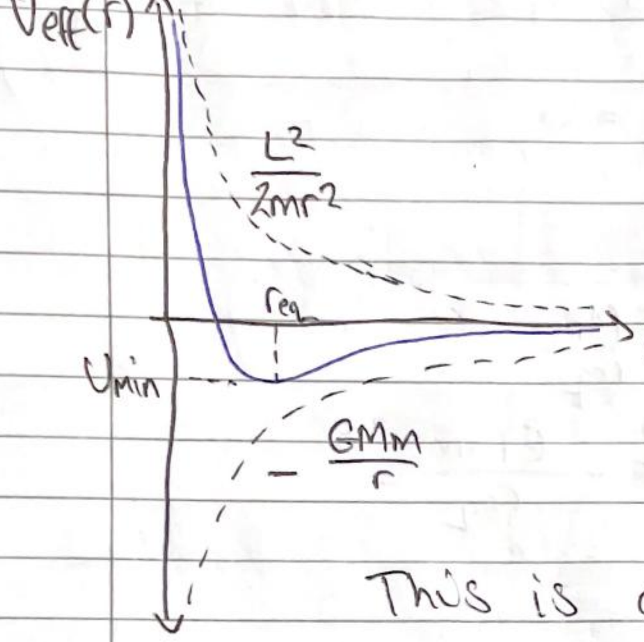
Classical Mechanics 16

Circular, Elliptical & Hyperbolic Orbits



Circular Orbits

For a given L, the smallest possible value of $E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$ occurs when

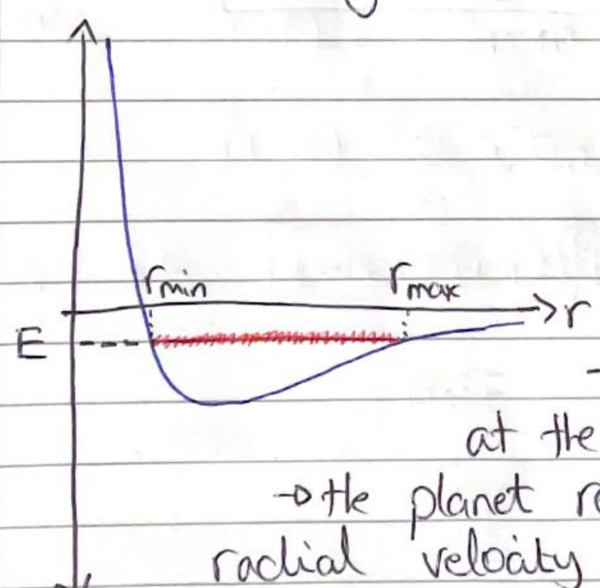
(1) $\dot{r} = 0$

(2) $r = r_{eq}$

and hence $E = U_{min}$

This is a circular orbit of radius r_{eq}.

A circular orbit is the lowest (most negative) energy orbit of angular momentum L.



