Complex Variables for SHM

$\psi(t) = A\cos(\omega t + \phi)$ Real

$\tilde{\psi}(t) = \tilde{A}e^{i\omega t}$ Complex

$\hookrightarrow \tilde{A} = Ae^{i\varphi}$

$$\tilde{\psi}(t) = Ae^{i(\omega t + \varphi)}$$
$$= A\left[\cos(\omega t + \phi) + i\sin(\omega t + \phi)\right]$$

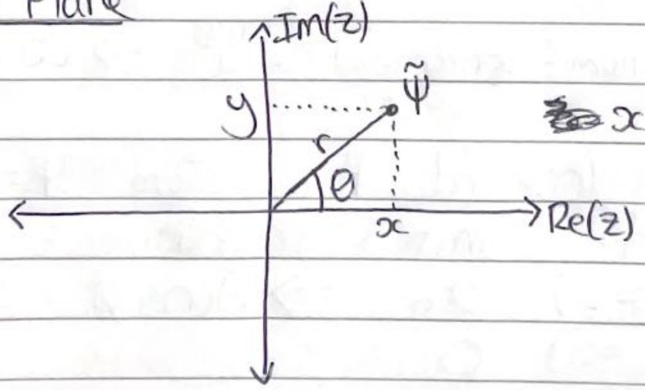
$$\underline{\underline{Re(\tilde{\psi}) = \psi}}$$

$$\dot{\psi} = -A\omega\sin(\omega t + \phi)$$
$$\dot{\tilde{\psi}} = i\omega\tilde{A}e^{i\omega t} = i\omega\tilde{A}$$
$$= \omega A\left[i\cos(\omega t + \phi) - \sin(\omega t + \phi)\right]$$

$$\underline{\underline{Re(\dot{\tilde{\psi}}) = \dot{\psi}}}$$

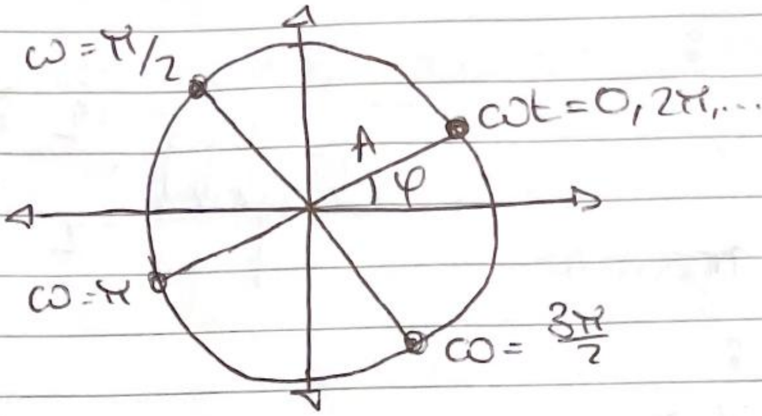
we've now unlocked a new ability. Instead of having to differentiate ψ(t) could be very hard instead we multiply by iω.

Complex Plane



$x + iy = re^{i\theta} = \tilde{\psi}$

→ moves in a circle of radius A in the complex plane with a constant speed. Also called a phasor.

