Fourier Transforms

A fourier transform is the generalisation of the fourier series to an interval of infinity.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n\pi}{\ell}\right)x} \qquad C_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{-i\left(\frac{n\pi}{\ell}\right)x}\,dx$$

Lets define the scaling parameter $k_0 = \pi/\ell$. It is an inverse scale parameter.

$$2\ell C_n = \int_{-\ell}^{\ell} f(x) e^{-ink_0x}\,dx$$

As $\ell \to \infty$

continous function discretely sampled

$$2\ell C_n \to g(nk_0)$$

$$g(nk_0) = \int_{-\infty}^{\infty} f(x) e^{-ink_0x}\,dx$$

KEY IDEA: As $\ell \to \infty$, the coefficients C_n become a continous function sampled discretely to get C_n.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ink_0x} = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} 2\ell C_n e^{ink_0x}$$

$$= \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} g(nk_0) e^{ink_0x} \qquad \left(\frac{1}{2\ell} = \frac{k_0}{2\pi}\right)$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} g(nk_0) e^{ink_0x} k_0$$

Lets introduce a new variable: $k = nk_0$

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} g(k)e^{ikx}k_0$$

$\Delta k = \Delta n k_0$ but $\Delta n = 1$ $\therefore$ $k_0 = \Delta k$ $\qquad (n = k/k_0)$

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} g(k)e^{ikx}\Delta k$$

there will be a scaling factor but as $-\infty$ to $\infty \rightarrow$ ignore.

$$\hookrightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)e^{ikx}dk$$

We can now continously sample k, whereas before we had to do it discretely. $\qquad C_n \rightarrow g(k)$

$$g(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

The fourier transform shows how $f(x) \rightarrow g(k)$.

$F[f(x)]$: fourier transform of $f(x)$
$F^{-1}[g(k)]$: 'inverse' fourier transform of $g(k)$

<u>Change Convention</u>:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(k)e^{-ikx}dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

x : spacial coordinate

e^{ikx} : spacial fourier transform

K : reciprocal variable. $\qquad K = \frac{2\pi}{\lambda}$

"wavenumber" (spacial frequency)

wavelength of e^{-ikx}

Second Representation: (temporal)

t : variable (time)

ω : reciprocal variable - angular frequency $\qquad \omega = 2\pi f$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} \, d\omega$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega t} \, dt$$

Fourier Transform of the Gaussian

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad \text{gaussian func.}$$

Normalized: $\int_{-\infty}^{\infty} f(x) \, dx = 1$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{t^2}{2\sigma^2}} e^{i\omega t} \, dt$$

lets consider the derivative:

$$\frac{dg(\omega)}{d\omega} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2} it e^{i\omega t} dt$$

now if we integrate by parts:

$$u = e^{i\omega t} \qquad \frac{dv}{dt} = it e^{-t^2/2\sigma^2}$$

$$\frac{du}{dt} = i\omega e^{i\omega t} \qquad v = -i\sigma^2 e^{-t^2/2\sigma^2}$$

$$\int_{-\infty}^{\infty} e^{-t^2/2\sigma^2} it e^{i\omega t} dt = \cancel{\left[-i\sigma^2 e^{i\omega t} e^{-t^2/2\sigma^2}\right]_{-\infty}^{\infty}}^{\to 0} - \int_{-\infty}^{\infty} -i^2\sigma^2 \omega e^{i\omega t} e^{-t^2/2\sigma^2} dt$$

$$= -\omega\sigma^2 \int_{-\infty}^{\infty} e^{i\omega t} e^{-t^2/2\sigma^2} dt$$

$$\hookrightarrow \frac{dg(\omega)}{d\omega} = -\omega\sigma^2 g(\omega)$$

$$\therefore g(\omega) = g(0) e^{-\frac{\sigma^2\omega^2}{2}}$$

$$g(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2} \cancel{e^{i\omega t}}^{\omega=0} dt$$

$$g(0) = \frac{1}{\sqrt{2\pi}} \qquad \therefore g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2\sigma^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2\rho^2}}$$

$$\rho = 1/\sigma$$

normalisation is non-standard, but the F.T of a gaussian is a gaussian.

