is a point of stable equilibrium: the force is zero at the minimum and pushes particles towards the minimum from either side. Any maximum of $U(x)$ is a point of unstable equilibrium: the force is zero at the maximum but pushes particles away on either side.

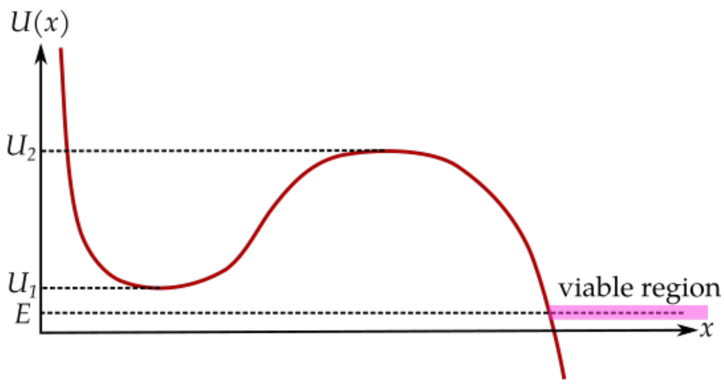
Since this force field was obtained by differentiating a potential function, it must be conservative, so the total energy must be conserved:

$$E = K + U = \text{const.}$$

We can use this and the simple observation that $K = \frac{1}{2}mv^2$ is always greater than or equal to zero to learn quite a lot about the motion of a particle in a potential $U(x)$ without doing any mathematics at all.

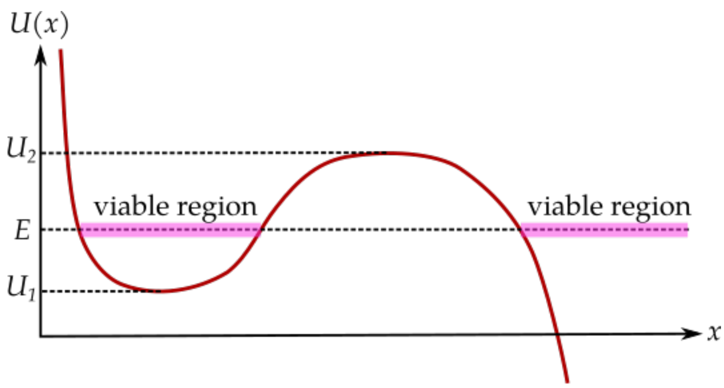
Cases

$\boldsymbol{E < U_1}$**:** Since $E = K + U(x)$ and K is always greater than or equal to zero, the particle is restricted to the viable region shown in pink in the diagram below.



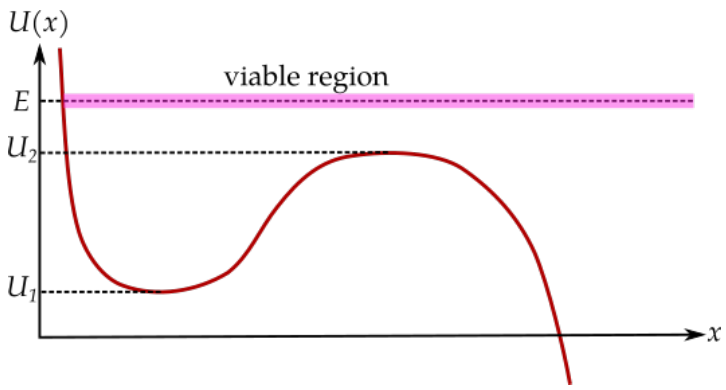
A particle moving to the right in the viable region accelerates right. A particle moving left decelerates until it runs out of kinetic energy at the point where $E = U(x)$ and $K = E - U(x) = 0$. It then turns around starts accelerating off to the right, "bouncing off" the potential barrier.

$\boldsymbol{U_1 < E < U_2}$**:** Now there are two viable regions.



A particle that starts in the well oscillates back and forth, turning around every time it reaches one of the ends of the viable region. A particle that starts in the right-hand viable region behaves like the particle in the case when $E < U_1$. The particle never has enough kinetic energy to surmount the potential barrier and move from one viable region to the other.

$\boldsymbol{E > U_2}$**:** The particle now has enough kinetic energy to move past the potential barrier and there is only one viable region again.



The behaviour of the particle is similar to the case when $E < U_1$, except it slows down and then speeds up again when travelling past the hump.

Ball on a Hill Analogy

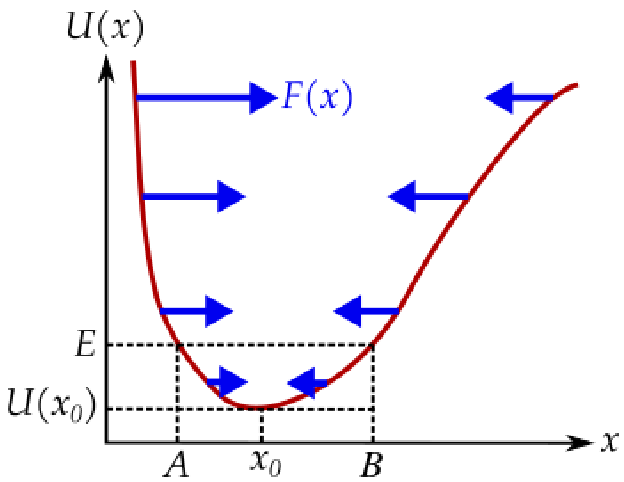
When you feel uncertain about the behaviour of a body moving in a potential $U(x)$, think what would happen to a ball rolling on a hilly landscape with height $h(x) = U(x)/mg$. The gravitational potential energy of the ball, $mgh(x)$, is then equal to $U(x)$.

The analogy is imperfect because the ball on the hill is moving up or down the hillside instead of along the x axis, but it is close enough to give you the right idea.

3.9 Small Oscillations about Equilibrium

We have shown that a particle bound in a potential well oscillates between the points A and B where $E = U(x)$ (see the diagram on the next page). If A and B are close enough to the minimum of the potential function at x_0, which we can always arrange by reducing the total energy E towards $U(x_0)$, almost any smooth potential function is approximately quadratic in the region $A \leq x \leq B$ explored by the particle. Imagine zooming in on the region near x_0. The more you zoom in, the more quadratic the potential function appears to be.

This can be shown mathematically by expanding $U(x)$ in a Taylor series



about its minimum at $x = x_0$:

$$U(x) = U(x_0) + \cancel{U'(x_0)(x - x_0)} + \frac{1}{2!}U''(x_0)(x - x_0)^2 + \cancel{\frac{1}{3!}U'''(x_0)(x - x_0)^3} + \ldots$$

The $U'(x_0)$ term is zero because $dU/dx = 0$ at the minimum of $U(x)$. The cubic and higher order terms are not zero, but are small enough to be ignored when $x - x_0$ is very small. If these terms are not small enough, all you have to do is reduce the energy, so reducing the maximum value of $|x - x_0|$ and making them smaller.

There are functions that do not become approximately quadratic when you zoom in on the minimum enough times — a V-shaped notch is one example — but any function with a Taylor expansion about its minimum (such functions are said to be "analytic at x_0") and a non-zero value of $U''(x_0)$ is approximately quadratic when viewed on a small enough scale.

Let us assume that our potential well is one such function, so that

$$U(x) \approx U(x_0) + \frac{1}{2}U''(x_0)(x - x_0)^2$$

when x is close enough to x_0. Since $F(x) = -dU(x)/dx$, this is equivalent to

$$F(x) = -U''(x_0)(x - x_0),$$

which you should recognise as Hooke's law, $F = -s\Delta x$, with spring constant $s = U''(x_0)$. It follows that:

Almost all small oscillations are simple harmonic!

This surprising result has all sorts of interesting applications.

Molecules exposed to infra-red light feel an oscillating electric field. If the atoms in the molecules are charged, the oscillating electric field excites simple harmonic vibrations of the molecule at the same frequency. The resonant frequencies of the molecule can be used as a fingerprint, allowing unknown molecules to be identified by measuring the infra-red frequencies they absorb best. This is the basis of infra-red vibrational spectroscopy, used in chemistry laboratories all over the world.

Engineering structures vibrate in the wind because they shed eddies at regular time intervals. In most cases the vibrations are small and simple harmonic, but the famous Tacoma Narrows Bridge disaster shows that this need not always be the case.

Pendulum: Although the pendulum is perhaps the most famous example of simple harmonic motion, it is only simple harmonic when the amplitude is small.

A piano string undergoes something close to simple harmonic motion after being struck by the hammer. Because the amplitude of vibration has to be zero where the string is clamped at either end, only standing waves that "match" the length of the string are allowed. (The string vibrates at its lowest possible frequency when the wavelength is twice the length of the string, so that half a wavelength fits along the string.) The frequency depends on the string tension, allowing the piano to be tuned.

Simple Harmonic Motion

The aim of this section is to show that the solutions of Newton's second law for a simple harmonic oscillator are sine-like oscillations.

We start with Newton's second law:

$$m\frac{dv}{dt} = -sx \qquad \Leftrightarrow \qquad \frac{dv}{dt} = -\omega^2 x, \quad \text{where } \omega^2 \triangleq s/m.$$

Method 1

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

- Guess $x(t) = A\sin(\omega t + \phi)$, where A and ϕ are arbitrary constants.
- Plug the guess ("trial solution") in to Newton's second law (exercise for the reader). It works!
- Newton's second law of motion is a second-order differential equation (the highest derivative it contains is a second derivative), so solving it requires two integrations. The most general solution should therefore contain two arbitrary constants. Our trial solution must be the general solution.

Method 2 (for interest only)

Start from the conservation of energy:

$$\begin{aligned} & \frac{1}{2}mv^2 + \frac{1}{2}sx^2 = E \\ \Rightarrow\quad & \frac{1}{2}v^2 + \frac{1}{2}\omega^2 x^2 = \frac{E}{m} \\ \Rightarrow\quad & v^2 = \omega^2 A^2 - \omega^2 x^2, \end{aligned}$$

where A^2 is just a new name for $2E/(m\omega^2)$.

Taking the square root and rewriting v as dx/dt gives

$$\begin{aligned} & \frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \\ \Rightarrow\quad & \int \frac{dx}{\sqrt{A^2 - x^2}} = \omega \int dt \\ \Rightarrow\quad & \sin^{-1}(x/A) = \omega t + \phi \\ \Rightarrow\quad & x = A\sin(\omega t + \phi), \end{aligned}$$

where ϕ is an arbitrary constant of integration. (Strictly, the square root should have been preceded by a $\pm$ sign, but both roots turn out to give equivalent solutions.)

Both methods give the same result:

$$\boxed{x(t) = A\sin(\omega t + \phi)}$$

The constants A and ϕ are fixed by the initial conditions:

$$\begin{aligned} A &= \text{"amplitude"} = \text{maximum value of } x(t), \\ \phi &= \text{phase at } t = 0. \end{aligned}$$

The angular frequency ω is related to the frequency f and time period T via $\omega = 2\pi f = 2\pi/T$.

Chapter 4

Two-Body Dynamics

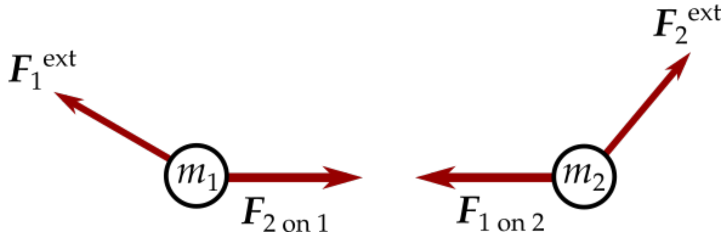
Until now, most of this course has been about how single bodies move under the influence of externally applied forces. We now move on to look at pairs of bodies interacting with each other. The main topic will be collisions, but we will also look at how rockets use momentum conservation to accelerate by ejecting gas at high speed. The ejected gas is not a single body, so rocket motion is not quite a two-body problem, but it is similar in many respects.

4.1 The Centre of Mass and the Reduced Mass

The idea of the centre of mass, introduced here for two interacting particles, is very general, applying to assemblies of arbitrarily many particles, including people and trees. The idea of reduced mass is much more limited, applying only to two-body systems, but is useful enough to be worth knowing about.

Centre-of-Mass Motion

Think about two bodies exerting forces on each other whilst also affected by external forces.



Starting from Newton's second law for both bodies,

$$\begin{aligned} m_1\ddot{\boldsymbol{r}}_1 &= \boldsymbol{F}_{2\,\text{on}\,1} + \boldsymbol{F}_1^{\text{ext}}, \\ m_2\ddot{\boldsymbol{r}}_2 &= \boldsymbol{F}_{1\,\text{on}\,2} + \boldsymbol{F}_2^{\text{ext}}, \end{aligned}$$

and adding the two equations gives

$$m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_{2\,\text{on}\,1} + \boldsymbol{F}_{1\,\text{on}\,2} + \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{F}_2^{\text{ext}}.$$

The internal forces, $\boldsymbol{F}_{1\,\text{on}\,2}$ and $\boldsymbol{F}_{2\,\text{on}\,1}$, cancel out by Newton's third law, and the two external forces add to give the total external force, $\boldsymbol{F}^{\text{ext}}$, so we get

$$\begin{aligned} & m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}^{\text{ext}} \\ \Rightarrow \quad & (m_1+m_2)\left(\frac{m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2}{m_1+m_2}\right) = \boldsymbol{F}^{\text{ext}} \\ \Rightarrow \quad & M\frac{d^2}{dt^2}\left(\frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M}\right) = \boldsymbol{F}^{\text{ext}} \qquad (M \triangleq m_1 + m_2). \end{aligned}$$

If we now define

The Centre-of-Mass Position

$$\boldsymbol{R} \triangleq \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2}$$

the result above can be written in a form that looks like Newton's second law:

$$\boxed{M\ddot{\boldsymbol{R}} = \boldsymbol{F}^{\text{ext}}}$$

As long as the internal forces obey Newton's third law, the centre of mass accelerates like a point particle of mass $M = m_1 + m_2$, feeling only the total external force.

Notes

- If $\boldsymbol{F}^{\text{ext}} = \boldsymbol{0}$, then $M\ddot{\boldsymbol{R}} = \boldsymbol{0}$ and the centre-of-mass momentum, $M\dot{\boldsymbol{R}}$, is constant.

- It is easy to show (try it!) that an analogous result holds for N interacting bodies. The centre of mass, defined by

$$\boldsymbol{R} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{M}, \qquad M = \sum_{i=1}^{N} m_i,$$

evolves according to Newton's second law,

$$M\ddot{\boldsymbol{R}} = \boldsymbol{F}^{\text{ext}},$$

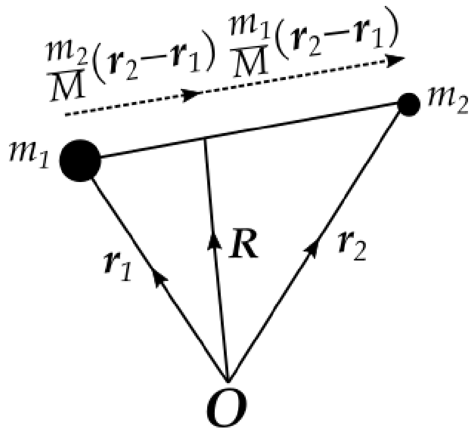
feeling only the total external force,

$$\boldsymbol{F}^{\text{ext}} = \sum_{i=1}^{N} \boldsymbol{F}_i^{\text{ext}}.$$

- The centre of mass is the mass-weighted average position. To help understand this statement, it is useful to rewrite the definition of $\boldsymbol{R}$, adding and subtracting $m_2\boldsymbol{r}_1$ to and from the numerator:

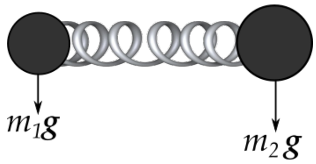
$$\boldsymbol{R} = \frac{(m_1+m_2)\boldsymbol{r}_1 + m_2(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{M} = \boldsymbol{r}_1 + \frac{m_2}{M}(\boldsymbol{r}_2 - \boldsymbol{r}_1).$$

The diagram below illustrates this vector equation:



  - $\boldsymbol{R}$ lies on the line from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$.
  - If $m_1 > m_2$, $\boldsymbol{R}$ is closer to $\boldsymbol{r}_1$.
  - If $m_2 > m_1$, $\boldsymbol{R}$ is closer to $\boldsymbol{r}_2$.
  - If $m_1 = m_2$, $\boldsymbol{R}$ lies midway between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

Example: Two Masses on a Spring



Since $\boldsymbol{F}^{\text{ext}} = m_1\boldsymbol{g} + m_2\boldsymbol{g} = M\boldsymbol{g}$, Newton's second law for the centre of mass reads:

$$M\ddot{\boldsymbol{R}} = M\boldsymbol{g} \quad \Rightarrow \quad \ddot{\boldsymbol{R}} = \boldsymbol{g}.$$

The centre of mass free falls, regardless of the internal forces.

Internal Motion

The equation governing the internal motion of a two-body system can also be made to look like Newton's second law. Return to the two-masses-on-a-spring example above but concentrate on the separation vector $\boldsymbol{r}_2 - \boldsymbol{r}_1$ instead of the centre-of-mass position $\boldsymbol{R}$. Newton's second law for the two masses,

$$\begin{aligned} m_1\ddot{\boldsymbol{r}}_1 &= \boldsymbol{F}_{2\,\text{on}\,1} + m_1\boldsymbol{g} = -\boldsymbol{F}_{1\,\text{on}\,2} + m_1\boldsymbol{g}, \\ m_2\ddot{\boldsymbol{r}}_2 &= \boldsymbol{F}_{1\,\text{on}\,2} + m_2\boldsymbol{g}, \end{aligned}$$

can be rearranged to give

$$\begin{aligned} \ddot{\boldsymbol{r}}_1 &= \frac{-\boldsymbol{F}_{1\,\text{on}\,2}}{m_1} + \boldsymbol{g}, \\ \ddot{\boldsymbol{r}}_2 &= \frac{\boldsymbol{F}_{1\,\text{on}\,2}}{m_2} + \boldsymbol{g}. \end{aligned}$$

Subtracting the first equation from the second then gives

$$\frac{d^2}{dt^2}(\boldsymbol{r}_2 - \boldsymbol{r}_1) = \boldsymbol{F}_{1\,\text{on}\,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

If we re-express this in terms of the position of the second body relative to the first,

$$\boldsymbol{r} \triangleq \boldsymbol{r}_2 - \boldsymbol{r}_1,$$

and the *reduced mass* μ defined by

Reduced Mass

$$\frac{1}{\mu} \triangleq \frac{1}{m_1} + \frac{1}{m_2} \qquad \left(\Leftrightarrow \ \mu = \frac{m_1 m_2}{m_1 + m_2}\right)$$

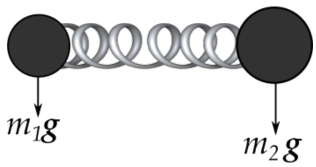
it looks like Newton's second law:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\boldsymbol{F}_{1\,\text{on}\,2}}{\mu} \qquad \Rightarrow \qquad \mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_{1\,\text{on}\,2}.$$

Notes

- The internal motion (of $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$) is governed by the internal forces and the reduced mass μ.
- Warning: if the external forces cause the two bodies to accelerate at different rates, the internal motion is more complicated. The concept of reduced mass is only useful in two-body systems experiencing gravitational or zero external forces.
- Unlike the centre of mass, the concept of reduced mass does not generalise easily to more than two particles.

Example: Two Masses on a Spring



The internal coordinate $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ evolves according to

$$\mu\ddot{\boldsymbol{r}} = -s(\boldsymbol{r} - \boldsymbol{r}_{\text{eq}}),$$

where s is the spring constant and $\boldsymbol{r}_{\text{eq}}$ is the end-to-end vector of the spring at its natural length. The motion is one-dimensional, so we drop the vector notation from now on. Introducing a new variable $d \triangleq r - r_{\text{eq}}$, and noting that $\ddot{d} = \ddot{r}$, the equation of motion becomes

$$\mu\ddot{d} = -sd.$$

This is the equation of simple harmonic motion.

- If $m_1 = m_2$, $\mu = m_1/2$ and

  $$\ddot{d} = -\frac{2s}{m_1}d.$$

  The factor of two on the right-hand side arises because the two masses feel equal and opposite internal forces and *both* accelerate. This makes the acceleration of the length of the spring twice as large as you might at first expect.

- If $m_2 \gg m_1$, $\mu \approx m_1$ and

  $$\ddot{d} \approx -\frac{s}{m_1}d.$$

  The factor of two has vanished because the second body is so massive that it hardly moves. It is like attaching m_1 via a spring to a wall.

Summary

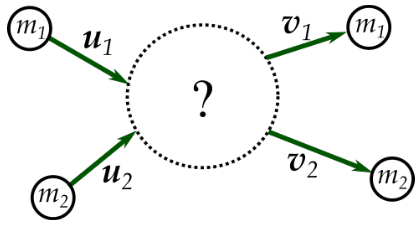
External forces determine the motion of the centre of mass. The relevant mass is $M = m_1 + m_2$.

Internal forces determine the internal motion. The relevant mass is $\mu = m_1 m_2/(m_1 + m_2)$.

4.2 Collisions

Collisions are brief interactions of two or more bodies mediated by forces. Our aim here is to relate the situation well before the collision, when the bodies are far enough apart that the forces between them are almost zero, to the situation well after the collision, when the bodies are again far apart and again non-interacting. Gravity is so long ranged that it is hard to move two masses far enough apart to make the attraction between them nearly zero, but most of the other interactions between everyday objects die off reassuringly rapidly with distance.

A general two-body collision looks like this:



In this diagram: $\begin{cases} \boldsymbol{u}_1 \text{ and } \boldsymbol{u}_2 \text{ are the velocities before the collision,} \\ \boldsymbol{v}_1 \text{ and } \boldsymbol{v}_2 \text{ are the velocities after the collision.} \end{cases}$

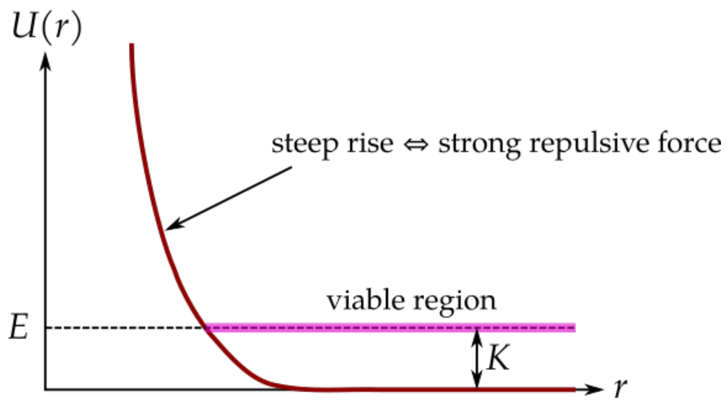
The details of the collision — the question mark in the diagram — are not very important, but one thing we know for sure is that momentum is conserved at all times before, during and after the collision:

$$m_1 \dot{\boldsymbol{v}}_1 = \boldsymbol{F}_{2\,\text{on}\,1} = -\boldsymbol{F}_{1\,\text{on}\,2} = -m_2 \dot{\boldsymbol{v}}_2,$$

so

$$m_1 \dot{\boldsymbol{v}}_1 + m_2 \dot{\boldsymbol{v}}_2 = \frac{d}{dt}\left(m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2\right) = \boldsymbol{0}.$$

If the interaction is conservative, so that $F(r) = -dU(r)/dr$, the total energy $E = K + U(r)$ is also conserved.

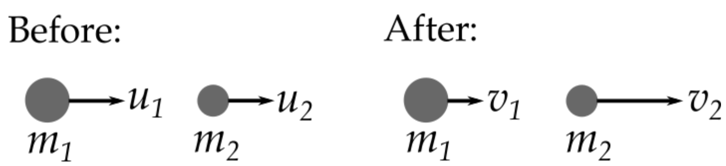


Beyond the range of the interaction, $U(r)$ is a constant (usually chosen equal to zero), so the values of the kinetic energy well before and well after the collision must be the same. The kinetic energy varies during the collision, but returns to its initial value once the collision is over. Collisions in which the incoming and outgoing kinetic energies are the same are said to be **elastic**. As we have just seen, elastic collisions occur when the forces between the bodies are conservative.

In most collisions between everyday objects (although not between fundamental particles), some energy is lost as heat and the outgoing kinetic energy is smaller than the incoming kinetic energy. These collisions are called **inelastic**.

4.3 Elastic Collisions in One Dimension

We start with the simplest possible case. A particle of mass m_1 and velocity u_1 undergoes a head-on elastic collision with a particle of mass m_2 and velocity u_2. The velocities of the two particles after the collision are v_1 and v_2.



Any of u_1, u_2, v_1, and v_2 may be negative (the only constraints are $u_1 > u_2$, so that the collision happens in the first place, and $v_2 > v_1$, so that the particles do not pass through each other), but I find it easier to get the signs right when the vector arrows in the diagram all point right.

The statements of conservation of momentum and kinetic energy,

$$\begin{aligned} m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2, \\ \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2, \end{aligned}$$

can be rearranged into

$$\begin{aligned} m_1(u_1 - v_1) &= m_2(v_2 - u_2), \\ m_1(u_1^2 - v_1^2) &= m_2(v_2^2 - u_2^2). \end{aligned}$$

Dividing the second equation by the first,

$$\frac{u_1^2 - v_1^2}{u_1 - v_1} = \frac{v_2^2 - u_2^2}{v_2 - u_2},$$

and remembering that $a^2 - b^2 = (a - b)(a + b)$ gives

$$\frac{\cancel{(u_1 - v_1)}(u_1 + v_1)}{\cancel{u_1 - v_1}} = \frac{\cancel{(v_2 - u_2)}(v_2 + u_2)}{\cancel{v_2 - u_2}}$$

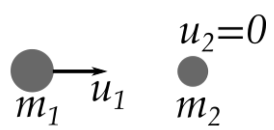
and so

$$v_2 - v_1 = -(u_2 - u_1).$$

This says that the relative velocity of particle 2 (with respect to particle 1) changes sign during the collision, while the relative speed stays the same. The vector equivalent of this relationship between velocities is *not* true in three-dimensional elastic collisions, but the relative speed is still conserved: $|\boldsymbol{v}_2 - \boldsymbol{v}_1| = |\boldsymbol{u}_2 - \boldsymbol{u}_1|$.

Example: Elastic Collision with a Stationary Target

Suppose that m_2 is a stationary target.



