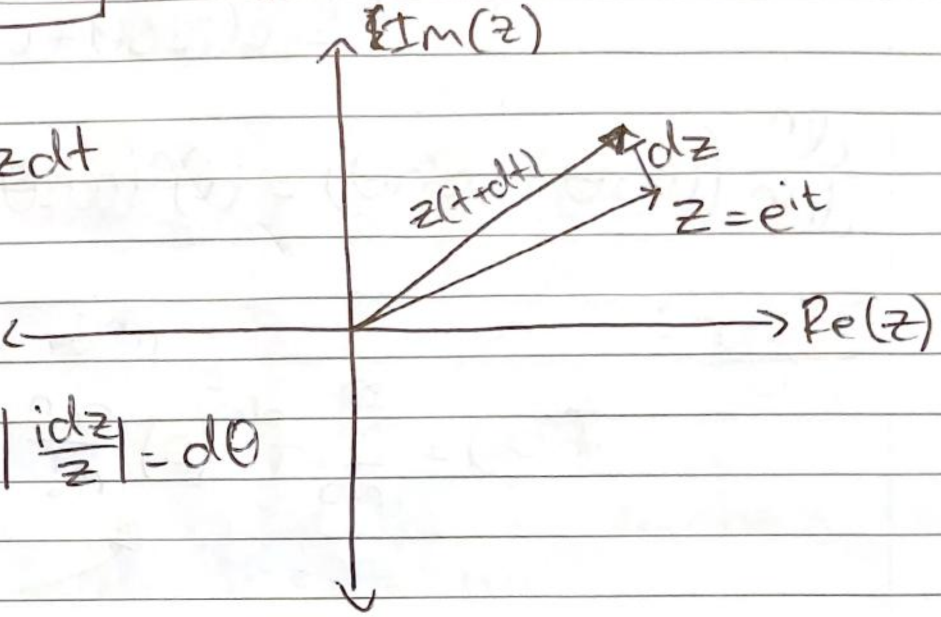
Lecture 3

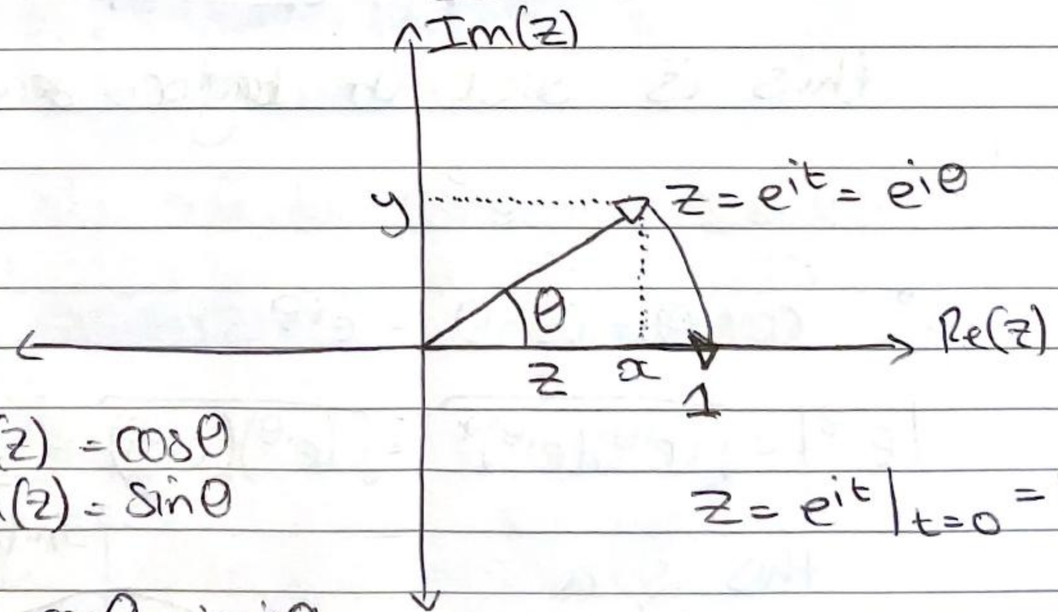
$\boxed{e^{it} = ?}$ Last lecture

$dz = iz\,dt$



$|dt| = \left|\frac{i\,dz}{z}\right| = d\theta$

Conjecture: the same kind of rotation in complex plain occurs also for a finite t or θ.



