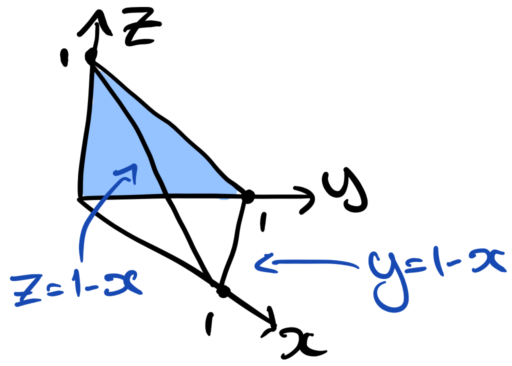
3D Integrals

integrate $f(x,y,z) = \alpha x$ in the region bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$.

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \alpha x \, dz\,dy\,dx$$



$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\alpha x z\right]_0^{1-x-y} dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \alpha x - \alpha x^2 - \alpha x y \, dy\,dx = \int_{x=0}^{1} \left[\alpha x y - \alpha x^2 y - \tfrac{1}{2}\alpha x y^2\right]_0^{1-x} dx$$

$$= \int_{x=0}^{1} \tfrac{1}{2}\alpha x (1-x)^2 \, dx \qquad = \left[\tfrac{1}{4}\alpha x^2 - \tfrac{1}{3}\alpha x^3 + \tfrac{1}{8}\alpha x^4\right]_0^1 = \frac{\alpha}{24}$$

$$\frac{(1-x)^2}{2} = \text{cross sectional area}$$

Jacobians in 3D

$$d\underline{r} = \frac{\partial \underline{r}}{\partial u} du + \frac{\partial \underline{r}}{\partial v} dv + \frac{\partial \underline{r}}{\partial w} dw = d\underline{r_u} + d\underline{r_v} + d\underline{r_w}$$

$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\underline{r_u} = \frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j} + \frac{\partial z}{\partial u}\hat{k}$$

$$dV_{uvw} = \left(\frac{\partial x}{\partial u}\hat{\imath} + \frac{\partial y}{\partial u}\hat{\jmath} + \frac{\partial z}{\partial u}\hat{k}\right) \cdot \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \cdot \begin{matrix} du \\ dv \\ dw \end{matrix}$$

1D Jacobian

$$dV_{uvw} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \cdot du\,dv\,dw$$

2D Jacobians

3D Jacobian

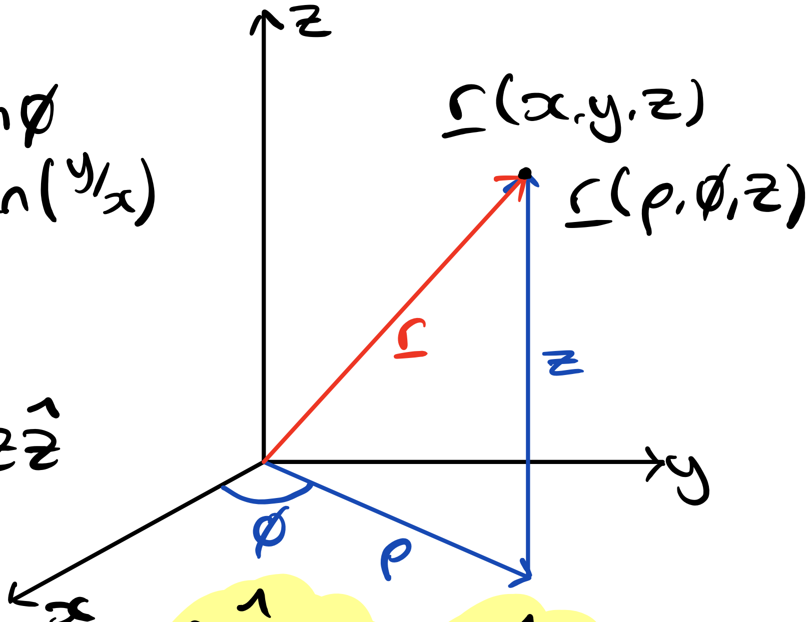
Cyclindrical Polar Coordinates

Same as plane polar but with a z axis.

$x = \rho\cos\phi \quad y = \rho\sin\phi$

$\rho = \sqrt{x^2 + y^2} \quad \phi = \arctan(y/x)$

$z = z$



$$\underline{r}(\rho,\phi,z) = \rho\hat{\rho}(\phi) + z\hat{z}$$

N.B. $\hat{\rho}$ & $\hat{\phi}$ depend upon ϕ.

$$\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{\rho} \qquad \frac{\partial\hat{\rho}}{\partial\phi} = \hat{\phi}$$

$$dx\,dy\,dz = |J|\,d\rho\,d\phi\,dz = dV \qquad J = \rho$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho$$