Substituting the velocity reversal equation just derived,

$$v_2 - v_1 = -(u_2 - u_1) = u_1 \qquad \Rightarrow \qquad v_1 = v_2 - u_1,$$

into the conservation of momentum equation,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 \qquad \Rightarrow \qquad v_1 + \frac{m_2}{m_1} v_2 = u_1,$$

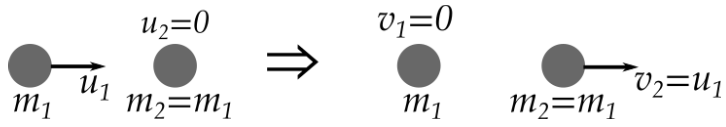
gives

$$\left(1+\frac{m_2}{m_1}\right)v_2 = 2u_1 \qquad \Rightarrow \qquad \boxed{v_2 = \frac{2m_1}{m_1+m_2}u_1}$$

and

$$v_1 = \frac{2m_1}{m_1+m_2}u_1 - u_1 \qquad \Rightarrow \qquad \boxed{v_1 = \frac{m_1-m_2}{m_1+m_2}u_1}$$

- If $m_1 = m_2$, then $v_1 = 0$ and $v_2 = u_1$:



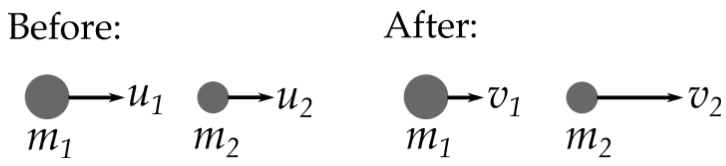
  The incoming particle stops and the outgoing particle picks up all of the momentum, kinetic energy and velocity.

- If $m_1 > m_2$, then v_1 and v_2 are both positive. After the collision, both particles are moving to the right.

- If $m_1 < m_2$, then $v_1 < 0$ (the incoming particle bounces back and moves off to the left) and $v_2 > 0$ (the target particle moves right).

- If $m_1 \ll m_2$ (think of a ball bouncing on the Earth), then $v_1 \approx -u_1$ and $v_2 \approx 0$. The incoming particle bounces off the target, reversing its momentum and retaining almost all of its speed and kinetic energy; the final speed of the target particle is very small.

4.4 Inelastic Collisions in One Dimension



For elastic collisions, we showed using momentum and kinetic energy conservation that

$$v_2 - v_1 = -(u_2 - u_1).$$

This is no longer true when the collision is inelastic, but the empirical law

$$v_2 - v_1 = -e(u_2 - u_1)$$

holds in many cases. The constant e (which has nothing at all to do with the base of natural logarithms $e = 2.7182818\ldots$) is called the **coefficient of restitution**.

- If $e = 1$, the collision is elastic.
- If $e = 0$, then $v_2 - v_1 = 0$. This means that the two bodies stick together like lumps of putty and move off together. The internal contribution to the kinetic energy is zero after the collision because the internal coordinate $r = r_2 - r_1$ is no longer changing; but the conservation of total momentum requires the centre-of-mass kinetic energy to stay the same.
- If $e > 1$, the kinetic energy after the collision is larger than the kinetic energy before. Perhaps the collision of the two bodies caused an explosion or released energy previously stored in a coiled spring.

If the target is initially stationary ($u_2 = 0$), one can show (see question 4 of problem sheet 6) that

$$v_1 = \frac{m_1 - em_2}{m_1 + m_2} u_1, \qquad v_2 = \frac{(1+e)m_1}{m_1 + m_2} u_1.$$

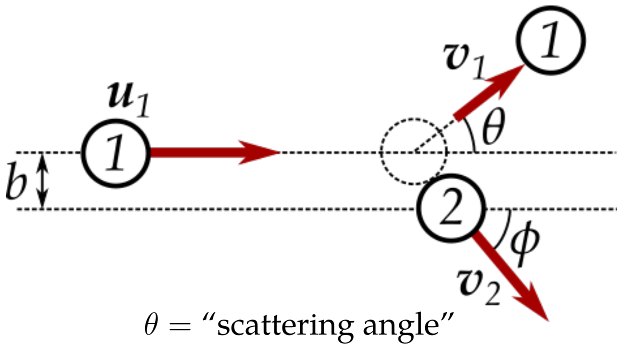
When $e = 1$, these equations reduce to the corresponding results for elastic collisions derived in Sec. 4.3.

4.5 Introduction to Collisions in Three Dimensions

Analysing a general three-dimensional collision is quite hard, so we start with a simple example.

Example: Elastic Collision of Equal Masses, One Initially Stationary

If the two masses are elastic balls of equal mass, $m_1 = m_2 = m$, the collision might look something like this.



$\theta =$ "scattering angle"

$\phi =$ "recoil angle"

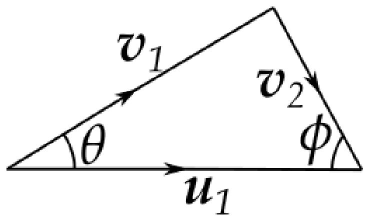
$b =$ "impact parameter"

If the masses are positive point charges they will never touch (that would cost infinite energy), but the incoming and outgoing trajectories will be approximately straight far from the collision zone and we can still describe the collision using the labels $\boldsymbol{u}_1$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, b, θ, ϕ. Remember that we are primarily interested in the trajectories well before and well after the collision; the details of what happens during the collision are less important.

As in one dimension, the physics is determined by momentum conservation and kinetic energy conservation. The momentum conservation law tells us that

$$
\begin{aligned}
& m\boldsymbol{u}_1 + m\boldsymbol{0} = m\boldsymbol{v}_1 + m\boldsymbol{v}_2 \\
\Rightarrow \quad & \boldsymbol{u}_1 = \boldsymbol{v}_1 + \boldsymbol{v}_2,
\end{aligned}
$$

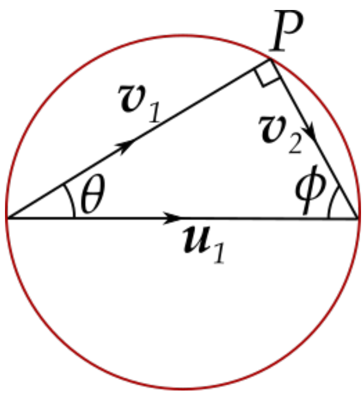
implying that $\boldsymbol{u}_1$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ form a vector triangle.



The kinetic energy conservation law tells us that:

$$
\begin{aligned}
& \frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \\
\Rightarrow \quad & u_1^2 = v_1^2 + v_2^2.
\end{aligned}
$$

This is Pythagoras's theorem, so the triangle must be right angled.



If follows that the angle between the scattered and recoiling trajectories is always $\pi/2$,

$$\theta + \phi = \frac{\pi}{2},$$

and that the right-angled corner of the velocity triangle lies somewhere on the red circle of radius $u_1/2$. The exact location of the corner depends on the impact parameter b. Real collision experiments rarely attempt to fix b, so scattering is seen at angles all around the circle.

Example: Elastic Collision of Unequal Masses, One Initially Stationary

This is a harder problem that will take up most of the rest of the chapter. The first difficulty is that $\boldsymbol{u}_1$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ no longer form a vector triangle. Momentum is still conserved, so the momentum vectors $\boldsymbol{p}_1 \triangleq m_1\boldsymbol{u}_1$, $\boldsymbol{q}_1 \triangleq m_1\boldsymbol{v}_1$, and $\boldsymbol{q}_2 \triangleq m_2\boldsymbol{v}_2$ form a triangle,

$$\boldsymbol{p}_1 = \boldsymbol{q}_1 + \boldsymbol{q}_2,$$

but this is a harder result to use.

The second difficulty is that the momentum triangle is not right-angled, so $\theta + \phi$ is no longer equal to $\pi/2$. The right-angled nature of the equal-mass velocity triangle (or, equivalently, the equal-mass momentum triangle) arose because cancelling the common factor of $m_1 = m_2$ from the conservation of kinetic energy left Pythagoras's theorem. Now that the masses are different, this no longer works.

The aim of the rest of this chapter is to provide a careful solution of the problem of elastic collisions of unequal masses. Since the impact parameter b is hard to measure, we avoid it by asking question such as:

Given $\boldsymbol{u}_1$, m_1, m_2 and ϕ (or θ), what are $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and θ (or ϕ)?

Answering questions like these is much easier if you work in the centre-of-mass frame.

4.6 The Centre-of-Mass Frame

Starting from the definition of the centre of mass,

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{M},$$

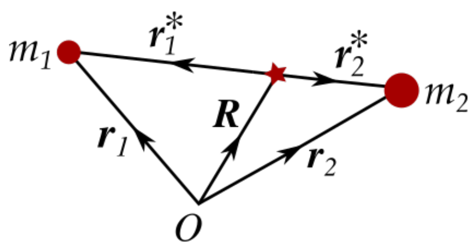
and differentiating twice with respect to time gives

$$\ddot{\boldsymbol{R}} = \frac{m_1\ddot{\boldsymbol{r}}_1 + m_2\ddot{\boldsymbol{r}}_2}{M} = \frac{\boldsymbol{F}_{2\,\mathrm{on}\,1} + \boldsymbol{F}_{1\,\mathrm{on}\,2}}{M} = \boldsymbol{0},$$

by Newton's third law. The acceleration of the centre of mass is zero, so the centre-of-mass velocity $\dot{\boldsymbol{R}}$ must be constant.

Assuming that the laboratory frame is inertial, it follows that the moving frame with its origin on the centre of mass is also inertial: Newton's laws hold in both frames and energy and momentum are conserved in both frames. In more human terms, an observer sitting on the centre of mass has just as much right to argue that she is "at rest" as an observer in the laboratory frame.

The diagram below shows two masses, their coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ relative to a fixed origin O in the laboratory frame, and their coordinates $\boldsymbol{r}_1^*$ and $\boldsymbol{r}_2^*$ relative to the centre of mass $\boldsymbol{R}$, which is indicated by a red star.



The diagram shows that $\boldsymbol{r}_1^* = \boldsymbol{r}_1 - \boldsymbol{R}$ and $\boldsymbol{r}_2^* = \boldsymbol{r}_2 - \boldsymbol{R}$. Since the centre of mass always lies on the line between m_1 and m_2 (see Sec. 4.1), $\boldsymbol{r}_1^*$ and $\boldsymbol{r}_2^*$ are always anti-parallel.

Momentum in the Centre-of-Mass Frame

The centre-of-mass momentum vectors $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ are easy enough to work out:

$$\begin{aligned}
\boldsymbol{p}_1^* &= m_1\dot{\boldsymbol{r}}_1^* \\
&= m_1(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{R}}) \\
&= m_1\dot{\boldsymbol{r}}_1 - \frac{m_1(m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2)}{M} \\
&= \frac{m_1(M-m_1)}{M}\dot{\boldsymbol{r}}_1 - \frac{m_1m_2}{M}\dot{\boldsymbol{r}}_2 \\
&= \frac{m_1m_2}{M}(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2) \\
&= -\mu\dot{\boldsymbol{r}},
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{p}_2^* &= m_2\dot{\boldsymbol{r}}_2^* \\
&= m_2(\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{R}}) \\
&= m_2\dot{\boldsymbol{r}}_2 - \frac{m_2(m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2)}{M} \\
&= \frac{m_2(M-m_2)}{M}\dot{\boldsymbol{r}}_2 - \frac{m_1m_2}{M}\dot{\boldsymbol{r}}_1 \\
&= \frac{m_1m_2}{M}(\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{r}}_1) \\
&= \mu\dot{\boldsymbol{r}},
\end{aligned}$$

where $\mu = m_1m_2/M$ is the reduced mass and $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the position of m_2 relative to m_1. Since

$$\boldsymbol{p}_1^* + \boldsymbol{p}_2^* = -\mu\dot{\boldsymbol{r}} + \mu\dot{\boldsymbol{r}} = \boldsymbol{0},$$

the centre-of-mass frame is also the zero-momentum frame. (See question 4 of problem sheet 5 for a more general proof of this result.)

Kinetic Energy in the Centre-of-Mass Frame

It is useful to work out the relationship between the kinetic energies in the laboratory and centre-of-mass frames. Remembering that $|\boldsymbol{a}+\boldsymbol{b}|^2 = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b}) = a^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + b^2$, we get

$$\begin{aligned}
K &= \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 \\
&= \frac{1}{2}m_1\left|\dot{\boldsymbol{r}}_1^* + \dot{\boldsymbol{R}}\right|^2 + \frac{1}{2}m_2\left|\dot{\boldsymbol{r}}_2^* + \dot{\boldsymbol{R}}\right|^2 \\
&= \frac{1}{2}m_1\left[(\dot{r}_1^*)^2 + 2\dot{\boldsymbol{r}}_1^*\cdot\dot{\boldsymbol{R}} + \dot{R}^2\right] + \frac{1}{2}m_2\left[(\dot{r}_2^*)^2 + 2\dot{\boldsymbol{r}}_2^*\cdot\dot{\boldsymbol{R}} + \dot{R}^2\right] \\
&= \frac{1}{2}m_1(\dot{r}_1^*)^2 + \frac{1}{2}m_2(\dot{r}_2^*)^2 + \cancel{(m_1\dot{\boldsymbol{r}}_1^* + m_2\dot{\boldsymbol{r}}_2^*)}\cdot\dot{\boldsymbol{R}} + \frac{1}{2}(m_1+m_2)\dot{R}^2 \\
&= K^* + \frac{1}{2}M\dot{R}^2.
\end{aligned}$$

The cross term vanished because the total momentum in the centre-of-mass frame is zero: $\boldsymbol{p}_1^* + \boldsymbol{p}_2^* = \boldsymbol{0}$.

In words, we have just shown that:

Separation of the Kinetic Energy

(KE in laboratory frame) = (KE in centre-of-mass frame) + (KE of centre of mass)

This important result also holds for collections of arbitrarily large numbers of particles or bodies.

The expression for K^*, the KE in the centre-of-mass frame, can be simplified as follows:

$$\begin{aligned}
K^* &= \frac{1}{2}m_1\left(\dot{r}_1^*\right)^2 + \frac{1}{2}m_2\left(\dot{r}_2^*\right)^2 \\
&= \frac{\left(p_1^*\right)^2}{2m_1} + \frac{\left(p_2^*\right)^2}{2m_2} && \left[\text{as } \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}\right] \\
&= \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(\mu\dot{r})^2 && [\text{as } \boldsymbol{p}_2^* = -\boldsymbol{p}_1^* = \mu\dot{\boldsymbol{r}}] \\
&= \frac{1}{2}\mu\dot{r}^2. && \left[\text{as } \frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}\right]
\end{aligned}$$

Expressed in words, this says that

Kinetic Energy in the Centre-of-Mass Frame

$$\frac{1}{2}(\text{reduced mass}) \times (\text{relative velocity})^2$$

4.7 Elastic Scattering in the Centre-of-Mass Frame

When you work in the centre-of-mass frame, collision problems are *much* simpler. The main advantage is that the incoming momenta,

$$\boldsymbol{p}_1^* = \boldsymbol{p}^* \qquad \text{and} \qquad \boldsymbol{p}_2^* = -\boldsymbol{p}^*,$$

are equal and opposite. The total momentum in the centre-of-mass frame starts off equal to zero and must remain equal to zero at all times. It follows that the outgoing momenta $\boldsymbol{q}_1^*$ and $\boldsymbol{q}_2^*$ are also equal and opposite:

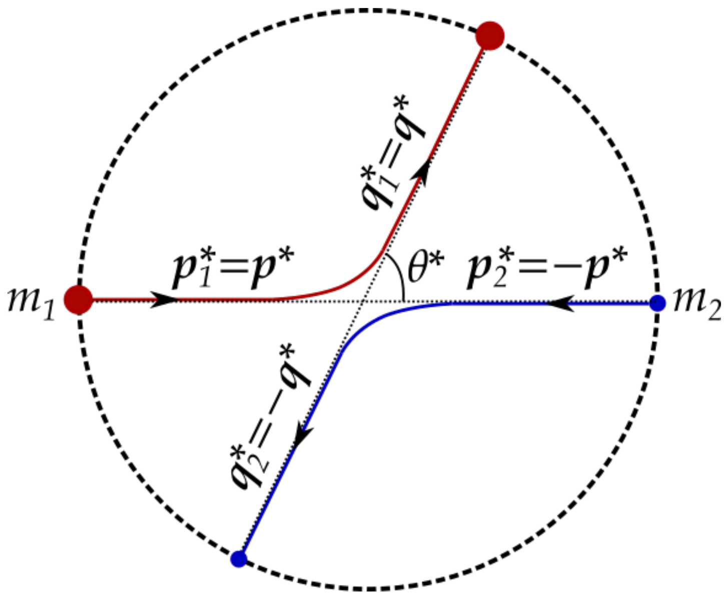
$$\boldsymbol{q}_1^* = \boldsymbol{q}^* \qquad \text{and} \qquad \boldsymbol{q}_2^* = -\boldsymbol{q}^*.$$

The vectors $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ are not the same as each other, but elastic collisions conserve kinetic energy, so

$$\begin{aligned} & \frac{(p_1^*)^2}{2m_1} + \frac{(p_2^*)^2}{2m_2} = \frac{(q_1^*)^2}{2m_1} + \frac{(q_2^*)^2}{2m_2} \\ \Rightarrow \quad & \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(p^*)^2 = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(q^*)^2 \\ \Rightarrow \quad & p^* = q^*. \end{aligned}$$

Although $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ may point in different directions, they must have the same length.

As seen in the centre-of-mass frame, then, all collisions are simple.



To characterise a collision completely, all we need to do is specify $\boldsymbol{p}^*$ and θ^*. The centre-of-mass-frame scattering angle θ^* depends on the impact parameter b and will be different for different collisions, even when $\boldsymbol{p}^*$ is the same.

4.8 Transforming Back to the Laboratory Frame

Characterising collisions in the centre-of-mass frame is easy, but we need to transform the vectors and angles back into the laboratory frame before we can compare them with experiment. Given $\boldsymbol{p}^*$ and θ^*, how do we find $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, θ and ϕ?

This turns out to be awkward, but it is manageable if you work carefully, step by step. It is easiest to derive expressions for the vectors ($\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$) before going after the angles (θ and ϕ).

Vectors

(a) m_2 was initially stationary (in the laboratory frame), so

$$\boxed{\boldsymbol{p}_2 = \boldsymbol{0}} \tag{4.8.1}$$

(b) The next step is to find the velocity $\dot{\boldsymbol{R}}$ of the centre of mass. Once we have that, everything else becomes much easier. One way to derive an expression for $\dot{\boldsymbol{R}}$ uses the observation that $\dot{\boldsymbol{r}}_2 = \boldsymbol{0}$ before the collision:

$$\begin{aligned} \boldsymbol{p}_2^* &= m_2\dot{\boldsymbol{r}}_2^* && \text{(by definition)} \\ &= m_2(\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{R}}) && \text{(frame transformation)} \\ &= -m_2\dot{\boldsymbol{R}}. && \text{(because } \dot{\boldsymbol{r}}_2 = \boldsymbol{0}\text{)} \end{aligned}$$

Since $\boldsymbol{p}_2^* = -\boldsymbol{p}^*$ (see the diagram at the end of Sec. 4.7), this gives

$$\boxed{\dot{\boldsymbol{R}} = \frac{\boldsymbol{p}^*}{m_2}} \tag{4.8.2}$$

(c) We can find $\boldsymbol{p}_1$ by differentiating the definition of $\boldsymbol{R}$,

$$(m_1 + m_2)\dot{\boldsymbol{R}} = m_1\dot{\boldsymbol{r}}_1 + \cancel{m_2\dot{\boldsymbol{r}}_2} = \boldsymbol{p}_1,$$

and using Eq. (4.8.2):

$$\boxed{\boldsymbol{p}_1 = \left(\frac{m_1}{m_2} + 1\right)\boldsymbol{p}^*} \tag{4.8.3}$$

We now know everything about the situation before the collision.

(d) After the collision,

$$\begin{aligned} && \dot{\boldsymbol{r}}_1 &= \dot{\boldsymbol{r}}_1^* + \dot{\boldsymbol{R}} && \text{(frame transformation)} \\ \Rightarrow && m_1\dot{\boldsymbol{r}}_1 &= m_1\dot{\boldsymbol{r}}_1^* + m_1\dot{\boldsymbol{R}}. \end{aligned}$$

The laboratory-frame and centre-of-mass-frame momenta of particle 1 after the collision are $\boldsymbol{q}_1$ and $\boldsymbol{q}_1^*$, so this becomes:

$$\boxed{\boldsymbol{q}_1 = \boldsymbol{q}^* + \frac{m_1}{m_2}\boldsymbol{p}^*} \qquad \text{(using Eq. (4.8.2) again)} \tag{4.8.4}$$

(e) We can use the same approach to find $\boldsymbol{q}_2$:

$$\begin{aligned} \dot{\boldsymbol{r}}_2 &= \dot{\boldsymbol{r}}_2^* + \dot{\boldsymbol{R}} && \text{(frame transformation)} \\ \Rightarrow \qquad m_2\dot{\boldsymbol{r}}_2 &= m_2\dot{\boldsymbol{r}}_2^* + m_2\dot{\boldsymbol{R}}, \end{aligned}$$

and hence

$$\boxed{\boldsymbol{q}_2 = -\boldsymbol{q}^* + \boldsymbol{p}^*} \tag{4.8.5}$$

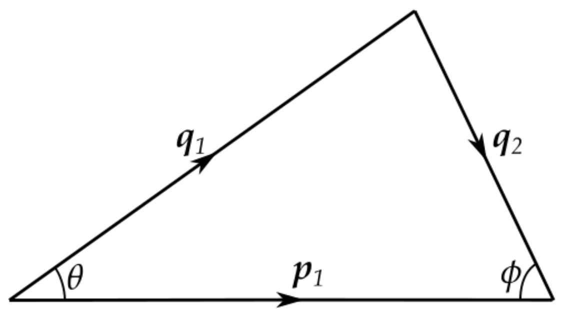
We now have expressions for the laboratory-frame momentum vectors of both particles before and after the collision in terms of $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$. Before going on to work out the angles, it is worth making a sanity check:

$$\begin{aligned} \boldsymbol{q}_1 + \boldsymbol{q}_2 &= \left(\frac{m_1}{m_2} + 1\right)\boldsymbol{p}^* && \text{[using Eqs. (4.8.4) and (4.8.5)]} \\ &= \boldsymbol{p}_1. \quad \checkmark && \text{[using Eq. (4.8.3)]} \end{aligned}$$

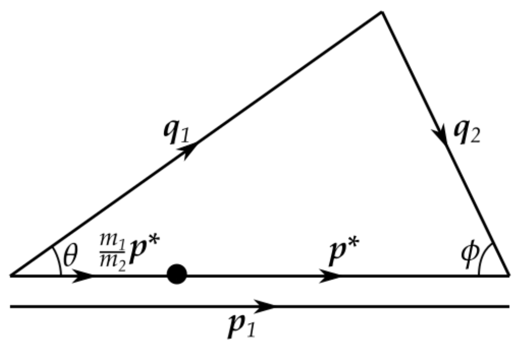
Angles

The easiest way to find the angles is to use the vector equations derived above to draw a vector diagram and then use trigonometry.

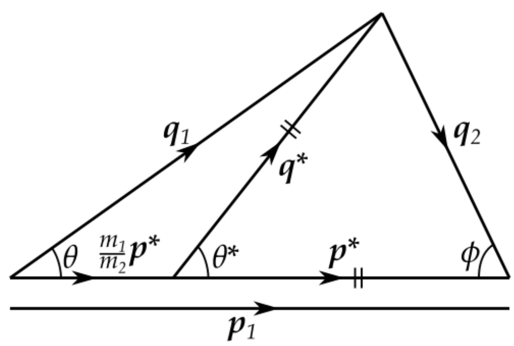
(a) First draw the $\boldsymbol{p}_1 = \boldsymbol{q}_1 + \boldsymbol{q}_2$ momentum-conservation vector triangle.



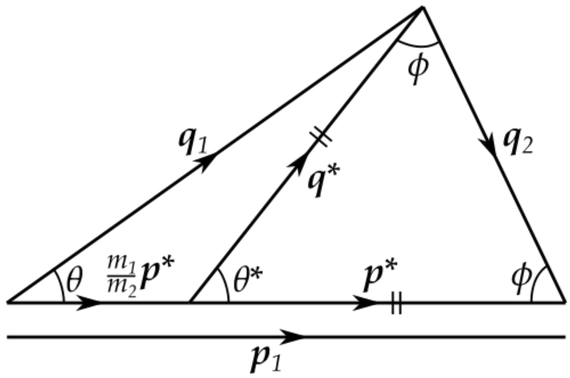
(b) Now use $\boldsymbol{p}_1 = \left(\frac{m_1}{m_2} + 1\right) \boldsymbol{p}^*$ to divide $\boldsymbol{p}_1$ into two parts.



(c) Next use $\boldsymbol{q}_1 = \frac{m_1}{m_2}\boldsymbol{p}^* + \boldsymbol{q}^*$ to add $\boldsymbol{q}^*$ and then θ^* (which is the angle between $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$) to the diagram.



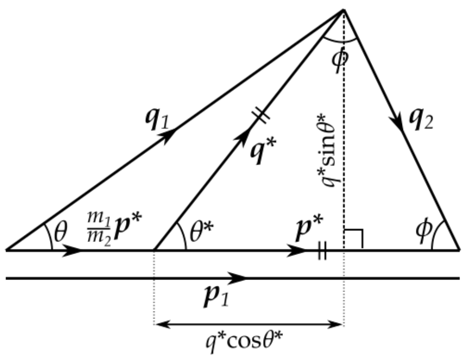
(d) Since $q^* = p^*$, the triangle on the right-hand side is an isosceles triangle and the third angle within it is also ϕ.



This allow us to relate the recoil angle ϕ to the centre-of-mass scattering angle θ^*:

$$\boxed{\phi = \frac{1}{2}(\pi - \theta^*)}$$

(e) To find an equation for θ, it is helpful to drop a vertical line from the top corner of the triangle. Two right-angled triangles are created to the left of the vertical line.



Applying trigonometry to these triangles gives an equation for $\tan\theta$.

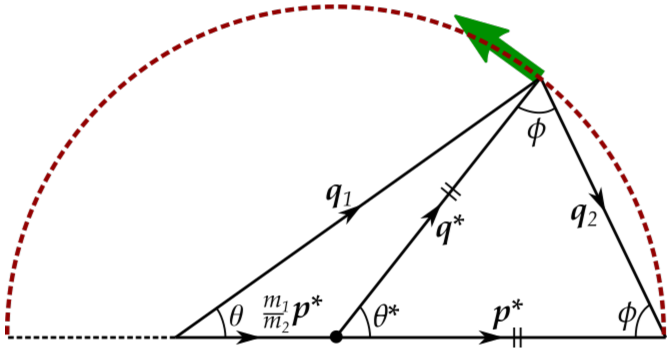
$$\boxed{\tan\theta = \frac{q^*\sin\theta^*}{\frac{m_1}{m_2}p^* + q^*\cos\theta^*} = \frac{\sin\theta^*}{\frac{m_1}{m_2} + \cos\theta^*}}$$

We have now succeeded in our aim of expressing all of the laboratory-frame scattering vectors and angles in terms of the centre-of-mass-frame parameters $\boldsymbol{p}^*$, $\boldsymbol{q}^*$ and θ^*.

Maximum Scattering Angle

Imagine gradually increasing θ^* from 0 to π, keeping $p^* = q^*$ constant. When is the scattering angle θ largest?

Case 1: $m_1 < m_2$ (as assumed in all of the diagrams drawn so far).