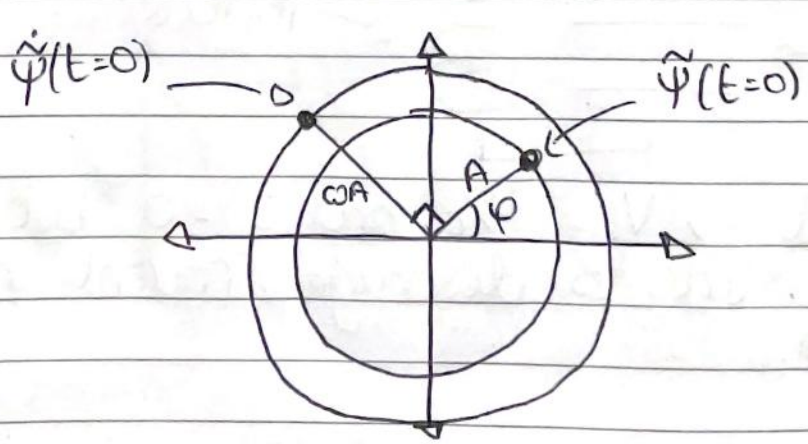


time period $T = \frac{2\pi}{\omega}$

ψ is visualised as the projection onto the real axis of a point moving in a circle of radius A and with period equal to the period of the SHM.

$$\dot{\tilde{\psi}} = i\omega\tilde{\psi} \qquad i = e^{i\frac{\pi}{2}}$$
$$= e^{i\frac{\pi}{2}} \cdot \omega \cdot e^{i(\omega t + \varphi)} \cdot A$$
$$= \omega A e^{(\omega t + \varphi + \pi/2)}$$

This point moves in a circle of radius ωA and π/2 ahead of $\tilde{\psi}$.

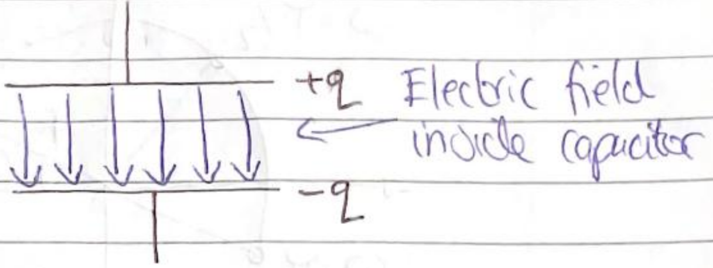


Circuits

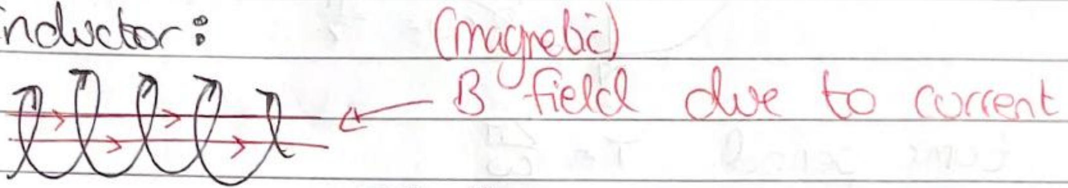
We will look at a circuit involving a capacitor and an inductor.

Capacitor:

$$\underset{\text{voltage}}{V} = \underset{\text{capacitance}}{\frac{q}{c}}$$



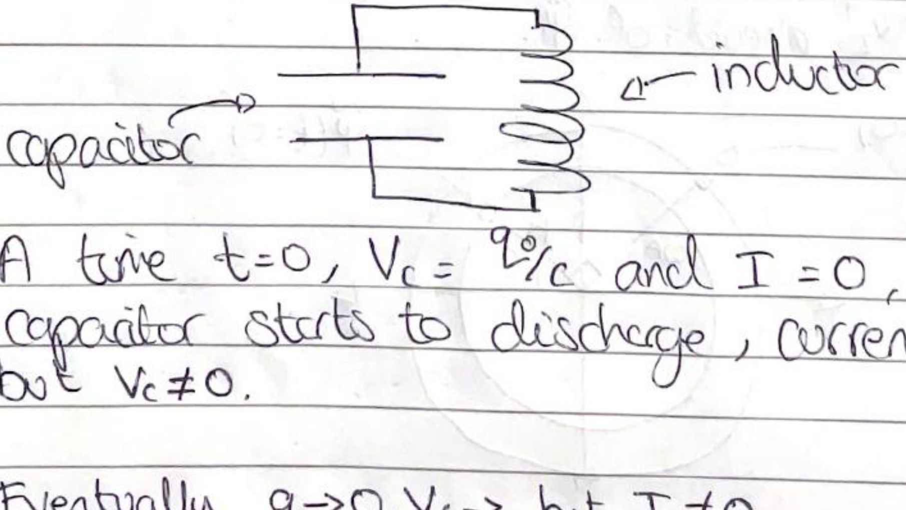
Inductor:



