Power in Steady State

In steady state,

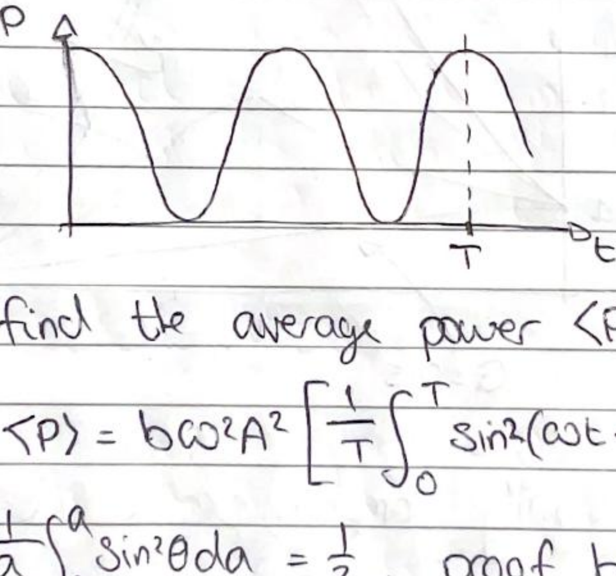
$$P = \text{power in (from driver)}$$
$$= \text{power out (due to damping)}$$

$$P = -\underline{F_d} \cdot \underline{v}$$

damping force $= -b\underline{v}$

$$v = -\omega A \sin(\omega t + \varphi)$$

$$P = b|\underline{v}|^2$$
$$= b\omega^2 A^2 \sin^2(\omega t + \varphi)$$



lets find the average power $\langle P \rangle$.

$$\langle P \rangle = b\omega^2 A^2 \left[\frac{1}{T} \int_0^T \sin^2(\omega t + \varphi)\, dt \right]$$

N.B. $\frac{1}{a}\int_0^a \sin^2\theta \, da = \frac{1}{2}$, proof below...

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $\langle \sin^2\theta \rangle + \langle \cos^2\theta \rangle = 1$

$\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle$ $\quad$ $\langle \sin^2\theta \rangle = \frac{1}{2}$

$$\langle P \rangle = \frac{1}{2} b\omega^2 A^2$$

$$A^2 = \frac{A_0^2}{(1-W^2)^2 - W^2/Q^2}$$

$$b = m\gamma$$
$$= \frac{m\omega_0}{Q}$$

$\omega^2 = W^2 \omega_0^2$

$$A_0^2 = \frac{F_0^2}{k^2} = \frac{F_0^2}{m^2\omega_0^4}$$

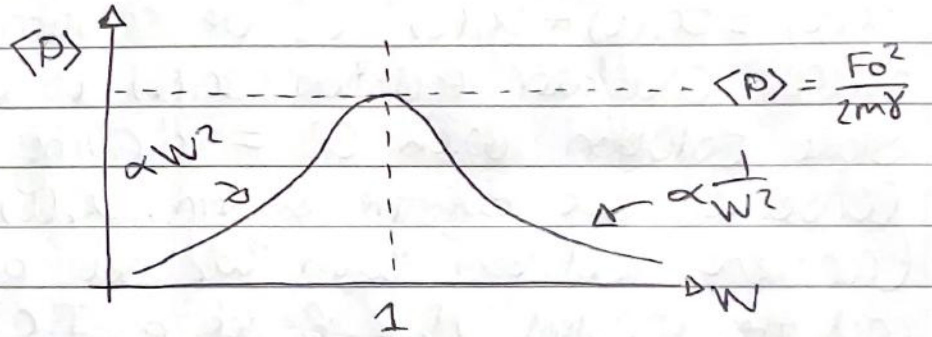
$$\langle P \rangle = \frac{1}{2}\frac{m\omega_0}{Q}\omega_0^2 \frac{F_0^2}{m^2\omega_0^4}\frac{1}{(1-W^2)^2 - W^2/Q^2}W^2$$

$$= \frac{1}{2}\frac{F_0^2}{Qm\omega_0}\frac{W^2}{(1-W^2)^2 - W^2/Q^2}$$

$$= \frac{1}{2}\cdot\frac{F_0^2}{Qm\omega_0}\cdot\frac{1}{(1/W - W)^2 + 1/Q^2}$$

$\langle P_{max}\rangle$ will occur when $\frac{1}{W} - W^2 = 0$ ie. $W = 1$. $\langle P \rangle_{MAX} = F_0^2/2m\gamma$

- $W \ll 1$: $\langle P \rangle \propto W^2$
- $W \gg 1$: $\langle P \rangle \propto 1/W^2$



Transient

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

The general solution to the forced oscillation equation can be obtained by combing the steady-state and transient state.

$$\underbrace{x(t) = x_1(t)}_{\text{steady-state}} + \underbrace{x_2(t)}_{\text{transient}}$$

to find the transient, we begin by looking at the homogeneous equation

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$$

The solution of which is

$$x(t) = x_2(t) = Re(A_2 e^{i\varphi_2} e^{i\omega_2 t})$$

where A_2 and φ_2 can be found by boundary conditions.

$$\omega_2 = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

$x(t) = x_1(t) + x_2(t)$ is the solution of the forced oscillator equation. $x_1(t)$ is the steady state solution with ω = driving frequency and constants set by the system. $x_2(t)$ is the transient solution with ω_2 set by the system and the constants A_2 & φ_2 set by I.C.