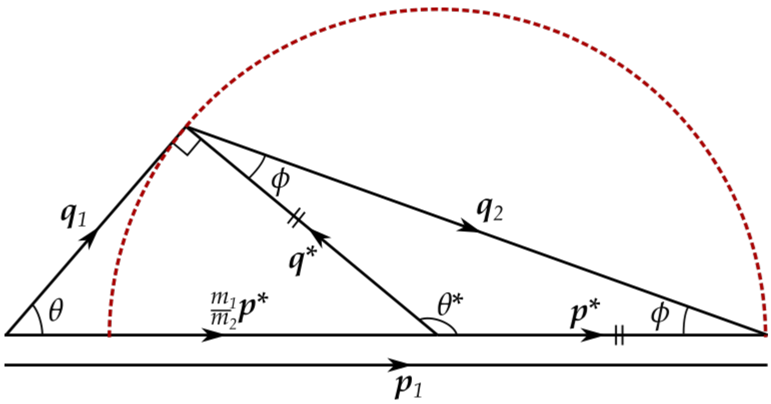


The diagram shows that, as θ^* increases from 0 to π, so does θ. The maximum scattering angle is π.

$$\boxed{\theta_{\text{max}} = \pi}$$

This should come as no surprise as we reached the same conclusion in Sec. 4.3, where we discovered that a smaller mass "bounces back" after a head-on $(b = 0)$ collision with a larger mass.

Case 2: $m_1 < m_2$

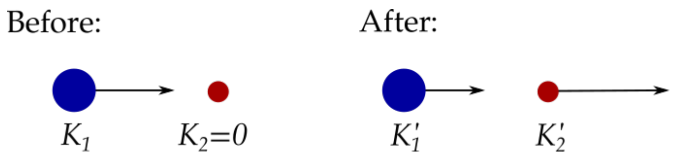


The scattering angle θ is now largest when $\boldsymbol{q}_1$ is tangent to the circle of radius $p^* = q^*$. Using trigonometry on the right-angled triangle shown in the diagram gives:

$$\boxed{\sin\theta_{\text{max}} = \frac{q^*}{\frac{m_1}{m_2}p^*} = \frac{m_2}{m_1}}$$

Kinetic Energy Transfer

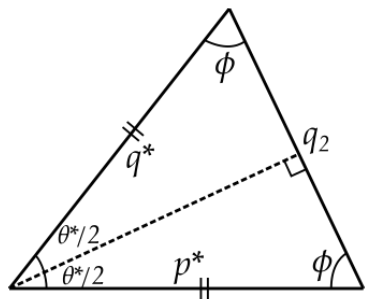
What fraction of the incident kinetic energy is transferred to a stationary target in an elastic collision?



To work this out, we need to find K_2'/K_1. The incident kinetic energy K_1 is easy:

$$K_1 = \frac{p_1^2}{2m_1} = \frac{\left(\frac{m_1}{m_2}+1\right)(p^*)^2}{2m_1}.$$

To find K_2', we redraw the triangle on the right-hand side of of the momentum diagram.



Trigonometry gives $q_2 = 2p^*\sin(\theta^*/2)$, leading us to

$$K_2' = \frac{(q_2)^2}{2m_2} = \frac{4(p^*)^2\sin^2(\theta^*/2)}{2m_2}.$$

We are now ready to work out the fraction of the incident kinetic energy transferred to the target:

$$\begin{aligned}
\frac{K_2'}{K_1} &= \frac{4\cancel{(p^*)^2}\sin^2\left(\theta^*/2\right)/\cancel{2}m_2}{\left(\frac{m_1}{m_2}+1\right)^2\cancel{(p^*)^2}/\cancel{2}m_1} \\
&= \frac{4m_1\sin^2\left(\theta^*/2\right)}{m_2\left(\frac{m_1}{m_2}+1\right)^2} \\
&= \frac{4m_1 m_2\sin^2\left(\theta^*/2\right)}{(m_1+m_2)^2}.
\end{aligned}$$

The fraction of energy transferred is largest when $\sin(\theta^*) = 1$ (or, equivalently, $\theta^* = \pi$):

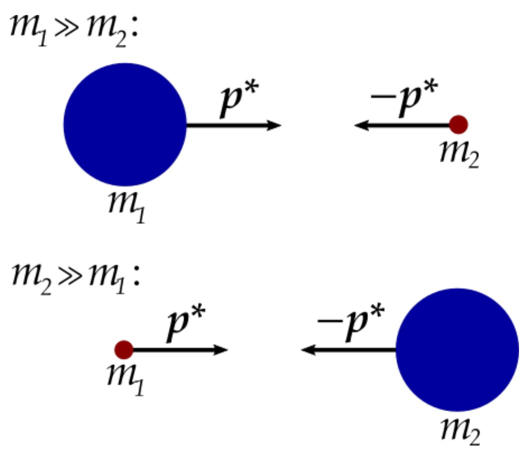
$$\left(\frac{K_2'}{K_1}\right)_{\text{max}} = \frac{4m_1 m_2}{(m_1+m_2)^2}.$$

- If $m_1 = m_2$, $(K_2'/K_1)_{\text{max}} = 1$, so the incident particle stops and the target particle picks up 100% of the incident kinetic energy. We showed this earlier in Sec. 4.3 when considering one-dimensional collisions.

- If $m_1 \gg m_2$, $(K_2'/K_1)_{\text{max}} \approx 4m_2/m_1$ is very small. The heavy target hits the light particle and starts it moving, but the speed of the target is always less than twice the speed of the incoming projectile [see Sec. 4.3, where we showed that $v_2 = 2m_1u_1/(m_1+m_2)$] and its kinetic energy is small because it has such a small mass.

- If $m_2 \gg m_1$, $(K_2'/K_1)_{\text{max}} \approx 4m_1/m_2$ is very small. The light projectile bounces back from the heavy target, transferring twice its initial momentum but very little energy.

When seen in the centre-of-mass frame, the collisions with $m_1 \gg m_2$ and $m_2 \gg m_1$ are mirror images of each other:



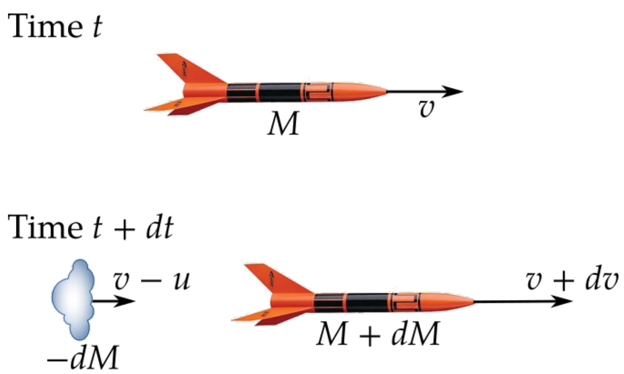
This explains the $m_1 \leftrightarrow m_2$ symmetry in the results for $m_1 \gg m_2$ and $m_2 \ll m_1$.

4.9 The Rocket Equation

Rockets work by ejecting mass at high speed. Because the total momentum of the rocket and ejected fuel must remain constant, pushing fuel out of the back makes the rocket accelerate forwards. The only purpose of the rocket engine, and the only reason it burns the fuel, is to make sure that the fuel is ejected as fast as possible.

Supposed that the current speed of the rocket is $v(t)$ and that its current mass (including the fuel it is carrying but has not yet burnt) is $M(t)$. The exhaust velocity — the speed at which fuel is ejected relative to the rocket — is u. The rocket loses mass, so $dM/dt < 0$. Since $dt > 0$, it follows that $dM < 0$.

$$\text{In time } dt \quad \begin{cases} \text{change in mass of rocket:} & dM < 0 \\ \text{mass of fuel ejected:} & -dM > 0 \end{cases}$$



If the rocket is not yet moving fast, $v - u < 0$ and the fuel is moving left (despite the direction of the arrow drawn in the diagram). If the rocket is moving fast, v might be greater than u.

Referring to the diagram and using momentum conservation gives

$$\begin{aligned} & (M + dM)(v + dv) + (-dM)(v - u) = Mv \\ \Rightarrow \quad & \cancel{Mv} + \cancel{vdM} + Mdv + \cancel{dMdv} - \cancel{vdM} + udM = \cancel{Mv}. \end{aligned}$$

The $dMdv$ term was crossed out not because it cancels another term but because it is proportional to $(dt)^2$. In the limit as $dt \to 0$, it becomes smaller and smaller in comparison with the Mdv and udM terms, which are proportional to dt. When dt is small enough, the $dMdv$ term can be neglected.

After this simplification, the conservation of momentum equation reads:

$$\begin{aligned} & Mdv = -udM \\ \Rightarrow \quad & \int_{v_0}^{v(t)} dv' = -u \int_{M_0}^{M(t)} \frac{dM'}{M'} \\ \Rightarrow \quad & v(t) - v_0 = -u \left[\ln M'\right]_{M_0}^{M(t)} \\ \Rightarrow \quad & v(t) = v_0 - u \ln\left(\frac{M(t)}{M_0}\right), \end{aligned}$$

with M_0 the initial mass and v_0 the initial speed of the rocket. Since $M(t) < M_0$ (fuel has been burnt), the logarithm is negative. This is awkward, so we choose to rewrite the result as:

The Rocket Equation

$$v(t) = v_0 + u \ln\left(\frac{M_0}{M(t)}\right)$$

It is interesting to see that the rocket can end up going faster than the exhaust speed u.

The problem with the idea of using rockets to travel across the galaxy is the logarithm, which rises very slowly as $M_0/M(t)$ gets large. Suppose, for example, that a rocket starts at rest, already having escaped Earth's gravity, and that 99% of its mass is fuel. The mass of the astronauts, the payload, the rocket motor, and the structure that contains the fuel accounts for the other 1%. A reasonable estimate for u is $3.4\,\text{km}{\cdot}\text{s}^{-1}$, which is the exhaust velocity of Space X's "Falcon 9 Full Thrust" rocket. The final velocity of the rocket, after all of the fuel has been burnt, is

$$v_{\text{final}} = u \ln\left(\frac{M_0}{0.01M_0}\right) \approx 3.4\ln(100) \approx 15.7\,\text{km}\cdot\text{s}^{-1}.$$

The speed of light $c \approx 3 \times 10^8\,\text{m}{\cdot}\text{s}^{-1}$, so

$$v_{\text{final}} \approx 5.2 \times 10^{-5} c.$$

The 4.22 light-year journey to Proxima Centauri would take 81,000 years! This may be why we are not regularly visited by aliens.

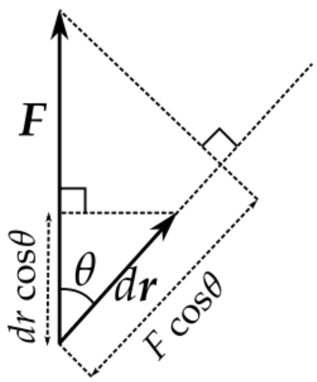
Chapter 5

Three-Dimensional Motion

5.1 Work in Three Dimensions

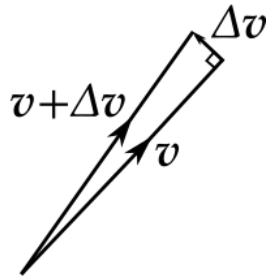
The work dW done by a force $\boldsymbol{F}$ on a body that moves by $d\boldsymbol{r}$ is

$$\boldsymbol{F} \cdot d\boldsymbol{r} \quad = \quad \underbrace{(F\cos\theta)}_{\text{component of } \boldsymbol{F} \text{ along } d\boldsymbol{r}} dr \quad = \quad F\underbrace{(dr\cos\theta)}_{\text{component of } d\boldsymbol{r} \text{ along } \boldsymbol{F}}$$



Because of the dot product, components of $\boldsymbol{F}$ perpendicular to the displacement vector $d\boldsymbol{r} = \boldsymbol{v}dt$ do no work. This makes intuitive sense.

There is another way to understand why components of $\boldsymbol{F}$ perpendicular to $\boldsymbol{v}$ do no work. Since $\boldsymbol{F} = md\boldsymbol{v}/dt$, applying a force $\boldsymbol{F}$ for a time Δt changes the velocity by $\Delta\boldsymbol{v} \approx \boldsymbol{F}\Delta t/m$. The change in velocity is parallel to $\boldsymbol{F}$, and is perpendicular to $\boldsymbol{v}$ if $\boldsymbol{F}$ is perpendicular to $\boldsymbol{v}$.



Pythagoras's theorem then tells us the length $|\boldsymbol{v}|$ of the vector $\boldsymbol{v}$ changes by

$$\begin{aligned}
&\sqrt{v^2 + (\Delta v)^2} - v \\
&\quad = v\left[1 + \left(\frac{\Delta v}{v}\right)^2\right]^{1/2} - v \\
&\quad = v\left[1 + \frac{1}{2}\left(\frac{\Delta v}{v}\right)^2 + \ldots\right] - v \qquad \text{(binomial expansion)} \\
&\quad = \frac{(\Delta v)^2}{2v} + \ldots,
\end{aligned}$$

which is *second order* in $\Delta v = |\Delta \boldsymbol{v}|$. Thus, although the acceleration

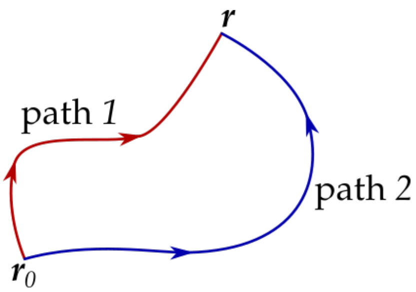
$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta \boldsymbol{v}}{\Delta t}\right)$$

is finite, the rate of change of the speed is zero:

$$\frac{d|\boldsymbol{v}|}{dt} = \lim_{\Delta t \to 0}\left(\frac{(\Delta v)^2}{2v\Delta t}\right) = 0.$$

Since the speed cannot change, neither can the kinetic energy. The work-energy theorem says that the work done is the change in kinetic energy, and the change in kinetic energy is zero, so perpendicular forces cannot do any work.

In three dimensions, the work done when a body under the influence of a position-dependent force $\boldsymbol{F}(\boldsymbol{r})$ moves from some arbitrary reference point $\boldsymbol{r}_0$ to any chosen final point $\boldsymbol{r}$ normally depends on the path taken. This differs from one dimension, where any force field $F(x)$ that depends only on position is conservative.



$$\int_{\text{path 1}} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \neq \int_{\text{path 2}} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

Because of the path dependence of the work done, you cannot in general define a potential $U(\boldsymbol{r})$ via

$$\cancel{U(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}'}$$

as we did in one dimension. (If you tried, you would find that different paths from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ gave different results. Which would you say is the value of $U(\boldsymbol{r})$?)

Fortunately, almost all of the important forces in physics are *central*, and position-dependent central force fields are always conservative, even in three dimensions.

5.2 Central Forces

The force $\boldsymbol{F}(\boldsymbol{r})$ between two bodies or particles separated by $\boldsymbol{r}$ is called central if:

1. It always acts parallel or anti-parallel to the vector $\boldsymbol{r}$ between the two bodies.
2. Its magnitude only depends on the distance $r = |\boldsymbol{r}|$ between the two bodies.

The formula for a central force field looks like this:

$$\boldsymbol{F}(\boldsymbol{r}) = F(r)\hat{\boldsymbol{r}}.$$

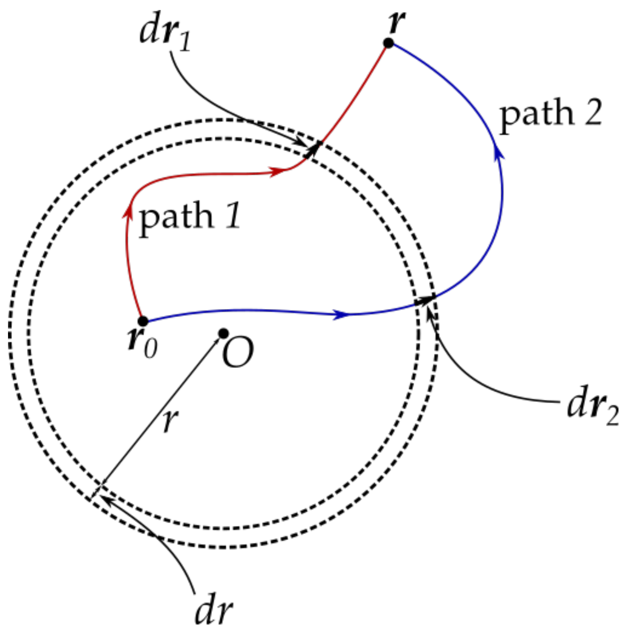
The value of the function $F(r)$ is a number, not a vector, and its argument $r = |\boldsymbol{r}|$ is the distance between the two bodies, not the vector between them. The direction of $\boldsymbol{r}$ does not affect the magnitude of the force.

The most famous central force field is gravity,

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GMm}{r^2}\hat{\boldsymbol{r}},$$

for which $F(r) = -GMm/r^2$. The electrostatic forces that bind electrons to atoms and atoms into molecules and solids are also central. The only non-central force commonly encountered in everyday physics is the magnetic Lorentz force, $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$.

As can be understood by thinking about the diagram below, the work done by a central force field does *not* depend on the path taken.



The vectors $d\boldsymbol{r}_1$ and $d\boldsymbol{r}_2$ are the segments of paths 1 and 2 spanning the gap between the circle of radius r and the circle of radius $r+dr$. To find the contributions these segments make to the work done, we need to evaluate

$$dW_1 = \boldsymbol{F}(\boldsymbol{r}_1) \cdot d\boldsymbol{r}_1 \quad \text{and} \quad dW_2 = \boldsymbol{F}(\boldsymbol{r}_2) \cdot d\boldsymbol{r}_2,$$

where $\boldsymbol{r}_1$ is the position vector (measured from the origin O) of path segment $d\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is the position vector of path segment $d\boldsymbol{r}_2$. The force field is central, so

$$\boldsymbol{F}(\boldsymbol{r}_1) = F(r)\hat{\boldsymbol{r}}_1 \quad \text{and} \quad \boldsymbol{F}(\boldsymbol{r}_2) = F(r)\hat{\boldsymbol{r}}_2,$$

with $\hat{\boldsymbol{r}}_1$ a unit vector pointing radially outwards at the position of $d\boldsymbol{r}_1$ and $\hat{\boldsymbol{r}}_2$ a unit vector pointing radially outwards at the position of $d\boldsymbol{r}_2$. Since both path segments are the same distance r from the origin, the magnitude $F(r)$ of the central force is the same for both.

Combining the equations above gives

$$dW_1 = F(r)\hat{\boldsymbol{r}}_1 \cdot d\boldsymbol{r}_1 = F(r)dr \quad \text{and} \quad dW_2 = F(r)\hat{\boldsymbol{r}}_2 \cdot d\boldsymbol{r}_2 = F(r)dr.$$

The dot product of the radial unit vector $\hat{\boldsymbol{r}}_1$ with $d\boldsymbol{r}_1$ yields the radial component of $d\boldsymbol{r}_1$, which is just the distance dr between the two circles. The same applies to $\hat{\boldsymbol{r}}_2$, of course, so both dot products give the same result. The conclusion is that dW_1 and dW_2 are equal.

By drawing lots of concentric circles centred on the origin, with radii ranging from $|\boldsymbol{r}_0|$ to $|\boldsymbol{r}|$, it is possible to pair the contributions to the two work integrals segment by segment. The contributions made by the two segments in every pair are the same, so the two integrals must also be the same:

$$\int_{\text{path 1}} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{\text{path 2}} \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{r} = \int_{r_0}^{r} F(r')dr'.$$

Notice that the final integral in the equation above is an ordinary one-dimensional integral from r_0 to r and contains no vectors. This is a great simplification.

We have just shown that all central force fields are conservative, so we can now choose a reference point $\boldsymbol{r}_0$ and define a potential function $U(\boldsymbol{r})$ in the usual way:

$$U(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}' = -\int_{r_0}^{r} F(r')dr'.$$

No path need be specified because all paths from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ give the same answer. The fact that the answer can also be expressed as a simple one-dimensional radial integral tells us that the potential $U(\boldsymbol{r})$ only depends on the distance r from the origin: $U(\boldsymbol{r}) = U(|\boldsymbol{r}|) = U(r)$. The direction of $\boldsymbol{r}$ is irrelevant.

The fundamental theorem of calculus states that

$$\frac{d}{dr}\int_{r_0}^{r} F(r')dr' = F(r),$$

so the force can be found from the potential using the standard one-dimensional formula:

$$F(r) = -\frac{dU(r)}{dr}.$$

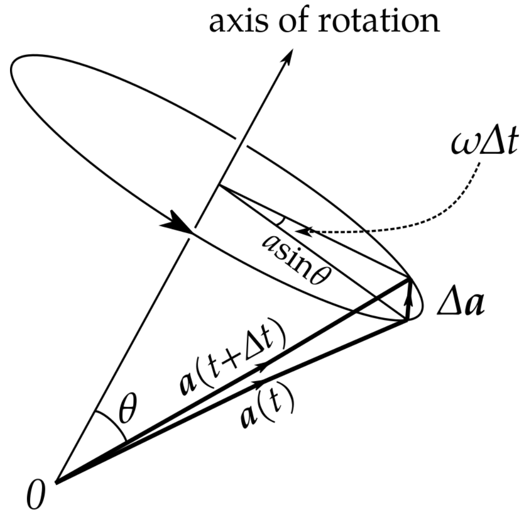
(Note: We made use of the theory described here in Section 3.7, when we worked out the gravitational potential energy far from the Earth.)

5.3 Rotating Vectors

When working in three dimensions, one has to take account of the possibility that vectors may change their direction as well as their magnitude. Rotating vectors crop up frequently in the rest of this course, so now is a good time to get to grips with them. They are most easily described using the angular velocity vector and the vector cross product.

Rotating Vectors of Fixed Length

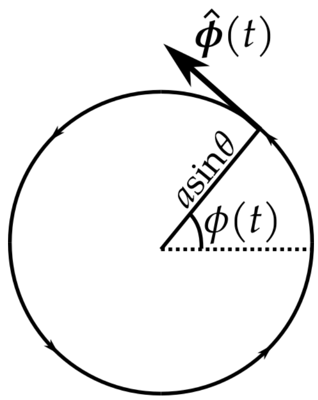
For simplicity, it is best to start with a rotating vector $\boldsymbol{a}(t)$ of fixed length: $|\boldsymbol{a}(t)| = a =$ constant.



- The first step in learning to describe $\boldsymbol{a}(t)$ mathematically is to imagine moving its tail to the origin and keeping it there. As you may remember from Professor Vvedensky's lectures, mathematical vectors have no idea of their location — the three components describe the position of the head relative to the tail without specifying the position of the tail itself — so moving the tail to the origin does nothing at all. Nevertheless, I find it aids intuition to imagine doing it.

- The diagram on the previous page shows a rotating vector with its tail on the origin. The head of $\boldsymbol{a}(t)$ sweeps around the circle of radius $a \sin\theta$ at rate ω Radians per second, where ω is known as the angular velocity. We are not assuming that the rotation rate is constant, so ω may depend on time. The direction of the axis of rotation may also change with time.

- As $\Delta t \to 0$
  - $|\Delta \boldsymbol{a}| \to$ arc length $= (a \sin\theta)(\omega \Delta t)$.
  - The direction of $\Delta \boldsymbol{a}$ becomes tangential to the circle. We call this the $\hat{\boldsymbol{\phi}}$ direction from now on. Looking down on the circle along the rotation axis, the $\hat{\boldsymbol{\phi}}$ vector looks like this:



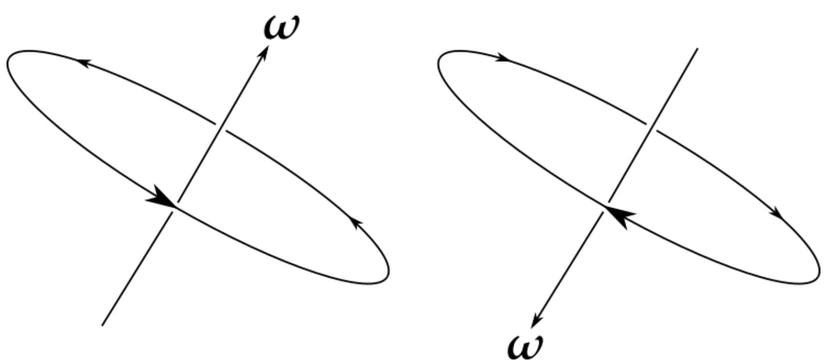
  - Hence

$$\frac{d\boldsymbol{a}}{dt} = \lim_{\Delta t \to 0} \left(\frac{\Delta \boldsymbol{a}}{\Delta t} \right) = \lim_{\Delta t \to 0} \left(\frac{(a \sin\theta)(\omega \Delta t) \hat{\boldsymbol{\phi}}}{\Delta t} \right) = \omega a \sin\theta \, \hat{\boldsymbol{\phi}}.$$

- Describing $\hat{\boldsymbol{\phi}}$ in words is awkward, so it is fortunate that there is a better way. Define the **angular velocity vector**

$$\boldsymbol{\omega} = \omega \hat{\boldsymbol{\omega}},$$

where $\hat{\boldsymbol{\omega}}$ is a unit vector along the axis of rotation, with its direction (up or down) chosen using the right-hand rule.



One way to work out the direction of the angular velocity vector is to point the thumb of your right hand along the rotation axis. If your fingers naturally curl in the direction of rotation, your thumb is pointing in the $\boldsymbol{\omega}$ direction. If not, turn your hand over so that the thumb is again pointing along the rotation axis but in the opposite direction. Your fingers will now be curling in the direction of rotation and your thumb will be pointing along $\boldsymbol{\omega}$.

- Referring back to the diagram on the page 88 shows that $\boldsymbol{\omega} \times \boldsymbol{a} = |\boldsymbol{\omega}||\boldsymbol{a}| \sin\theta\, \hat{\boldsymbol{\phi}}$, so we get

Rate of Change of a Vector of Fixed Length
$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{\omega} \times \boldsymbol{a}$$

Rotating Vectors of Variable Length

If the length of $\boldsymbol{a}$ depends on time, write $\boldsymbol{a}(t) = a(t)\hat{\boldsymbol{a}}(t)$, where $a(t) = |\boldsymbol{a}(t)|$ and $\hat{\boldsymbol{a}}(t)$ is a unit vector in the direction of $\boldsymbol{a}(t)$. Since unit vectors have fixed length, we already know that $d\hat{\boldsymbol{a}}/dt = \boldsymbol{\omega} \times \hat{\boldsymbol{a}}$. Differentiating $\boldsymbol{a}(t)$ using the product rule gives

$$\frac{d\boldsymbol{a}}{dt} = \frac{da}{dt}\hat{\boldsymbol{a}} + a\frac{d\hat{\boldsymbol{a}}}{dt} = \dot{a}\hat{\boldsymbol{a}} + a\boldsymbol{\omega} \times \hat{\boldsymbol{a}} = \dot{a}\hat{\boldsymbol{a}} + \boldsymbol{\omega} \times \boldsymbol{a}.$$

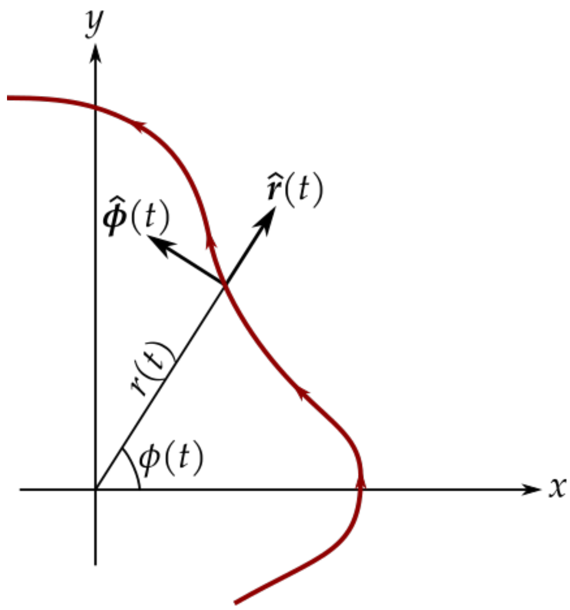
Rate of Change of a Vector of Variable Length
$$\frac{d\boldsymbol{a}}{dt} = \dot{a}\hat{\boldsymbol{a}} + \boldsymbol{\omega} \times \boldsymbol{a}$$

The rate of change has two perpendicular parts: the $\dot{a}\hat{\boldsymbol{a}}$ term is parallel to $\boldsymbol{a}(t)$ and the $\boldsymbol{\omega} \times \boldsymbol{a}$ term is perpendicular to $\boldsymbol{a}(t)$.

