Classical Mechanics 1S

Motion in a Gravitation Field

Kepler's laws

I) Planets move in ellipses with the sun at one focus.

II) The radius vector from the Sun to the planet sweeps out equal area at all times

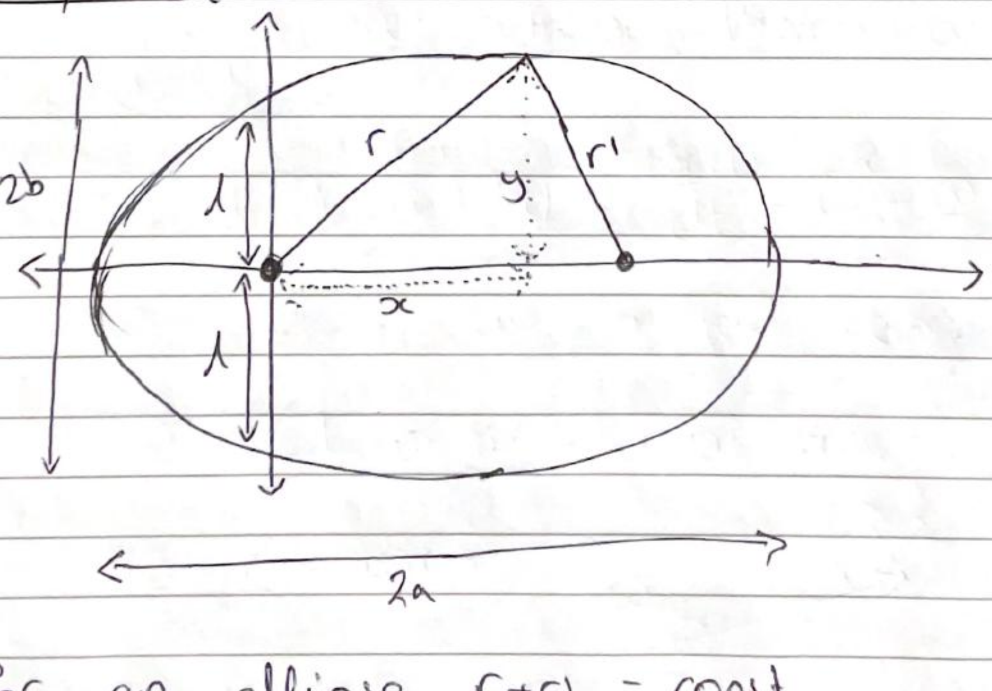
III) For all planets the ratio:

$$\frac{(\text{orbital period})^2}{(\text{semi-major axis})^3} = \frac{4\pi^2}{GM}$$

is the same.

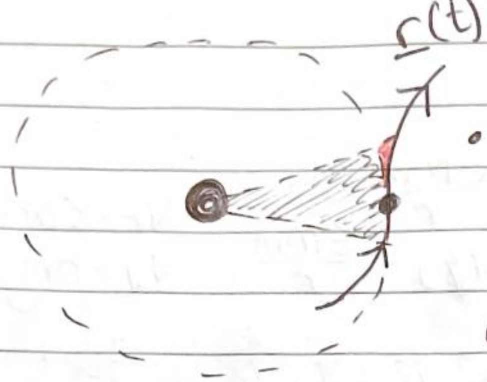
Newton derived all 3 of these laws from his law of gravitation.

Ellipses (KI)



for an ellipse, $r + r' = \text{const}$.
where the two black dots are the foci.
l = semi-latus rectum

Area (KII)



- Red area proportional to $(\Delta t)^2$ and becomes negligible in comparison to grey area as $\Delta t \to 0$.

$$\Delta A \approx \left(\frac{V_\phi \Delta t}{2\pi r}\right)\pi r^2 = \frac{1}{2} V_\phi r \Delta t = \frac{L}{2m}\Delta t$$

$$\Delta A \propto \Delta t$$

T^2/r^3 (KIII)

easy to show for circular orbits, for ellipses, see hand-out.

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}$$

$$\frac{4\pi^2}{T^2} = \frac{GM}{r^3} \Rightarrow T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

Conservation of Energy

$$E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Conservation of Angular Momentum

$$\underline{L} = \underline{r} \times \underline{p} = \underline{r} \times m\underline{v} = m r v_\phi \underline{\hat{\omega}}$$

$\therefore$ orbits are planar ($\underline{\hat{\omega}}$ perp. to plane)

Orbital Energy & Effective Potential

Orbital Energy

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$
$$= \frac{1}{2}m(v_r^2 + v_\phi^2) - \frac{GMm}{r}$$

v_r = radial velocity
v_ϕ = angular velocity

$v_r = \dot{r}$ ← not a vector, a length!

$$= \frac{1}{2}m\dot{r}^2 + \frac{(mv_\phi r)^2}{2mr^2} - \frac{GMm}{r}$$

This gives an equation for total orbital energy of

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r}$$

Aside: $v^2 = v_r^2 + v_\phi^2$

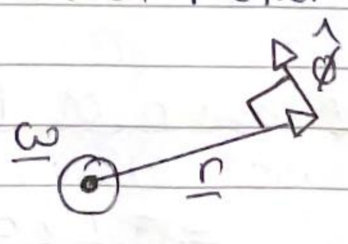
This is just pythagoras theorem, but is is useful to see,

$$\underline{r}(t) = r(t)\hat{r}(t) \qquad \frac{d\underline{r}(t)}{dt} = \dot{r}\hat{r} + \underline{\omega} \times \underline{r}$$

we showed this a few lectures ago.