Differential Equations

CA

$$ay'' + by' + cy = 0$$

is a second-order linear ODE with constant coefficients where a,b,c are all real.

In O&W, we have

$$\frac{d^2\psi}{dt^2} + \gamma \frac{d\psi}{dt} + \omega_0^2 \psi = 0$$

we have assumed that we have small displacements so the DE is linear.

in CA we used a trail solution of $y = e^{\alpha x}$ to form the characteristic equation

$$a\alpha^2 + b\alpha + c = 0$$

solve quadratically.

In OEW, our trail solution was $\psi = Ae^{i\omega t}$

$$\omega^2 - i\gamma\omega - \omega_0^2 = 0$$

$$\omega = \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

In CA, there were 3 cases : $b^2 - 4ac > 0$ we had two distinct real roots

$$y = Ae^{\alpha x} + Be^{\beta x}$$

$b^2 - 4ac = 0$, we have repeated root

$$y = (A + Bx)e^{\alpha x}$$

$b^2 - 4ac < 0$, we had complex conjugate roots

$$y = Ae^{\alpha x}(B\cos\beta x + iC\sin\gamma x)$$

in OEW, for light damping ($Q > 0.5$), it was equivalent to case 3 ($b^2 - 4ac < 0$) with two complex conjugates as solutions.

$$\omega_0^2 - \frac{\gamma^2}{4} > 0.$$

for heavy damping ($Q < 0.5$), it's equivalent to Case 1, $b^2 - 4ac > 0$, 2 distinct imaginary roots - no real value - $e^{i\omega t}$ purley decaying - no oscillation.

$$\omega_0^2 - \frac{\gamma^2}{4} < 0$$

for critical damping ($Q = 0.5$), it's equivalent to case 2 ($b^2 - 4ac = 0$)

$$\omega_0^2 - \frac{\gamma^2}{4} = 0$$

In <u>CA</u> a 2^{nd} order linear ODE inhomogeneous was of the form

$$ay'' + by' + cy = f(x)$$

In <u>O&W</u>, it took the form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

In <u>CA</u>, the solution took the form

$$y(x) = Ay_1(x) + By_2(x) + y_p(x)$$

where $y_1(x)$ & $y_2(x)$ are the solutions of the homogeneous equation. $y_p(x)$ is the particular solution.

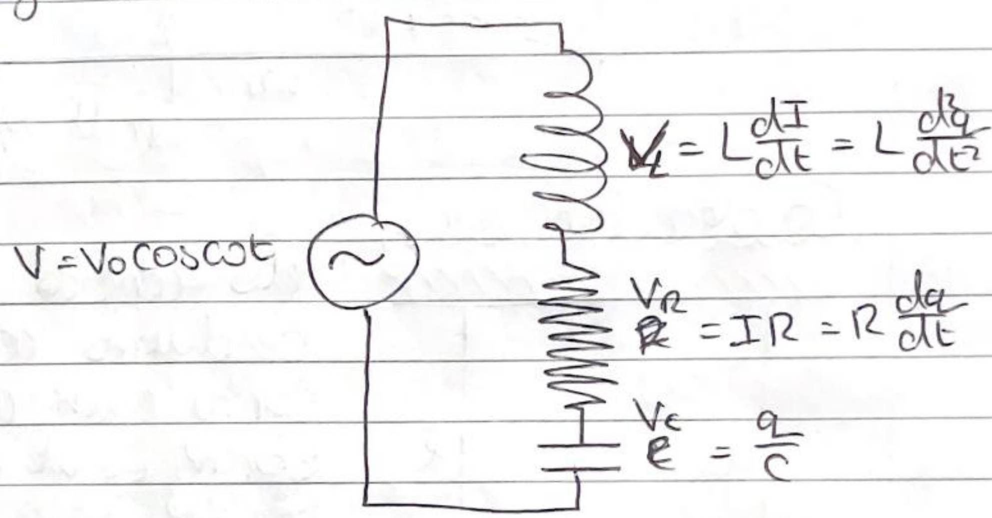
In <u>O&W</u>, the solution took the form

$$x(t) = x_1(t) + x_2(t)$$

$x_1(t)$ is the steady state solution (equivalent to particular solution) and $x_2(t)$ is the transient which is equivalent to $Ay_1(x) + By_2(x)$.

Circuits Again

Now we'll consider a circuit made by a capacitor, inductor & resistor along with an AC voltage source.



$$V_L + V_R + V_C = V_0 \cos\omega t$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0\cos\omega t$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_0}{L}\cos\omega t$$

Very similar to the forced oscillator equation.

$$\gamma = R/L \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Comparing the Oscillator (O) and circuit (C).

- Energy come in from driver for O and the source for C.
- The power dissipated for O is $b|\dot{x}|^2$ and for C is $R|\dot{q}|^2 = I^2R$

first order derivative

→ when the driving frequency is close to ω_0, in O you get a large amplitude in x & $\dot{x}$. In C you get a large amplitude in q & I.