Example: Rotating Vectors in Two Dimensions

Working in two dimensions may seem artificial, but we will see in the next section that motion under a central force field is always planar, so two dimensions are enough to describe planetary orbits. Because we are interested in separating the radial and angular components of the trajectory, it is best to work in plane polar coordinates.

Suppose a particle moves along the trajectory $\boldsymbol{r}(t)$ as shown in the diagram below.



The angular velocity vector $\boldsymbol{\omega}(t) = \dot{\phi}(t)\hat{\boldsymbol{k}}$ is perpendicular to the xy plane in which the trajectory lies. In the diagram above, $\boldsymbol{\omega}$ is pointing out of the paper because $\dot{\phi}(t) > 0$.

Because the vector $\boldsymbol{r}(t)$ with plane-polar coordinates $(r(t), \phi(t))$ depends on t, so do the unit vectors $\hat{\boldsymbol{r}}(t)$ and $\hat{\boldsymbol{\phi}}(t)$ glued to the end of $\boldsymbol{r}(t)$:

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{r}} = \omega\hat{\boldsymbol{\phi}},$$
$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{\phi}} = -\omega\hat{\boldsymbol{r}}.$$

The $\boldsymbol{\omega}$ vector is always perpendicular to $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$, so the "$\sin\theta$" terms in the cross products are always equal to one. The right-hand rule tells us that $\boldsymbol{\omega} \times \hat{\boldsymbol{r}}$ points in the $\hat{\boldsymbol{\phi}}$ direction and that $\boldsymbol{\omega} \times \hat{\boldsymbol{\phi}}$ points in the $-\hat{\boldsymbol{r}}$ direction.

Velocity

We are now ready to derive general expressions for the velocity and acceleration of the particle. Differentiating $\boldsymbol{r}(t) = r(t)\hat{\boldsymbol{r}}(t)$ gives

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = \dot{r}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt} = \dot{r}\hat{\boldsymbol{r}} + \omega r\hat{\boldsymbol{\phi}},$$

showing that the velocity can always be split into radial and angular parts:

$$v_r = \dot{r}, \qquad\qquad v_\phi = \omega r$$

(Note that the angular part of the velocity vector is not the same thing as the angular velocity vector. Sorry!)

Acceleration

There are no surprises so far, but the expression for the acceleration is more interesting:

$$\begin{aligned}
\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} &= \left(\ddot{r}\hat{\boldsymbol{r}} + \dot{r}\frac{d\hat{\boldsymbol{r}}}{dt}\right) + \left(\dot{\omega}r\hat{\boldsymbol{\phi}} + \omega\dot{r}\hat{\boldsymbol{\phi}} + \omega r\frac{d\hat{\boldsymbol{\phi}}}{dt}\right) \\
&= \left(\ddot{r}\hat{\boldsymbol{r}} + \omega\dot{r}\hat{\boldsymbol{\phi}}\right) + \left([\dot{\omega}r + \omega\dot{r}]\hat{\boldsymbol{\phi}} - \omega^2 r\hat{\boldsymbol{r}}\right) \\
&= \left(\ddot{r} - \omega^2 r\right)\hat{\boldsymbol{r}} + \left(2\omega\dot{r} + \dot{\omega}r\right)\hat{\boldsymbol{\phi}}.
\end{aligned}$$

The radial component, $\ddot{r} - \omega^2 r$, has two contributions. The $\ddot{r}$ term describes changes in the radial component of the velocity and is non-zero even when the particle is moving directly towards or away from the origin and $\omega = 0$. The other term,

$$-\omega^2 r = -\frac{v^2}{r}, \qquad\qquad \text{(because } v = r\omega\text{)}$$

is the centripetal acceleration you met at school. The difference is that the derivation we have just been through is not restricted to circular motion. Whenever the motion of the particle has a rotational part, there is a centripetal contribution to the radial acceleration.

Angular Momentum

The form of the angular component of the acceleration is less easy to explain intuitively but has important consequences when the force acting on the particle is central. In that case Newton's second law reads

$$F(r)\hat{\boldsymbol{r}} = m(\ddot{r} - \omega^2 r)\hat{\boldsymbol{r}} + m(2\omega\dot{r} + \dot{\omega}r)\hat{\boldsymbol{\phi}},$$

and equating the radial and angular components gives:

$$F(r) = m\ddot{r} - m\omega^2 r, \qquad\qquad 0 = 2m\omega\dot{r} + m\dot{\omega}r.$$

We are going to revisit the radial equation of motion in a couple of lectures when discussing orbits. Multiplying and dividing the angular equation by r gives

$$\begin{aligned} & \frac{1}{r}\left(2m\omega r\dot{r} + m\dot{\omega}r^2\right) = 0 \\ \Rightarrow \quad & \frac{1}{r}\frac{d}{dt}\left(m\omega r^2\right) = 0 \\ \Rightarrow \quad & m\omega r^2 = mv_\phi r = \text{constant} \qquad\qquad (\text{using } v_\phi = r\omega). \end{aligned}$$

This shows that $mv_\phi r = m\omega r^2$, which is called the **angular momentum** and usually denoted using a capital L, is conserved whenever a particle is moving through a central force field. Discovering a new conservation law is always a big deal in physics, so this is an important result. In the next section we generalise the idea of angular momentum to three dimensions, but it is worth learning the two-dimensional definition before moving on.

Angular Momentum in Two Dimensions

$$L \triangleq mv_\phi r = m\omega r^2$$

5.4 Angular Momentum and Torque in Three Dimensions

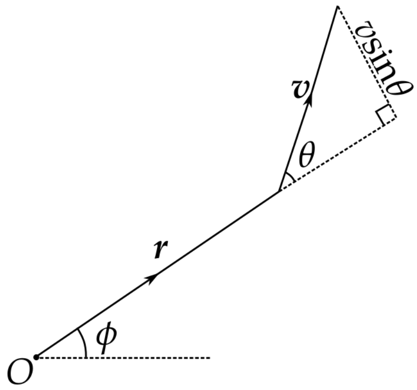
Angular Momentum

In three dimensions, the angular momentum of a particle is a vector.

Angular Momentum in Three Dimensions

$$\boldsymbol{L} \triangleq \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}$$

This definition is most easily understood using a diagram.



At any given time, the position and velocity vectors $\boldsymbol{r}$ and $\boldsymbol{v}$ define a plane, which I have chosen to make the plane of the paper. The orientation of the $\boldsymbol{r}$-$\boldsymbol{v}$ plane may of course depend on time.

- Because it is a cross product, the angular momentum vector $\boldsymbol{r} \times m\boldsymbol{v}$ is always perpendicular to the $\boldsymbol{r}$-$\boldsymbol{v}$ plane. In the diagram above, the right-hand rule tells us that $\boldsymbol{L}$ points out of the page.
- The magnitude of $\boldsymbol{L}$ is $mvr \sin\theta = mv_\phi r$, where v_ϕ is the rotational component of $\boldsymbol{v}$.

If the motion is confined to the x-y plane, as it was at the end of Section 5.3, the angular momentum vector always points along the z axis:

$$\boldsymbol{L} = mv_\phi r\hat{\boldsymbol{k}}.$$

Because the direction is fixed (to within a sign, which we can incorporate into the definition of v_ϕ), it is often omitted, reproducing the two-dimensional scalar version of angular momentum discussed at the end of Section 5.3.

Rate of Change of Angular Momentum

Angular momentum is useful because its rate of change has a very simple and intuitive form:

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) \\
&= \cancel{\frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p}} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} && \text{(using the product rule)} \\
&= \boldsymbol{r} \times \boldsymbol{F}. && \text{(using Newton's second law)}
\end{aligned}$$

The $(d\boldsymbol{r}/dt) \times \boldsymbol{p}$ term vanished because $\boldsymbol{v} = d\boldsymbol{r}/dt$ is parallel to $\boldsymbol{p} = m\boldsymbol{v}$ and their cross product is always zero.

The $\boldsymbol{r} \times \boldsymbol{F}$ term is called the **torque**

Torque

$$\boldsymbol{G} \triangleq \boldsymbol{r} \times \boldsymbol{F}$$

and the equation for the rate of change of the angular momentum is normally written

Angular Version of Newton's Second Law

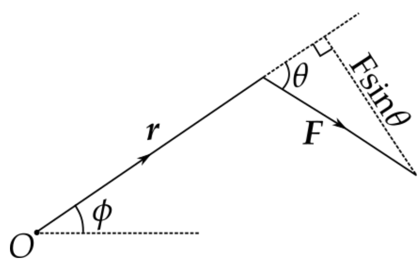
$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{G}$$

The resemblance of this equation to Newton's second law, $d\boldsymbol{p}/dt = \boldsymbol{F}$, explains why angular momentum (which does not even have the same dimensions as linear momentum) got its name.

The derivation of $d\boldsymbol{L}/dt = \boldsymbol{G}$ used nothing more than Newton's second law and a couple of definitions, so there is no new physics here, but that does not mean that the angular version of Newton's second law is not useful. Whenever you study anything able to rotate — and that is a category that includes almost everything — you will find yourself using it.

Torque

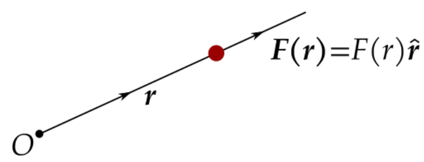
Torque is a three-dimensional generalisation of the "couple" or "moment" you met at school.



- Because it is a cross product, the torque $\boldsymbol{r} \times \boldsymbol{F}$ is always perpendicular to the $\boldsymbol{r}$-$\boldsymbol{F}$ plane. In the diagram above, the torque points into the page. (The angular momentum is perpendicular to the $\boldsymbol{r}$-$\boldsymbol{v}$ plane, which need not be the same as the $\boldsymbol{r}$-$\boldsymbol{F}$ plane.)
- The magnitude of $\boldsymbol{G}$ is $Fr\sin\theta = F_\phi r$, where F_ϕ is the rotational component of $\boldsymbol{F}$.

Conservation of Angular Momentum

Central forces, $\boldsymbol{F}(\boldsymbol{r}) = F(r)\hat{\boldsymbol{r}}$, are always parallel to $\boldsymbol{r}$, so the torque they exert, $\boldsymbol{r} \times \boldsymbol{F}$, is always zero. This makes physical sense because central forces accelerate masses directly towards or away from the origin, without causing any rotation.



Because $d\boldsymbol{L}/dt = \boldsymbol{G}$, a force that exerts no torque can never change the angular momentum.

Angular Momentum Conservation
Motion in central force fields conserves angular momentum.

A simple but important consequence of this result is that any particle moving in a central force field is confined to a plane. Since the angular momentum vector is perpendicular to the $\boldsymbol{r}$–$\boldsymbol{v}$ plane, and since the angular momentum vector is conserved, the orientation of the $\boldsymbol{r}$–$\boldsymbol{v}$ plane is fixed: the particle can never leave its initial plane of motion.

The Torque Equation for Assemblies of Particles

We have shown that the rate of change of the angular momentum of a single particle is equal to torque applied, but this result would not be very important if it did not also hold for systems of many particles interacting via central forces. Here we consider two particles only; the N-particle case is dealt with in question 6 of problem sheet 7.

Suppose that two masses m_1 and m_2 are moving under the influence of externally applied forces $\boldsymbol{F}_1^{\text{ext}}$ and $\boldsymbol{F}_2^{\text{ext}}$. Newton's second law tells us that

$$\begin{aligned}
\dot{\boldsymbol{p}}_1 &= m_1 \ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{F}_{2\,\text{on}\,1}, \\
\dot{\boldsymbol{p}}_2 &= m_2 \ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_2^{\text{ext}} + \boldsymbol{F}_{1\,\text{on}\,2}.
\end{aligned}$$

Since the interaction forces $\boldsymbol{F}_{2\,\text{on}\,1}$ and $\boldsymbol{F}_{1\,\text{on}\,2}$ are central, they are parallel or anti-parallel to the vector $\boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ from particle 1 to particle 2. By Newton's third law, they are also equal and opposite:

$$\boldsymbol{F}_{1\,\text{on}\,2} = F_{12}(r_{12})\hat{\boldsymbol{r}}_{12} = -\boldsymbol{F}_{2\,\text{on}\,1}.$$

The total angular momentum of the two particles is

$$\boldsymbol{L} = \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2,$$

so its rate of change is

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= \cancel{\dot{\boldsymbol{r}}_1 \times \boldsymbol{p}_1} + \boldsymbol{r}_1 \times \dot{\boldsymbol{p}}_1 + \cancel{\dot{\boldsymbol{r}}_2 \times \boldsymbol{p}_2} + \boldsymbol{r}_2 \times \dot{\boldsymbol{p}}_2 \\
&= \boldsymbol{r}_1 \times \left(\boldsymbol{F}_1^{\text{ext}} + \boldsymbol{F}_{2\,\text{on}\,1}\right) + \boldsymbol{r}_2 \times \left(\boldsymbol{F}_2^{\text{ext}} + \boldsymbol{F}_{1\,\text{on}\,2}\right) \\
&= \left(\boldsymbol{r}_1 \times \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{r}_2 \times \boldsymbol{F}_2^{\text{ext}}\right) + \left(\boldsymbol{r}_1 \times \boldsymbol{F}_{2\,\text{on}\,1} + \boldsymbol{r}_2 \times \boldsymbol{F}_{1\,\text{on}\,2}\right) \\
&= \boldsymbol{G}^{\text{ext}} + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{F}_{1\,\text{on}\,2},
\end{aligned}$$

because $\boldsymbol{F}_{2\,\text{on}\,1} = -\boldsymbol{F}_{1\,\text{on}\,2}$. Since $\boldsymbol{F}_{1\,\text{on}\,2}$ is always parallel or anti-parallel to $\boldsymbol{r}_2 - \boldsymbol{r}_1$, the cross product involving $\boldsymbol{F}_{1\,\text{on}\,2}$ vanishes and we are left with

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{G}^{\text{ext}}.$$

The internal contributions to the rate of change of the total angular momentum vanish. This allows the angular version of Newton's second law to be used to find out how bodies made of many particles bound together react to externally applied torques without having to worry about all of the internal forces.

5.5 Motion in a Gravitational Field

Isaac Newton (born Christmas Day 1643; died 1727) was the first person to realise that the forces making the Moon orbit the Earth, the Earth orbit the Sun, and an apple fall were the same.

Newton's Law of Gravitation

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$$

Before Newton, most scientists did not even understand that a force was needed to keep the planets in orbit. Most of the astronomical observations that enabled Newton's work had been taken by Tycho Brahe (1546–1601; the last great naked-eye astronomer) roughly a century earlier. (Science moved slowly back then.) Newton's task was made easier by Johannes Kepler (1571–1630), who turned Brahe's tables of numbers into three laws of planetary motion.

Kepler's 1st Law: Planets move in ellipses with the Sun at one focus.

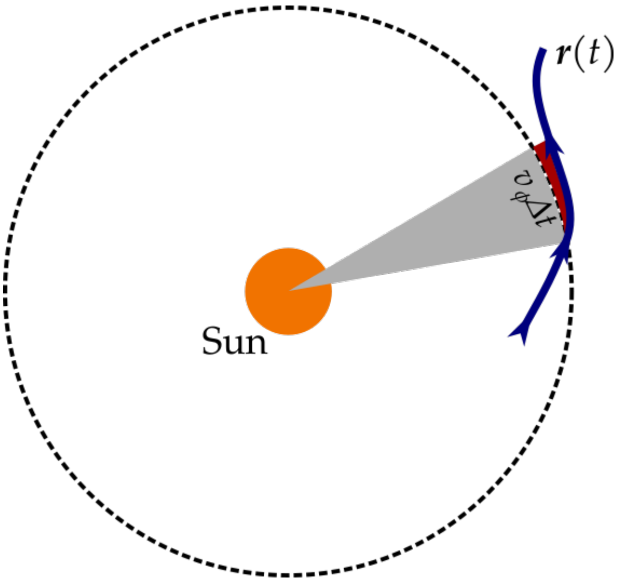
Kepler's 2nd Law: The radius vector from the Sun to the planet sweeps out equal areas in equal times.

Kepler's 3rd Law: For all bodies orbiting the Sun, $\frac{(\text{orbital period})^2}{(\text{semi-major axis})^3}$ is the same. We now know that this ratio is equal to $4\pi^2/GM$, where G is Newton's gravitational constant and M is the mass of the Sun.

[Note: The properties of ellipses, including the meanings of the terms *focus* and *semi-major axis*, will be discussed later in this chapter and are covered in more detail in an optional, non-examinable, handout available on Blackboard.]

Kepler's laws follow directly from Newton's law of gravitation, but they were important historically and deriving them was one of Newton's triumphs. We will only prove the second law in these notes, but you should at least know that they exist. The 3rd law is sometimes useful for solving orbital mechanics problems.

Kepler's second law is a simple consequence of angular momentum conservation. The diagram below shows the track of a body around the Sun over a brief interval of time Δt.



The red area outside the circle is proportional to $(\Delta t)^2$ and becomes negligible in comparison with the area of the grey slice, which is proportional to Δt, in the limit as $\Delta t \to 0$. The grey slice accounts for $v_\phi \Delta t$ of the $2\pi r$ circumference of the circle, so its area is

$$\Delta A \approx \frac{v_\phi \Delta t}{2\pi r} \pi r^2 = \frac{1}{2} v_\phi r \Delta t.$$

Hence

$$\frac{dA}{dt} = \text{area swept out per second} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} v_\phi r = \frac{L}{2m}.$$

If the angular momentum is conserved during the orbit (which does not look likely to be the case in my diagram but is true for real orbits), $dA/dt =$ constant. This is Kepler's second law.

Kepler's third law is tricky to prove in the general elliptical case but easy to verify for circular orbits. Applying Newton's second law in the radial direction, equating the gravitational force to the mass times the centripetal acceleration, gives

$$\begin{aligned} & \frac{mv^2}{r} = \frac{GMm}{r^2} \\ \Rightarrow \quad & \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r} \qquad \left(\text{because } v = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{T}\right) \\ \Rightarrow \quad & \frac{T^2}{r^3} = \frac{4\pi^2}{GM}. \end{aligned}$$

This is Kepler's third law.

Gravitational Orbits

Much more important than Kepler's laws are the following three statements. Gravitational orbits:

1. conserve energy:

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r};$$

2. conserve angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v} = mv_\phi r\hat{\boldsymbol{\omega}},$$

where $\hat{\boldsymbol{\omega}}$ is a unit vector perpendicular to the plane of the orbit;

3. are planar.

Most of the orbital mechanics problems you will meet in this course amount to little more than applying these rules. (Sadly, this can be harder than it sounds.) When dealing with elliptical orbits, it is often useful to focus on the point when the planet is nearest to the sun (called the perihelion) and the point when it is farthest from the Sun (called the aphelion). The radial distance r is at its minimum at perihelion and its maximum at aphelion, so $\dot{r} = 0$ at both points.

5.6 Orbital Energy

One tactic that greatly simplifies orbital mechanics is to split the kinetic energy into radial and angular parts:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\left(v_r^2 + v_\phi^2\right) - \frac{GMm}{r},$$

where v_r is the radial part of the velocity and v_ϕ is the rotational part. Since $L = mv_\phi r$, the v_ϕ^2 term can be rewritten in terms of the angular momentum to get

The Energy Equation

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

The value of this reformulation becomes apparent when you realise that L is conserved. The angular kinetic energy term, $\frac{1}{2}mv_\phi^2$, now looks (and acts) like an r-dependent potential, leaving an expression for the total energy that depends on the radial position and the radial velocity only. Almost everything you need to know about the angular motion is hidden in the fixed value of L. As long as you remember that the constant value of L differs for different orbits, you can forget about the angular motion and concentrate on the radial motion.

The potential-like angular kinetic energy term,

$$U_\phi(r) \triangleq \frac{L^2}{2mr^2},$$

is called the **centrifugal potential**. To understand its physical origins, imagine a planet approaching the Sun. The planet's angular momentum, $L = mv_\phi r$, is conserved, so its angular speed v_ϕ must increase as r decreases. This in turn increases the angular contribution to the kinetic energy:

$$\frac{1}{2}mv_\phi^2 = \frac{L^2}{2mr^2}.$$

The extra rotational kinetic energy is not lost forever and can be reclaimed if the planet later moves away from the Sun, so it acts like a store of potential energy.

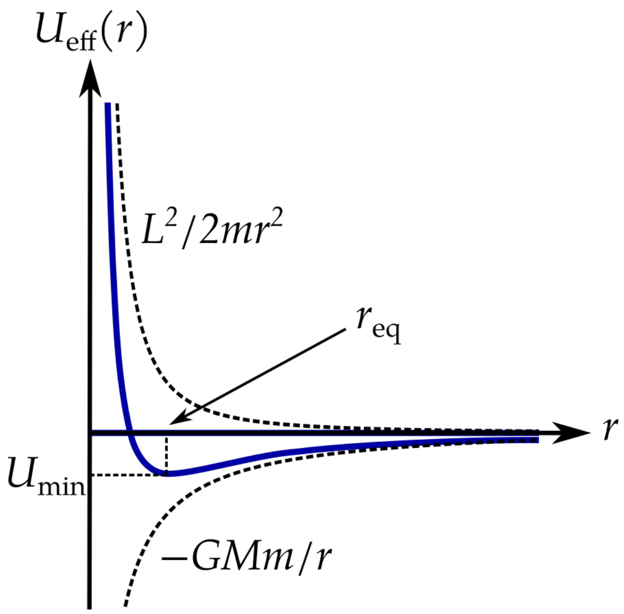
The gravitational and centrifugal potential terms are often combined into an effective potential

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r},$$

and the total energy written as

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r).$$

The effective potential looks like this:



The minimum occurs where

$$\frac{dU_{\text{eff}}(r)}{dr} = \frac{-L^2}{mr^3} + \frac{GMm}{r^2} = 0 \qquad \Rightarrow \qquad r_{\text{eq}} = \frac{L^2}{GMm^2}$$

and the minimum value is

$$U_{\text{min}} = \frac{L^2}{2mr_{\text{eq}}^2} - \frac{GMm}{r_{\text{eq}}} = \frac{GMm^2 r_{\text{eq}}}{2mr_{\text{eq}}^2} - \frac{GMm}{r_{\text{eq}}} = -\frac{GMm}{2r_{\text{eq}}}.$$

Although there is no doubt that the energy can be written as $\frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$, it is not obvious that $U_{\text{eff}}(r)$ can be treated as an effective potential. Can the corresponding effective force,

$$F_{\text{eff}}(r) \triangleq -\frac{dU_{\text{eff}}(r)}{dr} = \frac{L^2}{mr^3} - \frac{GMm}{r^2},$$

really be used in a radial version of Newton's second law? If we assume it can, we get

$$\begin{aligned} m\ddot{r} &= \frac{L^2}{mr^3} - \frac{GMm}{r^2} \\ &= \frac{(mr^2\omega)^2}{mr^3} - \frac{GMm}{r^2} \qquad \text{(because } L = mv_\phi r = mr^2\omega) \\ &= m\omega^2 r - \frac{GMm}{r^2}. \end{aligned}$$

We derived exactly this equation (with a general radial force $F(r)$ in place of the gravitational force $-GMm/r^2$ and the $m\omega^2 r$ term on the left-hand side, where it really belongs) in the "Angular Momentum" subsection of the long example on two-dimensional rotating vectors at the end of Section 5.3, so yes, it works. The effective force can safely be used in a radial version of Newton's second law.

5.7 Circular Orbits

One way to derive the properties of circular orbits is to start from the statement of conservation of energy,

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) = \text{constant along the orbit},$$

and the definition of the effective potential,

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

- The smallest possible value of $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$ occurs when $\dot{r} = 0$ (minimising the radial kinetic energy term) and $r = r_{\text{eq}}$ (minimising the effective potential term), in which case $E = U_{\text{eff}}(r_{\text{eq}}) = U_{\text{min}}$. The orbit is circular (because $\dot{r} = 0$) with radius

  $$r_{\text{eq}} = \frac{L^2}{GMm^2}$$

  and total energy

  $$E = U_{\text{min}} = -\frac{GMm}{2r_{\text{eq}}}.$$

  (These expressions for r_{eq} and U_{min} were derived towards the end of Section 5.6.)

- This circle is the lowest (most negative) energy orbit of angular momentum L.

We can check these results using techniques you learnt at school. For a circular orbit of radius r_{eq}, the radial velocity $v_r = \dot{r}$ is zero and the gravitational force must be equal to the mass times the centripetal acceleration:

$$\frac{GMm}{r_{\text{eq}}^2} = \frac{mv_\phi^2}{r_{\text{eq}}} \qquad \Rightarrow \qquad mv_\phi^2 = \frac{GMm}{r_{\text{eq}}}.$$

Multiplying through by mr_{eq}^2 and remembering that $L = mv_\phi r$ gives

$$L^2 = GMm^2 r_{\text{eq}} \qquad \Rightarrow \qquad r_{\text{eq}} = \frac{L^2}{GMm^2}.$$

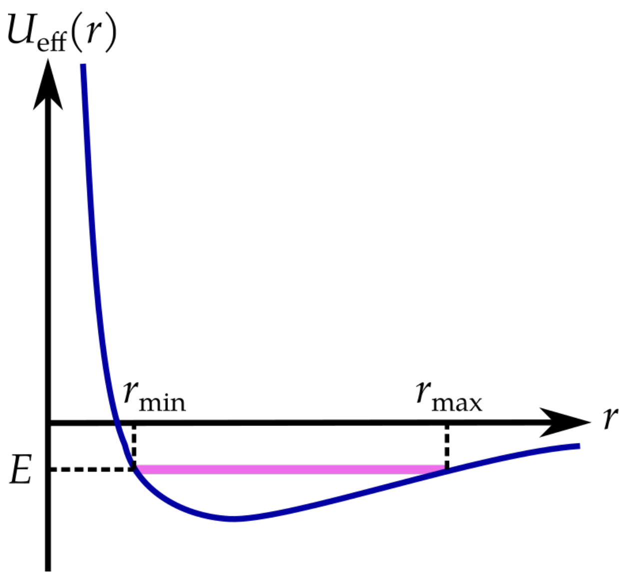
The total energy E is the sum of the kinetic energy and the potential energy:

$$E = \frac{1}{2}mv_\phi^2 - \frac{GMm}{r_{\text{eq}}^2} = \frac{GMm}{2r_{\text{eq}}} - \frac{GMm}{r_{\text{eq}}} = -\frac{GMm}{2r_{\text{eq}}}.$$

This argument has successfully reproduced the expressions for r_{eq} and E found using the effective potential.

