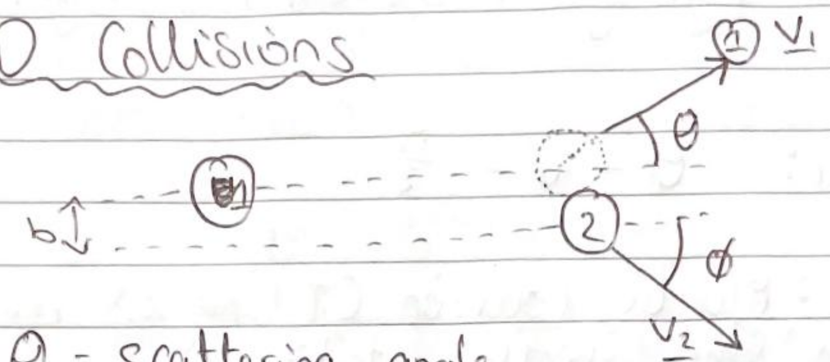
Classical Mechanics: 9

3D Collisions



(fig 1.

θ = scattering angle
ϕ = recoil angle
b = impact parameter

The impact parameter is hard to measure. We often avoid it by asking questions without it.

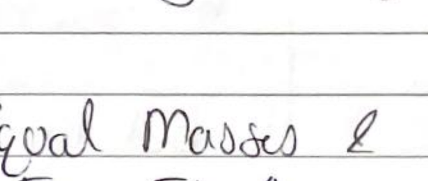
<u>Example:</u> Elastic Collision Of Equal Masses & One itially Stationary For Fig. 1.

- We solved the b=0 case earlier: ball 1 stops and ball 2 picks up all the kinetic energy.

<u>Conservation of Momentum</u>

$$m\underline{u}_1 + m0 = m\underline{v}_1 + m\underline{v}_2$$

$$\underline{u}_1 = \underline{v}_1 + \underline{v}_2$$



<u>KE Conservation</u>

$$\frac{1}{2}mu_1^2 + \frac{1}{2}m0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

$$u^2 = v_1^2 + v_2^2$$

this is pythagorean theorem, $\therefore$ right angle triangle.

hence: $\theta + \phi = \frac{\pi}{2}$

Example: Elastic Collision of Unequal Masses w. One itially stationary.

- $\underline{u}_1, \underline{v}_1, \underline{v}_2$ no longer form a vector triangle
- $\underline{p}_1 \triangleq m_1\underline{u}_1$, $\underline{q}_1 \triangleq m_1\underline{v}_1$, $\underline{q}_2 \triangleq m_2\underline{v}_2$ form a triangle as

$$\underline{p}_1 = \underline{q}_1 + \underline{q}_2$$

but the triangle is not right-angled. $\theta + \phi \neq \frac{\pi}{2}$.

When facing tough scattering problem, it is easier to work in the centre of mass frame.

Centre of Mass Frame

- In the absence of external forces, the CoM moves at a constant velocity:

$$\ddot{\underline{R}} = \frac{d^2}{dt^2}\left(\frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1+m_2}\right) = \frac{m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2}{m_1+m_2} = 0$$

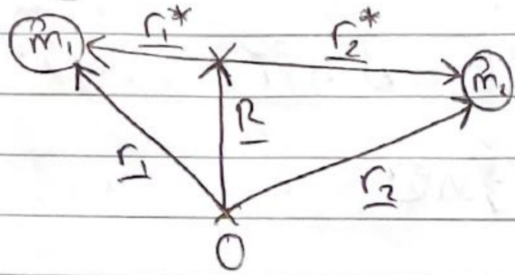
from Newton3.

The moving frame with its origin on the CoM is therefore inertial:

- $\rightarrow$ Newton's laws hold
- $\rightarrow$ momentum is conserved
- $\rightarrow$ if collision elastic, energy is conserved.

Centre of Mass Coordinates

Coordinates in the centre of mass frame are indicated by stars:



$$\boxed{\begin{aligned} \underline{r}_1^* &= \underline{r}_1 - \underline{R} \\ \underline{r}_2^* &= \underline{r}_2 - \underline{R} \end{aligned}}$$

Momentum?

$$\underline{p}_1^* = m_1 \dot{\underline{r}}_1^*$$

$$= m_1(\dot{\underline{r}}_1 - \dot{\underline{R}})$$

$$= m_1 \dot{\underline{r}}_1 - \left[\frac{m_1 \dot{\underline{r}}_1 + m_2 \dot{\underline{r}}_2}{m_1 + m_2}\right] m_1 \qquad M = m_1 + m_2$$

$$= \frac{m_1(M - m_1)\dot{\underline{r}}_1}{M} - \frac{m_1 m_2 \dot{\underline{r}}_2}{M}$$

$$= \frac{m_1 m_2}{M}(\dot{\underline{r}}_1 - \dot{\underline{r}}_2)$$

$$\boxed{\underline{p}_1^* = -\mu \dot{\underline{r}}}$$ (reduced mass) $\underline{r} =$ difference vector

$\underline{r} \triangleq \underline{r}_2 - \underline{r}_1$

$$\boxed{\underline{p}_2^* = \mu \dot{\underline{r}}}$$

The centre of mass frame is also the zero-momentum frame

Total KE is CoM Frame

$$K = \frac{1}{2} m_1 \dot{\underline{r}}_1 + \frac{1}{2} m_2 \dot{\underline{r}}_2$$

$$= \frac{1}{2} m_1 (\dot{\underline{r}}_1^* + \dot{\underline{R}})^2 + \frac{1}{2} m_2 (\dot{\underline{r}}_2^* + \dot{\underline{R}})^2$$

$$= \frac{1}{2} m_1 \left[(\dot{\underline{r}}_1^*)^2 + 2\dot{\underline{r}}_1^* \cdot \dot{\underline{R}} + (\dot{\underline{R}})^2\right] + \frac{1}{2} m_2 \left[(\dot{\underline{r}}_2^*)^2 + 2\dot{\underline{r}}_2^* \cdot \dot{\underline{R}} + (\dot{\underline{R}})^2\right]$$

$$= \frac{1}{2} m_1 \dot{\underline{r}}_1^{*2} + \frac{1}{2} m_2 \dot{\underline{r}}_2^{*2} + \underbrace{(m_1 \dot{\underline{r}}_1^* + m_2 \dot{\underline{r}}_2^*)}_{=0 \ (\underline{p}_1^* + \underline{p}_2^* = 0)} \cdot \dot{\underline{R}} + \frac{1}{2} M \dot{\underline{R}}^2$$

$$= \frac{1}{2} m_1 (\dot{r}_1^*)^2 + \frac{1}{2} m_2 (\dot{r}_2^*)^2 + \frac{1}{2} M \dot{R}^2$$

$$K = K^* + \frac{1}{2} M \dot{R}^2$$

KE = KE in CoM frame + KE of CoM

KE in CoM frame

$$K^* = \frac{1}{2} m_1 (\dot{r}_1^*)^2 + \frac{1}{2} m_2 (\dot{r}_2^*)^2$$

$$= \frac{(\underline{p}_1^*)^2}{2m_1} + \frac{(\underline{p}_2^*)^2}{2m_2} \qquad (\underline{p}_2^* = -\underline{p}_1^* = \mu \dot{\underline{r}})$$

$$= \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(\mu \dot{r})^2 \qquad \left(\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\mu}\right)$$

$$= \frac{1}{2} \mu \dot{r}^2$$

KE in CoM Frame = $\frac{1}{2}$ x reduced mass x revlotiicy velocity^2