Lagrangian Mechanics

The 'Liveliness' of a System

Let's invent a new value. We'll call it the liveliness of a particle which we will describe as follows.

$$Liveliness = Kenetic\ Energy - Potential\ Energy$$

- A particle with a high 'liveliness' will have a high velocity, but it cannot go much faster.
- A particle with low 'liveliness' will have a high velocity, but it can go faster.
- A particle with negative 'liveliness' will have a low velocity, but it can go faster.

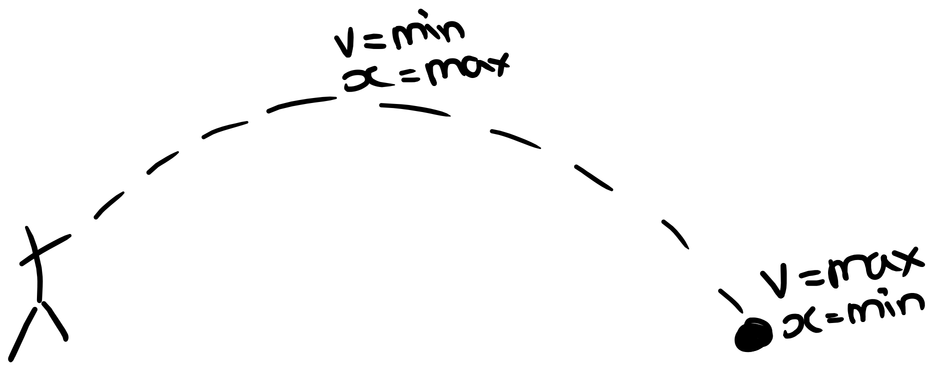
The 'liveliness' of a system give us a rough understanding of the current velocity and potential future velocity.

$$L(x,v) = T(v) - V(x) \tag{1.1}$$

Where *L(x,v)* is the 'liveliness', *T(v)* is the kinetic energy and *V(x)* is the potential energy.

Oh, FYI 'liveliness' = lagrangian.

Though Experiment 1 – Throwing a Tennis Ball



The lagrangian of a system will be equal to:

$$L(x,v) = \frac{1}{2}mv^2 - mgx \tag{1.2}$$

As soon as you throw the ball, the velocity will be at its max and the displacement (height) will be at its min.

The Action of a System

Continuing with the tennis ball thought experiment, we can now sum all of the value of the lagrangian between two points in time.

$$A = \int_{t_0}^{t_1} L\, dt \tag{1.3}$$

This means the action of the tennis ball is equal to:

$$Action = \int_{t_0}^{t_1} (\frac{1}{2}mv^2 - mgx)\, dt$$

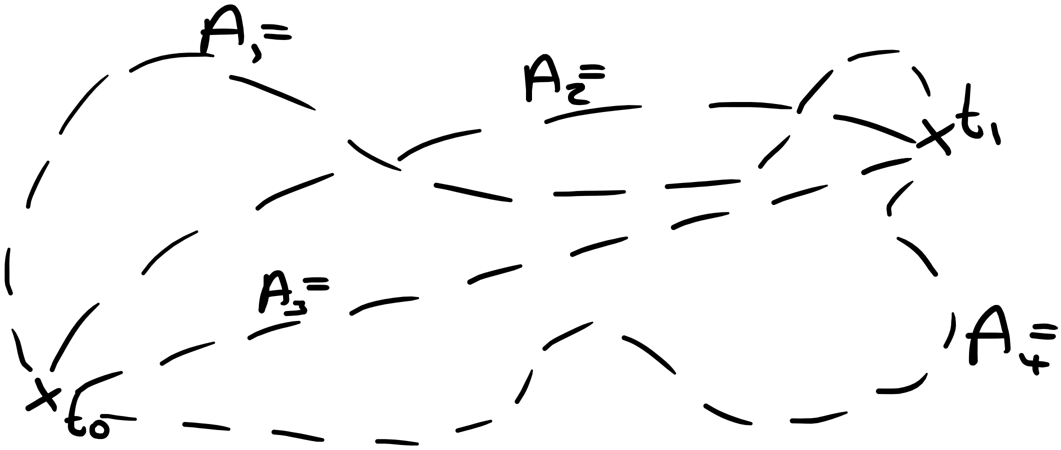
The Principle of Least Action / The Universal Law

The universal law of nature is very simple. It's:

Nature is lazy

The big issue in physics is figure how to write this down mathematically.