5.8 Elliptical Orbits

As can be seen in the diagram below, a planet for which $U_{\text{min}} < E < 0$ moves between r_{min} and r_{max} as it orbits.



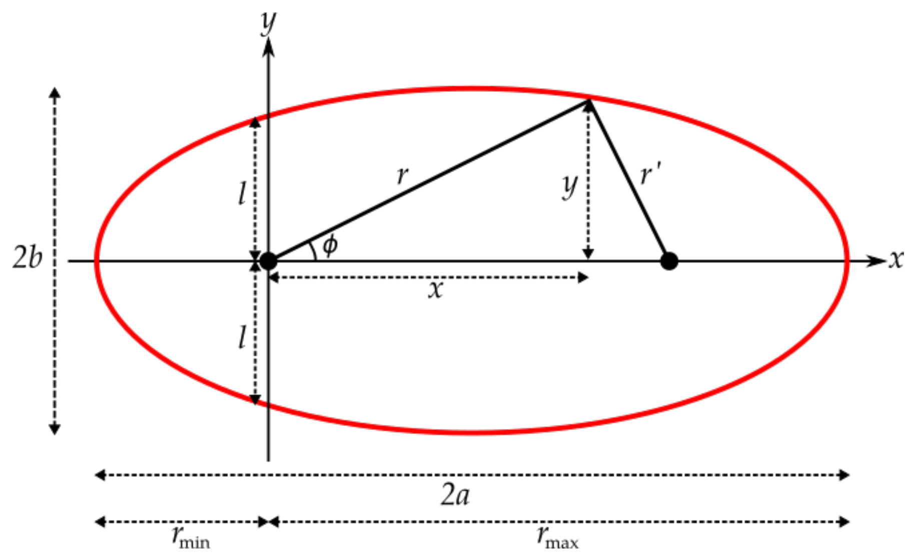
Since $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$, the radial velocity $\dot{r}$ is zero at the points r_{min} and r_{max} where $E = U_{\text{eff}}(r)$. As the radius of the orbit moves towards one of these values, the radial speed decreases. All of the radial kinetic energy has become (effective) potential energy by the time r reaches r_{min} or r_{max} and $\dot{r}$ is instantaneously zero there, but $F_{\text{eff}}(r) = -dU_{\text{eff}}(r)/dr$ is not zero and there is still a radial force accelerating the planet back into the viable (pink) region. The radial speed $\dot{r}$ does not stay zero and the planet immediately starts moving back into the viable region.

This argument shows that the radius of the planetary orbit oscillates between r_{min} and r_{max} as the planet orbits the Sun. It does not prove that the period of the radial oscillation matches the period of the orbit (which it does) or that the orbit is an ellipse (which it is), but it gets us part of the way.

If you want to see a full derivation of the elliptical orbit from Newton's laws, read the handout called "Derivation of Elliptical Orbit from Newton's Laws" on Blackboard. If you want to learn more about ellipses as mathematical objects, read the handout called "Definitions and Properties of Ellipses". Neither of these handouts is examinable.

Although we are not going to delve into the mathematics of ellipses and conic sections in this course, it is worth knowing the polar equation of an ellipse and some of the terminology used to describe them. The subject is so old that a few of the names are in Latin.

An ellipse looks like this:



and is described by the polar equation

Polar Equation of an Ellipse

$$r = \frac{l}{1 - e\cos\phi}$$

- The black blobs are called the foci of the ellipse. The Sun lies at one focus, with nothing at the other. Perhaps surprisingly, elliptical orbits are mirror symmetric about the plane bisecting the line between the two foci.

- The names of the constants appearing in the diagram and/or the polar equation are:

e	eccentricity
l	semi-latus rectum
a	semi-major axis
b	semi-minor axis
r_{min}	perihelion
r_{max}	aphelion

  For the orbit of the Moon about the Earth, the terms "perihelion" and "aphelion" are replaced by "perigee" and "apogee". For general orbits of one body about another, use "periapsis" and "apoapsis".

- The eccentricity e is a constant in the range $0 \leq e < 1$. If $e = 0$, the equation of the ellipse reduces to $r = l$, which is the equation of a circle.

- When $\phi = \pi/2$, $\cos\phi = 0$ and r is equal to the semi-latus rectum l (as shown on the diagram).

- To relate the values of r_{max} and r_{min} to the parameters l and e appearing in polar equation, set $\phi = 0$ and $\phi = \pi$:

$$r_{\text{max}} = r(\phi = 0) = \frac{l}{1-e}, \qquad r_{\text{min}} = r(\phi = \pi) = \frac{l}{1+e}.$$

  As $e \to 1$, $r_{\text{min}} \to l/2$ but $r_{\text{max}} \to \infty$. The ellipse becomes infinitely long in the x direction. A longer and thinner ellipse is said to be more "eccentric".

- The semi-major axis a is half of the longest diameter of the ellipse. (For a circle, a is equal to the radius.) The diagram shows that $a = \frac{1}{2}(r_{\text{min}} + r_{\text{max}})$, so

$$a = \frac{1}{2}\left(\frac{l}{1+e} + \frac{l}{1-e}\right) = \frac{l}{(1+e)(1-e)}.$$

It is often helpful to use this result to express r_{min} and r_{max} in terms of a instead of l:

$$r_{\text{max}} = \frac{l}{1-e} = (1+e)a, \qquad r_{\text{max}} = \frac{l}{1+e} = (1-e)a.$$

Energy of an Elliptical Orbit

Many of the expressions involving elliptical orbits are disappointingly complicated, but there is a very simple formula for the total energy. A common trick in orbital mechanics is to compare the motion of the planet at its perihelion, $r_{\text{min}} = (1-e)a$, and its aphelion, $r_{\text{max}} = (1+e)a$. Since these are the minimum and maximum distances from the Sun, $\dot{r} = 0$ at both points. It follows that

$$E = \cancel{\frac{1}{2}m\dot{r}^2} + U_{\text{eff}}(r) = \frac{L}{2mr^2} - \frac{GMm}{r}$$

at both points:

$$E = \frac{L^2}{2m(1+e)^2a^2} - \frac{GMm}{(1+e)a}, \qquad E = \frac{L^2}{2m(1-e)^2a^2} - \frac{GMm}{(1-e)a}.$$

Our aim is to eliminate L to find E as a function of the semi-major axis a and the eccentricity e. To this end, multiply the two equations for E by $(1+e)^2$ and $(1-e)^2$, respectively, to obtain

$$(1+e)^2E = \frac{L^2}{2ma^2} - \frac{(1+e)GMm}{a}, \quad (1-e)^2E = \frac{L^2}{2ma^2} - \frac{(1-e)GMm}{a},$$

and subtract one equation from the other to get

$$\begin{aligned}
& [(1+e)^2 - (1-e)^2]E = [(1+e) - (1-e)]\left(\frac{-GMm}{a}\right) \\
\Rightarrow \qquad & 4eE = -\frac{2eGMm}{a} \\
\Rightarrow \qquad & \boxed{E = -\frac{GMm}{2a}}
\end{aligned}$$

The energy of an elliptical orbit depends on the length of the semi-major axis but not on the eccentricity. (I still find this surprising.)

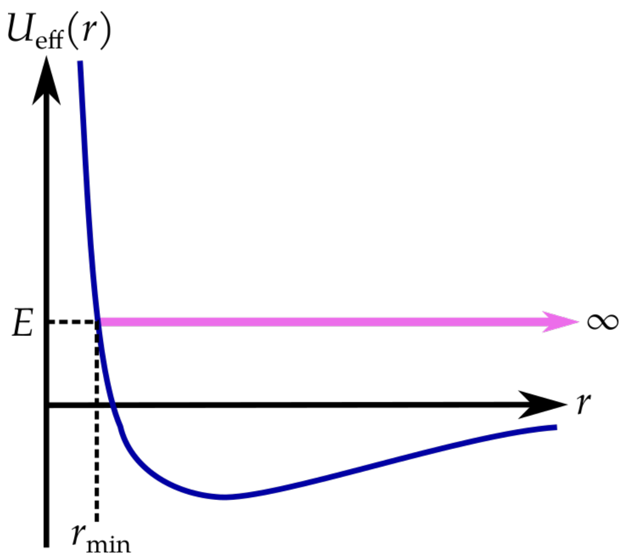
Working from the same starting point, one can also show that

$$a = \frac{L^2}{GMm^2(1+e)(1-e)} \qquad \text{and} \qquad e = \sqrt{1 + \frac{2EL^2}{G^2m^3M^2}}\,.$$

The algebra required is straightforward in principle but fiddly in practice. You are welcome to try it if you are interested, but these two equations are not examinable.

5.9 Hyperbolic Orbits

If $E > 0$, the planet has enough energy to escape to infinity (where the potential energy is zero) with some kinetic energy left over and will do so eventually. In other words, it is not really a planet.



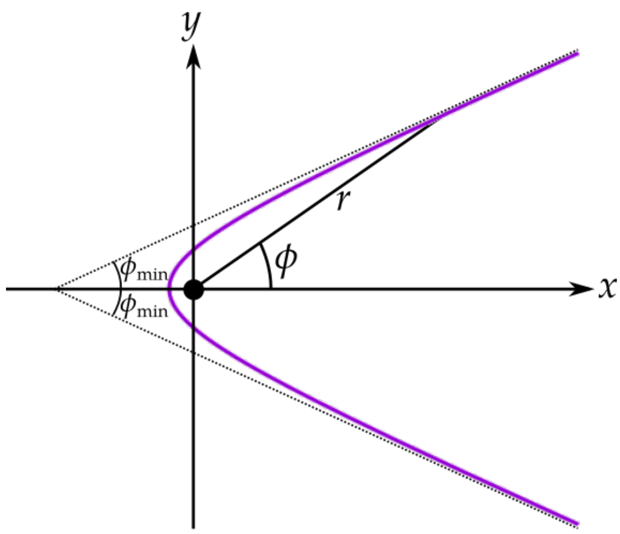
The surprise is that the (non-)planet cannot get closer to the Sun than the distance r_{min} for which $U(r_{\text{min}}) = E$. The collision is prevented by the repulsive centrifugal part of the effective potential,

$$U_{\text{eff}}(r) = \underbrace{\frac{L^2}{2mr^2}}_{\text{centrifugal potential}} - \underbrace{\frac{GMm}{r}}_{\text{gravitational potential}},$$

which dominates at small r. If L happens to be zero (the incoming planet is aimed at the heart of the Sun), the centrifugal potential vanishes and the planet plunges to a fiery death. If L is very small, so that r_{min} is less than the radius of the Sun, it suffers an equally bad fate. But as long as the incoming planet has sufficient angular momentum, it cannot collide with the Sun.

Plunging into the Sun is particularly difficult when the incoming planet or space ship starts off very far away. Unless v_ϕ is tiny, the angular momentum $L = mv_\phi r$ is large because r is large and the trajectory is almost bound to miss. Despite the many scary scenes in science fiction movies, it is actually quite difficult for a space ship to be sucked into the Sun (or to land on a planet from a long distance away). To fall into the Sun successfully, the ship is likely to have to use its engines to reduce its angular momentum relative to the target.

The trajectory followed by a lucky planet that misses the Sun is called a hyperbola and looks like this:



The polar equation is the same as for an ellipse,

$$r = \frac{l}{1 - e\cos\phi},$$

but the eccentricity e is now greater than 1. Since the radius r is always greater than zero, $e\cos\phi$ must be less than 1 even though e is greater than 1. This means the planet can never be found at polar angles ϕ close to zero, where $\cos\phi$ is close to 1 and $e\cos\phi$ is greater than 1. If we choose the domain of ϕ such that $-\pi < \phi \leq \pi$, the condition $e\cos\phi < 1$ translates to

$$|\phi| > \phi_{\text{min}} = \arccos(1/e).$$

The point from which the two dashed asymptotes radiate in the diagram above is what became of the centre of the ellipse, which headed off to

$x = +\infty$ as e approached 1 from below and reappeared on the left-hand side of the focus when e was greater than 1 and the ellipse turned into a hyperbola.

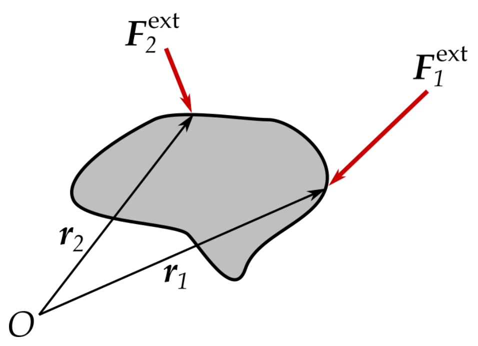
Chapter 6

Rigid-Body Dynamics

6.1 Newton's Laws for Extended Bodies

Newtonian mechanics describes the classical world of objects of finite size, not the quantum world of point particles, but this course has treated bodies — even planets — as point-like objects so far. Replacing planets by points is, to say the least, a little unrealistic. We have already shown that the centre of mass of an extended body moves like a point particle and will revisit that observation in the next section, but extended bodies can rotate as well as move. To understand them fully, we will need to think about torque and angular momentum as well as forces and linear momentum.

The diagram below shows an extended body acted on by two external forces, $\boldsymbol{F}_1^{\text{ext}}$ and $\boldsymbol{F}_2^{\text{ext}}$.



The total external force is

$$\boldsymbol{F}^{\text{ext}} = \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{F}_2^{\text{ext}},$$

and the total external torque about the origin O is

$$\boldsymbol{G}^{\text{ext}} = \boldsymbol{r}_1 \times \boldsymbol{F}_1^{\text{ext}} + \boldsymbol{r}_2 \times \boldsymbol{F}_2^{\text{ext}}.$$

For now, it is easiest to think of the body as an assembly of point or point-like particles (atoms, say) bound together by internal forces. This model of matter is easy to reason about and convenient for mathematical derivations, but impractical for calculating the properties of human-scale objects, which contain at least Avogadro's number of particles. Later, in Sec. 5.10, we switch to a more tractable continuum model in which the point particles are replaced by a mass density $\rho(\boldsymbol{r})$. The mass within a tiny box of volume dV at $\boldsymbol{r}$ is $\rho(\boldsymbol{r})dV$.

The N particles in an extended body feel internal and external forces:

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{\text{ext}} + \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \boldsymbol{F}_{j\,\text{on}\,i}\,, \qquad i = 1, 2, \ldots, N.$$

The interactions between the particles seem to complicate matters, but we have already shown that the internal forces make no contribution to the rate of change of the total momentum (see Sec. 2.3 for the case of two particles; the N-particle case was left as an exercise) and that the internal torques make no contribution to the rate of change of the total angular momentum (see Sec. 5.4 for the two-particle case and Q6 of Problem Sheet 7 for the N-particle case). Expressed mathematically:

$$\dot{\boldsymbol{P}} = \frac{d}{dt}\left(\sum_i \boldsymbol{p}_i\right) = \boldsymbol{F}^{\text{ext}}, \qquad \dot{\boldsymbol{L}} = \frac{d}{dt}\left(\sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i\right) = \boldsymbol{G}^{\text{ext}},$$

with

$$\boldsymbol{F}^{\text{ext}} = \sum_i \boldsymbol{F}_i^{\text{ext}}, \qquad \boldsymbol{G}^{\text{ext}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{\text{ext}}.$$

The internal forces and torques *do* affect the momenta and angular momenta of individual particles within the body, but not the momentum or angular momentum of the body as a whole. The almost magical cancellation of the internal contributions explains why momentum and angular momentum are such useful ideas.

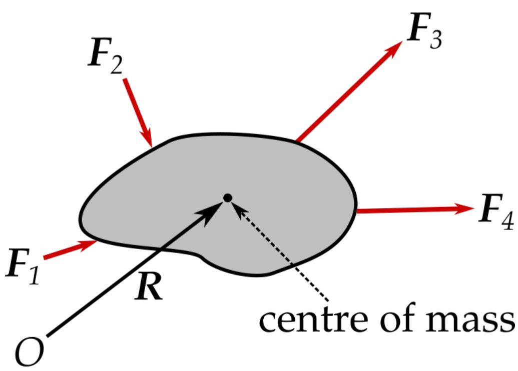
Newton's first and third laws obviously work just as well for extended bodies as for point particles, so we already know everything we need to know. In a sense there is nothing more to do — we already have our "grand unified theory" — but writing down the laws of motion is only the first step towards understanding the world around us. We still have to work out how to solve them.

Since the internal forces cancel out, they will not appear in the rest of this chapter. All of the forces we need to think about are external. This makes the "ext" labels are unnecessary and we rarely use them from now on.

6.2 Centre-of-Mass Motion

This section shows that the centre of mass of an extended body moves *exactly* like a point particle. The equivalence applies not only to its linear motion — an idea we met earlier in the course — but also to its rotation about the origin of coordinates.

The potato in the diagram below is being pushed and pulled by four external forces, each acting at a different point on the surface.



Since $M\boldsymbol{R} \triangleq \sum_i m_i \boldsymbol{r}_i$, it follows that

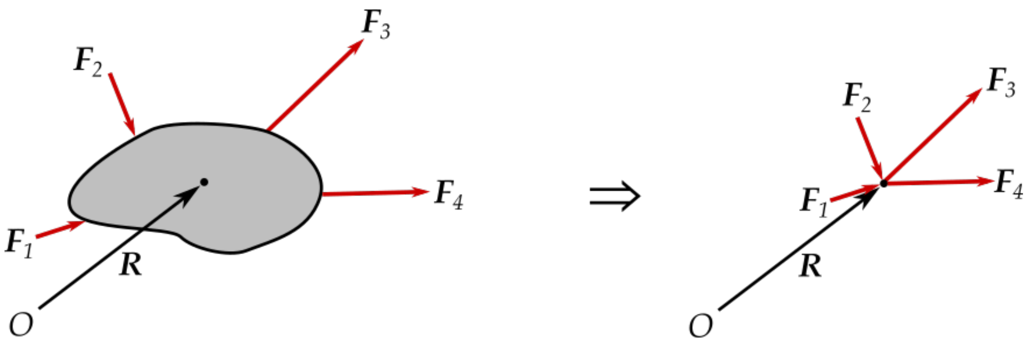
$$\frac{d}{dt}\left(M\dot{\boldsymbol{R}}\right) = \frac{d}{dt}\left(\sum_i m_i \dot{\boldsymbol{r}}_i\right) = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4.$$

The last step made use of the fact that the rate of change of the total momentum equals the total external force. We can derive a similar result for the rate of change of $\boldsymbol{R} \times M\dot{\boldsymbol{R}}$:

$$\frac{d}{dt}\left(\boldsymbol{R} \times M\dot{\boldsymbol{R}}\right) = \cancel{\dot{\boldsymbol{R}} \times M\dot{\boldsymbol{R}}} + \boldsymbol{R} \times \frac{d}{dt}\left(M\dot{\boldsymbol{R}}\right) = \boldsymbol{R} \times \left(\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4\right).$$

If we call $M\dot{\boldsymbol{R}}$ the "momentum of the centre of mass" and $\boldsymbol{R} \times M\dot{\boldsymbol{R}}$ the "angular momentum of the centre of mass", which seems natural, these equations can be stated very simply in words:

> The momentum and angular momentum of the centre of mass change *exactly* as if all of the external forces were acting directly on a point particle of mass M positioned at the centre of mass.



We implicitly assumed this when studying orbits, treating the Sun and the planets moving around it as point particles.

6.3 Separating the Centre-of-Mass and Internal Physics

The momentum, angular momentum, and kinetic energy of an extended body can always be divided into contributions associated with the motion of the centre of mass and contributions associated with the rotation about the centre of mass. Furthermore, these contributions satisfy Newton's laws independently. This holds even when the centre of mass is accelerating and the frame in which the centre-of-mass is stationary is not an inertial frame. In other words, the statement that the motion about the centre-of-mass satisfies Newton's laws is not just equivalent to the statement that the laws of physics must be the same in all inertial frames.

Before proving these results, it is helpful to recall that the position of particle i relative to the centre of mass is denoted $\boldsymbol{r}_i^*$ and given by

$$\boldsymbol{r}_i^* = \boldsymbol{r}_i - \boldsymbol{R}.$$

It follows that

$$\begin{aligned}
\sum_i m_i \boldsymbol{r}_i^* &= \sum_i m_i (\boldsymbol{r}_i - \boldsymbol{R}) \\
&= \sum_i m_i \boldsymbol{r}_i - \big(\sum_i m_i\big) \boldsymbol{R} \\
&= M\boldsymbol{R} - M\boldsymbol{R} \\
&= \boldsymbol{0},
\end{aligned}$$

and hence, by differentiating, that

$$\sum_i m_i \dot{\boldsymbol{r}}_i^* = \boldsymbol{0}.$$

The mass-weighted averages of the positions and velocities of the N particles in the body are zero in the centre-of-mass frame.

Momentum

Since $\boldsymbol{P} \triangleq \sum_i m_i \dot{\boldsymbol{r}}_i \triangleq M\dot{\boldsymbol{R}}$, the total momentum of an extended body made of N particles is equal to the momentum of its centre of mass.

Separation of the Momentum
Total momentum = Momentum of the centre of mass

There is no "internal momentum" because, as you showed in Q4 of Problem Sheet 5, the centre-of-mass frame is also the zero-momentum frame.

Kinetic Energy

Section 4.6 showed how to write the total kinetic energy of two interacting particles as the sum of a term associated with the motion of the centre of mass and a term associated with the motion about the centre of mass.

This also works for N particles:

$$\begin{aligned}
K &= \frac{1}{2}\sum_i m_i \dot{r}_i^2 \\
&= \frac{1}{2}\sum_i m_i \left(\dot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{R}}\right)^2 \\
&= \frac{1}{2}\sum_i m_i \left(\dot{r}_i^*\right)^2 + \cancel{\sum_i m_i \dot{\boldsymbol{r}}_i^*} \cdot \dot{\boldsymbol{R}} + \frac{1}{2}\left(\sum_i m_i\right)\dot{R}^2 \\
&= K^* + \frac{1}{2}M\dot{R}^2.
\end{aligned}$$

The step from the second line to the third line used the vector identity

$$(\boldsymbol{a}+\boldsymbol{b})^2 = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b}) = a^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + b^2.$$

We showed that $\sum_i m_i \dot{\boldsymbol{r}}_i^* = \boldsymbol{0}$ at the beginning of this section.

Separation of the Kinetic Energy		
Total kinetic energy	=	Kinetic energy in the centre-of-mass frame + Kinetic energy of the centre of mass

Torque

The externally applied torque can also be separated into the torque applied about the centre of mass and the torque applied to the centre of mass:

$$\begin{aligned}
\boldsymbol{G} &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i \\
&= \sum_i (\boldsymbol{r}_i^* + \boldsymbol{R}) \times \boldsymbol{F}_i \\
&= \sum_i \boldsymbol{r}_i^* \times \boldsymbol{F}_i + \boldsymbol{R} \times \sum_i \boldsymbol{F}_i \\
&= \boldsymbol{G}^* + \boldsymbol{R} \times \boldsymbol{F}.
\end{aligned}$$

Remember that only the externally applied forces contribute; we have already shown that the internal forces can be ignored.

Separation of the Externally Applied Torque

Total torque = Torque about the centre of mass
+ Torque acting on the centre of mass

When we work out the torque acting on the centre of mass, we treat it as a point particle, ignoring the possibility that the external forces may also be trying to make the body rotate about the centre of mass. We imagine that all of the external forces push directly on the centre of mass (even though they are really pushing on different parts of the body) and calculate the torque they exert at that single point relative to the fixed origin of coordinates. The result depends on the choice of origin, although the torque about the centre of mass does not.

Angular Momentum

It is less obvious, but equally true, that the angular momentum can be separated into a term associated with the rotation about the centre of mass and a term associated with the rotation of the centre of mass about the origin:

$$\begin{aligned}
\boldsymbol{L} &= \sum_i \boldsymbol{r}_i \times m_i \dot{\boldsymbol{r}}_i \\
&= \sum_i (\boldsymbol{r}_i^* + \boldsymbol{R}) \times m_i \left(\dot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{R}}\right) \\
&= \sum_i \left(\boldsymbol{r}_i^* \times m_i \dot{\boldsymbol{r}}_i^* + \boldsymbol{r}_i^* \times m_i \dot{\boldsymbol{R}} + \boldsymbol{R} \times m_i \dot{\boldsymbol{r}}_i^* + \boldsymbol{R} \times m_i \dot{\boldsymbol{R}}\right) \\
&= \boldsymbol{L}^* + \left(\cancel{\sum_i m_i \boldsymbol{r}_i^*}\right) \times \dot{\boldsymbol{R}} + \boldsymbol{R} \times \left(\cancel{\sum_i m_i \dot{\boldsymbol{r}}_i^*}\right) + \boldsymbol{R} \times M\dot{\boldsymbol{R}} \\
&= \boldsymbol{L}^* + \boldsymbol{R} \times M\dot{\boldsymbol{R}},
\end{aligned}$$

where the equation $\sum_i m_i \boldsymbol{r}_i^* = \boldsymbol{0}$ and its time derivative were used again.

Separation of the Angular Momentum

Total angular momentum =
Angular momentum about the centre of mass
+ Angular momentum of the centre of mass

The angular momentum of the centre of mass is also evaluated by assuming that the entire mass of the body is concentrated at one point. Unlike the angular momentum about the centre of mass, the angular momentum of the centre of mass depends on the location of the origin of the coordinate system.

Rotations about the Centre of Mass

We have just shown that the torque and angular momentum can be separated into contributions acting on the centre of mass and contributions acting about the centre of mass. Here we show that the rate of change of the angular momentum about the centre of mass is equal to the torque about the centre of mass. The surprise is that this is true even when the centre of mass is accelerating.

Start from the rotational version of Newton's second law,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{G} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i \,,$$

and the equation for the rate of change of the angular momentum of the centre of mass derived in Sec. 6.2:

$$\frac{d}{dt}\big(\boldsymbol{R} \times M\dot{\boldsymbol{R}}\big) = \boldsymbol{R} \times \sum_i \boldsymbol{F}_i \,.$$

Remembering that $\boldsymbol{L} = \boldsymbol{L}^* + \boldsymbol{R} \times M\dot{\boldsymbol{R}}$, rearranging to get $\boldsymbol{L}^* = \boldsymbol{L} - \boldsymbol{R} \times M\dot{\boldsymbol{R}}$, differentiating, and using the two equations above gives

$$\begin{aligned}
\frac{d\boldsymbol{L}^*}{dt} &= \frac{d\boldsymbol{L}}{dt} - \frac{d}{dt}\Big(\boldsymbol{R} \times M\dot{\boldsymbol{R}}\Big) \\
&= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i - \boldsymbol{R} \times \sum_i \boldsymbol{F}_i \\
&= \sum_i (\boldsymbol{r}_i - \boldsymbol{R}) \times \boldsymbol{F}_i \\
&= \sum_i \boldsymbol{r}_i^* \times \boldsymbol{F}_i \\
&= \boldsymbol{G}^*,
\end{aligned}$$

as we had hoped.

