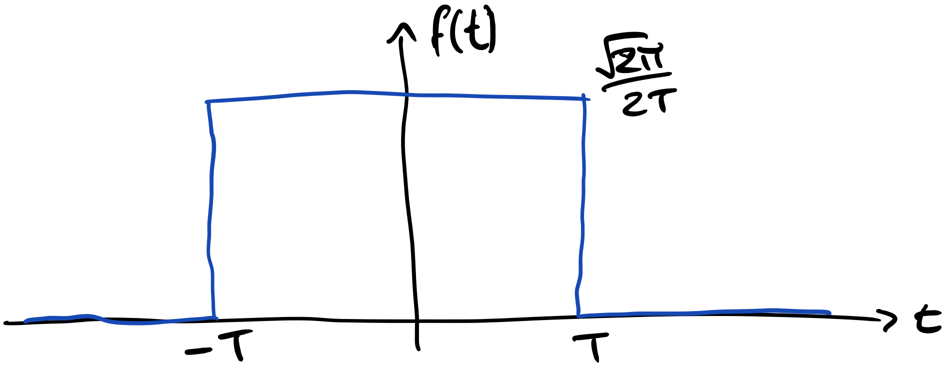
Top-Hat Function

$$f(t) = \begin{cases} \frac{\sqrt{2\pi}}{2T} & \text{if } t \leq |T| \\ 0 & \text{if } t \geq |T| \end{cases}$$



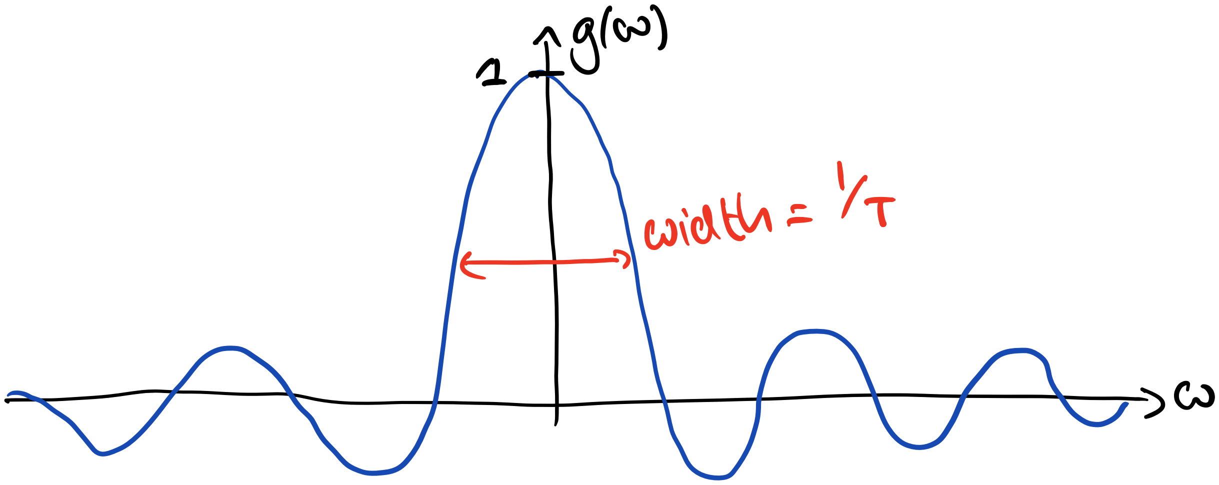
The fourier transform is

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{2T} \int_{-T}^{T} e^{i\omega t} dt$$

→ as $T \to \infty$, this is the limiting eq^n for delta function

$$= \frac{1}{2T} \left[\frac{1}{i\omega} e^{i\omega t} \right]_{-T}^{T} = \frac{1}{2T} \left[\frac{e^{i\omega T} - e^{-i\omega T}}{i\omega} \right] = \frac{\sin(\omega T)}{\omega T}$$

$$= \frac{\operatorname{sinc}(\omega T)}{\omega T}$$



The 'top-hat' and 'exponential' limiting functions of the dirac delta function are related by a fourier transform.

As $T \to \infty$ then $f(t) \to$ const & $g(\omega) \to \delta(\omega)$

$T \to 0$ then $f(t) \to \delta(t)$ & $g(\omega) \to$ const.

Partial Differential Equations

A fourier transform allows us to change a partial diff. equation into an ordinary diff eqn. ODEs are often easier to solve.

Lets look at the heat equation in 1D.

a: heat coefficient

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

u(x,t): temp

let's fourier transform with respect to the x coordinate (spacial)

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{ikx}\, dx$$

fourier transform of u(x,t)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t) e^{-ikx}\, dk$$

now, inserting these into our diff. eqn.