In classical mechanics we say that the ball will take the path with the lowest action.

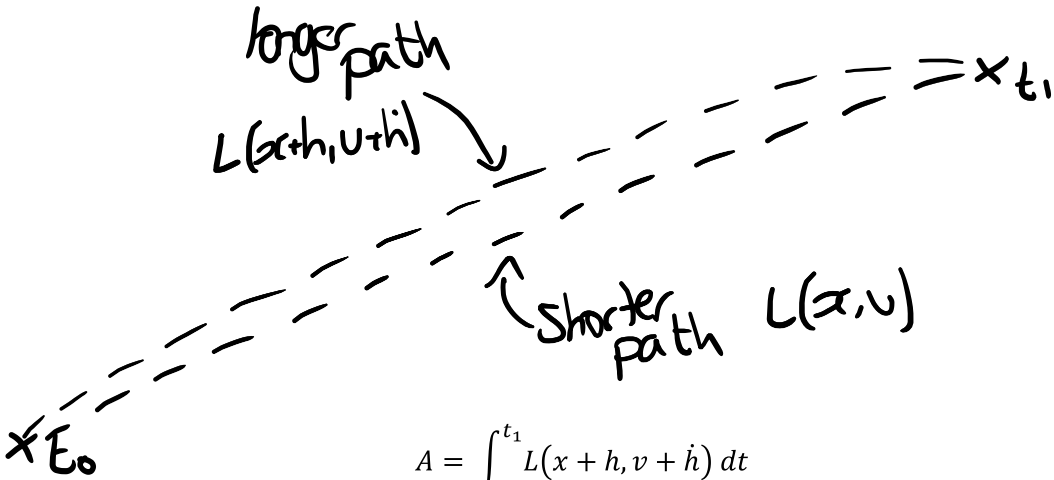


If we calculated the action of the 4 paths, we could find the route the particle took by finding the path which has the lowest action.

Deriving the Euler-Lagrange Equation

There are theoretically an infinite number of possible routes a particle could take between two points. So we need to find a way to calculate the lagrange with gives the lowest action.

Lets image we compare two paths. To get the second path we're going to add a small number(s) – h & $\dot{h}$ – the position and velocity.



$$A = \int_{t_0}^{t_1} L\left(x + h, v + \dot{h}\right) dt \tag{1.4}$$

Now we can use the taylor expansion.

$$A = \int_{t_0}^{t_1} L(x, v) + \frac{h}{1!}\frac{\partial L}{\partial x} + \frac{\dot{h}}{1!}\frac{\partial L}{\partial v} + \frac{h^2}{2!}\frac{\partial^2 L}{\partial x^2} + \frac{\dot{h}^2}{2!}\frac{\partial^2 L}{\partial v^2} \tag{1.5}$$

This expansion can continue to infinity, but this isn't important for us.

All we care about is the first section of the expansion. When we are at the path which has the lowest action, the first section will be equal to zero.

$$0 = \int_{t_0}^{t_1} h\frac{\partial L}{\partial x} + \dot{h}\frac{\partial L}{\partial v} \tag{1.6}$$

$$0 = \int_{t_0}^{t_1} h\frac{\partial L}{\partial x} + \frac{dh}{dt}\frac{\partial L}{\partial v}$$

Now we need to use integration by parts. Yay!

$$\int f(x)\frac{dg(x)}{dx} = f(x)g(x) - \int \frac{df(x)}{dx}g(x)$$

Therefore:

$$\int_{t_0}^{t_1} \frac{dh}{dt}\frac{\partial L}{\partial v} = h\frac{\partial L}{\partial v} - \int_{t_0}^{t_1} h\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)$$

As h tends to zero, $h\frac{\partial L}{\partial v}$ will also tend to zero. Therefore, we can conclude:

$$0 = \int_{t_0}^{t_1} h\frac{\partial L}{\partial x} - h\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)$$

$$0 = \int_{t_0}^{t_1} h\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)\right)$$

And as the integral is equal to zero, the following must also be equal to zero.

