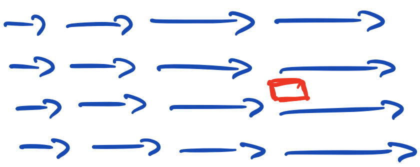
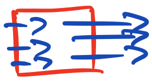
Helmholtz Theorem: (subject to appropiate boundy conditions)
if you know $\nabla \cdot \underline{B}$ & $\nabla \times \underline{B}$ everywhere, you know $\underline{B}$ everywhere.

<u>div</u>

$$\nabla \cdot \underline{B} = \lim_{V \to 0} \left(\frac{1}{V} \oiint_S \underline{B} \cdot d\underline{S} \right)$$

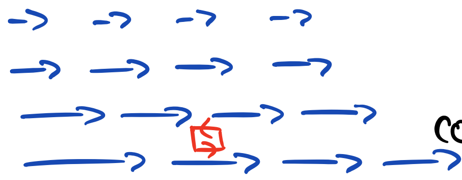




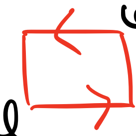
more flux flowing out then in ∴ +ve.

<u>curl</u>

$$\nabla \times \underline{B} \cdot \hat{n} = \lim_{A \to 0} \left(\frac{1}{A} \oint \underline{B} \cdot d\underline{s} \right)$$



represents rotation of a vector field



greater contribution on bottom leg than top leg.

Divergence

div $\underline{B}$ or $\nabla \cdot \underline{B}$

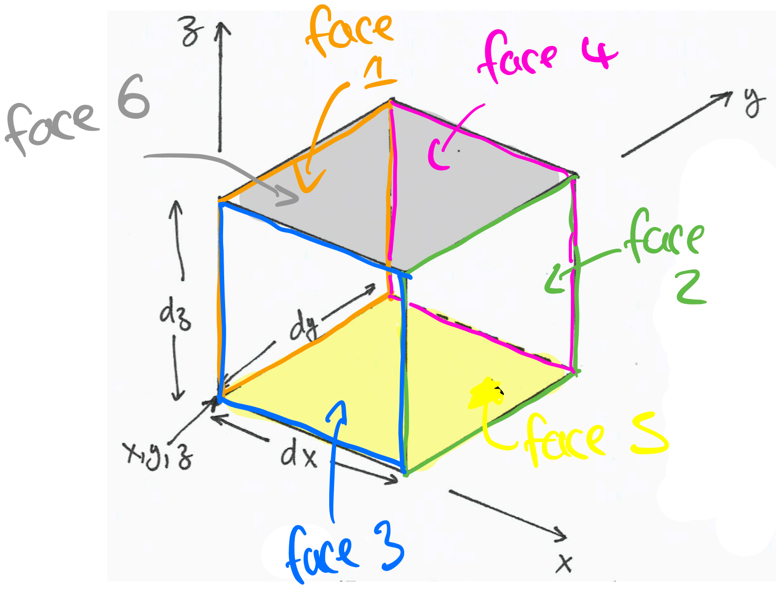
$$\nabla \cdot \underline{B} = \lim_{V \to 0} \left(\frac{1}{V} \oiint \underline{B} \cdot d\underline{s} \right)$$

- <u>closed</u> surface, $d\underline{s}$ is <u>outwards facing</u>
- ∴ divergence is 'flux density'

Lets derive geometrically:

$$\underline{B} = B_x \hat{\imath} + B_y \hat{\jmath} + B_z \hat{k}$$

Face	$d\mathbf{s}$
① $\Delta x = 0$	$-\hat{i}\,dy\,dz$
② $\Delta x = dx$	$\hat{i}\,dy\,dz$
③ $\Delta y = 0$	$-\hat{j}\,dx\,dz$
④ $\Delta y = dy$	$\hat{j}\,dx\,dz$
⑤ $\Delta z = 0$	$-\hat{k}\,dx\,dy$
⑥ $\Delta z = dz$	$\hat{k}\,dx\,dy$



We want $\sum_{i=1}^{6} F_i$ where F_i is the flux out of face i. We will compute F_i using the tangent plane approximation to $\underline{B}$ over each face. Then F_i becomes accurate as $dx, dy, dz \to 0$.

B_x varies with x & y. this is the tangent plane approximation.