Ellipical Orbits

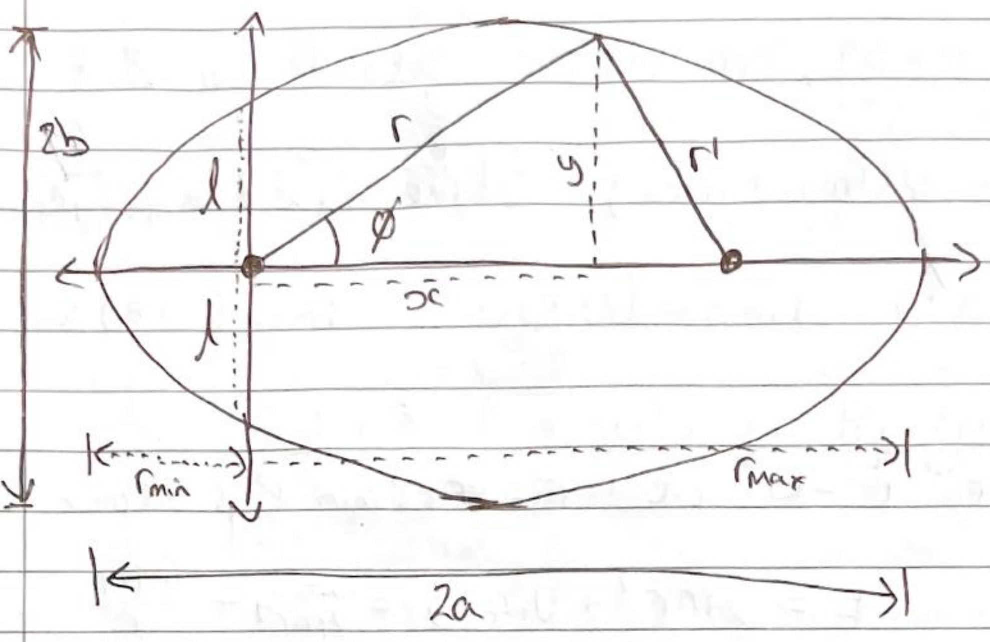
A planet with $U_{min} < E < 0$ moves between r_{min} and r_{max} as it orbits.

→ $E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$, so $\dot{r} = 0$ at the points r_{min} & r_{max}.

→ the planet reverses direction of its radial velocity at this point.

→ the orbit is an ellipse.

Basic Facts About Ellipses



Polar Equation of Ellipse

$$r = \frac{l}{1 - e\cos\phi}$$

e = 'eccentricity' is const. $(0 \leq e \leq 1)$ $\equiv e \neq 1$!!
l = 'semi-latus rectum'
a = 'semi-major axis'
b = 'semi-minor axis'
r_{min} = 'perihelion' (closest point to sun)
= 'perigee' (closest point to Earth)
= 'periapsis' (general)
r_{max} = 'aphelion' (farthest from sun)
= 'apogee' (farthest from Earth)
= 'apoapsis' (general)

$$l = r(\phi = \pi/2)$$
$$r_{max} = r(\phi = 0) = \frac{l}{1-e}$$
$$r_{min} = r(\phi = \pi) = l/_{1+e}$$

if $e = 0$, the ellipse becomes a circle.

As $e \to 1$, the ellipse gets longer & thinner.

$$a = \frac{1}{2}(r_{min} + r_{max}) = \frac{1}{2}\left(\frac{l}{1+e} + \frac{l}{1-e}\right) = \frac{l}{(1+e)(1-e)}$$

Hence, $r_{min} = (1-e)a$ $\quad r_{max}(1+e)a$

<u>Energy of an Ellipital Orbit</u>

Since $\dot{r} = 0$ at $r = r_{min}$ & $r = r_{max}$.

$$E = \cancel{\frac{1}{2}m\dot{r}^2} + V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

Hence,

$$E = \frac{L^2}{2m(1+e)^2a^2} - \frac{GMm}{(1+e)a}$$

$$E = \frac{L^2}{2m(1-e)^2a^2} - \frac{GMm}{(1-e)a}$$

then, $$(1+e)^2E = \frac{L^2}{2ma^2} - \frac{GMm(1+e)}{a}$$

$$(1-e)^2E = \frac{L^2}{2ma^2} - \frac{(1-e)GMm}{a}$$

Subtract $$\left[(1+e)^2-(1-e^2)\right]E = \left[(1+e)-(1-e)\right]\left(-\frac{GMm}{a}\right)$$