$Re(z) = \cos\theta$
$Im(z) = \sin\theta$

$z = e^{it}\big|_{t=0} = 1$

$e^{i\theta} = \cos\theta + i\sin\theta$

Euler's Formula

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta$$
$$= i(\cos\theta + i\sin\theta)$$

$$\frac{d^n}{d\theta^n}(\cos\theta + i\sin\theta) = (i)^n(\cos\theta + i\sin\theta)$$

Taylor:

nth derivative of f

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

$$\left.\frac{d^n f}{dx^n}\right|_{x=0}$$

this is what we really should write

$$f(\theta) = \cos\theta + i\sin\theta$$

$$f(\theta) = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots$$

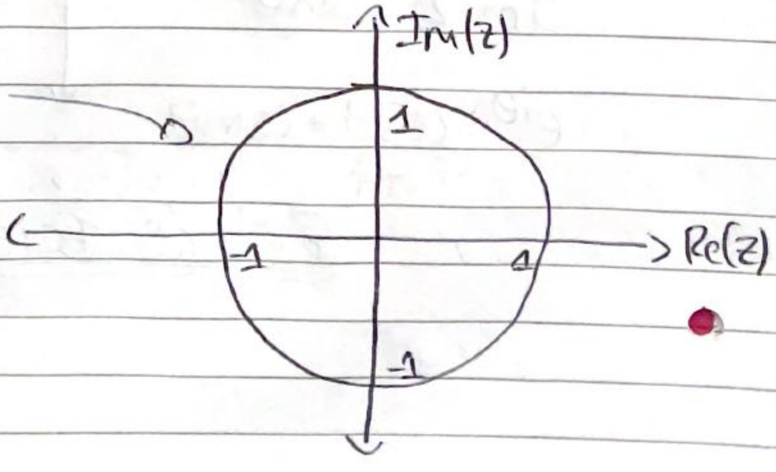
this is just the taylor series of

$e^{i\theta}$.

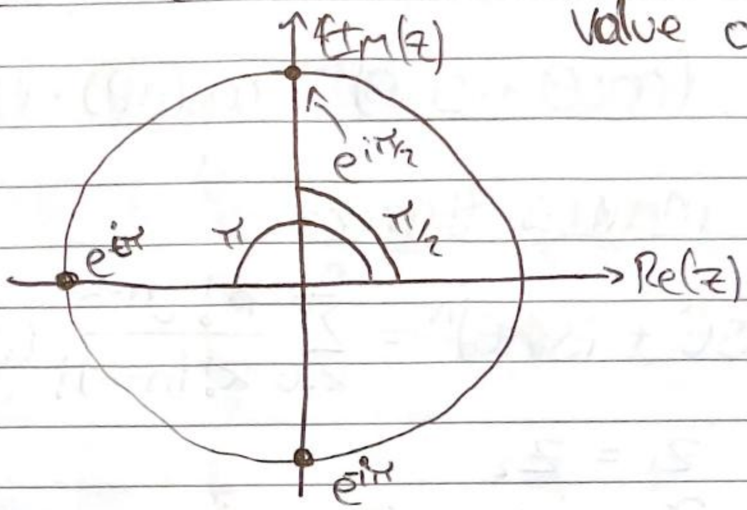
$$\therefore \cos\theta + i\sin\theta = e^{i\theta}$$

$$|e^{i\theta}| = \sqrt{(e^{i\theta})(e^{i\theta})^*} = \sqrt{(e^{i\theta})(e^{-i\theta})} = \underline{\underline{1}}$$

this is a unite circle.



eg $e^{i\frac{\pi}{2}}$ $e^{i\pi}$ $e^{-i\pi/2}$ ← all have absolute value of 1.



we can generalise this to numbers which don't lie on the unit circle.

$$z = |z|(\cos\theta + i\sin\theta)$$
$$= |z|e^{i\theta}$$

$$z_1 \cdot z_2 = |z_1|e^{i\theta_1} \cdot |z_2|e^{i\theta_2}$$
$$= |z_1||z_2|e^{i(\theta_1+\theta_2)}$$

we also note for complex conjugates:

$$z^* = |z|e^{-i\theta} \qquad (i \to -i)$$

Applications

1) $\cos(n\theta) = ?$ $\quad \sin(n\theta) = ?$

we can often do this for $n=2$, but how can we generalise this to any n?

$$(e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n \rightarrow$ binomial exp.