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t) e^{-ikx}\, dx\right) = a^2 \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(k,t) e^{-ikx}\, dk\right)$$

$$\int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a^2 U(k,t) \frac{\partial^2}{\partial x^2}(e^{-ikx})\, dx$$

$$\frac{\partial U}{\partial t} = -a^2 k^2 U(k,t)$$

This diff. eqn has only derivaties with respect to t so we can write it as an ODE.

$$\frac{dU(k,t)}{dt} = -a^2k^2U(k,t)$$

solt. at $t=0$
$C = U(k,t=0)$

$\therefore$ solution has the form $\quad U(k,t) = Ce^{-a^2k^2t}$

$$C = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{ixy}dx$$

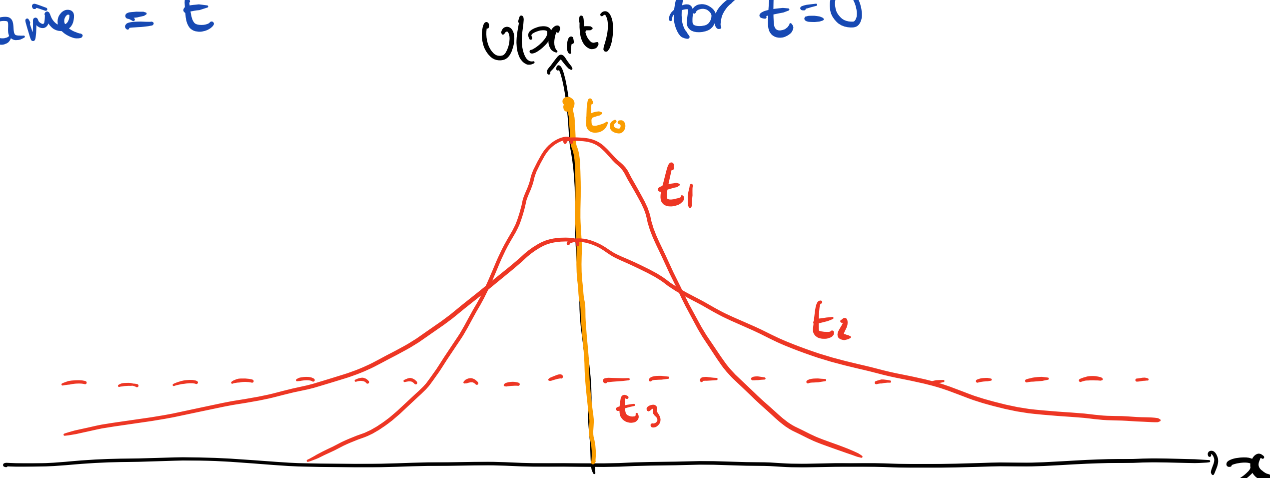
To get our solution in terms of x & t, we now do an inverse fourier transform.

gaussian in k

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Ce^{-a^2k^2t}e^{-ikx}\,dk$$

our solution at time $= t$

our initial solution for $t=0$



at $t=0$, only $x=0$ has $u(x,t) \neq 0$

As t increases, the gaussian $e^{-a^2tk^2}$ has a decreasing width, $\therefore$ $u(x,t)$ shows a gaussian with an increasing width as time goes on.

Inhomogeneous ODEs

$$\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + \omega_0^2 u = f(t)$$

We can use fourier transforms to write derivatives as algebraic factors. Lets assume f(t) has the form

$f(t) \in \mathbb{R} \rightarrow$

$$f(t) = 2|A|\cos(\Omega t + \phi)$$

angular freq.

phase diff.

amplitude

$$f(t) = Ae^{i\Omega t} + A^* e^{-i\Omega t} \qquad (A = |A|e^{i\phi})$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(Ae^{i\Omega t} + A^* e^{-i\Omega t})e^{i\omega t}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Ae^{i(\omega+\Omega)t} + A^* e^{i(\omega-\Omega)t}\,dt$$

This is just the exponential form of the delta function.

$$= \sqrt{2\pi}\left[A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)\right]$$

Now lets find the fourier transform of the LHS (the diff. eq^n).

$$u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}\,d\omega$$

$$\frac{du}{dt} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\,{-i\omega}e^{-i\omega t}\,d\omega = -i\omega u$$

$$F[-i\omega u(t)] = -i\omega g(\omega)$$

$$\frac{d^2u}{dt^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega) - \omega^2 e^{-i\omega t}\, d\omega = -\omega^2 u$$

$$F[-\omega^2 u(t)] = -\omega^2 g(\omega)$$

$$F[LHS] = -\omega^2 g(\omega) - i\omega\gamma g(\omega) + \omega_0^2 g(\omega)$$

$$-\omega^2 g(\omega) - i\omega\gamma g(\omega) + \omega_0^2 g(\omega) = \sqrt{2\pi}\left[A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)\right]$$

$$g(\omega) = \frac{\sqrt{2\pi}\left[A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)\right]}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

This is now solved in the frequency domain. Now we do an inverse fourier transform to get $u(t)$.

$$u(t) = \int_{-\infty}^{\infty}\left[\frac{A\delta(\omega+\Omega) + A^*\delta(\omega-\Omega)}{\omega_0^2 - \omega^2 - i\omega\gamma}\right] e^{-i\omega t}\, d\omega$$

The two delta functions collapse the integral apart from at $\omega = \mp\Omega$.

$$u(t) = \frac{Ae^{i\Omega t}}{\omega_0^2 - \omega^2 + i\omega\gamma} + \frac{A^* e^{-i\Omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

These are complex conjugates of each other. Writing

$$B = |B|e^{i\alpha} = \omega_0^2 - \omega^2 + i\Omega t$$

$$u(t) = \frac{A}{B}e^{i\Omega t} + \frac{A^*}{B^*}e^{-i\Omega t} = \frac{|A|}{|B|}\cos(\Omega t + \phi - \alpha)$$

$$|B| = \sqrt{(\omega_0^2 - \omega^2)^2 + \Omega^2\omega^2}$$ when $\Omega = \omega_0$, $|B| \to 0$

$u(t) \to \infty$