Classical Mechanics 17

Newton's Laws for Solid Bodies

So far, we have treated bodies as point-like objects. We should now begin to take account of the fact bodies are extended.

Forces can make bodies spin as well as move. ∴ We need to consider torque and angular momentum, as well as forces & linear momentum.

Internal & External Forces

- for now, continue to think of extended bodies as an assembly of particles.
- The particles feel internal and external forces

$$F_i = F_i^{ext} + \sum_{j(\neq i)} F_{j \text{ on } i}$$

- Fortunately, we have already shown that the internal forces cancel out to leave:

$$\dot{\underline{P}} = \frac{d}{dt}\left(\sum_i \underline{p_i}\right) = \underline{F}^{ext} = \sum_i \underline{F_i}^{ext}$$

$$\dot{\underline{L}} = \frac{d}{dt}\left(\sum_i \underline{r_i} \times \underline{p_i}\right) = \underline{G}^{ext} = \sum_i \underline{r_i} \times \underline{F_i}^{ext}$$

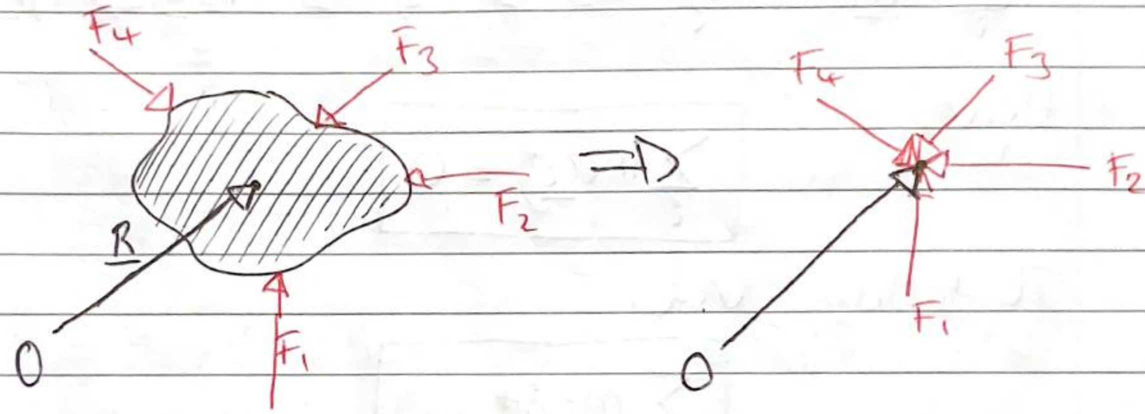
Since N1 & N3 are unchanged and we know the linear & rotational form of N2, there is nothing more to do..... if only!

From now on, we'll drop 'ext' as we'll no longer consider internal forces.

Centre-of-Mass Motion

$$\underline{R} = \frac{\sum_i m_i \underline{r_i}}{\sum_i m_i}$$

The momentum & angular momentum of the CoM change exactly as if all the external forces were acting on the CoM.



$$\dot{\underline{R}} = \frac{\sum_i m_i \dot{\underline{r_i}}}{\sum_i m_i}$$

$$M\dot{\underline{R}} = \sum_i m_i \dot{\underline{r_i}} = \underline{P}$$

the total force $= F_1 + F_2 + F_3 + F_4$

$$\frac{d}{dt}(M\dot{\underline{R}}) = \dot{\underline{P}} = \underline{F}$$

∴ Shown we consider all forces as if acting on CoM for linear momentum.

$$\frac{d}{dt}(\underline{R} \times M\dot{\underline{R}}) = \cancel{\dot{\underline{R}} \times M\dot{\underline{R}}} + \underline{R} \times \frac{d}{dt}(M\dot{\underline{R}})$$

$$= \underline{R} \times \underline{F}$$

and now we've shown it for angular momentum.

Separating the CoM & Internal Physics

Since,

$$\underline{r_i}^* = \underline{r_i} - \underline{R} \qquad M\underline{R} = \sum_i m_i \underline{r_i}$$

we get,

$$\sum_i m_i \underline{r_i}^* = \sum_i m_i(\underline{r_i} - \underline{R}) = \sum_i m_i \underline{r_i} - \cancel{M\underline{R}}\left(\sum_i m_i\right)\underline{R} = M\underline{R} - M\underline{R}$$

Hence,

$$\boxed{\sum_i m_i \underline{r_i}^* = 0}$$

It follows that,

$$\boxed{\sum_i m_i \dot{\underline{r_i}}^* = 0}$$

There is no 'internal' momentum because the CoM frame is also the zero-momentum frame.