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

de moivre's theorum

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} \frac{n!\, i^{n-k}}{k!\,(n-k)!} \cos^k\theta + i\sin^{n-k}\theta$$

if: $z_1 = z_2$

$x_1 + iy_1 = x_2 + iy_2$

then: $x_1 = x_2 \quad y_1 = y_2$

Example $n = 5$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + i5\cos^4\theta\sin\theta -$
$10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta$
$+ i\sin^5\theta$

$= \cancel{\sin(5\theta)} + \cos(5\theta) + i\sin(5\theta)$

Real Part

$\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10$
$\cos^3\theta + 5\cos^5\theta$
$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Complex part

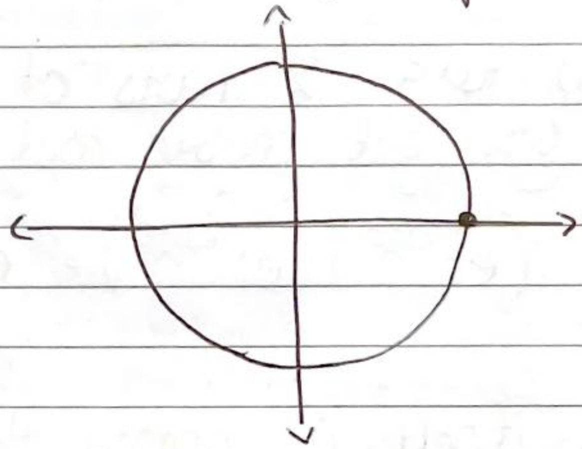
$i\sin(5\theta) = i5\cos^4\theta\sin\theta - i10\cos^2\theta\sin^3\theta + i\sin^5\theta$
$= i5\sin\theta - i10\sin^3\theta + i5\sin^5\theta - i10\sin^3\theta +$
$i10\sin^5\theta + i\sin^5\theta$
$= i16\sin^5\theta - i20\sin^3\theta + i5\sin\theta$

$\sin(5\theta) = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Roots of unity

$\sqrt[2]{1} = \pm 1$ $\quad$ $\sqrt[3]{1} = 1, \ldots$ ← only 1 root?

lets look at the complex plane...



$n=2$ is square root
$n=3$ is cubic root

$e^{i0}, e^{i2\pi}, e^{i4\pi}, \ldots$ all give solutions to $e^{i\theta} = 1$,
$\therefore$ we say that

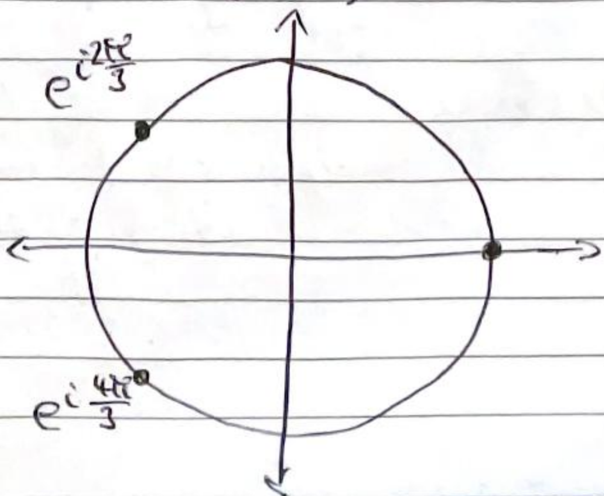
$$1 = e^{i2\pi k} \qquad k \in \mathbb{N}$$

← ~~integers~~ natural numbers

$$\left(e^{i2\pi k}\right)^{\frac{1}{n}} = e^{i\frac{2\pi k}{n}}$$

when we get $k=n$, we get the same answer as $k=0$, hence the answers repeat.

For example when were looking for the cubic root of 1 ($n=3$).



$\sqrt[n]{1}$ are the n roots:

$$e^{i\frac{2\pi k}{n}} \qquad k = 0, 1, 2 \ldots, n-1$$

Some interesting properties:

if you take 2 roots of unity and multiply them, you get another root of unity.

$$\left(e^{i\frac{2\pi k}{n}}\right)\left(e^{i\frac{2\pi k'}{n}}\right) = \underbrace{e^{i\frac{2\pi}{n}(k+k')}}_{\text{a root of unity}}$$

Unity itself is among the roots. $1 e^{i\frac{2\pi}{n} \times 0}$

if $e^{i\frac{2\pi k}{n}}$ is a root of unity, then its complex conjugate is also a root of unity.

$e^{-i\frac{2\pi k}{n}}$ is also a root of unity.

Proof if we take the n^{th} power of the root it should equal 1.

$$\left(e^{-i\frac{2\pi}{n}k}\right)^n = e^{-i2\pi k} = 1 \quad ✓$$

roots come in conjugate pairs. (group theory)