Angular Version of Newton's Second Law for Rotations about the Centre of Mass
Torque about the centre of mass = Rate of change of angular momentum about the centre of mass

Summary

Sec. 6.3 has been long enough to be worth summarising. It showed that the motions of bodies consisting of many interacting particles can always be split into two parts. The centre of mass moves like a point particle of mass $M = \sum_i m_i$, obeying Newton's laws for point particles in every respect. Even when the extended body has many forces acting on it, all in different places, the centre of mass responds to the sum of these, just as if they were all acting directly on the point mass M. Extended bodies can also rotate about their centre of mass, and the rotation rate changes in response to torques exerted by the external forces about the centre of mass. Even when the centre of mass is accelerating, the rate of change of the angular momentum $\boldsymbol{L}^*$ about the centre of mass is equal to the torque $\boldsymbol{G}^*$ acting about the centre of mass. This separation makes rotational dynamics much easier than one might at first expect.

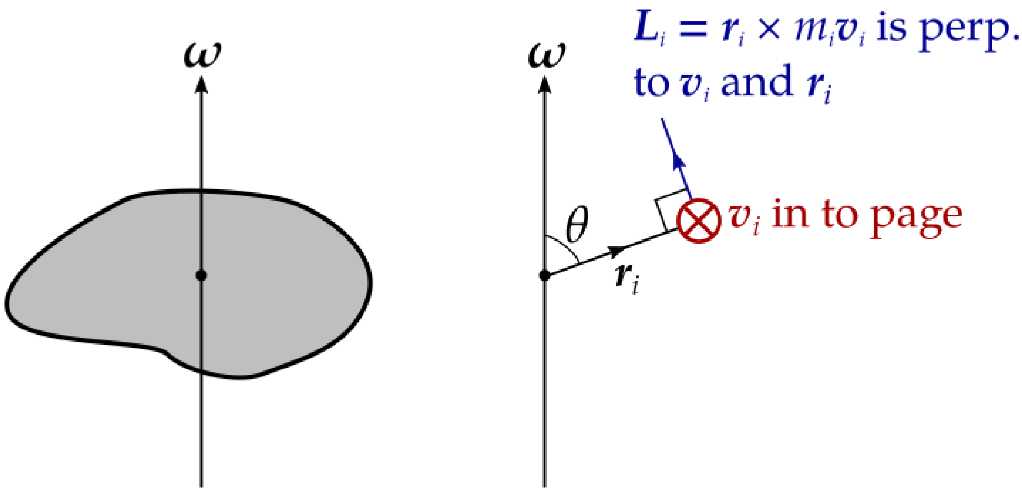
6.4 Rigid Bodies

Everything discussed in this chapter so far applies just as well to fluids and wobbly jellies as it does to diamonds, but bodies that deform as they move and spin are difficult to describe mathematically. In most cases, the only way to make progress is to solve the differential equations numerically using a computer.

To introduce the main ideas of rotational dynamics in the simplest possible setting, we assume from now on that the bodies we are studying are rigid. Although rigid bodies can move and rotate, the *relative* positions of the particles of which they are composed are fixed. It is as if the constituent masses m_i are held together by a rigid framework of light rods that prevent them from moving relative to each other. If we were to embed a set of coordinate axes within a rigid body, making those axes move and rotate with the body as a whole, the positions of the particles as measured along the embedded coordinate axes would never change. Unless the applied forces are large enough to cause extreme accelerations, it is reasonable to treat most solids as rigid.

6.5 The Angular Momentum of a Rigid Body

The diagram below shows a rigid potato rotating at angular velocity $\boldsymbol{\omega}$.



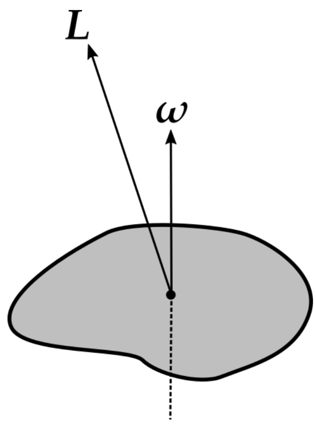
The origin of the coordinate system is at the centre of mass, although we will not add asterisks to the vectors to indicate this. The centre of mass may itself be moving or accelerating, but as the centre-of-mass motion can be treated separately we can forget about that possibility. Because the body is rigid, every mass m_i within it rotates at the same angular velocity $\boldsymbol{\omega}$. This is illustrated in the right-hand panel of the diagram.

Evaluating the vector cross product shows that

$$\begin{aligned}\boldsymbol{v}_i &= \boldsymbol{\omega} \times \boldsymbol{r}_i \\ &= \omega r_i \sin\theta \ \text{ into the page} \\ &= \omega r_i \sin\theta\, \hat{\boldsymbol{\phi}},\end{aligned}$$

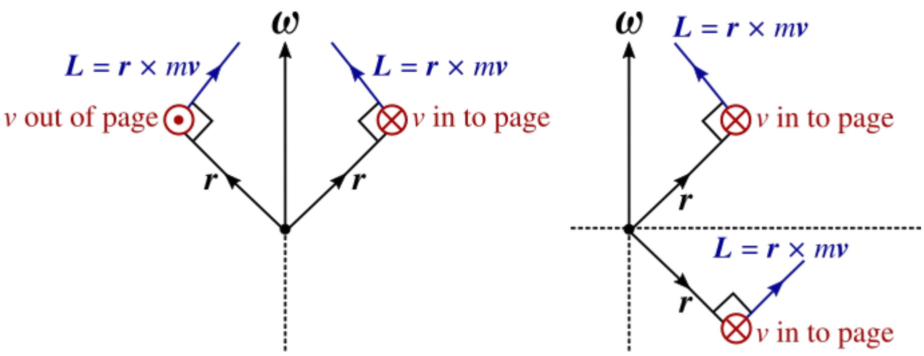
so $\boldsymbol{L}_i = \boldsymbol{r}_i \times m_i \boldsymbol{v}_i$ points as shown in the diagram. The thing to notice is that $\boldsymbol{L}_i$ is not parallel to $\boldsymbol{\omega}$. The angular momentum vector and the angular velocity vector point in different directions.

Even more unfortunate is that this is not only true for point particles; the same holds for wonky-shaped rigid bodies like the potato, made of lots of point masses.



In the absence of external forces, the angular momentum $\boldsymbol{L}$ is conserved, so the angular velocity $\boldsymbol{\omega}$ is not. (If $\boldsymbol{\omega}$ were constant, $\boldsymbol{L}$ would be rotating about $\boldsymbol{\omega}$ as if embedded in the body and $\boldsymbol{L}$ would vary with time.) This makes things complicated.

Fortunately, for bodies with (enough) symmetry, the components of $\boldsymbol{L}$ perpendicular to $\boldsymbol{\omega}$ cancel out.



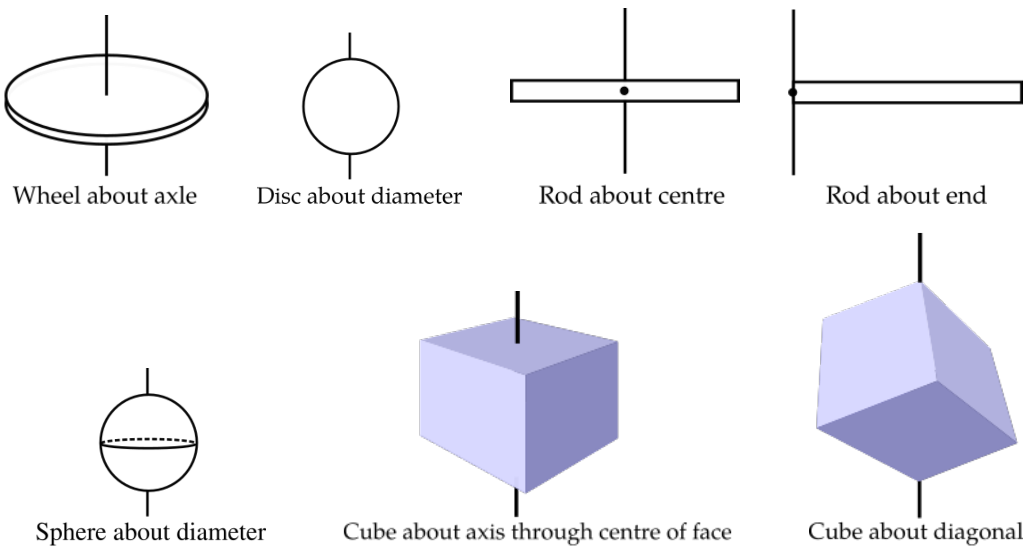
Two-fold rotational symmetry about $\boldsymbol{\omega}$ axis

Mirror plane perpendicular to $\boldsymbol{\omega}$

The figure on the left assumes that the body is unchanged if you rotate it by π radians about the $\boldsymbol{\omega}$ axis. For every mass on one side of the axis, there is an identical mass on the opposite side. These two masses contribute equal and opposite components of angular momentum perpendicular to the rotation axis, so the total angular momentum perpendicular to the rotation axis is zero and the angular momentum of the whole body is parallel to $\boldsymbol{\omega}$. Similar arguments can be applied to any body with an n-fold rotation axis parallel to $\boldsymbol{\omega}$ (that is to say, to any body that looks the same after a rotation by $2\pi/n$ radians), with n any integer greater than or equal to two.

The figure on the right assumes that the body is unchanged if you reflect it in the plane perpendicular to the rotation axis, passing through the centre of mass. Then, for every mass above the plane, there is an identical mass below the plane, and the two make equal and opposite contributions to the angular momentum perpendicular to the rotation axis. The angular momentum of the whole body is again parallel to $\boldsymbol{\omega}$.

One or other of these two arguments can be used to show that bodies of all of the types pictured below have $\boldsymbol{L}$ parallel to $\boldsymbol{\omega}$.



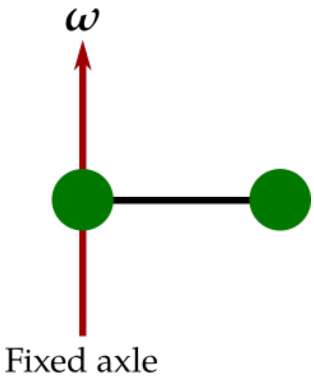
An alternative, simpler, argument works equally well for most of these examples. Suppose that the angular momentum vector $\boldsymbol{L}$ is tilted in some direction with respect to the angular velocity vector $\boldsymbol{\omega}$. If there are always other tilt directions equivalent to the first by symmetry, there are other equally good choices for the direction of $\boldsymbol{L}$. Since there is only one $\boldsymbol{L}$ and no grounds for preferring one of the equally good tilt directions over the others, the angular momentum cannot be tilted in any of them. The only remaining option is that it is parallel to $\boldsymbol{\omega}$.

Analysing the rotations of bodies for which $\boldsymbol{L}$ is not parallel to $\boldsymbol{\omega}$ is quite challenging and will not be covered in this course.

6.6 Rotations about Axles

An axle is a fixed rod about which a body rotates. Car and bicycle wheels rotate on axles, as do gyroscopes, doors, and door handles.

Imagine mounting a simple rigid body — a dumbbell consisting of two masses connected by a light stiff rod — to an axle as shown in the figure below.



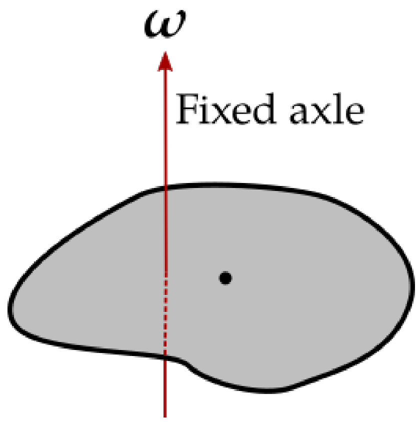
As the dumbbell spins around the axle at angular velocity $\boldsymbol{\omega}$, the axle supplies the centripetal force required to keep the outer mass on its circular orbit. If the dumbbell spins too fast, the axle will break.

More generally, an axle provide whatever forces are required to keep the object mounted on it in place in the directions perpendicular to the axle and whatever torques are required to stop the object rotating about any axis perpendicular to the axle. Most axle mountings also keep objects in place in the parallel direction. The only thing axles cannot do is apply a torque around their own axis (which is assumed to have been well greased).

Now imagine mounting the rigid potato on an axle. The axle need not pass through the centre of mass or point in any specific direction. As usual, the motion of the potato is determined by Newton's second law in its linear and rotational forms:

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F} = \boldsymbol{F}^{\text{ext}} + \boldsymbol{F}^{\text{axle}},$$
$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{G} = \boldsymbol{G}^{\text{ext}} + \boldsymbol{G}^{\text{axle}}.$$

The internal forces do not matter, but the potato feels forces and torques from the axle as well as those applied externally. The forces and torques



from the axle exactly counter the external forces and torques in all directions except one, keeping the potato in place relative to the axle and preventing it from rotating around any axis perpendicular to the axle. The only coordinate that can change is the angle of rotation of the potato around the axle. In that, parallel, direction only, the axle exerts no torque, $G_{\parallel}^{\text{axle}} = 0$, and the angular velocity of the potato changes in response to the externally applied torque only:

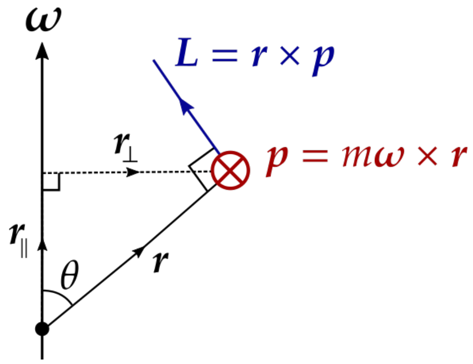
$$\frac{dL_{\parallel}}{dt} = G_{\parallel}^{\text{ext}}.$$

Notice that this is no longer a vector equation. As we only have one coordinate left, we are in effect working in one dimension and using vectors is unnecessary. The angular momentum of the potato may have perpendicular components, but the axle provides any torques required to keep these components rotating around the axle at angular velocity $\boldsymbol{\omega}$, just as it applies any forces required to stop the centre of mass moving in a straight line at constant linear velocity. Only the rotation around the axle is unconstrained, and this is the only motion we need to worry about.

6.7 The Moment of Inertia

This course only covers bodies with $\boldsymbol{L}$ parallel to $\boldsymbol{\omega}$ and bodies mounted on axles, so we only care about $L_{\parallel} = \hat{\boldsymbol{\omega}} \cdot \boldsymbol{L}$. This section explains how to evaluate $L_{\parallel}$ and how to express it in terms of the moment of inertia, which is a rotational equivalent of mass. Just as the mass m tells you how much force is required to make a body accelerate at 1 m·s^{-2}, the moment of intertia I tells you how much torque is required to produce an angular acceleration of 1 Radian·s^{-2}.

Suppose that m is a particle at position $\boldsymbol{r}$ in a rigid body rotating at angular velocity $\boldsymbol{\omega}$.



Then $\boldsymbol{p} = m\boldsymbol{\omega} \times \boldsymbol{r} = m\omega r \sin\theta$ into page $= m\omega r_\perp \hat{\boldsymbol{\phi}}$ and

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} = (\boldsymbol{r}_\parallel + \boldsymbol{r}_\perp) \times (m\omega r_\perp \hat{\boldsymbol{\phi}}) \\ &= m\omega r_\perp (\boldsymbol{r}_\parallel \times \hat{\boldsymbol{\phi}} \;+\; \boldsymbol{r}_\perp \times \hat{\boldsymbol{\phi}}) \\ &= m\omega r_\perp \left(-r_\parallel \hat{\boldsymbol{r}}_\perp \;+\; r_\perp \hat{\boldsymbol{\omega}}\right). \end{aligned}$$

The term we care about is the one parallel to the angular velocity, so the $\hat{\boldsymbol{r}}_\perp$ term is of no interest. The parallel component of the angular momentum of the mass point is

$$L_\parallel = m r_\perp^2 \omega = m r_\perp v_\phi,$$

just as one would expect. This is exactly the formula used for the angular momentum of a planet in a two-dimensional orbit around the Sun, although now derived in three dimensions.

To work out the angular momentum of the entire body we simply add the contributions from all of the mass points:

$$L_\parallel = \left(\sum_i m_i r_{i\perp}^2\right) \omega = I\omega,$$

where the moment of inertia I is defined by

Moment of Inertia

$$I \triangleq \sum_i m_i r_{i\perp}^2$$

Aside: The Moment of Inertia Tensor (not examinable)

Some of you may be frustrated by my neglect of components of the angular momentum perpendicular to the angular velocity, so here is a brief explanation of how to relate $\boldsymbol{L}$ to $\boldsymbol{\omega}$ for a general rigid body.

The momentum of a mass m at point $\boldsymbol{r}$ in a rigid body rotating at angular velocity $\boldsymbol{\omega}$ is $m\boldsymbol{\omega} \times \boldsymbol{r}$, so its angular momentum is

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}).$$

If you choose a set of coordinate axes and write $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ and $\boldsymbol{\omega} = \omega_x\hat{\boldsymbol{\imath}} + \omega_y\hat{\boldsymbol{\jmath}} + \omega_z\hat{\boldsymbol{k}}$, you can work out the x, y and z components of $m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ term by term. (This painful exercise can be simplified by using the vector triple product formula, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$, proved in one of your vectors and matrices problem sheets.) The result is three equations,

$$\begin{aligned}
L_x &= m(y^2 + z^2)\omega_x - mxy\omega_y - mxz\omega_z\,, \\
L_y &= -myx\omega_x + m(z^2 + x^2)\omega_y - myz\omega_z\,, \\
L_z &= -mzx\omega_x - mzy\omega_y + m(x^2 + y^2)\omega_z\,.
\end{aligned}$$

Adding the contributions to the total angular momentum from all of the mass points in the body and rewriting the three equations in matrix form gives:

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(z_i^2 + x_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

The 3×3 matrix that maps $\boldsymbol{\omega}$ to $\boldsymbol{L}$ is called the *moment of inertia tensor*. The diagonal elements have the same form, $\sum_i m_i r_{i\perp}^2$, as the moment of inertia used in this course.

Aside: Principal Axes (not examinable)

The last aside explained that the angular momentum and angular velocity of a general rigid body are related by a 3×3 real symmetric matrix, I, called the moment of inertia tensor: $\boldsymbol{L} = \mathrm{I}\boldsymbol{\omega}$. Your vectors and matrices course has explained how, given any real symmetric 3×3 matrix, you can always find three perpendicular vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, for which

$$\mathrm{I}\boldsymbol{e}_1 = \lambda_1 \boldsymbol{e}_1, \qquad \mathrm{I}\boldsymbol{e}_2 = \lambda_2 \boldsymbol{e}_2 \quad \text{and} \quad \mathrm{I}\boldsymbol{e}_3 = \lambda_3 \boldsymbol{e}_3.$$

If the matrix I is applied to one of these vectors, its only effect is to multiply the vector by a constant. The vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are called the eigenvectors of I and are normally chosen to have unit length; the corresponding numbers λ_1, λ_2 and λ_3 are the eigenvalues.

Suppose we start a rigid body rotating about an axis parallel to one of the eigenvectors of its moment of inertia tensor. For concreteness, imagine spinning the body so that $\boldsymbol{\omega} = \omega\hat{e}_1$, where $\hat{e}_1$ is a unit vector in the $\boldsymbol{e}_1$ direction. The angular momentum $\boldsymbol{L} = \mathrm{I}\boldsymbol{\omega} = \lambda_1\omega\hat{e}_1 = \lambda_1\boldsymbol{\omega}$ is then parallel to $\boldsymbol{\omega}$. When spun around one of the eigenvectors of its moment of inertia tensor, the angular velocity and angular momentum of even the wonkiest rigid body are parallel and the rotational motion can be treated using exactly the methods described in this chapter.

The eigenvectors of the moment of inertia tensor are called the *principal axes* of the body. Although you cannot use symmetry to help you guess the principal axes of a general wonky-shaped rigid body, they always exist and you can calculate them by using integration (numerical if necessary) to find the I matrix. This makes the approach to rotational dynamics taken in this course more general than you might at first think.

To work out the angular acceleration of the body about the axis of rotation, we start from the angular version of Newton's second law, $\boldsymbol{G} = d\boldsymbol{L}/dt$, take the parallel component of this vector equation to get $G_{\parallel} = dL_{\parallel}/dt$, and remember that $L_{\parallel} = I\omega$. This gives

Angular Acceleration of a Rigid Body

$$G_{\parallel} = I\frac{d\omega}{dt}$$

Assuming that the rigid body is rotating at angular velocity $\boldsymbol{\omega}$ and remembering that we can treat the kinetic energy of the centre of mass separately, the rotational part of the kinetic energy can also be written in terms of I:

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i (r_{i\perp}\omega)^2 = \frac{1}{2}\left(\sum_i m_i r_{i\perp}^2\right)\omega^2,$$

telling us that

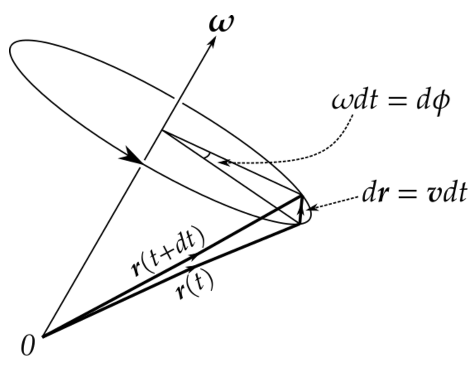
Angular Kinetic Energy

$$K = \frac{1}{2}I\omega^2$$

The close analogy between the angular equations $G_{\parallel} = I d\omega/dt$ and $K = \frac{1}{2}I\omega^2$ and the one-dimensional linear equations $F = m dv/dt$ and $K = \frac{1}{2}mv^2$ is obvious.

6.8 Rotational Work

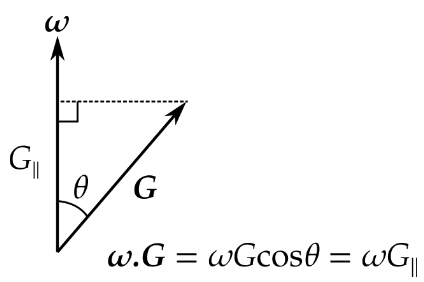
The diagram below shows a mass point in a solid body rotating at angular velocity $\boldsymbol{\omega}$.



The work done in time dt is

$$\begin{aligned} dW &= \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= \boldsymbol{F} \cdot (\boldsymbol{v}\, dt) \\ &= \boldsymbol{F} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) dt \\ &= \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \boldsymbol{F}) dt \\ &= \boldsymbol{\omega} \cdot \boldsymbol{G} dt \\ &= \omega G_{\|} dt \\ &= G_{\|} d\phi. \end{aligned}$$

The step from line 2 to line 3 works because the mass point at $\boldsymbol{r}$ in a rigid body has velocity $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. Changing $\boldsymbol{F} \cdot (\boldsymbol{\omega} \times \boldsymbol{r})$ to $\boldsymbol{\omega} \cdot (\boldsymbol{r} \times \boldsymbol{F})$ is allowed because of the scalar triple product formula: $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$. The replacement of $\omega \cdot \boldsymbol{G}$ by $\omega G_{\|}$ is explained by the following diagram:



We have now derived the rigid-body version of the

Rotational Work Formula
$dW = G_{\|} d\phi$

We can find a rotational equivalent of the work-energy theorem by starting from the rotational version of the second law, $G_\parallel = I d\omega/dt$, and following exactly the same steps as in Sec. 3.5:

$$\begin{aligned} G_\parallel &= I\frac{d\omega}{d\phi}\frac{d\phi}{dt} && \text{(chain rule)} \\ &= I\omega\frac{d\omega}{d\phi} && (\omega = d\phi/dt) \\ &= \frac{d}{d\phi}\left(\frac{1}{2}I\omega^2\right) && \text{(product rule or chain rule again)} \end{aligned}$$

Integrating both sides from the initial angle ϕ_i to the final angle ϕ_f gives:

Rotational Work-Energy Theorem

$$\frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \int_{\theta_i}^{\theta_f} G_\parallel(\theta)\, d\theta$$

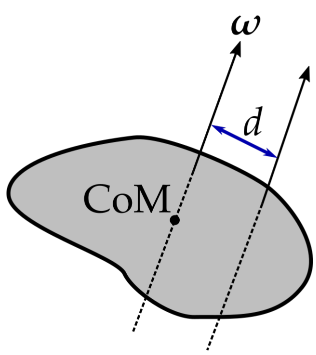
6.9 The Parallel Axis Theorem

The Parallel Axis Theorem

If I^* is the moment of inertia about an axis through the centre of mass, then

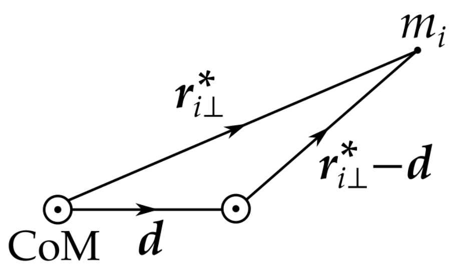
$$I = I^* + Md^2$$

is the moment of inertia about a parallel axis a distance d away.



Proof

If we look down on the two parallel axes, the mass m_i at position $\boldsymbol{r}_i$ and the perpendicular vectors from the two axes to that point look like this.



Looking down rotation axis

The moment of inertia about the centre of mass is given by

$$I^* = \sum_i m_i (r^*_{i\perp})^2$$

and the moment of inertia about the parallel axis is

$$\begin{aligned} I &= \sum_i m_i (\boldsymbol{r}^*_{i\perp} - \boldsymbol{d})^2 \\ &= \sum_i m_i \left[(r^*_{i\perp})^2 - 2\boldsymbol{d} \cdot \boldsymbol{r}^*_{i\perp} + d^2 \right] \\ &= I^* - 2\boldsymbol{d} \cdot \cancel{\sum_i m_i \boldsymbol{r}^*_{i\perp}} + \sum_i m_i d^2 \\ &= I^* + M d^2, \end{aligned}$$

which is the parallel axis theorem. The summation in the middle term is zero because it is the perpendicular component of $\sum_i m_i \boldsymbol{r}^*_i$ and we showed that this vanishes in Sec. 6.3.

