Functions II

The total Differential

When we have a function of a single variable $f(x)$ and we have a small change $x \to x + \delta x$, so that $f \to f + \delta f$ then $\delta f \simeq \frac{df}{dx} \delta x$

In the limits (small change $\to 0$),

$$df = \frac{df}{dx} dx.$$

Now for a function of two variables, small changes $x \to x + \delta x$, $y \to y + \delta y$ leads to $u(x,y) \to u + \delta u$. with

$$\delta u \simeq \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y$$

In the limit (small change $\to 0$),

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy}$$

This is called the total differential of $u(x,y)$.

Example $\quad u(x,y) = x^2 \sin y + y^3$

$$\delta u = (2x \sin y)\delta x + (x^2 \cos y + 3y^2)\delta y$$
$$\Rightarrow du = (2x \sin y)dx + (x^2 \cos y + 3y^2)dy.$$

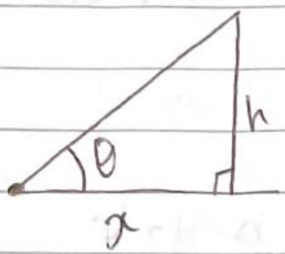
Example $\quad$ Area of a Rectangle $\quad A = xy$

$$\delta A = (x + \delta x)(y + \delta y) - xy$$
$$= (y\delta x + x\delta y) + \delta x \delta y$$

second order small ≈ 0.

$$\delta A = y\delta x + x\delta y \iff dA = x\,dy + y\,dx$$

<u>Example</u> Height of a building $h = x\tan\theta$



$x = 200m \pm 2m$
$\theta = 20^\circ \pm \frac{1}{2}^\circ$

$$\delta h \approx (\tan\theta)\delta x + (x\sec^2\theta)\delta\theta$$

central estimate $= 200\tan\left(\frac{\pi}{9}\right) = 72.8m$.

$\Rightarrow \delta h = (\tan 20)(2) + (200\sec^2 20)(0.008)$ ← 0.5° in rads.
$= 2.7m$

to calc. uncertainties replace $\delta x, \delta\theta$ with error.

$$\underline{\underline{h = 72.8 \pm 2.7m}}$$

<u>Function of a Function</u>

When we have had previously
$u = f(x)$ and $x = g(t)$
we have a consequence.

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} = f'(x)\cdot g'(t) = f'(g(t))g'(t)$$

Now consider $u = u(x,y)$ where $x = x(t)$ $y = y(t)$.
From our total differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

we can divide by dt.

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

Example volume of cylinder of radius r & height h

$$V = \pi r^2 h$$

If we're told that $r = 2t$ & $h = 1 + t^2$

$$\frac{dV}{dt} = (2\pi rh)(2) + (\pi r^2)(2t)$$

$$= 4\pi(2t)(1+t^2) + 2\pi(2t)^2 t$$

$$= 8\pi t + 8\pi t^3 + 8\pi t^3$$

$$= \underline{8\pi(t + 2t^3)}$$

Example $u = x^2 y$ $x = st$ $y = s + t$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2xy)(t) + (x^2)(1)$$

$$= 2(st)(s+t)(t) + (st)^2$$

$$= 2s^2t^2 + 2st^3 + s^2t^2 = \underline{3s^2t^2 + 2st^3}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= (2xy)(s) + (x^2)(1)$$

$$= 2(st)(s+t)(s) + (st)^2$$

$$= 2s^3t + 2s^2t^2 + s^2t^2 = \underline{3s^2t^2 + 2s^3t}$$