Face 1: $B_x + \frac{\partial B_x}{\partial x} dx + \frac{\partial B_x}{\partial y} dy$

As its planar, the average value is just the value at the centre.

avg.

$$F_1 = -\left(B_x + \frac{\partial B_x}{\partial x}\frac{dx}{2} + \frac{\partial B_x}{\partial y}\frac{dy}{2}\right) dy\,dz = -\bar{B}_{x_1} dy\,dz$$

$$F_2 = \left(\bar{B}_{x_1} + \frac{\partial \bar{B}}{\partial x} dx\right) dy\,dz$$

$$F_1 + F_2 = \frac{\partial}{\partial x}\left[B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2} + \frac{\partial B_x}{\partial z}\frac{dz}{2}\right] dx\,dy\,dz$$

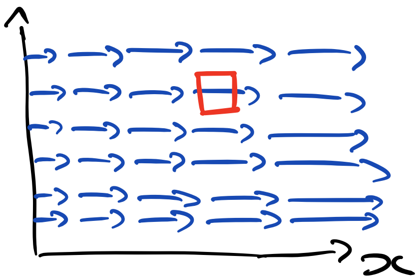
There are analogous terms for $F_3 + F_4$ & $F_5 + F_6$.

Now $\nabla \cdot \underline{B} = \lim_{dx,dy,dz \to 0} \sum_{i=1}^{6} \frac{F_i}{dx\,dy\,dz}$

$$\nabla \cdot B = \lim_{dx,dy,dz \to 0} \Big[\frac{\partial}{\partial x}\Big(B_x + \cancel{\frac{\partial B_x}{\partial y}\frac{dy}{2}} + \cancel{\frac{\partial B_x}{\partial z}\frac{dz}{2}}\Big) + \frac{\partial}{\partial y}\Big(B_y + \cancel{\frac{\partial B_y}{\partial x}\frac{dx}{2}} + \cancel{\frac{\partial B_y}{\partial z}\frac{dz}{2}}\Big) + \frac{\partial}{\partial z}\Big(B_z + \cancel{\frac{\partial B_z}{\partial x}\frac{dx}{2}} + \cancel{\frac{\partial B_z}{\partial y}\frac{dy}{2}}\Big)\Big]$$

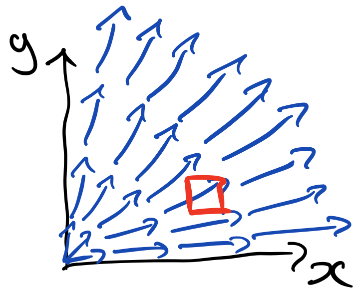
$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

The terms varying across the faces are unimportant, it's only the variation between the faces that matter.



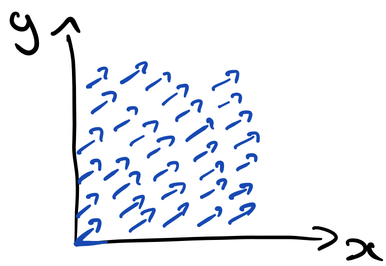
__Example 1__ $\underline{B} = \alpha x \hat{i}$

$$\nabla \cdot \underline{B} = \frac{\partial(\alpha x)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(0)}{\partial z} = \alpha$$



__Example 2__ $\underline{B} = x\hat{i} + y\hat{j} + z\hat{k} = \rho\hat{\rho}$

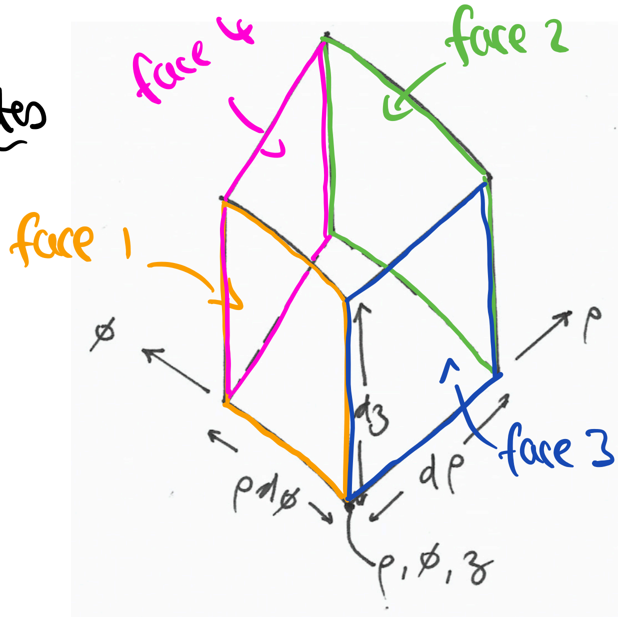
$$\nabla \cdot \underline{B} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$