6.10 Linear and Rotational Analogies

Linear Motion	**Rotational Motion**
v	ω
m	I
$p = mv$	$L_\parallel = I\omega$
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
$F = \frac{dp}{dt} = m\frac{dv}{dt}$	$G_\parallel = \frac{dL_\parallel}{dt} = I\frac{d\omega}{dt}$
$dW = F dx$	$dW = G_\parallel d\phi$
$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int_{x_i}^{x_f} F(x)dx$	$\frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 = \int_{\phi_i}^{\phi_f} G_\parallel(\phi)d\phi$

6.11 Into the Continuum

How can you work out sums like

$$M = \sum_i m_i\,, \qquad \boldsymbol{R} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i\,, \qquad I = \sum_i m_i r_{i\perp}^2\,,$$

for a body made of 6×10^{23} atoms? The answer is that you treat the body as a continuous object and turn the sums into integrals. The first step is to introduce the mass density $\rho(\boldsymbol{r})$, so that you can write:

Mass Density

$$dm \quad = \quad \begin{array}{c}\text{mass in volume}\\ \text{element } dV \text{ at } \boldsymbol{r}\end{array} \quad = \quad \rho(\boldsymbol{r})\, dV$$

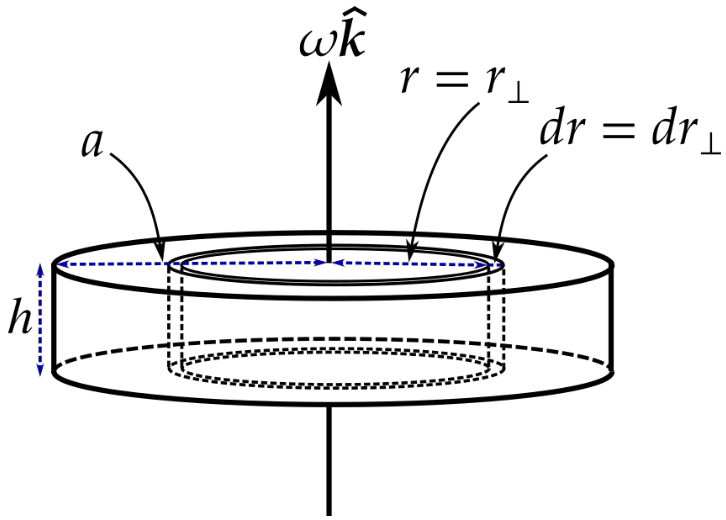
Then

$$\begin{aligned}
M &= \sum dm = \sum \rho(\boldsymbol{r})\, dV &\rightarrow&& M &= \int \rho(\boldsymbol{r})\, dV,\\
M\boldsymbol{R} &= \sum \boldsymbol{r} dm = \sum \boldsymbol{r}\rho(\boldsymbol{r})\, dV &\rightarrow&& M\boldsymbol{R} &= \int \boldsymbol{r}\rho(\boldsymbol{r})\, dV,\\
I &= \sum r_\perp^2 dm = \sum r_\perp^2 \rho(\boldsymbol{r})\, dV &\rightarrow&& I &= \int r_\perp^2 \rho(\boldsymbol{r})\, dV.
\end{aligned}$$

You have not been taught volume integration yet, so you are not yet equipped to work out integrals like these for rigid bodies of any shape. If the shape is simple enough, however, it is often possible to convert three-dimensional integrals into one-dimensional integrals you already know how to do.

Example: Uniform Cylinder

A uniform cylinder of constant density $\rho(\boldsymbol{r}) = \rho$, radius a, and height h is rotating about its axis.



To evaluate the total mass and moment of inertia, imagine splitting the cylinder in to narrow concentric rings of width $dr = dr_\perp$ as shown in the diagram. The area between the two circles that delineate the top of the ring of inner radius r is

$$dA = \pi(r + dr)^2 - \pi r^2 = 2\pi r\,dr + [\text{negligible } (dr)^2 \text{ terms}] \to 2\pi r\,dr.$$

This result could also have been obtained by defining $A = \pi r^2$ and writing

$$dA = \frac{dA}{dr}dr = 2\pi r\,dr,$$

or just by looking at the diagram. The volume of the ring is $h\,dA = h\,2\pi r\,dr$ and its mass is $\rho h\,dA = \rho h\,2\pi r\,dr$.

We can now work out the total mass of the cylinder (just to check that the method works) and the moment of inertia:

$$M = \sum dm = \sum_{\text{rings}} \rho h 2\pi r \, dr \quad \rightarrow \quad M = \int_0^a \rho h 2\pi r \, dr = \rho h \pi a^2, \quad \checkmark$$

$$I = \sum r_\perp^2 \, dm = \sum_{\text{rings}} r^2 \rho h 2\pi r \, dr \quad \rightarrow \quad I = \int_0^a \rho h 2\pi r^3 \, dr = \frac{1}{2}\rho h \pi a^4.$$

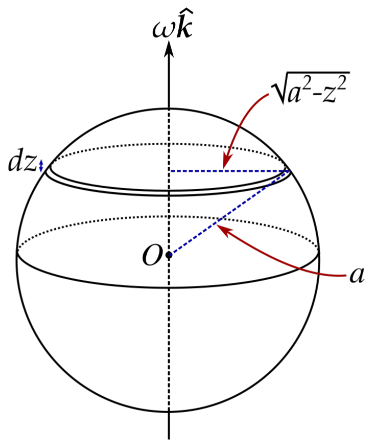
The moment of inertia is normally rewritten in terms of the total mass:

$$I = \frac{1}{2}\left(\rho h \pi a^2\right) a^2 = \frac{1}{2} M a^2.$$

We could use a similar method to work out the position of the centre of mass, but there is no need as it is obviously in the middle of the cylinder.

Example: Uniform Sphere

A uniform sphere of density ρ and radius a is rotating about a diameter.



To work out its total mass (as a check) and moment of inertia, split it into discs, each of height dz. The radius of the disk at height z is $\sqrt{a^2 - z^2}$.

Then

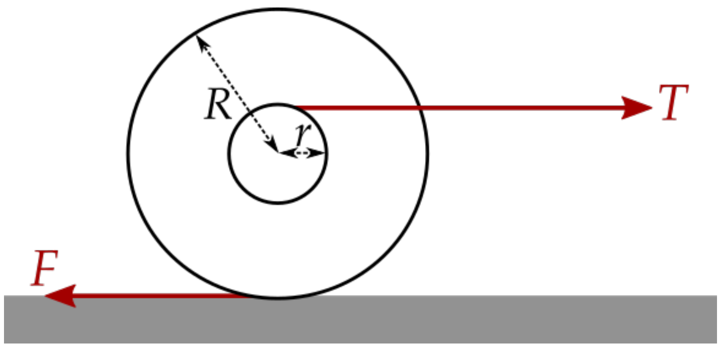
$$M = \sum_{\text{discs}} dm = \sum_{\text{discs}} \rho\pi(a^2 - z^2)\, dz \rightarrow \int_{-a}^{a} \rho\pi(a^2 - z^2)\, dz = \rho\frac{4}{3}\pi a^3 \quad \checkmark$$

and

$$
\begin{aligned}
I &= \sum_{\text{mass elements}} r_\perp^2\, dm \\
&= \sum_{\text{discs}} dI \qquad \text{(we have already done the sum for each disc)} \\
&= \sum_{\text{discs}} \underbrace{\rho \frac{1}{2}\pi \left(a^2 - z^2\right)^2 dz}_{\substack{\text{moment of inertia of disc of} \\ \text{radius } \sqrt{a^2-z^2} \text{ and height } dz}} \\
&\to \int_{-a}^{a} \rho \frac{1}{2}\pi \left(a^2 - z^2\right)^2 dz \\
&= \frac{8\pi\rho a^5}{15} \\
&= \frac{2}{5}\left(\rho\frac{4}{3}\pi a^3\right) a^2 \\
&= \frac{2}{5} M a^2.
\end{aligned}
$$

6.12 Applications

Spool of Thread



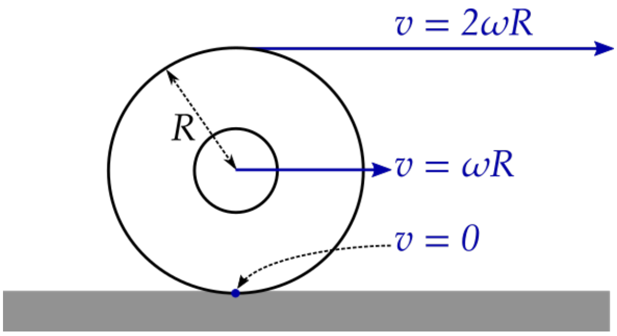
The diagram shows a spool of thread. The cotton is wound around the inner cylinder of radius r. Discs of radius $R > r$ are attached to both flat faces of the inner cylinder. These act as wheels, so that the spool rolls and accelerates if you pull on the thread with a force T. The mass of the spool, including the cotton, is M, and its moment of inertia is I. We assume that the spool rolls without slipping.

The spool feels two external forces. The first is the tension T in the thread, which depends on how hard you decide to pull and which we assume is known. The other is the frictional force F exerted by the floor on the spool. Since the spool is accelerating, $T > F$. The spool is rolling without slipping, so the point of the spool in contact with the floor is always stationary. This means that the force F is static friction, not dynamic friction; it cannot do work or dissipate energy because it acts over no distance.

If the spool has angular velocity ω Radians·s^{-1}, it moves a distance $2\pi R$ (one circumference) in time $T = 2\pi/\omega$. The speed v of its centre of mass is given by

$$v = \frac{\text{distance}}{\text{time}} = \frac{2\pi R}{2\pi/\omega} = \omega R.$$

The speed of the centre of mass is the same as the speed of a point on the rim relative to the centre of mass. The point in contact with the ground is not moving at all and the point at the top of the spool is moving at speed $2v = 2\omega R$.



Method 1: Use Newton's Second Law

As always, the centre of mass accelerates as if all of the external forces were acting directly on that point:

$$M\frac{dv}{dt} = T - F.$$

The rotation about the centre of mass is given by the rotational equivalent of Newton's second law:

$$G = rT + RF = \frac{dL}{dt} = I\frac{d\omega}{dt}$$

Both contributions to the torque about the centre of mass are trying to rotate the spool in the clockwise direction, so both act into the page and both have the same sign. Since $v = \omega R$, the angular version of Newton's second law can be rewritten as:

$$rT + RF = \frac{I}{R}\frac{dv}{dt}.$$

Eliminating the unknown force F from this equation using Newton's second law for the centre of mass, $M\,dv/dt = T - F$, gives

$$\begin{aligned}
& rT + R\left(T - M\frac{dv}{dt}\right) = \frac{I}{R}\frac{dv}{dt} \\
\Rightarrow \quad & \left(MR + \frac{I}{R}\right)\frac{dv}{dt} = (r+R)T \\
\Rightarrow \quad & \boxed{\frac{dv}{dt} = \frac{R(r+R)T}{MR^2 + I}}
\end{aligned}$$

The acceleration is constant (assuming T is constant) so we can now use the familiar SUVAT equations: $v = at$, $x = \frac{1}{2}at^2$, and $v^2 = 2ax$. The problem is solved.

Method 2: Use the Work-Energy Theorem

As already discussed, the point of the spool in contact with the ground is (instantaneously) stationary, so the static frictional force F does no work. When the spool rolls forward by ϕ radians, its centre of mass moves a distance $R\phi$. The work done by the tension force T is:

$$W = \text{force} \times \text{distance} = T(r\phi + R\phi).$$

The $r\phi$ is the length of thread that has unrolled from the spool and the $R\phi$ is the distance the centre of mass of the spool has moved towards the person pulling the end of the thread. The amount of thread pulled in (which is the distance required to calculate the work done by the person who is pulling) is the sum of these two. Since $x = R\phi$, we can rewrite the work as

$$W = T\left(\frac{r}{R} + 1\right)x.$$

Another way to do this is to remember that the work done can be divided into the work done on the centre of mass and the work done about the centre of mass. To calculate the work done on the centre of mass, we imagine as usual that all of the external forces (in this case T and F) are acting directly on the centre of mass. The work done about the centre of mass is entirely rotational and is given by $G\phi$, where G is the torque about the centre of mass. Putting everything together gives

$$
\begin{aligned}
W &= \text{work done on CoM} + \text{work done about CoM} \\
&= (T-F)x + G\phi \\
&= (T-F)x + \underbrace{(rT+RF)}_{G}\,\underbrace{(x/R)}_{\phi} \\
&= \left(1+\frac{r}{R}\right)Tx
\end{aligned}
$$

as before.

The hard part is now done. Applying the work-energy theorem gives

$$
\begin{aligned}
\left(1+\frac{r}{R}\right)Tx &= \underbrace{\frac{1}{2}Mv^2}_{\text{KE of CoM}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{KE about CoM}} \\
&= \frac{1}{2}\left(M+\frac{I}{R^2}\right)v^2 \qquad\qquad (\text{using } \omega = v/R)
\end{aligned}
$$

which rearranges to give

$$\boxed{v^2 = \frac{2R(r+R)T}{MR^2+I}\,x}$$

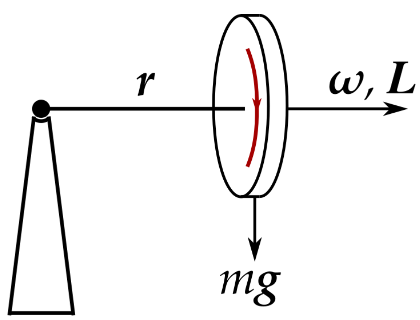
This is equivalent to the SUVAT equation $v^2 = 2ax$, with the acceleration a given by

$$a = \frac{R(r+R)T}{MR^2+I},$$

just as we found using Newton's second law.

Gyroscope

A gyroscope is a spinning flywheel subject to a torque.



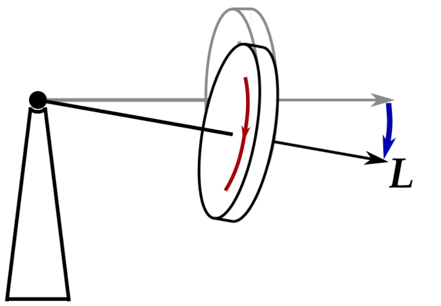
For simplicity, assume that the rotation axis is horizontal. (Treating a tilted rotation axis is only a little more difficult. Why not try it?) Since the flywheel is heavy and rotating rapidly, the angular momentum $\boldsymbol{L}$ is large. Because the flywheel is a disc, its angular momentum is also parallel to its angular velocity:

$$\boldsymbol{L} = I\boldsymbol{\omega}.$$

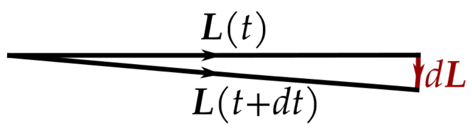
The torque about the pivot point at the top of the stand is

$$\boldsymbol{G} = \boldsymbol{r} \times m\boldsymbol{g} = mgr \text{ into page} = mgr\hat{\boldsymbol{\phi}}.$$

What happens when you let the gyroscope go? You might expect it to start falling,

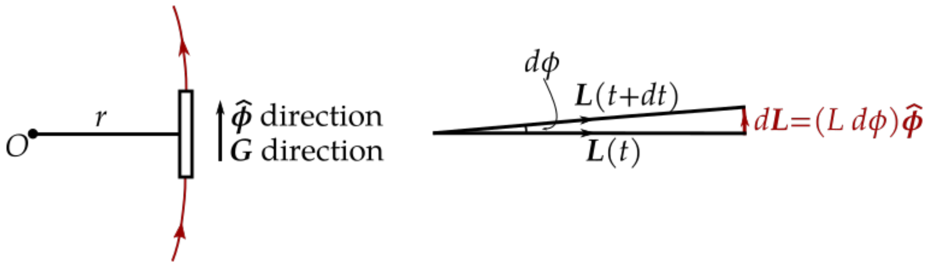


but then $d\boldsymbol{L}$ would point down:



Since $d\boldsymbol{L} = \boldsymbol{G}dt$ and $\boldsymbol{G}$ is into the page (in the $\hat{\boldsymbol{\phi}}$ direction), $d\boldsymbol{L}$ must be into the page too. Despite our preconceptions, the rotational version of Newton's second law is telling us that the gyroscope cannot start falling.

Since $\boldsymbol{G}$ points into the page, $d\boldsymbol{L} = \boldsymbol{G}dt$ must also point into the page. This means that the gyroscope must precess about the vertical axis. The figure below, which shows the gyroscope from above, explains how this works.



If the gyroscope is precessing as shown, the angular version of Newton's second law requires that

$$d\boldsymbol{L} = (L\,d\phi)\hat{\boldsymbol{\phi}} = \boldsymbol{G}\,dt = mgr\hat{\boldsymbol{\phi}}\,dt.$$

The directions of $d\boldsymbol{L}$ and $\boldsymbol{G}\,dt$ now agree, so the precessing solution — which was nothing more than a guess — seems to have been a good guess. Equating the lengths of $d\boldsymbol{L}$ and $\boldsymbol{G}\,dt$ gives

$$L\,d\phi = mgr\,dt$$

and so

$$\boxed{\frac{d\phi}{dt} = \frac{mgr}{L} = \frac{mgr}{I\omega}}$$

This tells us the angular frequency of the precession, $d\phi/dt$, in Rad·s^{-1}. If you try the same analysis with a tilted gyroscope, you should find that $d\phi/dt$ is unchanged.

One interesting feature of this result is that the precession frequency increases as friction reduces the spin rate ω of the gyroscope about its axis. Another interesting feature is that the precession of creates a small vertical component of the angular momentum, which we have neglected so far. As the torque has no vertical component (if we ignore the small frictional forces from the stand), the vertical component of the total angular momentum cannot change, so the gyroscope has no choice but to drop just a little, until the vertical component of its axial angular momentum $\boldsymbol{L}$ exactly cancels the vertical angular momentum resulting from the precession. As the gyroscope slows down and $\boldsymbol{L}$ reduces in magnitude, it has to drop more and more to keep the vertical component of angular momentum equal to zero, until eventually it falls.

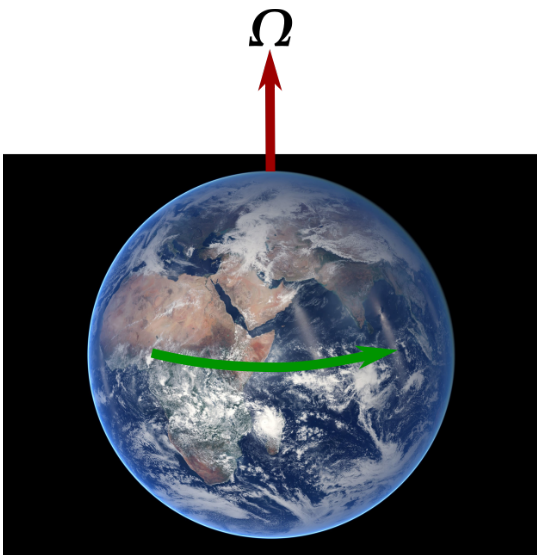
Chapter 7

Rotating Frames

7.1 Our Spinning World

One can argue that understanding the fictitious forces that (appear to) influence the motion of bodies in rotating frames is not very important. Isn't it better to work in inertial frames where all of the forces are real? And if we are going to study mechancs in non-inertial frames, why not tackle the general case in which the acceleration is more than a simple rotation? I am sympathetic to both of these viewpoints, but we happen to live on a spinning world and the fictitious forces described in this chapter have real effects on us.

The diagram below shows the sense of rotation and angular velocity vector of the Earth.

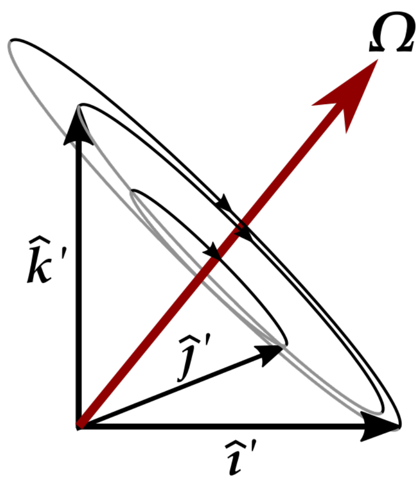


The Sun rises in the east and sets in the west, telling us that the Earth rotates from west to east. Its angular velocity vector points north.

7.2 Inertial and Rotating Coordinate Systems

As a first step towards understanding the fictitious forces that affect people living on rotating worlds, we are going to compare measurements made in two different coordinate system: an inertial frame with fixed unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, and a rotating frame with rotating unit vectors $\hat{\boldsymbol{\imath}}'(t)$, $\hat{\boldsymbol{\jmath}}'(t)$ and $\hat{\boldsymbol{k}}'(t)$. The angular velocity of rotation is $\boldsymbol{\Omega}$. It might help to imagine $\hat{\boldsymbol{\imath}}'(t)$, $\hat{\boldsymbol{\jmath}}'(t)$ and $\hat{\boldsymbol{k}}'(t)$ buried at the centre of the Earth, rotating once a day as the Earth in which they are embedded spins.

Because the Earth's angular velocity $\boldsymbol{\Omega}$ is (almost) constant, the rotating unit vectors spin steadily around it, tracing out cones.



We showed in Sec. 5.3 that the rate of change of a vector $\boldsymbol{a}(t)$ rotating at angular velocity $\boldsymbol{\Omega}$ whilst also changing length is given by:

$$\frac{d\boldsymbol{a}}{dt} = \dot{a}\hat{\boldsymbol{a}} + \boldsymbol{\Omega} \times \boldsymbol{a} = \dot{a}\hat{\boldsymbol{a}} + a\boldsymbol{\Omega} \times \hat{\boldsymbol{a}}.$$

The rotating unit vectors always have unit length, so $a = 1$, $\dot{a} = 0$, and their derivatives are given by

$$\boxed{\frac{d\hat{\boldsymbol{\imath}}'}{dt} = \boldsymbol{\Omega} \times \hat{\boldsymbol{k}}' \qquad \frac{d\hat{\boldsymbol{\jmath}}'}{dt} = \boldsymbol{\Omega} \times \hat{\boldsymbol{\jmath}}' \qquad \frac{d\hat{\boldsymbol{k}}'}{dt} = \boldsymbol{\Omega} \times \hat{\boldsymbol{k}}'}$$

Positions, Velocities and Accelerations

The position vector $\boldsymbol{r}(t)$ of a mass m can be written in terms of either set of basis vectors:

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}},$$
$$\boldsymbol{r}(t) = x'(t)\hat{\boldsymbol{\imath}}'(t) + y'(t)\hat{\boldsymbol{\jmath}}'(t) + z'(t)\hat{\boldsymbol{k}}'(t).$$

The coordinates of the mass as measured by the inertial observer are $x(t)$, $y(t)$ and $z(t)$, while the rotating observer measures $x'(t)$, $y'(t)$ and $z'(t)$. Both observers draw exactly the same arrow from the origin (which I am assuming for simplicity is shared by both coordinate systems) to the mass at all times, but they describe that arrow using different numbers.

Suppose, for example, that the mass is stationary in the inertial frame. The inertial observer measures coordinates x, y and z, all of which are independent of time. The rotating observer see the mass orbiting in a circle with angular velocity $-\boldsymbol{\Omega}$ and assigns time-dependent coordinates $x'(t)$, $y'(t)$ and $z'(t)$. The time dependence of the rotating observer's coordinates "undoes" the time dependence of their unit vectors, reproducing the time-independent arrow seen from the inertial frame.

The rotating observer, unaware that their unit vectors are rotating, *thinks* that the velocity and acceleration of the mass are:

$$\boldsymbol{v}'(t) = \dot{x}'(t)\hat{\boldsymbol{\imath}}'(t) + \dot{y}'(t)\hat{\boldsymbol{\jmath}}'(t) + \dot{z}'(t)\hat{\boldsymbol{k}}'(t),$$
$$\boldsymbol{a}'(t) = \ddot{x}'(t)\hat{\boldsymbol{\imath}}'(t) + \ddot{y}'(t)\hat{\boldsymbol{\jmath}}'(t) + \ddot{z}'(t)\hat{\boldsymbol{k}}'(t).$$

To measure the particle's velocity in the x' direction, for example, the rotating observer would measure its x' coordinate as a function of time and work out $\dot{x}'$ from the slope of the graph of x' against t. After finding $\dot{y}'$ and $\dot{z}'$ in an analogous way, they would say that $\boldsymbol{v}'(t) = \dot{x}'(t)\hat{\boldsymbol{\imath}}'(t) + \dot{y}'(t)\hat{\boldsymbol{\jmath}}'(t) + \dot{z}'(t)\hat{\boldsymbol{k}}'(t)$. This is *not* the same arrow as the inertial observer's velocity vector $\boldsymbol{v}(t) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}} + \dot{z}(t)\hat{\boldsymbol{k}}$ because the rotating observer has not taken the time dependence of $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$ and $\hat{\boldsymbol{k}}'$ into account.

Our aim in this section is to find the fictitious force $\boldsymbol{F}_{\text{fictitious}}$ required to make Newton's second law work in the rotating frame:

$$\boldsymbol{F}_{\text{true}} + \boldsymbol{F}_{\text{fictitious}} = m\boldsymbol{a}'.$$

Since Newton's second law already holds in the inertial frame, $\boldsymbol{F}_{\text{true}} = m\boldsymbol{a}$, and we can rewrite this as

$$\boldsymbol{F}_{\text{fictitious}} = m\boldsymbol{a}' - \boldsymbol{F}_{\text{true}} = m\boldsymbol{a}' - m\boldsymbol{a}.$$

We already know that $\boldsymbol{a}' \neq \boldsymbol{a}$, so $\boldsymbol{F}_{\text{fictitious}} \neq \mathbf{0}$.

What do we gain by adopting this rather artificial approach? Once we have introduced the fictitious forces required to make Newton's laws work again, we can transfer all of our intuitive and mathematical understanding of Newtonian mechanics in inertial frames to rotating frames without reinventing the subject from scratch. This is a great simplification.

Relationship between Rotating and Inertial Velocity Vectors

Before we can relate $\boldsymbol{a}'$ to $\boldsymbol{a}$, we need to relate $\boldsymbol{v}'$ to $\boldsymbol{v}$. We start from the one thing we know for certain, which is that $x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$ and $x'(t)\hat{\boldsymbol{\imath}}'(t) + y'(t)\hat{\boldsymbol{\jmath}}'(t) + z'(t)\hat{\boldsymbol{k}}'(t)$ are two different ways of the describing exactly the same position vector. Beginning with the velocity $\boldsymbol{v} = d\boldsymbol{r}/dt$ as measured in the inertial frame, we get

$$\begin{aligned}
\boldsymbol{v} &= \frac{d}{dt}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right) \\
&= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' + z'\hat{\boldsymbol{k}}'\right) && (\boldsymbol{r} \text{ is the same in both frames}) \\
&= \left(\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + \dot{z}'\hat{\boldsymbol{k}}'\right) + \left(x'\frac{d\hat{\boldsymbol{\imath}}'}{dt} + y'\frac{d\hat{\boldsymbol{\jmath}}'}{dt} + z'\frac{d\hat{\boldsymbol{k}}'}{dt}\right) && (\text{product rule}) \\
&= \boldsymbol{v}' + x'\left(\boldsymbol{\Omega} \times \hat{\boldsymbol{\imath}}'\right) + y'\left(\boldsymbol{\Omega} \times \hat{\boldsymbol{\jmath}}'\right) + z'\left(\boldsymbol{\Omega} \times \hat{\boldsymbol{k}}'\right) && \left(\frac{d\hat{\boldsymbol{\imath}}'}{dt} = \Omega \times \hat{\imath}', \ldots\right) \\
&= \boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r}.
\end{aligned}$$

The final result

Velocity in the Rotating Frame
$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{\Omega} \times \boldsymbol{r}$

makes good physical sense: a stationary object in the inertial frame ($\boldsymbol{v} = \mathbf{0}$) appears to be rotating at angular velocity $-\boldsymbol{\Omega}$ as seen from a frame with coordinate axes rotating at $+\boldsymbol{\Omega}$.

