Functions 12

Cartesians $\to$ Polar

$$u(x,y) \equiv \bar{u}(r,\theta)$$ both are othogonal systems.

$$x = r\cos\theta \qquad r = (x^2+y^2)^{\frac{1}{2}}$$
$$y = r\sin\theta \qquad \theta = \arctan(y/x)$$

We need to be careful!

$$\underbrace{\frac{\partial x}{\partial r}}_{\text{keeping } \theta \text{ constant}} = \cos\theta \qquad \underbrace{\frac{\partial r}{\partial x}}_{\text{keeping } y \text{ constant.}} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$$

We should not be tempted

$$\frac{\partial x}{\partial r} \neq \frac{1}{\left[\frac{\partial r}{\partial x}\right]}$$

The cartesian and polar versions of our functions have the same value but are described differently.

Chain Rule

$$\frac{\partial u}{\partial x} = \frac{\partial \bar{u}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \bar{u}}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad (u(r,\theta))$$

$$= (\cos\theta)\frac{\partial \bar{u}}{\partial r} + \left(-\frac{\sin\theta}{r}\right)\frac{\partial \bar{u}}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial \bar{u}}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \bar{u}}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= (\sin\theta)\frac{\partial \bar{u}}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial \bar{u}}{\partial \theta}$$

We have therefore partial differential opperators!

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Which relates the rates of change in the two different coordinate systems.

Example $u(x,y) = x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = \bar{u}(r,\theta)$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x = 2r\cos\theta \\ \frac{\partial u}{\partial y} &= -2y = -2r\sin\theta \end{aligned}\right\}$$

using our above results we can check our solution.

Example We can express the following equation in plane polar coordinates.

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$\left[\cos\theta \frac{\partial \bar{u}}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \bar{u}}{\partial \theta}\right]^2 + \left[\sin\theta \frac{\partial \bar{u}}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \bar{u}}{\partial \theta}\right]^2$$

$$\equiv \left(\frac{\partial \bar{u}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \bar{u}}{\partial \theta}\right)^2$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 \bar{u}}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \bar{u}}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial \bar{u}}{\partial r}$$
$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \bar{u}}{\partial \theta}$$

similarly for $\frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \bar{u}}{\partial r \partial \theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \bar{u}}{\partial \theta}$$
$$+ \frac{\cos^2\theta}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial \bar{u}}{\partial r}.$$

Hence (!),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta}$$

Implicit Functions

If we have a function defined implicity

$$F(x,y) = 0$$

then F does not change as x and y do so.

The total differential then is:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

So the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)}$$

Example $F(x,y) = x^2\sin y + xy - 1 = 0$

$$\frac{dy}{dx} = -\frac{2x\sin y + y}{x^2\cos y + x}$$

If we have an implicit function of 3 variables

$$F(x,y,z) = 0.$$

this constraints our point (x,y,z) to a particular surface. We can regard

$$x = x(y,z) \quad y = y(x,z) \quad z = z(x,y)$$

Now, no change in F <u>on</u> this surface

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0$$

at constant y: $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}$ $\left(F_x = \frac{\partial F}{\partial x}\right)$

at constant x: $\left(\frac{\partial z}{\partial y}\right)_x = -\frac{F_y}{F_z}$

at constant z: $\left(\frac{\partial y}{\partial x}\right)_z = -\frac{F_x}{F_y}$

N.B. $\left(\frac{\partial z}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial z}\right)_y}$

As we're holding the same variable constant, they are recipricals of one another.

Ideal Gas Law

$$PV = nRT$$

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1.$$