Lecture 8

Derivative of $f(z)$

Operations on ~~$f(x)$~~ $z \to f(z)$

$$\frac{df(z)}{dz} \qquad\qquad \cancel{\int f(z)\,dz} \text{ next year.}$$

$$\frac{df(x)}{dx} \longleftrightarrow \text{continuity of } f(x).$$

$\Rightarrow f(z) \longleftrightarrow$ continuity?

$f(z)$ is continous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0) \qquad (*)$$

I) $\lim_{z \to z_0} f(z)$ exists

II) $f(z_0)$ exists

III) I = II

The meaning of $*$:

For each $\varepsilon > 0$, there is $\delta > 0$ (δ dependent on ε) such that $|f(z) - f(z_0)| < \varepsilon$ wherever $|z - z_0| < \delta$.

Properties

a) if $f(z)$, $g(z)$ are continous at z_0, so is also $f(z)+g(z)$.

b) same for product $f(z)g(z)$

c) same for $f(z)/g(z)$ if $g(z_0) \neq 0$.

d) $f(g(z))$ - composition of f & g - is continous $g(z)$ is continous at $z = f(z_0)$

$$|g(f(z)) - g(f(z_0))| < \varepsilon \quad \text{when } |f(z) - f(z_0)| < \gamma.$$

but $f(z)$ is also continous:

$|f(z) - f(z_0)| < \gamma$ when $|z - z_0| < \delta$

Example $f(z) = z^2$

$$\lim_{z \to z_0} (f(z)) = \lim_{\delta \to 0} (z_0 + \delta)^2 =$$

$$= \lim_{\delta \to 0} (z_0^2 + 2z_0\delta + \delta^2) = z_0^2$$

ϵ, δ:

$|(z_0+\delta)^2 - z_0^2| < \varepsilon$ when δ becomes smaller & smaller.

$|2z_0\delta + \delta^2| = |\delta||2z_0 + \delta|$

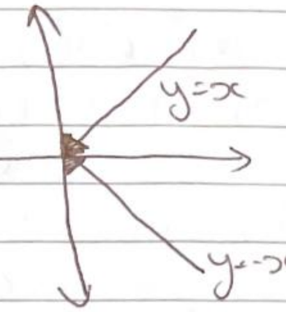
tends to zero as $\delta \to 0$.

Example $f(z) = \frac{z^*}{z} - \frac{z}{z^*}$

$$f(z) = \frac{x-iy}{x+iy} - \frac{x+iy}{x-iy} = \frac{(x-iy)^2 - (x+iy)^2}{x^2+y^2}$$

$$= \frac{4ixy}{x^2+y^2}$$

$$\lim_{z\to 0} f(z) = ?$$



We can approach our limit from any direction.

$$f(z)\Big|_{y=x} = -2i$$

$$f(z)\Big|_{y=-x} = +2i$$

$$\therefore \lim_{z\to 0} f(z) \text{ does not exist.}$$

<u>Derivates of complex numbers</u>

If $\displaystyle\lim_{\delta z\to 0} \frac{f(z+\delta z)-f(z)}{\delta z}$ $\qquad z, \delta z \in \mathbb{C}$

Calling $f(z+\delta z) - f(z) = \delta f$.

$$\lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \frac{df}{dz}.$$

If $f(z)$ has a derivative $\frac{df}{dz}$ in a region R. Then $f(z)$ is analytic in R.

<u>Example</u> $\qquad f(z) = z^2 \qquad \frac{df}{dz} = ?$

$$\lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \lim_{\delta z\to 0} \frac{(z+\delta z)^2 - z^2}{\delta z}$$

$$= \lim_{\delta z\to 0} \frac{2z\delta z + \delta z^2}{\delta z} = \lim_{\delta z\to} 2z + \delta z = \underline{\underline{2z}}$$

<u>Example</u> $\quad f(z) = |z|^2$

$$\frac{\delta f}{\delta z} = \frac{|z+\delta z|^2 - |z|^2}{\delta z} = \frac{(z+\delta z)(z^* + \delta z^*) - zz^*}{\delta z}$$

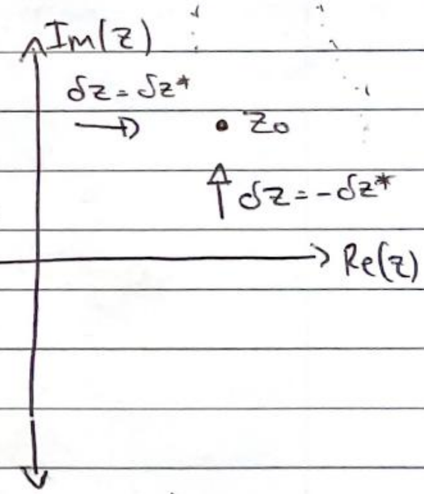
$$= \frac{zz^* + \delta z z^* + \delta z^* z + \delta z \delta z^* - zz^*}{\delta z}$$

$$= z^* + \delta z^* + z\frac{\delta z^*}{\delta z}$$

<u>Case 1</u> $\quad z = 0 = z^*$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} \delta z^* = 0.$$

<u>Case 2</u> $\quad z \neq 0$



We can approach z_0 from both the imaginary and real direction. Approaching from the real direction gives us:

$$\delta z = \delta z^* \quad (iy = 0)$$

Approaching from the imaginary axis gives us

$$\delta z = -\delta z^* \quad (x = 0)$$

<u>Approach from Re</u>

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} (z^* + \delta z + z)$$

$$= \underline{z + z^*}$$

<u>Approach from Im</u>

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} (z^* - \delta z - z)$$

$$= \underline{z^* - z}.$$

Conclusion

$\frac{df}{dz}$ does not exist

when $f = |z|^2$

not every function is differentiable in the complex plain, we should approach a point from two orthogonal points.