Functions 8

Change of variable

$$\int x e^{-x^2} dx$$

Let $u = x^2$ $\qquad du = 2x\,dx$

$$\int x e^{-x^2} dx = \frac{1}{2}\int e^{-u} du = -\frac{1}{2}e^{-u} + c$$
$$= -\frac{1}{2}e^{-x^2} + c$$

For trigonometric functions we can often try:

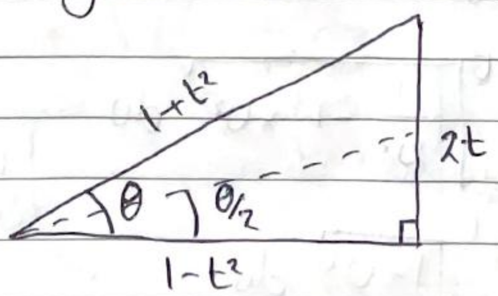
$$\boxed{t = \tan\left(\frac{\theta}{2}\right)}$$

so that: $\boxed{\cos\theta = \frac{1-t^2}{1+t^2}}$ $\quad \boxed{\sin\theta = \frac{2t}{1+t^2}}$

$\boxed{\tan\theta = \frac{2t}{1-t^2}}$ $\quad \boxed{\dot{\theta} = \frac{2}{1+t^2}}$ $\left(= \frac{d\theta}{dt}\right)$

N.B. $\quad (1+t)^2 = (2t)^2 + (1-t)^2$

this gives us a triangle:



$$1 = \sec^2\left(\frac{\theta}{2}\right)\cdot\left(\frac{1}{2}\frac{d\theta}{dt}\right)$$

$$\frac{d\theta}{dt} = \frac{2}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{2}{1+t^2}$$

Example $\quad \displaystyle\int \frac{dx}{2+\cos x}$

$$= \int \frac{dx}{dt}\frac{1}{\left(2+\frac{1-t^2}{1+t^2}\right)} dt$$

$$= \int \frac{2}{(1+t^2)} \cdot \frac{(1+t^2)}{(3+t^2)} \, dt$$

$$= 2\int \frac{dt}{3+t^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan\left(\frac{x}{2}\right)\right) + c$$

Example $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \, dx$

At first glance this does not look like trig, but a good strategy is to 'do something' to remove the square/cubic roots

$\frac{dx}{du} = -4u$

let $x = 1-2u^2$

$2u^2 = 1-x$

$u = \left(\frac{1-x}{2}\right)^{\frac{1}{2}}$

$= \cancel{\int_{-1}^{1}\sqrt{\frac{2-2u^2}{2u^2}} \int_{-1}^{1}\sqrt{\frac{1}{u^2}-1}}$

N.B. limits change when doing by substitution

$$= \int_{1}^{0} \sqrt{\frac{2-2u^2}{2u^2}} \, -4u \, du$$

$$= -4 \int_{1}^{0} \sqrt{\left(\frac{1}{u^2} - 1\right) \times u^2} \, du$$

$$= 4 \int_{0}^{1} \sqrt{1-u^2} \, du$$

minus sign swaps limits

let $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$$= 4 \int_{0}^{\pi} \cos^2\theta \, d\theta$$

$$= 4 \int_{0}^{\pi} \frac{1}{2}(1+\cos 2\theta) \, d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{0}^{\pi} = \pi$$

Intergration by parts

$$\int a\dot{b} = ab - \int \dot{a}b$$ inverse of product rule

Example $\int xe^x dx$

$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + c$$
$$= e^x(x-1) + c$$

We have to be quite careful with our choice of a & b. If we had chosen the opposite a & b:

$$= \frac{1}{2}x^2e^x - \int \frac{1}{2}e^x dx$$

This has gotten us further from the answer, not closer.

Example $\int \ln x \, dx$

$$= \int 1 \times \ln x \, dx$$

$$= x\ln x - \int x \frac{1}{x} dx$$
$$= x\ln x - x$$
$$= x(\ln x - 1)$$

Example $\int \arctan x = \int 1 \times \arctan x \, dx$

$$= x\arctan x - \int \frac{x}{1+x^2} dx$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + c$$

Mean Value

Consider a function $f(x)$ over a specified interval $[a, b]$.

The mean value 'average' of $f(x)$ over the interval

$$\boxed{\bar{f} \equiv \frac{1}{b-a} \int_a^b f(x)\, dx}$$

A modern notation from dirac is $\langle f \rangle$.