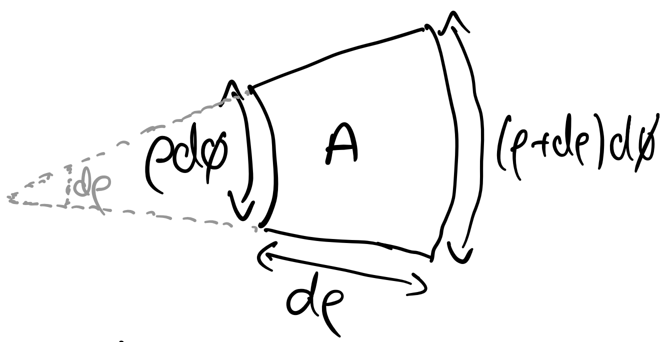
__Example 3__ $\underline{B} = \hat{i} + \hat{j} + \hat{k}$

$$\nabla \cdot \underline{B} = \frac{\partial(1)}{\partial x} + \frac{\partial(1)}{\partial y} + \frac{\partial(1)}{\partial z} = 0$$

Divergence in Cyclindrical Coordinates



Face	dS
① $\Delta\rho = 0$	$-\rho\, d\phi\, dz\, \hat{\rho}$
② $\Delta\rho = d\rho$	$(\rho + d\rho)\, d\phi\, dz\, \hat{\rho}$
③ $\Delta\phi = 0$	$-d\rho\, dz\, \hat{\phi}$
④ $\Delta\phi = d\phi$	$d\rho\, dz\, \hat{\phi}$
⑤ $\Delta z = 0$	$-\left(1 + \frac{d\rho}{2\rho}\right)\rho\, d\phi\, d\rho\, \hat{z}$
⑥ $\Delta z = dz$	$\left(1 + \frac{d\rho}{2\rho}\right)\rho\, d\phi\, d\rho\, \hat{z}$



$$A = \frac{1}{2}(\rho + d\rho)^2 d\phi - \frac{1}{2}\rho^2 d\phi$$

$$= \frac{[2\rho d\rho + d\rho^2]\, d\phi}{2}$$

$$= \rho\, d\phi\, d\rho \left(1 + \frac{d\rho}{2\rho}\right)$$

From before, we can ignore the variations over the face.

$$F_1 = -B_\rho\, \rho\, d\phi\, dz$$

$$F_2 = \left(B_\rho + \frac{\partial B_\rho}{\partial \rho} d\rho\right)(\rho + d\rho)\, d\phi\, dz$$

$$F_1 + F_2 = \left[\frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{\partial B_\rho}{\partial \rho}\frac{d\rho}{\rho}\right] \rho\, d\rho\, d\phi\, dz$$

$$F_3 = -B_\phi\, d\rho\, dz \qquad F_4 = \left[B_\phi + \frac{\partial B_\phi}{\partial \phi} d\phi\right] d\rho\, dz$$

$$F_3 + F_4 = \frac{\partial B_\phi}{\partial \phi}\, d\rho\, d\phi\, dz$$

$$F_5 = -B_z \left[1 + \frac{d\rho}{2\rho}\right] \rho\, d\phi\, dz$$

$$F_6 = \left[B_z + \frac{\partial B_z}{\partial z} dz\right] \rho\, d\phi\, d\rho \left[1 + \frac{d\rho}{2\rho}\right]$$

$$F_5 + F_6 = \frac{\partial B_z}{\partial z}\left(1 + \frac{\partial \rho}{2\rho}\right)\rho\, d\rho\, d\phi\, dz$$

$$\nabla \cdot \underline{B} = \lim_{dx,dy,dz \to 0} \frac{1}{V}\sum_{i=1}^{6} F_i = \frac{\sum_{i=1}^{6} F_i}{\rho\, d\rho\, d\phi\, dz} = \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

$$\nabla \cdot \underline{B} = \frac{1}{\rho}\frac{\partial(\rho B_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

Without working, in spherical polar:

$$\nabla \cdot \underline{B} = \frac{1}{\rho^2}\frac{\partial(\rho^2 B_\rho)}{\partial \rho} + \frac{1}{\rho\sin\theta}\frac{\partial B_\phi}{\partial \phi} + \frac{1}{\rho\sin\theta}\frac{\partial(\sin\theta\, B_\theta)}{\partial \theta}$$