Orthogonal & Orthonormal Function

The vector $\underline{a}$ & $\underline{b}$ are orthogonal if $\underline{a} \cdot \underline{b} = \sum_i a_i b_i = 0$. Called 'dot product', 'scaler product' or 'inner product'.

We can extend this to a set of vectors.

$$\{\underline{V_n}\} \rightarrow \underline{V_n} \cdot \underline{V_m} = 0 \quad \text{if } n \neq m$$

a complete orthogonal set is when the number of vectors equals the dimensions (span) of the space.

Orthonormal: $\quad V_n \cdot V_m = \delta_{nm} = \begin{cases} 1 & \text{if } n \neq m \\ 0 & \text{if } n = m \end{cases}$ KRONECKER DELTA

Given a complete orthonormal set of vectors, we can write an arbitrary vector as

$$\underline{A} = \sum_{n=1}^{N} a_n \underline{V_n} = a_1 \underline{V_1} + a_2 \underline{V_2} + a_3 \underline{V_3} + \dots$$

where N is the dimension of the space spanned by $\underline{V_n}$. We can also decompose any vector:

$$a_n = \underline{A} \cdot \underline{V_n}$$

We call the set $\{\underline{V_n}\}$ an 'orthonormal set of basis vectors'.

We can extend this idea to complex vectors:

$$\underline{V_n} \cdot \underline{V_m}^* = \delta_{nm}$$

We can also extend this concept to functions:

$$\langle f, g \rangle = \int_a^b f(x) g^*(x) dx \qquad f(x), g(x) \in \mathbb{C}$$

We call these an orthonormal set of functions if

$$\langle f_n, f_m \rangle = \delta_{nm}$$

$$\int_a^b f_n(x) f_m^*(x) \, dx = \delta_{nm}$$

If this set is complete, we now how an orthonormal basis set. We can then expand the arbitrary function f(x) on a closed interval as:

$$f(x) = \sum_{n \in \mathbb{Z}} a_n g_n(x)$$

$$a_n = \langle f, g_n \rangle = \int_a^b f(x) g_n^*(x) dx$$

Exercise 2.1 $\quad f(x) = \frac{1}{\sqrt{2\pi}} e^{in\theta} \quad n \in \mathbb{Z} \quad (-\pi, \pi)$

$$\langle f_n, f_m \rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{in\theta} \frac{1}{\sqrt{2\pi}} e^{-im\theta} \, d\theta$$

$$n=m \left\{ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{0} d\theta = \frac{1}{2\pi} \left[\theta \right]_{-\pi}^{\pi} = \frac{\pi}{2\pi} + \frac{\pi}{2\pi} = 1 \right.$$

$$n \neq m \left\{ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(n-m)} d\theta = \frac{i}{2\pi} \left[e^{n-m} e^{i\theta} \right]_{-\pi}^{\pi} = \right.$$

$$= -\frac{i}{2\pi} e^{n-m} + \frac{i}{2\pi} e^{n-m} = 0$$

$$\therefore \langle f_n, f_m \rangle = \delta_{nm}$$

In general, we define an inner product as a mapping which takes two vectors to a scaler. We can think of functions as a generalisation of vectors.

Useful Properties:

- ☐ conjugate symmetry $\langle x, y \rangle = - \langle y, x \rangle$
- ☐ linearity $\langle ax, y \rangle = a \langle x, y \rangle$
- ☐ additivity $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- ☐ positive-definiteness $\langle x, x \rangle \geq 0$
- ☐ $\langle x, x \rangle = 0 \iff x = 0$

Dirac Delta Function

$\delta(x)$ real function

Defined by the following critera:

$$\delta(x) = 0 \qquad \text{for} \quad x \neq 0$$

$$\int_a^b f(x)\,\delta(x)\,dx = f(0)$$

any arbitrary function

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

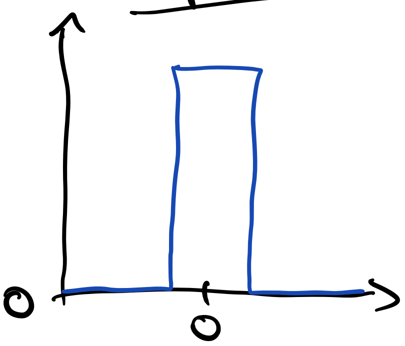
The delta function 'lives' inside integrals.

$$f(x) = \int_{-\infty}^{+\infty} f(t)\,\delta(t-x)\,dt \qquad \text{Sifting property}$$

There are many limiting sequences for the delta function.

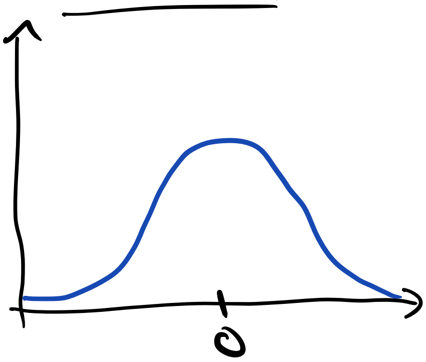
$$\delta(x) = \lim_{n \to \infty} \delta_n(x)$$

Top-Hat



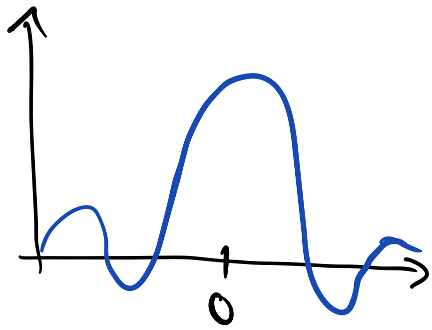
$$\delta_n(x) = \begin{cases} n & \text{if } |x| \leq \frac{1}{2n} \\ 0 & \text{if } |x| > \frac{1}{2n} \end{cases}$$

Gaussian



$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

Exponential



$$\delta_n(x) = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt}\,dt$$

These approximate the delta function as $n \to \infty$.

complex exponential form:

$$\delta_n(x) = \frac{1}{2\pi}\int_{-n}^{+n} e^{ixt}\,dt = \frac{\sin(nx)}{nx} = \operatorname{sinc}(nx)$$

if we substitute x for $t-x$,

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}\,d\omega$$