Gradient $(\nabla\Omega)$ 'del' or 'nabla'

For a scalar field $\Omega(x,y,z)$, the vector field

$$\underline{F} = \frac{\partial\Omega}{\partial x}\hat{\imath} + \frac{\partial\Omega}{\partial y}\hat{\jmath} + \frac{\partial\Omega}{\partial z}\hat{k} = \nabla\Omega$$

is conservative. $\oint \underline{F}\cdot d\underline{r} = 0.$

The symbol ∇ 'del' is an operator

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

operating on a scalar field producing a vector field.

Here Ω (often ϕ in physics) is the <u>potential</u> associated with the vector field $\underline{F}$.

N.B. Take care of the sign. eg. the gravitation field $g = -\nabla\phi$ ($\underline{F} = -m\underline{g}$). purly convention.

Using $\nabla\Omega$ we can calculate the derivative of Ω in any direction.

$$d\Omega = \frac{\partial\Omega}{\partial x}dx + \frac{\partial\Omega}{\partial y}dy + \frac{\partial\Omega}{\partial z}dz = \nabla\Omega\cdot d\underline{r}$$

What is the derivative in direction $\hat{a}$ if we move a distance ds.

$$d\Omega = \nabla\Omega \cdot ds \cdot \hat{a}$$

$$\frac{d\Omega}{ds} = \nabla\Omega \cdot \hat{a} \longleftarrow \text{directional derivative}$$

eg. if $\hat{a} = \hat{k}$

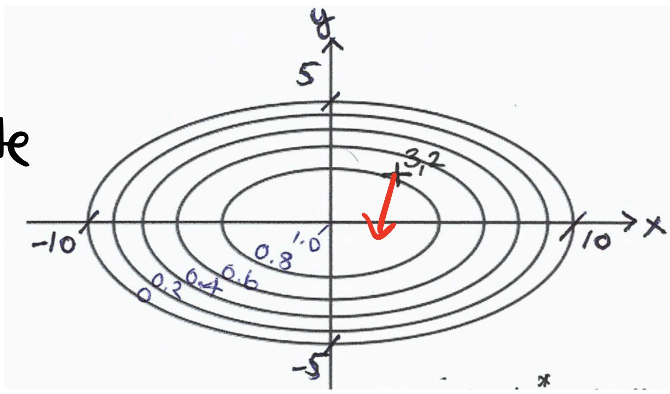
$$\frac{d\Omega}{ds} = \left(\frac{\partial\Omega}{\partial x}\hat{i} + \frac{\partial\Omega}{\partial y}\hat{j} + \frac{\partial\Omega}{\partial z}\hat{k}\right)\cdot\hat{k} = \frac{\partial\Omega}{\partial z}$$

At a particular point (x, y, z) in what direction $\hat{a}_{MAX}$ is $\frac{d\Omega}{ds}$ is maximum?

$$\hat{a}_{MAX} = \frac{\nabla\Omega}{|\nabla\Omega|} \qquad \nabla\Omega_{MAX} = |\nabla\Omega|$$

<u>Example</u> $\quad z = 1 - \frac{x^2}{100} - \frac{y^2}{25}$

I) at the point (3,2), what is the derection and magnitude of max derivative?



$$\nabla z = -\frac{2x}{100}\hat{i} - \frac{2y}{25}\hat{j}$$

$$\nabla z(3,2) = -\frac{6}{100}\hat{i} - \frac{4}{25}\hat{j} = -0.06\hat{i} - 0.16\hat{j}$$

magnitud: $\sqrt{0.06^2 + 0.16^2} = 0.17$

direction: $-\frac{0.06}{0.17}\hat{i} - \frac{0.16}{0.17}\hat{j}$

II) at (3,2), what is the derivative in the direction $\hat{i} + \hat{j}$?

$$\hat{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\nabla z \cdot \hat{a} = (-0.06\hat{i} - 0.16\hat{j}) \cdot \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$= -\frac{0.06}{\sqrt{2}} - \frac{0.16}{\sqrt{2}} = -\frac{0.22}{\sqrt{2}} = -0.155 \text{ (downhill)}$$

$\nabla\Omega$ in Other Coordinates

Generally $d\Omega = \nabla\Omega \cdot d\underline{r}$

Let's derive $\nabla\Omega$ in cyclindrical polar coords:

$$d\Omega = \frac{\partial\Omega}{\partial\rho}d\rho + \frac{\partial\Omega}{\partial\phi}d\phi + \frac{\partial\Omega}{\partial z}dz$$

$$d\underline{r} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$$

We need to find the function when dot producted with $d\underline{r}$ gives $d\Omega$.

$$\nabla\Omega(\rho,\phi,z) = \frac{\partial\Omega}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Omega}{\partial\phi}\phi + \frac{\partial\Omega}{\partial z}\hat{z}$$

Now lets derive it in spherical coordinates:

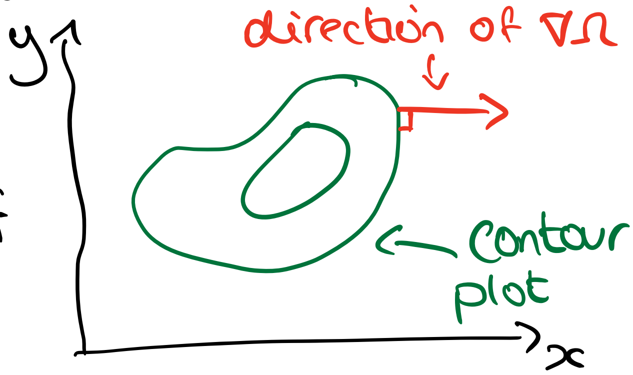
$$d\underline{r} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + \rho\sin\theta\, d\theta\,\hat{\theta}$$

$$\nabla\Omega(\rho,\phi,\theta) = \frac{\partial\Omega}{\partial\rho}\hat{\rho} + \frac{1}{\rho\sin\theta}\frac{\partial\Omega}{\partial\phi}\hat{\phi} + \frac{1}{\rho}\frac{\partial\Omega}{\partial\theta}\hat{\theta}$$

$\nabla\Omega$ & Normal to the Surface

In 2D, $\frac{\partial\Omega}{\partial s} = \nabla\Omega \cdot \hat{a}$. Along a contour $\frac{\partial\Omega}{\partial s} = 0$.

$\therefore \nabla\Omega$ is perpendicular to a line of constant Ω.



In 3D, $\Omega(x,y,z)$, $\nabla\Omega$ is perpendicual to a surface of constant Ω.

Let's consider a 3D surface described by $z=f(x,y)$. Suppose we want to find the direction normal to a point on the surface.

lagrangian?

Key Step: define $\Omega(x,y,z) = f(x,y) - z$. Then the surface is the surface $\Omega = 0$. A vector normal to the point (x,y,z) is

$$\underline{n} = \nabla\Omega(x,y,z) \qquad \hat{n} = \frac{\nabla\Omega}{|\nabla\Omega|}$$

<u>Example</u> $z = xy$

define $\Omega(x,y,z) = f(x,y) - z = xy - z$

$$\nabla\Omega = y\hat{i} + x\hat{j} - 1\hat{k}$$

Same as other method but ambiguity of sign.