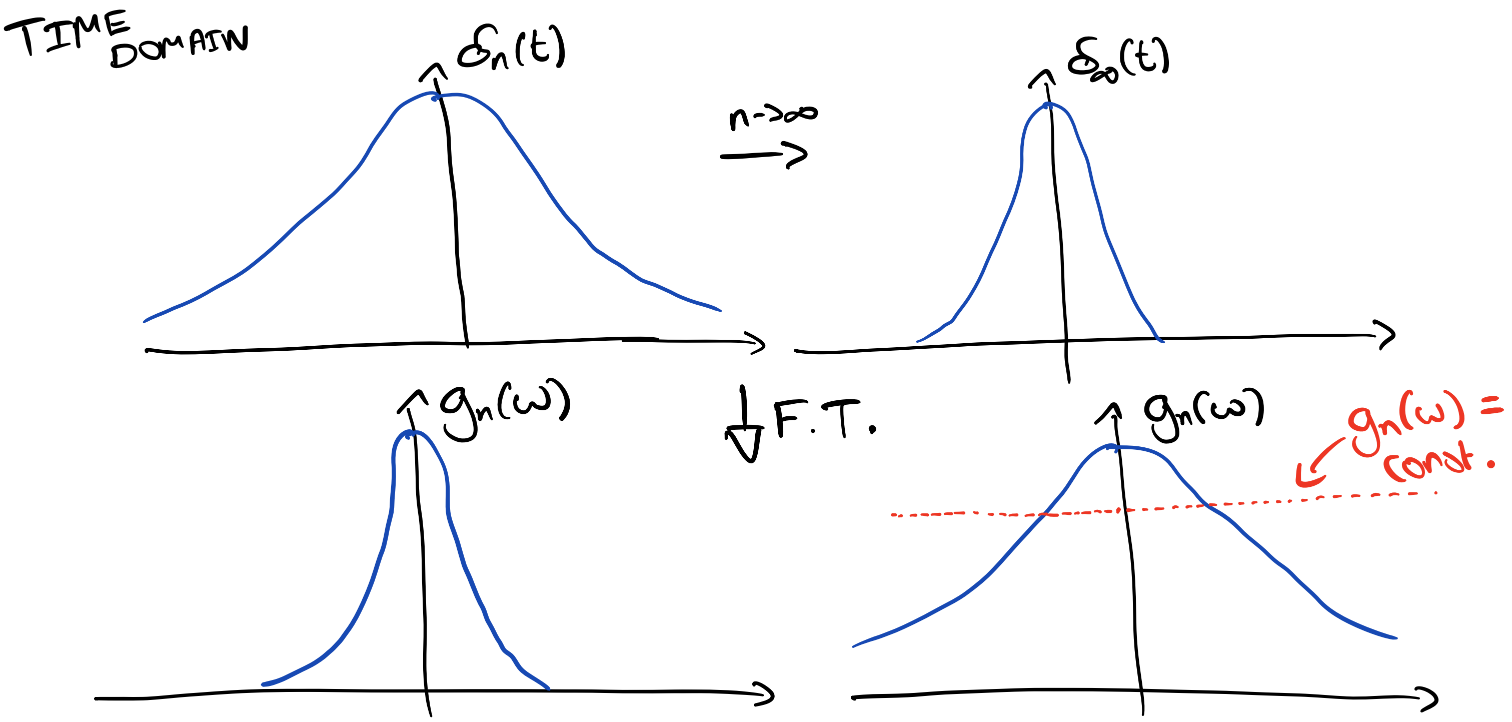
TIME DOMAIN $\longleftrightarrow$	FREQ DOMAIN
'wide'	'narrow'
'narrow'	'wide'

Fourier Transform of Delta Function

$F[\delta(t)]$ $\qquad g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt$

lets consider the gaussian limiting function:

$$\delta(t) = \lim_{n \to \infty} \frac{n}{\sqrt{\pi}} e^{-n^2 t^2}$$



As $\delta_n(t) \to \delta_\infty(t)$, $g_n(\omega) \to$ const.

consider the inverse fourier transform with $g(\omega) = \frac{1}{\sqrt{2\pi}}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} d\omega$$

recall another limiting function for $\delta(t)$:

$$\delta(t-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega(t-x)} \, d\omega$$

$$f(x) = -\delta(t) = \delta(t)$$

$$F[\delta(t)] = 1/\sqrt{2\pi} = g(\omega)$$

The fourier transform of a $\delta(t)$ is a const. All frequencies contribute equally.

Properties of F.T.

Linearity: $F[\alpha f_1(t) + \beta f_2(t)] = \alpha F[f_1(t)] + \beta F[f_2(t)]$

Sign Reversal: $F[f(-t)] = g(-t)$

phase shift of $e^{i\omega t_0}$

Translation: $F[f(t-t_0)] = e^{i\omega t_0} g(\omega)$

Scaling: $F[f(\alpha t)] = \frac{1}{|\alpha|} g(\frac{\omega}{\alpha})$

Derivative: $F[f'(t)] = i\omega g(\omega)$

Conjugation: $F[f^*(t)] = g^*(-\omega)$

Parseval's Identity:

$$\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \int_{-\infty}^{\infty} F(\omega) G^*(\omega) \, d\omega$$

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$