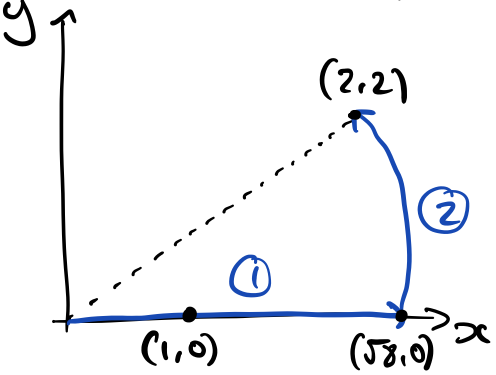
Example 4: plane polar coordinates, same end points, different path



1: $\phi = 0, \quad 1 \leq \rho \leq \sqrt{8}$

2: $\rho = 0, \quad \underbrace{0 \leq \phi \leq \pi/4}_{\text{limits}}$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi$$

$$dx = \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \phi} d\phi$$

$$\underline{F} \cdot d\underline{r} = 2xy\,dx + x^2 dy$$

Working not shown...

$$= 3\cos^2\phi \sin\phi \, \rho^2 d\rho + \cos\phi(1 - 3\sin^2\phi)\rho^3 d\phi$$

$$\int_{1} \underline{F} \cdot d\underline{r} = \int_1^{\sqrt{8}} \cancel{3\cos^2\phi \sin\phi \, \rho^2 d\rho} \xrightarrow{} \sin\phi = 0 \; + \int \cancel{\cos\phi(1-3\sin^2\phi)\rho^3 d\phi} \xrightarrow{} d\phi = 0$$

$$\int_{2} \underline{F} \cdot d\underline{r} = \int \cancel{3\cos^2\phi \sin\phi \, \rho^2 d\rho} \xrightarrow{} d\rho = 0 \; + \int_{\phi=0}^{\pi/4} \cos\phi(1-3\sin^2\phi)\rho^3 d\phi$$

$$\int \underline{F} \cdot d\underline{r} = \rho^3 \int_{\phi=0}^{\pi/4} \cos\phi - 3\sin^2\phi\cos\phi = 8\sqrt{8}\left[\sin\phi - \sin^3\phi\right]_0^{\pi/4}$$

$$= 8\sqrt{8}\left(\frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right)^3\right) = 8\sqrt{8}\left(\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)$$

$$= 8\sqrt{8}\left(\frac{2}{\sqrt{8}} - \frac{1}{\sqrt{8}}\right) = 8\sqrt{8} \times \frac{1}{\sqrt{8}} = 8$$

Independant of Path!

Line integrals, $\int \underline{F} \cdot d\underline{r}$, between the same end-points are independent of path.

The Big Idea:

If $\underline{F} \cdot d\underline{r}$ is an exact differential (check using partial derivates), then there will be a parent function $\Omega(x,y)$. Eg.

$$\int_C \underline{F} \cdot d\underline{r} = \int_C 2xy\,dx + \int_C x^2 dy = \int_C d\Omega(x,y)$$

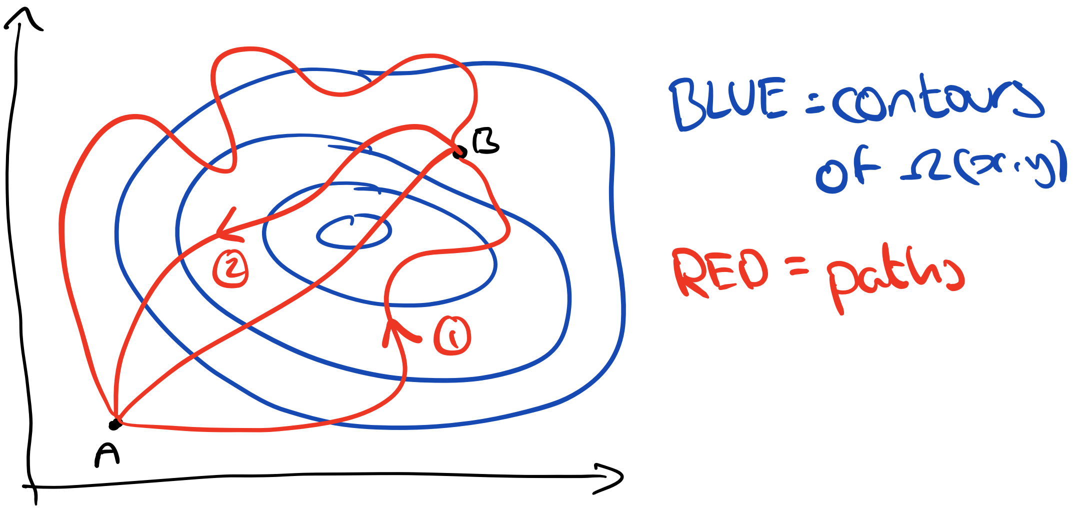
The path does not matter, only the end points matter.

$$d\Omega(x,y) = \frac{\partial \Omega}{\partial x} dx + \frac{\partial \Omega}{\partial y} dy$$

$$\therefore \Omega(x,y) = x^2 y + C$$

$$\int_C d\Omega(x,y) = \Omega_A(x,y) - \Omega_B(x,y)$$

The line integral calculates the 'change in height' of the function $\Omega(x,y)$ between the points A & B.



$$\int_{①} \underline{F} \cdot d\underline{r} + \int_{②} \underline{F} \cdot d\underline{r} = 0 \qquad \oint_c \underline{F} \cdot d\underline{r} = 0$$

closed path/loop

The integral of a closed path of loop of an exact differential is zero. We call $\underline{F}$ conservative, no work is done around any loop.

<u>Equivalent Statements:</u>

- $\underline{F} \cdot d\underline{r}$ is an exact differential
- $\underline{F} = \frac{\partial \Omega}{\partial x}\hat{i} + \frac{\partial \Omega}{\partial y}\hat{j} + \frac{\partial \Omega}{\partial z}\hat{k}$
- $\int_c \underline{F} \cdot d\underline{r}$ from A to B does not depend on path
- $\oint_c \underline{F} \cdot d\underline{r} = 0$
- $\underline{F}$ is conservative