CoM kinetic Energy

$$K = \frac{1}{2}\sum_i m_i |\dot{\underline{r_i}}|^2$$

$$= \frac{1}{2}\sum_i m_i \left(\dot{\underline{r_i}}^* + \dot{\underline{R}}\right) \cdot \left(\dot{\underline{r_i}}^* + \dot{\underline{R}}\right)$$

$$= \frac{1}{2}\sum_i m_i \dot{\underline{r_i}}^* \cdot \dot{\underline{r_i}}^* + \underbrace{\cancel{\sum_i m_i \dot{\underline{r_i}}^* \cdot \dot{\underline{R}}}}_{=0} + \frac{1}{2}\left(\sum_i m_i\right)\dot{\underline{R}} \cdot \dot{\underline{R}}$$

$$= K^* + \frac{1}{2}M\dot{\underline{R}}^2$$

Total KE = KE of CoM + KE about CoM.

Torque

$$\underline{G} = \sum_i \underline{r_i} \times \underline{F_i}$$

$$= \sum_i (\underline{r_i^*} + \underline{R}) \times \underline{F_i}$$

$$= \sum_i \underline{r_i^*} \times \underline{F_i} + \underline{R} \times \sum_i \underline{F_i}$$

$$= \underline{G^*} + \underline{R} \times \underline{F}$$

Total Torque = Torque on CoM + Torque about CoM.

Angular Momentum

$$\underline{L} = \sum_i \underline{r_i} \times m_i \underline{\dot{r}_i}$$

$$= \sum_i (\underline{r_i^*} + \underline{R}) \times m_i (\underline{\dot{r}_i^*} + \underline{\dot{R}})$$

$$= \sum_i \underline{r_i^*} \times m_i \underline{\dot{r}_i^*} + \underline{r_i^*} \times m_i \underline{\dot{R}} + \underline{R} \times m_i \underline{\dot{r}_i^*} + \underline{R} \times m_i \underline{\dot{R}}$$

$$= \underline{L^*} + \underbrace{\left(\sum_i m_i \underline{r_i^*}\right)}_{=0} \times \underline{\dot{R}} + \underline{R} \times \underbrace{\left(\sum_i m_i \underline{\dot{r}_i^*}\right)}_{=0} + \underline{R} \times \sum_i m_i \underline{\dot{R}}$$

$$= \underline{L^*} + \underline{R} \times M \underline{\dot{R}}$$

Total Ang. Mom. = Ang. Mom. of CoM + Ang. Mom. about CoM.

Rate of Change of Ang. Momentum

We know that,

$$\frac{d\underline{L}}{dt} = \underbrace{\underline{G}}_{\text{torque}} = \sum_i \underline{r_i} \times \underline{F}$$

$$\frac{d}{dt}(\underline{R} \times M\dot{\underline{R}}) = \underline{R} \times \underline{F} = \underline{R} \times \sum_i \underline{F_i}$$

Hence, $$\frac{d\underline{L}^*}{dt} = \frac{d}{dt}\left(\underline{L} - \underline{R} \times M\dot{\underline{R}}\right)$$

$$= \sum_i \underline{r_i} \times \underline{F_i} - \underline{R} \times \sum_i \underline{F_i}$$

$$= \sum_i (\underline{r_i} - \underline{R}) \times \underline{F_i}$$

$$= \sum_i \underline{r_i}^* \times \underline{F_i} = \underline{G}^*$$

Rate of change of Ang. Momen = Torque about CoM.

<u>Summary</u>

- momentum & angular momentum of CoM change <u>exactly</u> as if all external forces were acting directly on CoM.
- total momentum = CoM momentum
- total KE = KE of CoM + KE of CoM
- total torque = torque of CoM + torque about CoM
- total A. Momentum = A. momentum of CoM + A. momentum about CoM
- rate of change of A. momentum about CoM = torque about CoM.