Vectors 7

Simultaneous Equations

Solve simultaneously:

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \text{linear equations}$$

take ① $\times b_2$: $a_1b_2x + b_1b_2y = c_1b_2$
take ② $\times b_1$: $a_2b_1x + b_1b_2y = c_2b_1$

$$b_2① - b_1② = (a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

take ① $\times a_2$: $a_1a_2x + b_1a_2y = c_1a_2$
take ② $\times a_1$: $a_1a_2x + b_2a_1y = c_2a_1$

$$a_1② - a_2① = (b_2a_1 - b_1a_2)y = c_2a_1 - c_1a_2$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

In general:

$$x = \frac{\Delta_1}{\Delta} \qquad y = \frac{\Delta_2}{\Delta}$$

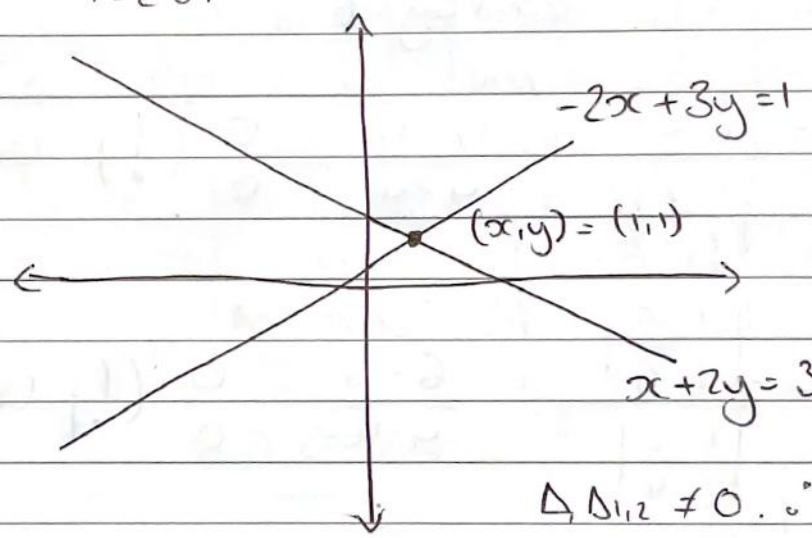
$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Graphical Representation

Example
$$x + 2y = 3$$
$$-2x + 3y = 1$$

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}} = \frac{9-2}{3+4} = \frac{7}{7} = 1$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}} = \frac{1+6}{4+3} = \frac{7}{7} = 1$$

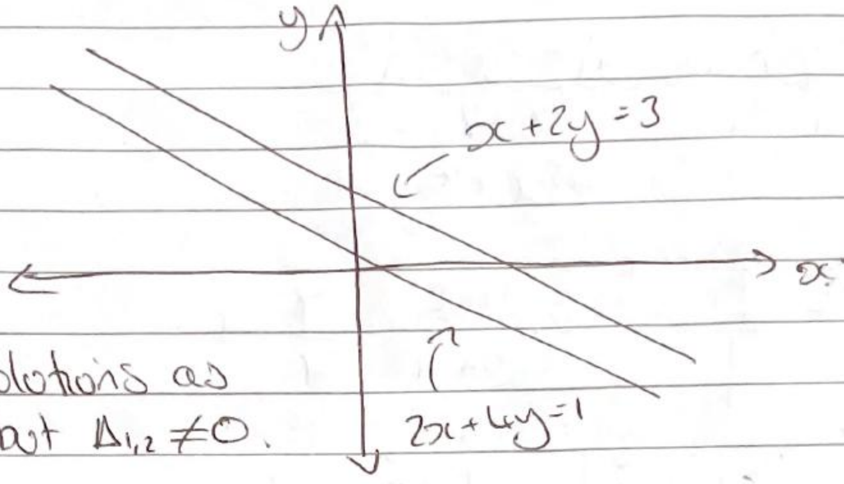


$\Delta, \Delta_{1,2} \neq 0$. ∴ unique solution

Example
$$x + 2y = 3$$
$$2x + 4y = 1$$

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{12-2}{4-4} = \frac{10}{0}$$ (!) undefined

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{1-6}{4-4} = \frac{-5}{0}$$ (!) undefined.

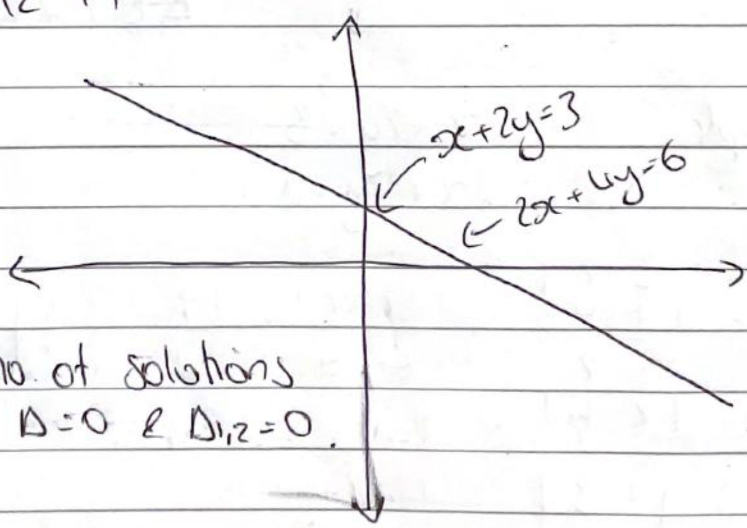


no solutions as
$\Delta = 0$ but $\Delta_{1,2} \neq 0$.

Example $\quad x + 2y = 3$
$\quad\quad\quad\quad 2x + 4y = 6$

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{12-12}{4-4} = \frac{0}{0} \; (!) \text{ undefined}$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}} = \frac{6-6}{4-4} = \frac{0}{0} \; (!) \text{ undefined.}$$



infinity no. of solutions
as both $\Delta = 0$ & $\Delta_{1,2} = 0$.

Cramer's Rule

Solve simultaneously:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

consider $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$$

combinate rule

$$= \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1y & b_1 & c_1 \\ b_2y & b_2 & c_2 \\ b_3y & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} c_1z & b_1 & c_1 \\ c_2z & b_2 & c_2 \\ c_3z & b_3 & c_3 \end{vmatrix}$$

multiple rule

$$= x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + y\cancel{\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}} + z\cancel{\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}}$$

cancel rule.

$$\Delta_1 = x\Delta \iff x = \frac{\Delta_1}{\Delta}$$

$$\boxed{x = \frac{\Delta_1}{\Delta} \qquad y = \frac{\Delta_2}{\Delta} \qquad z = \frac{\Delta_3}{\Delta}}$$

Cramer's Rule.

Example

$$x + y + z = 1$$
$$x + 2y - z = -1$$
$$x - y + z = 2$$

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix}$ $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}$ $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$ $\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$\Delta = -4$ $\quad \Delta_1 = -3$ $\quad \Delta_2 = 2$ $\quad \Delta_3 = -3$

$x = \frac{\Delta_1}{\Delta} = \frac{-3}{-4} = \frac{3}{4}$ $\quad y = \frac{\Delta_2}{\Delta} = \frac{2}{-4} = -\frac{1}{2}$ $\quad z = \frac{\Delta_3}{\Delta} = \frac{-3}{-4} = \frac{3}{4}$.

not included for briefity.