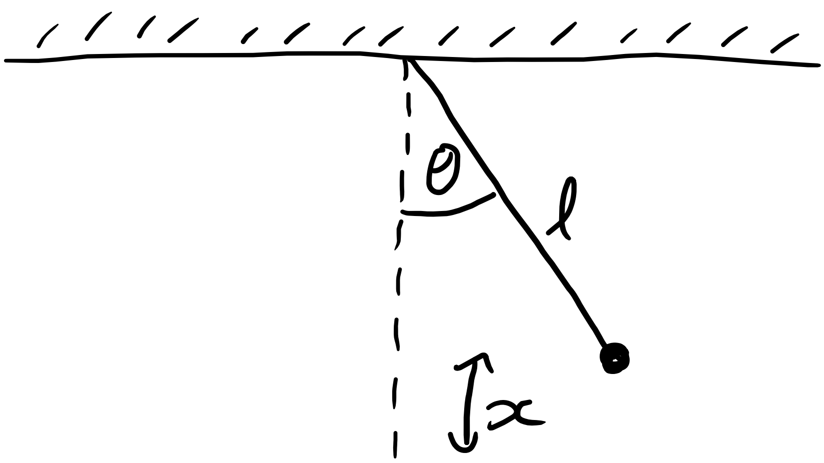
$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) - \frac{\partial L}{\partial x} = 0 \tag{1.7}$$

And this is the Euler-Lagrange equation.

One of the many benefits of the euler-lagrange equation over f=ma is that we can use both polar coordinated and cartesian coordinates.

A Pendulum

Let's us image a simple pendulum with length l and angle to the rest position θ.



Firstly, let's write x in terms of length & θ:

$$x = l - l\cos\theta$$

Now, let's write velocity in terms of angular velocity:

$$v = \omega l$$

If we substitute these into equation (1.2) we get:

$$L = \frac{1}{2}m\omega^2 l^2 - mgl + mgl\cos\theta \tag{1.8}$$

Now if we find the values for the following:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \omega}\right) = m\dot{\omega}l^2$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

We can sub these into the euler-lagrange equation to get:

$$m\dot{\omega}l^2 + mgl\sin\theta = 0 \tag{1.9}$$

We can divide by m & l to get.

$$\dot{\omega}l + g\sin\theta = 0$$

Quick rearrange:

$$\dot{\omega} = -\frac{g\sin\theta}{l} \tag{2.0}$$

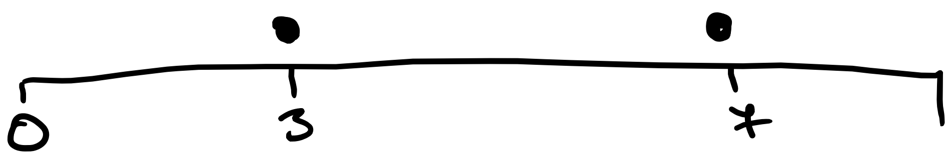
We can also approach this from the Newtonian perspective as well - we get the same equation.

Configuration Space

One of the things we overlooked so far is the difference between physical space and configuration space.

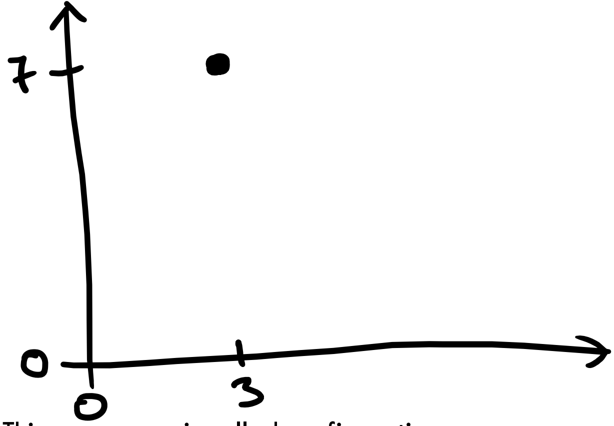
- The world we live in is physical space. Think newtons equations.
- The world in which we do lagrangian mechanics is configuration space.

Imagine we have a ruler which has two particles on.



To describe these particles we use two 1D vectors: (3) & (7).

Or we can use one 2D vector: $\begin{pmatrix}3\\7\end{pmatrix}$



This new space is called configuration space.

E.g. If we had 7 3D particles, we would need 21D configuration space to describe it.

So what is the use of this higher-dimensional space?

- Well remember the basic law of the universe: nature is lazy. We can use this to describe physicists. Physicists are lazy. They would rather solve 1 equation instead of 7.
- Also, we can use polar coordinates in the euler-lagrange equations as well as cartesian. With newton we can only use cartesian.

Force & Potential Energy

Force and potential energy are in fact linked in a very cool way.

$$F(x) = -\frac{dV(x)}{dx}$$