Lecture 11

Second Order ODE

lets start by looking at (I) linear (II) homogeneous second order differential equations.

$$a(x)y''(x) + b(x)y'(x) + c(x)y = 0 \qquad (*)$$

I) Superposition Principle

let $y_i(x)$ and $y_j(x)$ be solution of the diff eq^n $(*)$, then $[Ay_i(x) + By_j(x)]$ are also solutions. A & B are constants.

Proof

$a(x)[Ay_i''(x) + By_j''(x)] + b(x)[Ay_i' + By_j'] + c(x)[Ay_i + By_j] = 0$

$= A[a(x)y_i''(x) + b(x)y_i'(x) + c(x)y_i(x)] + B[a(x)y_j''(x) + b(x)y_j'(x) + c(x)y_j(x)]$

$= A[0] + B[0]$

$= 0$ ∎

II) Initial Conditions $\qquad y''(x) = f(x)$

$\Rightarrow y'(x) = \int^x f(t)\,dt + C_1$

$y(x) = \int^x \left[\int^u f(t)\,dt + C_1\right] du + C_0$

$y''(x)$ leads to 2 constants of intergration to be determined by 2 initial conditions.

III) General Solution

$$y = Ay_1(x) + By_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly independent

IV) Existence & Uniqueness

$$a(x)\,y''(x) + b(x)\,y'(x) + c(x)\,y = 0$$

$$+ \; 2\;I.C. \quad y(x_0) = y_1 \quad y'(x_0) = y'_0$$

has a solution that is unique if: $a(x), b(x), c(x)$ are continous and $a(x) \neq 0$.

V) Linear Independenance

By $y_1(x)$ and $y_2(x)$ are linearly independant we mathematically mean $y_1(x) \neq \lambda y_2(x)$.

To check for linear independence we use the Wronskian Evaluation.

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$ ← determinant.

$$= y_1(x)\,y_2'(x) - y_2(x)\,y_1'(x)$$

How does this work?

Let $y_1(x) = \lambda y_2(x)$ (linearly dependant)

$$W(y_1, y_2) = \begin{vmatrix} \lambda y_2(x) & y_2(x) \\ \lambda y_2'(x) & y_2'(x) \end{vmatrix} = 0$$

If solutions are linearly dependent, then Wronskian is zero.

If $W(y_1, y_2) = 0$, then $y_1(x)$ and $y_2(x)$ are linearly independant if they are analytical.

Solving $a(x)y'' + b(x)y' + c(x)y = 0$.

Lets start with the simplist case when we just have constants coefficents.

$$a(x) = a \quad b(x) = b \quad c(x) = c$$

$$\therefore \boxed{ay'' + by' + c = 0}$$

lets simplify and look at first order first:

$$ay' + by = 0$$

suggest a trial solution of $e^{\alpha x}$

$$a\alpha e^{\alpha x} + be^{\alpha x} = 0$$

$$a\alpha + b = 0$$

$$\alpha = -b/a$$

$$y(x) = ce^{-b/a}$$

let's use this same idea for our second order equations. $y = e^{\alpha x}$

$$a\alpha^2 e^{\alpha x} + b\alpha e^{\alpha x} + ce^{\alpha x} = 0$$

$$a\alpha^2 + b\alpha + c = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y_1(x) = e^{\alpha_1 x} \qquad y_2(x) = e^{\alpha_2 x}$$

General: $\underline{\underline{y(x) = Ae^{\alpha_1 x} + Be^{\alpha_2 x}}}$

Case 1: $\alpha_1 \neq \alpha_2$ both $\mathbb{R}$

α_1 and α_2 are real and distinct.

$$y = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{\alpha_1 x} & e^{\alpha_2 x} \\ \alpha_1 e^{\alpha_1 x} & \alpha_2 e^{\alpha_2 x} \end{vmatrix}$$

$$= \alpha_2 e^{(\alpha_1+\alpha_2)x} - \alpha_1 e^{(\alpha_1+\alpha_2)x}$$

$$= (\alpha_2 - \alpha_1) e^{(\alpha_1+\alpha_2)x} \neq 0.$$

Particular Solution for I.C. $y(0) = y_0$ $y'(0) = y'_0$,
Substitution into the general solution

$$y(0) = A \cdot 1 + B \cdot 1 = \boxed{A + B = y_0}$$

$$y'(0) = A \cdot \alpha_1 \cdot 1 + B \cdot \alpha_2 \cdot 1 = \boxed{A\alpha_1 + B\alpha_2 = y'_0}$$

$$A = \frac{\alpha_2 y_0 - y'_0}{\alpha_2 - \alpha_1} \qquad B = \frac{\alpha_1 y_0 - y'_0}{\alpha_1 - \alpha_2}$$

$$\boxed{y(x) = \frac{\alpha_2 y_0 - y'_0}{\alpha_2 - \alpha_1} e^{\alpha_1 x} + \frac{\alpha_1 y_0 - y'_0}{\alpha_1 - \alpha_2} e^{\alpha_2 x}}$$