$$4eE = -\frac{2eGMm}{a}$$

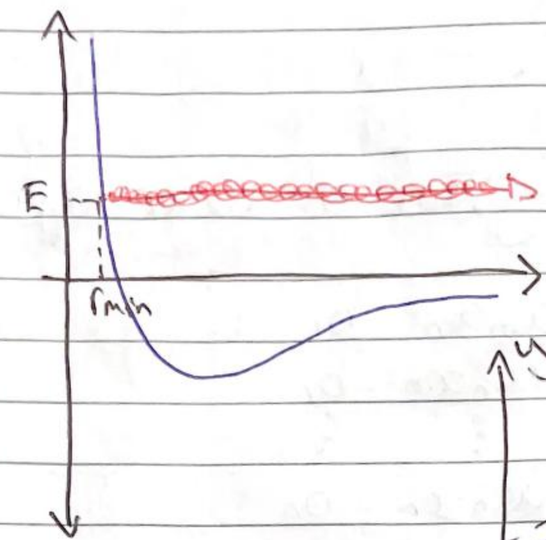
$$\boxed{E = -\frac{GMm}{2a}}$$ energy of an ellipical orbit.

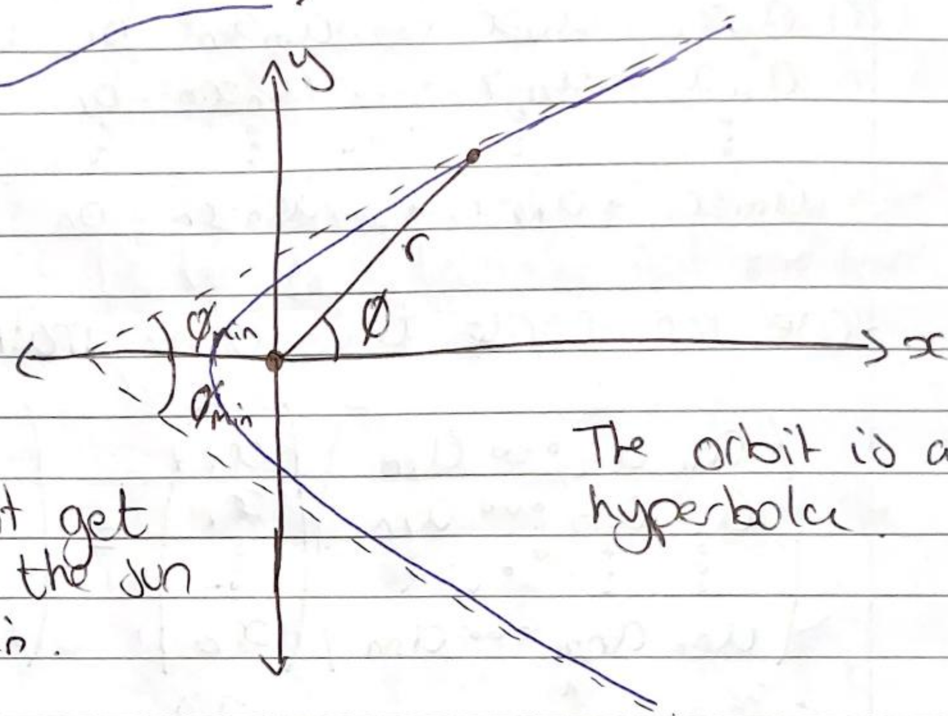
from the same starting point:

$$a = \frac{L^2}{GMm^2(1-e^2)} \qquad e = \sqrt{1 + \frac{2EL^2}{Gm^3M^2}}$$

Hyperbolic Orbits

if $E > 0$, the planet cannot be in a closed orbit.





The orbit is a hyperbola.

It cannot get closer to the sun than r_{min}.

For a hyperbola, the polar equation is still

$$r = \frac{\ell}{1 - e\cos\phi}$$

but now $e > 1$. Since r cannot be negative, require $\cos\phi < 1/e$. in the range $-\pi \leq \phi < \pi$:

$$|\phi| > \phi_{min} = \arccos\left(\frac{1}{e}\right)$$

as $|\phi|$ reduces towards ϕ_{min}, the distance from the origin $r \to \infty$.

N.B. hitting the sun from far-away is very difficult. Most likely hyperbolic trajectory & miss.