So $\bar{f}$ is the height of a rectangle whose area is the same as the area under the curve over the domain $[a,b]$

Root Mean Square

The root mean square value of $f(x)$ over $[a,b]$ is

$$f_{rms} = \sqrt{\frac{1}{b-a} \int_a^b f^2(x)\, dx}$$

in dirac notation is $\langle f^2 \rangle^{\frac{1}{2}}$.

We can view the difference between the mean and the root mean squared value as a guide to how 'spread' the data is.

A note on sine waves

$$\langle \sin x \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sin x\, dx = 0 = \langle \cos x \rangle$$

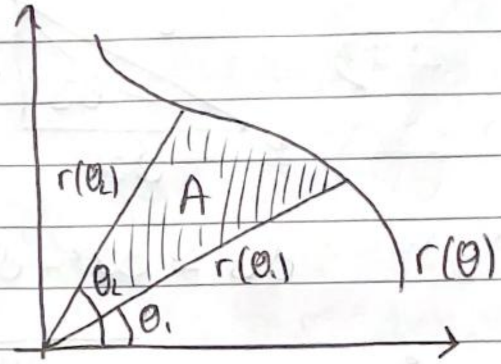
$$\langle \sin^2 x \rangle = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 x\, dx = \frac{1}{2}$$

$$\langle \sin^2 x \rangle^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \quad (\neq 0).$$

Area in Polars

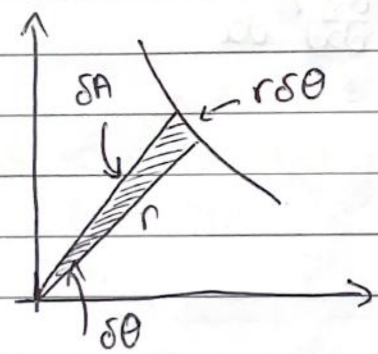
The geometry of a problem may lead us to consider using plane polar coordinates! (r, θ)

Consider:



A = Area of wedge.

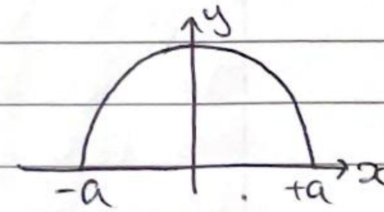
If we look at an infinitesimal wedge section:



$$\delta A \approx \frac{1}{2} r (r\delta\theta)$$

$$\text{Area } A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2(\theta)\, d\theta$$

Example   Area of a semi-circle



Cartesian

$$A = \int_{-a}^{a} y\,dx = \int_{-a}^{a} (a^2-x^2)^{\frac{1}{2}} dx$$

$$= 2\int_0^a \sqrt{a^2-x^2}\, dx \qquad \text{let } x = a\sin\theta$$

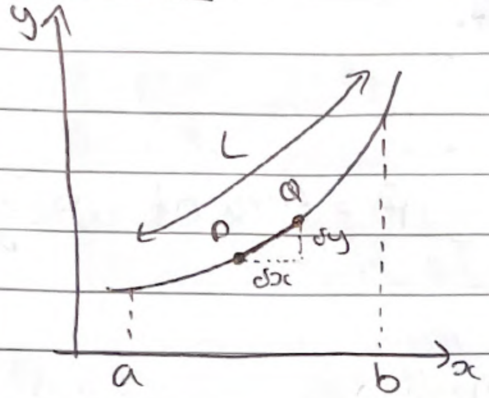
$$= 2\int_0^{\pi/2} a^2(1-\sin^2\theta)\, d\theta = 2\int_0^{\pi/2} a^2\cos^2\theta\, d\theta = \frac{\pi a^2}{2}$$

Polar

$$A = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2}a^2 \int_0^{\pi} d\theta = \frac{\pi a^2}{2}$$

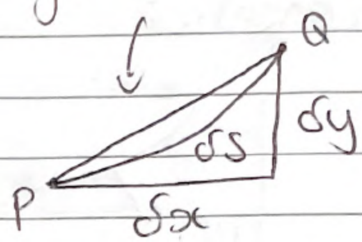
Path Length

Cartesian



$\Longrightarrow$

gets more accurate as $\delta s \to 0$.



$$\delta s^2 \simeq \delta x^2 + \delta y^2$$

$$\therefore \delta s = \delta x \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}$$

$$\therefore L = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$