Relationship between Rotating and Inertial Acceleration Vectors

We can find the acceleration as measured by the rotating observer by differentiating the relationship between the velocities in the inertial and rotating frames:

$$\begin{aligned}
\boldsymbol{a} &= \frac{d\boldsymbol{v}}{dt} \\
&= \frac{d}{dt}\left(\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + \dot{z}'\hat{\boldsymbol{k}}' + \boldsymbol{\Omega} \times \boldsymbol{r}\right) \\
&= \left(\ddot{x}'\hat{\boldsymbol{\imath}}' + \ddot{y}'\hat{\boldsymbol{\jmath}}' + \ddot{z}'\hat{\boldsymbol{k}}'\right) + \left(\dot{x}'\frac{d\hat{\boldsymbol{\imath}}'}{dt} + \dot{y}'\frac{d\hat{\boldsymbol{\jmath}}'}{dt} + \dot{z}'\frac{d\hat{\boldsymbol{k}}'}{dt}\right) + \boldsymbol{\Omega} \times \boldsymbol{v} \\
&= \boldsymbol{a}' + \left(\dot{x}'(\boldsymbol{\Omega} \times \hat{\boldsymbol{\imath}}') + \dot{y}'(\boldsymbol{\Omega} \times \hat{\boldsymbol{\jmath}}') + \dot{z}'(\boldsymbol{\Omega} \times \hat{\boldsymbol{k}}')\right) + \boldsymbol{\Omega} \times (\boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \boldsymbol{a}' + 2\boldsymbol{\Omega} \times \boldsymbol{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}),
\end{aligned}$$

where I have used $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r}$ twice and $d\hat{\boldsymbol{\imath}}'/dt = \boldsymbol{\Omega} \times \hat{\boldsymbol{\imath}}'$, $d\hat{\boldsymbol{\jmath}}'/dt = \boldsymbol{\Omega} \times \hat{\boldsymbol{\jmath}}'$ and $d\hat{\boldsymbol{k}}'/dt = \boldsymbol{\Omega} \times \hat{\boldsymbol{k}}'$ once.

The final result is

Acceleration in the Rotating Frame
$\boldsymbol{a}' = \boldsymbol{a} - 2\boldsymbol{\Omega} \times \boldsymbol{v}' - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$

Notice that I have chosen to express the right-hand side in terms of $\boldsymbol{v}'$ instead of $\boldsymbol{v}$. Since $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\Omega} \times \boldsymbol{r}$ it is easy to switch from one description to the other, but the rotating observer believes the velocity to be $\boldsymbol{v}'$, not $\boldsymbol{v}$, and we are seeking equations suitable for use by the rotating observer, so expressing everything in terms of $\boldsymbol{v}'$ is sensible. There is no need to worry about whether to use $\boldsymbol{r}$ or $\boldsymbol{r}'$ because both observers agree on the position vector: $\boldsymbol{r}' = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' + z'\hat{\boldsymbol{k}}' = \boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$.

7.3 Centrifugal and Coriolis Forces

Multiplying the expression for $\boldsymbol{a}'$ through by m gives

$$\begin{aligned}
m\boldsymbol{a}' &= m\boldsymbol{a} - 2m\boldsymbol{\Omega} \times \boldsymbol{v}' - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\
&= \boldsymbol{F}_{\text{true}} - 2m\boldsymbol{\Omega} \times \boldsymbol{v}' - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}).
\end{aligned}$$

As explained earlier, the fictitious forces are defined by the equation $m\boldsymbol{a}' = \boldsymbol{F}_{\text{true}} + \boldsymbol{F}_{\text{fictitious}}$, so the second and third terms on the right-hand side are both fictitious forces.

Newton's Second Law in the Rotating Frame

$$m\boldsymbol{a}' = \boldsymbol{F}_{\text{true}} + \big(\underbrace{-2\boldsymbol{\Omega} \times \boldsymbol{v}'}_{\text{Coriolis force}}\big) + \big(\underbrace{-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})}_{\text{centrifugal force}}\big)$$

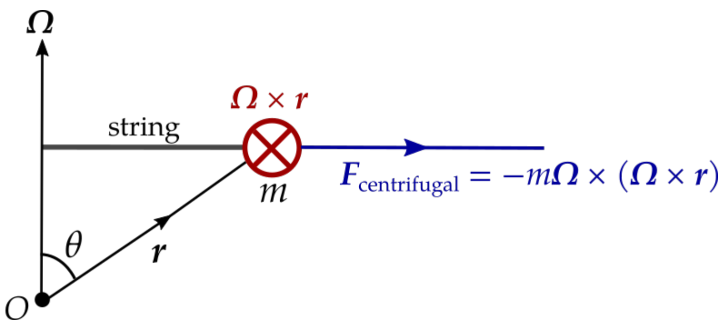
Seen from the rotating frame, the mass accelerates as if it experiences the true applied force $\boldsymbol{F}_{\text{true}}$ plust two fictitious forces, known as the Coriolis force and the centrifugal force.

The Centrifugal Force

The Centrifugal Force

$$\boldsymbol{F}_{\text{centrifugal}} = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$$

The diagram below shows a mass m attached to a string, rotating around an axis at angular velocity $\boldsymbol{\Omega}$. Think of the conker-on-a-string example we looked at much earlier in the course.



Because the mass is rotating in the inertial frame, it is stationary as seen from a frame that is also rotating with angular velocity $\boldsymbol{\Omega}$ about the same axis. In that rotating frame, $\boldsymbol{v}' = \boldsymbol{0}$ and the Coriolis force vanishes. The only fictitious force acting on the mass is the centrifugal force.

The first (innermost) cross product, $\boldsymbol{\Omega} \times \boldsymbol{r}$, gives a vector of magnitude $\Omega r \sin\theta = \Omega r_\perp$ pointing in the $\hat{\boldsymbol{\phi}}$ direction (into the page). Evaluating $-m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ then produces a vector of magnitude $\Omega^2 r \sin\theta = \Omega^2 r_\perp$ along the perpendicular vector from the rotation axis to m. Since $\boldsymbol{\Omega} \times \boldsymbol{r}$ is perpendicular to $\boldsymbol{\Omega}$, the second (outermost) cross product does not give any more $\sin\theta$ factors. The centrifugal force acts outwards, directly away from the rotation axis, and is equal and opposite to the centripetal force required to make a mass m orbit the axis at angular velocity $\boldsymbol{\Omega}$ in the inertial frame.

The mass is orbiting in the inertial frame, so the string is exerting a real centripetal force on it at all times, pulling it in towards the origin. As seen from the rotating frame, however, the mass is stationary. The rotating observer can see that the string is under tension and pulling the mass towards the origin, so why isn't the stationary mass accelerating in that direction? It is clear that something is wrong. Unaware that the mass is orbiting and therefore accelerating even though it looks stationary, the rotating observer argues (incorrectly) that the string tension is balanced by a hitherto unknown outward force, the centrifugal force.

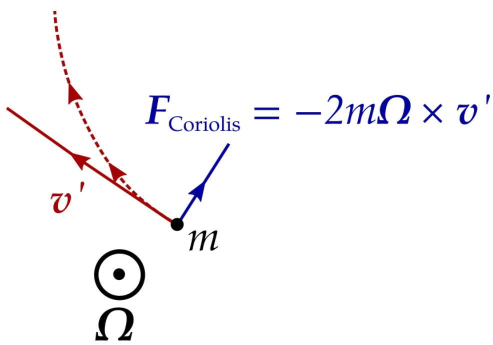
The Coriolis Force

The origin of the Coriolis force

The Coriolis Force

$$\boldsymbol{F}_{\text{Coriolis}} = -2m\boldsymbol{\Omega} \times \boldsymbol{v}'$$

can be explained using the diagram below.



This time we are looking down the axis of a frame rotating with angular velocity $\boldsymbol{\Omega}$. The mass m is no longer attached to a string and no forces are acting on it. It currently has velocity $\boldsymbol{v}'$ in the rotating frame.

Before trying to understand the behaviour of a mass moving in the rotating frame, it helps to think about a mass the rotating observer sees as stationary. As seen from the inertial frame, this mass must be orbiting anti-clockwise (in the example pictured above) about the rotation axis with the same angular velocity as the rotating observer. If the mass moves further away from the rotation axis but remains in the same direction as seen by the rotating observer, its anti-clockwise orbital speed $\Omega r_\perp$ has to increase. In the absence of real forces, however, there is nothing to create this anti-clockwise acceleration and masses moving radially outwards get "left behind" as seen by the rotating observer. Masses moving freely away from the axis appear to curve off to their right. Just as in the case of the centrifugal force, the rotating observer invents a fictitious force, the Coriolis force, to account for this apparent acceleration.

If you work through this argument for different velocities $\boldsymbol{v}'$, always looking down the rotation axis, you will find that the Coriolis force always acts to the right. If you reverse the direction of $\boldsymbol{\Omega}$ or look up instead of down the $\boldsymbol{\Omega}$ axis, it acts to the left. Fortunately, the cross product formula, $\boldsymbol{F}_{\text{Coriolis}} = -2m(\boldsymbol{\Omega} \times \boldsymbol{v}')$, takes care of these confusing directional issues automatically.

Example

Understanding how the centrifugal and Coriolis forces combine and why both are required to describe the motion of objects in rotating frames can be tricky. This simple example might help.

The mass in the image is stationary in the inertial frame and rotating at angular frequency $-\boldsymbol{\Omega}$ in the rotating frame. This is the reverse of the example used in the discussion of the centrifugal force above, where the mass was orbiting in the inertial frame but stationary in the rotating frame. The red line in the diagram below shows the track of the mass as seen in the rotating frame.

The centrifugal force, $\boldsymbol{F} = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = m\Omega^2 r\hat{\boldsymbol{r}}$, acts away from the origin O. The Coriolis force, $\boldsymbol{F} = -2m\boldsymbol{\Omega} \times \boldsymbol{v}' = -2m\Omega^2 r\hat{\boldsymbol{r}}$, acts towards O. Between them, they provide the centripetal force,

$$\boldsymbol{F}_{\text{centripetal}} = -m\Omega^2 r\hat{\boldsymbol{r}},$$

required to keep the mass (apparently) in orbit in the rotating frame. The mass is not orbiting in the inertial frame, so the centripetal force required to explain its orbit in the rotating frame is entirely fictitious. Fortunately, as long as the fictitious centrifugal and Coriolis forces are both included, we can use Newtonian mechanics in the rotating frame just as we would in an inertial frame.

7.4 The Earth as a Rotating Frame

Centrifugal Force on Earth

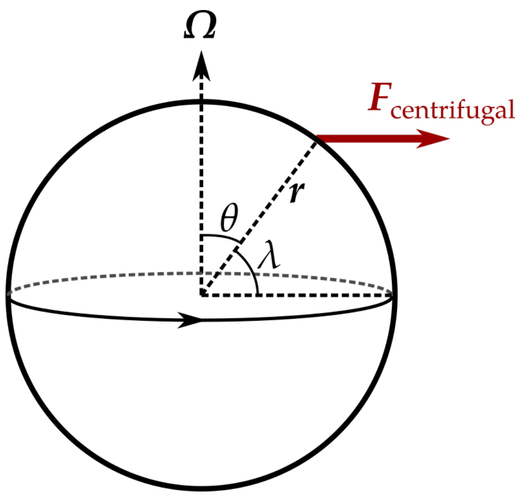
The Centrifugal Force
$\boldsymbol{F}_{\text{centrifugal}} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$

The diagram on the next page shows a point on the surface of the Earth at roughly the latitude of London (51.5°N). The latitude λ is measured up from the equator and is equal to $\pi/2 - \theta$, where θ (the polar angle used in the spherical polar coordinate system) is measured down from the north pole.

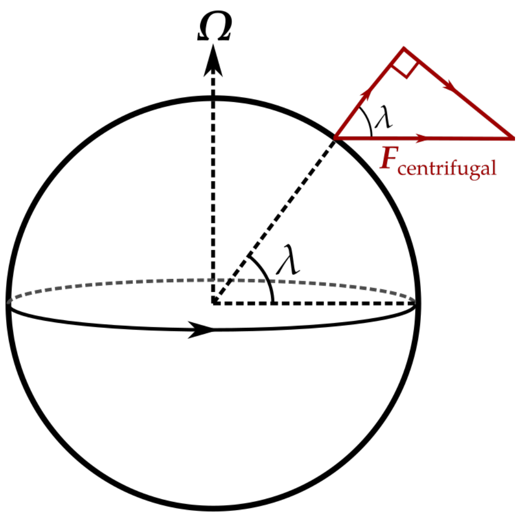
When $\boldsymbol{r}$ is as shown in the diagram, $\boldsymbol{\Omega} \times \boldsymbol{r}$ has magnitude $\Omega \cos\lambda$ ($= \Omega \sin\theta$) and points in to the page, perpendicular to $\boldsymbol{\Omega}$ and $\boldsymbol{r}$. It follows that $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ has magnitude $\Omega^2 r \cos\lambda$ and points to the left, directly towards the rotation axis. The centrifugal force $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ points directly away from the rotation axis and lies in the plane perpendicular to the rotation axis.

$$\boxed{F_{\text{centrifugal}} = m\Omega^2 r \cos\lambda \ \text{ directly away from the rotation axis}}$$

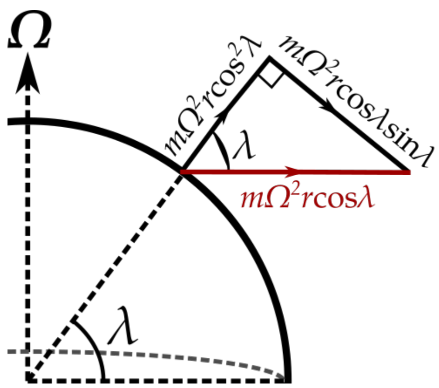
The centrifugal force is zero at the poles and maximum at the equator.



It is often helpful to resolve the centrifugal force into radial and tangential components, as shown in the figure below. The radial component $F_{\text{centrifugal}} \sin \lambda$ points vertically upwards, while the tangential component $F_{\text{centrifugal}} \cos \lambda$ points due south, parallel to the Earth's surface.



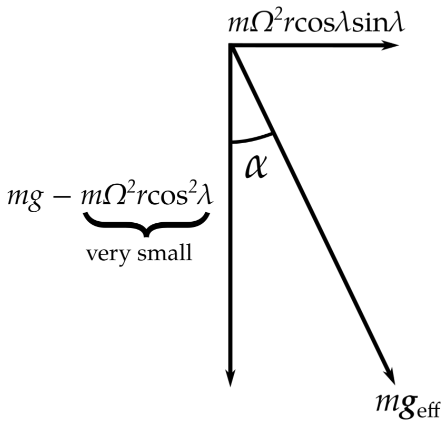
The components of the centrifugal force $F_{\text{centrifugal}} = m\Omega^2 r \cos \lambda$ are shown in more detail in the next figure.



The strength and direction of the centrifugal force depend on where you are, but the world is large and you have to travel a long way to see any changes. For most purposes, the centrifugal force can be fully taken into account by slightly modifying the effective value of $\boldsymbol{g}$, defining

$$m\boldsymbol{g}_{\text{eff}} = m\boldsymbol{g} + \boldsymbol{F}_{\text{centrifugal}}$$

as illustrated (from the point of view of a person standing in London) in the figure below.



The vertical component of the centrifugal force acts upwards, so the effective gravitational force is a little weaker than the true gravitational force. The tangential component also pulls masses to the south, so the effective gravitational force is slightly tilted away from the vertical. The force of gravity in London is not directed towards the centre of the Earth but towards a points just south of the centre. The deflection angle α, given by

$$\tan \alpha = \frac{\Omega^2 r \cos \lambda \sin \lambda}{g - \Omega^2 r \cos^2 \lambda},$$

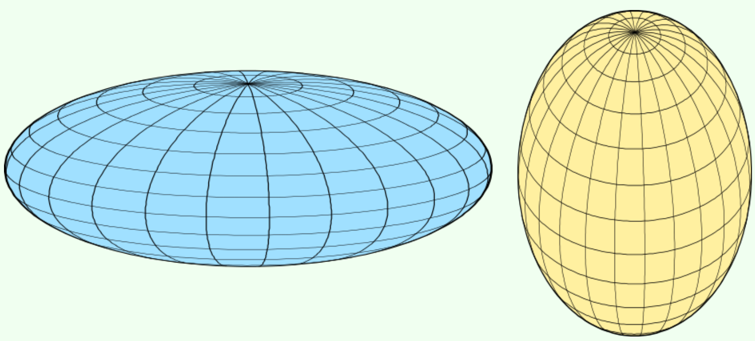
varies with latitude but is around 0.1° in London.

The centrifugal force may be fictitious but this is a real effect: the string of a plumb line in London really does point $\sim 0.1^\circ$ south of the centre of the Earth. You might think this would have obvious effects. Balls on flat ground in London and boats on the northern oceans ought to move steadily south, as if by magic. None of this happens because the Earth's surface and oceans also feel the centrifugal force and are perpendicular to $\boldsymbol{g}_{\text{eff}}$, not $\boldsymbol{g}$. This is easy to understand in the case of the ocean: if the ocean's surface were not perpendicular to $\boldsymbol{g}_{\text{eff}}$, the component of $\boldsymbol{g}_{\text{eff}}$ parallel to the surface would make water flow to remove the tilt, just as it does in a bucket. The ground is stiffer than the ocean and has hills and valleys, but is also perpendicular to $\boldsymbol{g}_{\text{eff}}$ on average. The ocean's surface and the ground are equipotentials (surfaces of constant potential energy) of the effective force field, including the ficitious forces.

An interesting consequence is that the world is not perfectly round: the equatorial radius of 6378 km is slightly larger than the polar radius of 6357 km. The world is actually an oblate spheroid.

Aside: Oblate and Prolate Spheroids

An oblate spheroid is the three-dimensional object you get when you take an ellipse (the outer rim of the diagams below) and rotate it around its minor axis; a prolate spheroid is the object you get when you rotate an ellipse around its major axis. The Earth is an oblate spheroid.

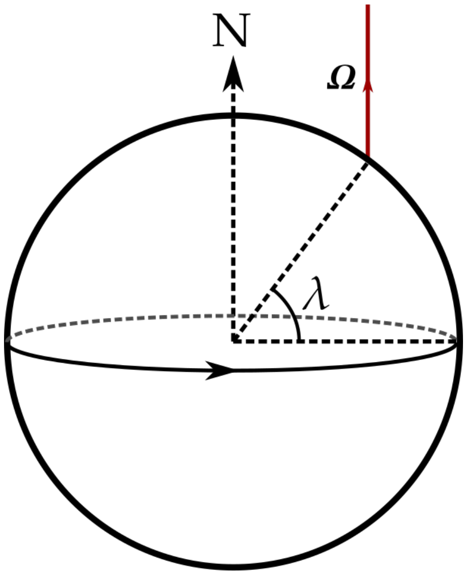


Oblate Prolate

Coriolis Force on Earth

The Coriolis Force

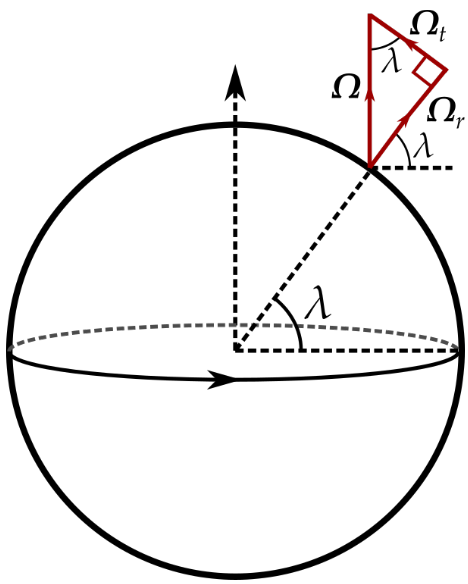
$$\boldsymbol{F}_{\text{Coriolis}} = -2m\boldsymbol{\Omega} \times \boldsymbol{v}'$$



The Coriolis force depends on the velocity $\boldsymbol{v}'$ as measured in the rotating frame, so it cannot be incorporated into an effective gravitational field. Its dependence on the direction of $\boldsymbol{v}'$ (up, down, east, west, north, south) makes understanding the Coriolis force tricky, so we are going to think separately about how it affects bodies moving horizontally (tangentially) and bodies moving vertically (radially). The force on a body moving both horizontally and vertically can be found by adding the forces due to the horizontal and vertical components of $\boldsymbol{v}'$.

Effects of the Coriolis Force on Bodies moving Horizontally

The effects of the Coriolis force on bodies moving horizontally are most easily understood by resolving $\boldsymbol{\Omega}$ into radial and tangential components, as shown in the diagram below.



This allows us to decompose $\boldsymbol{F}_{\text{Coriolis}}$ into horizontal and vertical components:

$$\boldsymbol{F}_{\text{Coriolis}} = -2m(\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_t) \times \boldsymbol{v}' = -2m\boldsymbol{\Omega}_t \times \boldsymbol{v}' - 2m\boldsymbol{\Omega}_r \times \boldsymbol{v}'.$$

The $-2m\boldsymbol{\Omega}_t \times \boldsymbol{v}'$ contribution is vertical because it is perpendicular to $\boldsymbol{\Omega}_t$ and $\boldsymbol{v}'$, both of which lie in the plane of the Earth's surface.

- If $\boldsymbol{v}'$ points west, $-2m\boldsymbol{\Omega}_t \times \boldsymbol{v}'$ points down.
- If $\boldsymbol{v}'$ points east, $-2m\boldsymbol{\Omega}_t \times \boldsymbol{v}'$ points up.
- If $\boldsymbol{v}'$ points north or south, $-2m\boldsymbol{\Omega}_t \times \boldsymbol{v}'$ is zero.

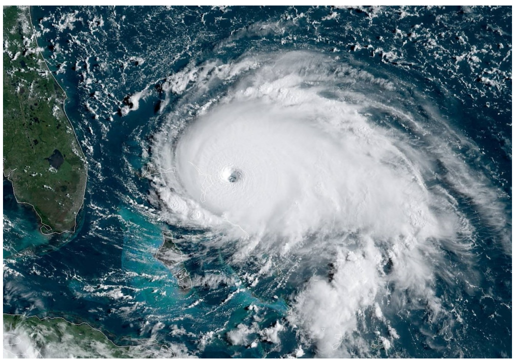
The tangential component of the angular velocity vector produces a Coriolis force that adds to or subtracts from the force of gravity, depending on whether the mass is moving west or east. Masses moving east are a little lighter than masses moving west! This is less surprising that it appears: a mass moving east, in the same direction as the rotation of the Earth's surface below it, behaves like a stationary mass on an Earth with a slightly larger angular velocity, which would experience a larger centrifugal force.

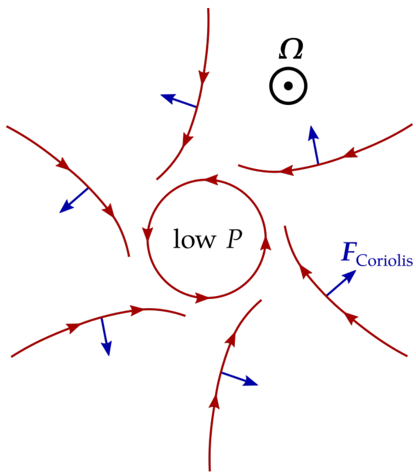
The term involving the radial component of the Earth's angular velocity is more interesting. The $-2m\boldsymbol{\Omega}_r \times \boldsymbol{v}'$ contribution to the Coriolis force must be perpendicular to $\boldsymbol{\Omega}_r$, which is radial, so it lies in the plane of the Earth's surface. It is also be perpendicular to $\boldsymbol{v}'$, the velocity vector as measured in the rotating frame. In the northern hemisphere, it pushes moving masses to their right; in the southern hemisphere, it pushes moving masses to their left. The magnitude of the horizontal Coriolis force,

$$|\boldsymbol{F}^{\text{horizontal}}_{\text{Coriolis}}| = |-2m\boldsymbol{\Omega}_r \times \boldsymbol{v}'| = 2mv'\Omega_r = 2mv'\Omega\sin\lambda,$$

is maximum at the poles and zero at the equator. This is the opposite of the centrifugal force, which was maximum at the equator and zero at the poles.

The effects of the horizontal component of the Coriolis force are surprisingly important. Shells fired from warships travel far enough at a high enough speed to make the deflection caused by the Coriolis force significant. When warships sail from the northern hemisphere to the southern hemisphere, or vice versa, they miss their targets unless they readjust their sights. A much more interesting consequence of the Coriolis force is its effect on weather systems. The figure below shows a photograph of a hurricane approaching Florida and a diagram of the air flow around the storm.

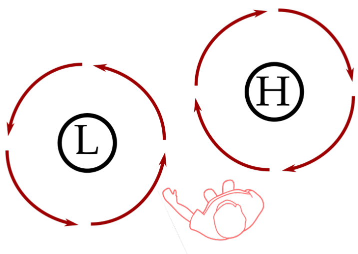




The air-flow diagram shows that hurricanes form around regions of low pressure. Air from the surrounding higher-pressure regions starts moving towards the centre, pushed by the pressure gradient. The Coriolis force then deflects the moving air to its right, creating an anti-clockwise circulation. This is why low-pressure weather systems in the northern hemisphere circulate anti-clockwise (and high-pressure systems circulate clockwise). In the southern hemisphere, where the Coriolis force deflects the moving air to its left, low-pressure systems circulate clockwise and hurricanes are called cyclones.

The Buys Ballot law, named after the nineteenth centry Dutch meteorologist Christophorus Henricus Diedericus Buys Ballot, states that

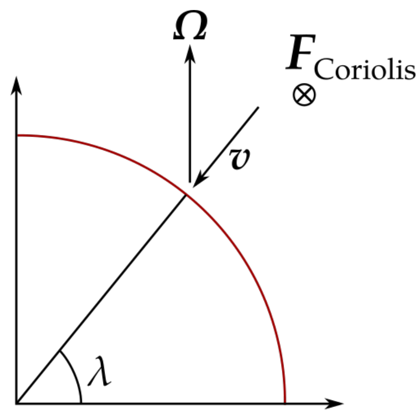
> In the northern hemisphere, if you turn your back to the wind, the low-pressure region is to your left and somewhat ahead of you.



As the diagram shows, this is a simple consequence of the circulation direction, which depends on the Coriolis force.

Effects of the Coriolis Force on Bodies moving Vertically

The diagram below shows a body falling vertically in the rotating frame.



The Coriolis force $\boldsymbol{F}_{\text{Coriolis}} = -2m\boldsymbol{\Omega} \times \boldsymbol{v}'$ points east (that is, in the $\hat{\boldsymbol{\phi}}$ direction), so falling objects in the northern hemisphere land a little to the east of where you might expect. The magnitude of the horizontal force, $|\boldsymbol{F}_{\text{Coriolis}}| = 2mv'\Omega\cos\lambda$, increases as the body drops and v' increases. This makes it necessary to integrate the horizontal component of the velocity with respect to time to find the total deflection. To complicate matters even more, the small horizontal component of the velocity produces an additional Coriolis force, which points radially outwards and has components in the $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ directions! This week's assessed problem analyses the deflection in detail.