If the current varies with time, so does the induced magnetic field. This time varying magnetic field could then induce an electric field.

$$\underset{\text{voltage}}{V} = \underset{\text{inductance}}{L}\overset{\text{current}}{\frac{dI}{dt}}$$

- charge the capacitor
- disconnect from voltage source at t=0
- connect inductor



A time t=0, $V_c = q_0/c$ and $I = 0$, but the capacitor starts to discharge, current increases but $V_c \neq 0$.

Eventually $q \to 0, V_c \to$ but $I \neq 0$.

The capacitor starts to charge in the opposite direction, the current decreases, the voltage across inductor will fall.

N.B. total voltage around the circuit is zero.

$$\frac{q}{c} + L\frac{dI}{dt} = 0 \qquad I = \frac{dq}{dt}$$

$$L\frac{d^2I}{dt^2} + \frac{q}{c} = 0 \qquad \underline{I.C.}\ q(t=0) = q_0$$

$$\dot{q}_0 = 0.$$

This is the same as newton's law but with different constants.

$$\omega_{circuit} = \sqrt{\frac{1}{LC}}\,.$$

Energy in SHM

lets look at an archtypal example; a spring.

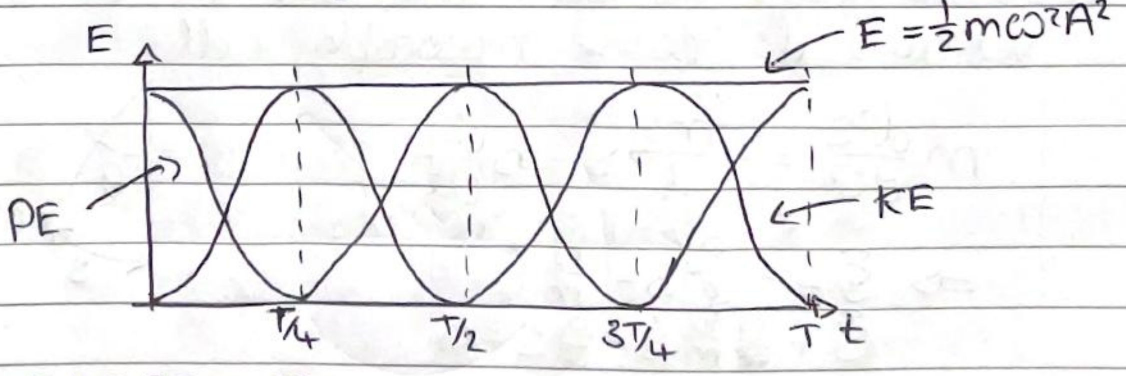
~~E=KE+PE=~~

$$PE = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\varphi) \qquad (k = m\omega^2)$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\varphi)$$

$$E = PE + KE = \frac{1}{2}m\omega^2A^2(\cos^2() + \sin^2())$$

$$= \frac{1}{2}m\omega^2A^2$$



- E, KE, PE $\propto A^2$
- E is const

if we look at our capacitor-inductor circuit, we find.

$PE = \frac{1}{2}\frac{1}{C}q^2$ $KE = \frac{1}{2}LI^2$

This is not the 'real' kinetic & potential energy. Here the PE is the energy stored in the electric field in the capacitor. The kinetic energy is the energy stored by the magnetic field in the inductor.

If we assume no dissipation or damping then the total energy will be constant.

But real systems do lose energy (eg. air resistance).

With damping, we'll guess that ✡ the amplitude decreases ✡ time period increases. In reality we'll see something more complicated.