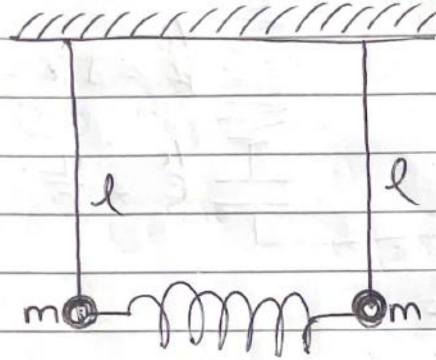
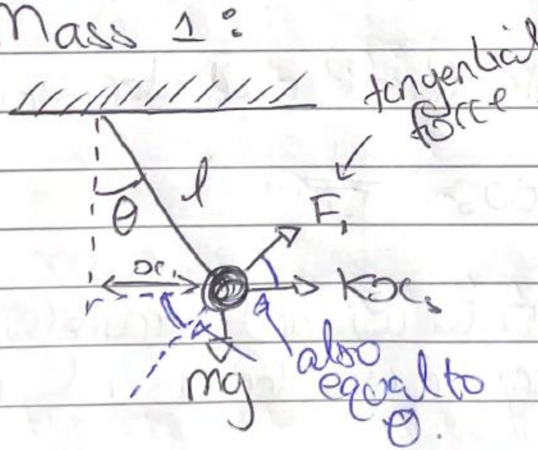
Coupled Pendulums



lets consider two identical pendulums coupled by a spring with constant k. The spring is at its natural length.

We are going to neglect damping & also assume small oscillations.

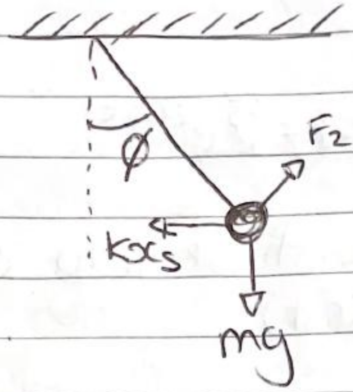
Mass 1:



$$F_1 = -mg\sin\theta + kx_1\cos\theta$$

$$\text{displacement} = l\theta \simeq l\sin\theta \simeq x_1$$

Mass 2:



$F_2 = -mg\sin\phi - kx_s\cos\phi$

displacement $\simeq x_2$

The total extension of the spring can be found by $x_s = x_2 - x_1$

$$\therefore \quad F_1 = -mg\underbrace{\sin\theta}_{\frac{x_1}{l}} + k(x_2 - x_1)\underbrace{\cos\theta}_{\simeq 1}$$

$$F_2 = -mg\sin\phi - k(x_2 - x_1)\cos\phi$$

$$F_1 = -mg\frac{x_1}{l} + k(x_2 - x_1)$$

$$\left(\frac{g}{l} = \omega_p^2\right)$$

$$m\ddot{x}_1 = -m\frac{g}{l}x_1 + k(x_2 - x_1)$$

$$\left(\frac{k}{m} = \omega_s^2\right)$$

$$\ddot{x}_1 = -\omega_p^2 x_1 + \omega_s^2(x_2 - x_1)$$

$$\hookrightarrow \quad \ddot{x}_2 = -\omega_p^2 x_2 - \omega_s^2(x_2 - x_1)$$

Normal Modes

To analyse oscillations of a complex system, we use normal modes. These are special motions in which all parts of the system oscillate at the same frequency.

Our coupled pendulums have two.

For our system, our (complex) solutions have the form.

$$\tilde{x}_1(t) = \tilde{A}e^{i\omega t} \qquad \tilde{x}_2(t) = \tilde{B}e^{i\omega t}$$

Now we can substitute these into our equations (let's do $\ddot{x}_1$ first).

$$-\omega^2\tilde{A}\cancel{e^{i\omega t}} = -\omega_p^2\tilde{A}\cancel{e^{i\omega t}} + \omega_s^2\tilde{B}\cancel{e^{i\omega t}} - \omega_s^2\tilde{A}\cancel{e^{i\omega t}}$$

$$-\omega^2 = -\omega_p^2 + \frac{\tilde{B}}{\tilde{A}}\omega_s^2 - \omega_s^2$$

$$\alpha = \tilde{B}/\tilde{A}$$

$$\omega^2 = \omega_p^2 - \alpha\omega_s^2 + \omega_s^2$$

now $\ddot{x}_2$

$$-\omega^2\tilde{B}\cancel{e^{i\omega t}} = -\omega_p^2\tilde{B}\cancel{e^{i\omega t}} - \omega_s^2\tilde{B}\cancel{e^{i\omega t}} + \omega_s^2\tilde{A}\cancel{e^{i\omega t}}$$

$$\omega^2 = \omega_p^2 + \omega_s^2 + \frac{\omega_s^2}{\alpha}$$

We know that ω must be the same. let's subtract the two.