$\underline{\omega}$ is $\perp$ to the plane of motion.

$\underline{\omega} \times \underline{r} = \omega r \hat{\phi}$



Hence,

$$v^2 = (\dot{r}\hat{r} + r\omega\hat{\phi})\cdot(\dot{r}\hat{r} + r\omega\hat{\phi})$$
$$= \dot{r}^2 + r^2\omega^2 \qquad (\hat{r}\cdot\hat{\phi} = 0 \text{ as } \hat{r}\perp\hat{\phi})$$
$$= v_r^2 + v_\phi^2$$

Centrifugal Potential

Since $L = \text{const}$,

$\frac{L^2}{2mr^2}$ is a function of r.

It acts like an addition potential know as the **centrifugal potential**.

Almost everything you need to know about the angular motion is hidden in the fixed value of L.

As a planet approaches the sun, its angular momentum $L = mrv_\phi$ is conserved. It follows that v_ϕ increase that r decreases. The angular part of the KE

$$\frac{1}{2}mv_\phi^2 = \frac{L^2}{2mr^2}$$

increases to.

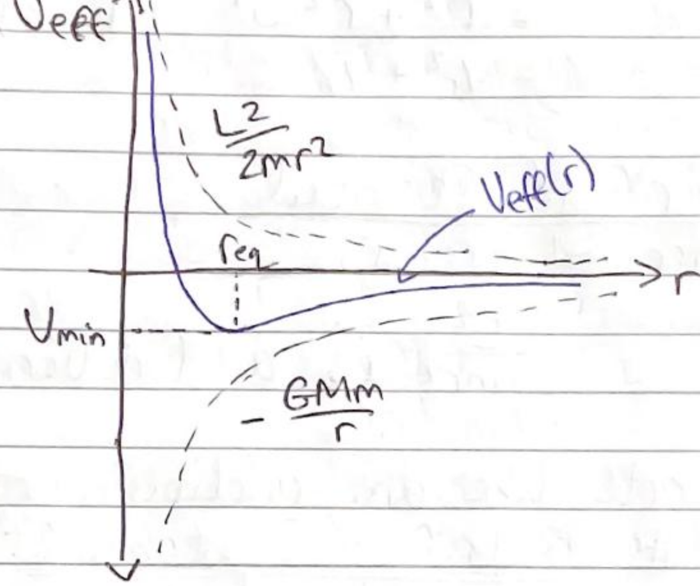
The total energy is conserved. Decreasing r and increasing $L^2/2mr^2$ 'sucks' energy out of the $\frac{1}{2}m\dot{r}^2$ & $\frac{GMm}{r}$ terms. As r increases again, the other degrees of freedom get the centrifugal potential energy back again.

Total Energy

$E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$

Effective Potential

$U_{eff}(r) = \frac{L^2}{2mr^2} + \frac{GMm}{r}$



Aside: Can we really treat $U_{eff}(r)$ as an effective radial potential? Is it true that:

$$m\ddot{r} = -\frac{dU_{eff}}{dr}$$

$$= \frac{L^2}{mr^3} - \frac{GMm}{r^2}$$

$$= \frac{(m\omega r^2)^2}{mr^3} + F(r)$$

$$= \underbrace{m\omega^2 r}_{\text{centripetal force}} + \underbrace{F(r)}_{\text{radial force}}$$

For a circular orbit, $\dot{r} = 0$ so $E = U_{eff}(r)$

For orbits in general, if the total energy is negative the orbit is bound. If total energy is positive they're not really orbits.

finding the radius:

$$\frac{dU_{eff}}{dr} = 0$$

$$-\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0$$

$$r_{eq} = \frac{L^2}{GMm^2}$$

$$U_{min} = \frac{L^2}{2mr_{eq}^2} - \frac{GMm}{r_{eq}}$$

$$= \frac{GMm^2 r_{eq}}{2mr_{eq}^2} - \frac{GMm}{r_{eq}}$$

$$= -\frac{GMm}{2r_{eq}}$$

$$\boxed{r_{eq} = \frac{L^2}{GMm^2} \qquad U_{min} = -\frac{GMm}{2r_{eq}}}$$