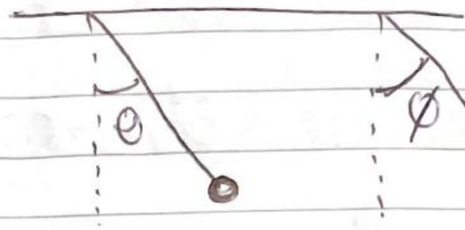
$$0 = \omega_s^2\left(-\alpha + \frac{1}{\alpha}\right)$$

$$\alpha = \frac{1}{\alpha} \Rightarrow \alpha^2 = 1$$

$$\alpha = \pm 1$$

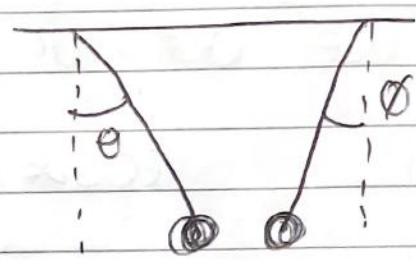
Therefore, we have two normal modes.

Mode 1: (when $\alpha = 1$)
$\tilde{B} = \tilde{A} \Rightarrow x_1 = x_2$



$\omega^2 = \omega_p^2 - \omega_s^2 + \omega_s^2$
$\omega^2 = \omega_p^2 \qquad \underline{\omega_1 = \omega_p}$

Mode 2: (when $\alpha = -1$)
$\tilde{B} = -\tilde{A} \Rightarrow x_1 = -x_2$



$\omega^2 = \omega_p^2 + \omega_s^2 + \omega_s^2$
$\omega = \sqrt{\omega_p^2 + 2\omega_s^2}$

Superposition of Normal Modes

The equations of motion are linear. This means any linear combination of solutions is also a solution.

(Notation:
x_{ij} = displacement of mass i in mode j.)

$x_{11} = A\cos(\omega_1 t) \qquad x_{21} = A\cos(\omega_1 t)$
$x_{12} = A\cos(\omega_2 t) \qquad x_{22} = -A\cos(\omega_2 t)$

We've assumed both normal modes have the same amplitude and that they're real. $\tilde{A} = A$.

Now, if we sum our solutions

$x_1 = x_{11} + x_{12} = A\cos(\omega_1 t) + A\cos(\omega_2 t)$ Mass 1
$x_2 = x_{21} + x_{22} = A\cos(\omega_1 t) - A\cos(\omega_2 t)$ Mass 2

Due to the principle of superposition, these must also be solutions to our DE.

$$\ddot{x}_1 = -\omega_p^2 x_1 + \omega_s^2 (x_2 - x_1)$$

Although not show, they do satisfy this equation.

We use the trig identities:

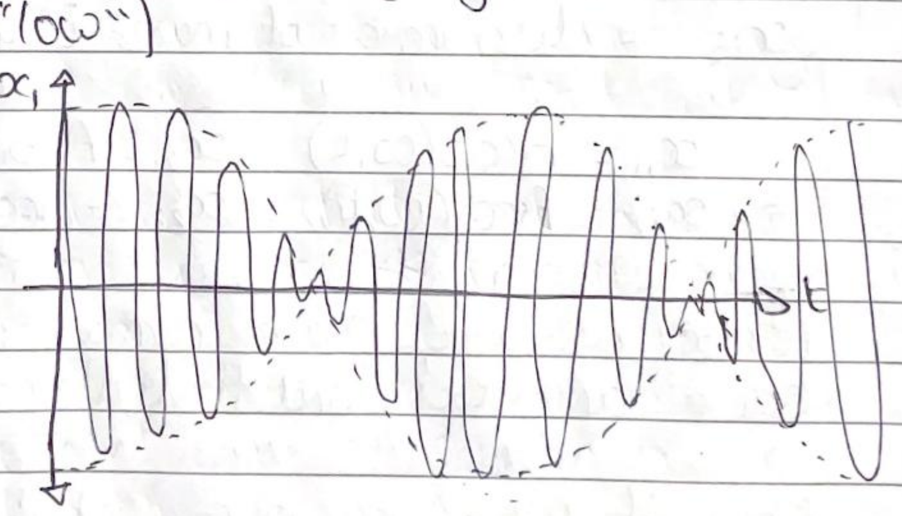
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$\therefore$

$$x_1 = 2A\cos(\omega_h t)\cos(\omega_e t)$$
$$x_2 = -2A\sin(\omega_h t)\sin(\omega_e t)$$

where $\omega_h = \frac{\omega_1 + \omega_2}{2}$ ("high") and $\omega_e = \frac{\omega_1 - \omega_2}{2}$ ("low")



The mass oscillates at ω_h, modulated by an envelope of ω_e.