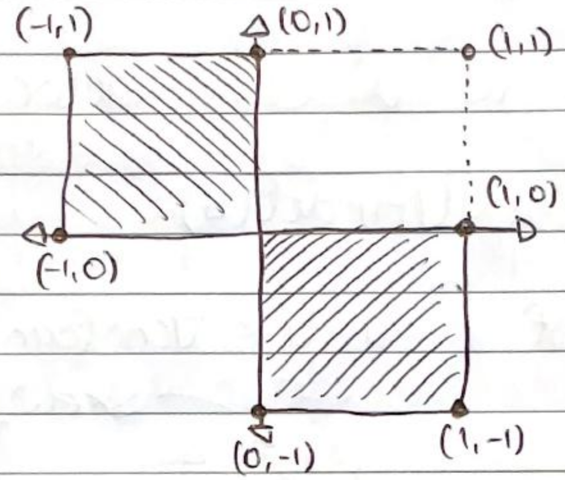
Vectors 13

Transformations

2D Transformations

transformations on shapes can be represented by $AX = B$ ← collection of vectors
↑ collection of vectors



$$X = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

✡ consider $A_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $A_x X = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$

reflection in x axis $(y \leftrightarrow -y)$

✡ consider $A_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_y X = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

reflection in y axis $(x \leftrightarrow -x)$

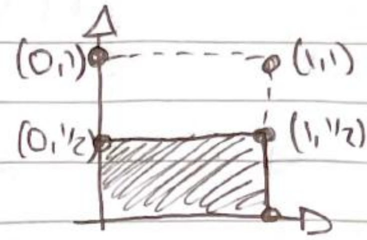
Basis vectors can be used to find a transformation matrix.

Example $A_{str} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$A_{str} X = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

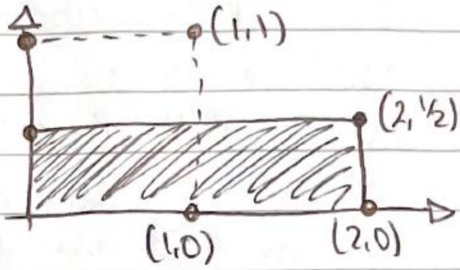
__Example__ $A_{shr} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

$A_{shr} X = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$



__Example__ $A_{sq} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

$A_{sq} X = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$



The determinant is a scale factor.

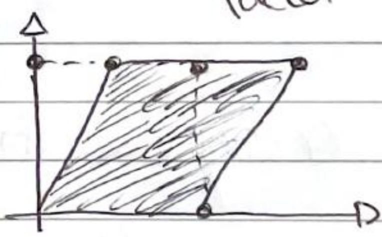
$$\det(AB) = \det(A)\det(B)$$

So operation of a matrix transform scales area by $\det(A)$.

skew factor

__Example__ $A_{skew} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

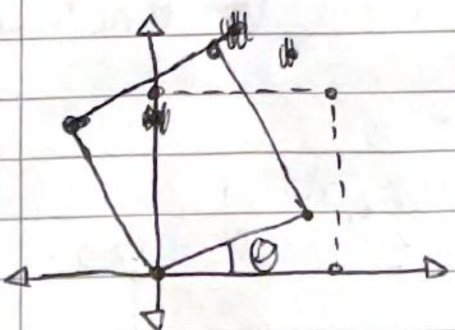
$A_{skew} X = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix}$



N.B. $\det(A_{skew}) = 1$ (area preserving)

__Example__ $A_{rot} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$A_{rot} X = \begin{pmatrix} 0 & \cos\theta & -\sin\theta & \cos\theta - \sin\theta \\ 0 & \sin\theta & \cos\theta & \cos\theta + \sin\theta \end{pmatrix}$



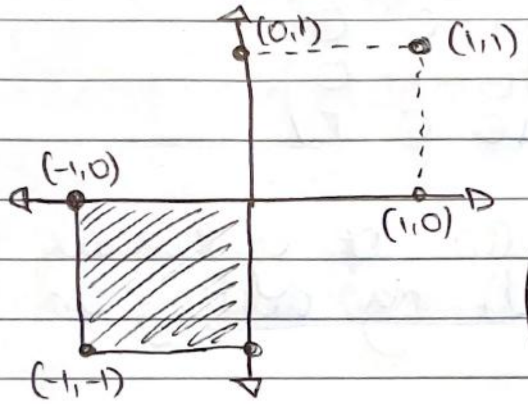
$\det(A_{rot}) = \cos^2\theta + \sin^2\theta = 1$.

area preserving

Combination of Transformations

for example $\quad A_x A_y X \quad$ combined into a single operation

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ reflection in both x & y axis



N.B. equivalent to rotation matrix

$$\begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Combination of rotations

$$A_\theta A_\phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \sin\phi\cos\theta & -\sin\theta\sin\phi + \cos\theta\cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix}$$

In 2D, combinations of rotations are equivalent to a single operation with angles added.

Transformations in 3D

To find transformations in 3D consider transformations on $\hat{i}, \hat{j}, \hat{k}$ which is just the identity matrix.

Example find reflection in yz $(x \leftrightarrow -x)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example find reflection in xz $(y \leftrightarrow -y)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 2D transform can be generalised to 3D case (if z unchanged) by adding an unchanging z.

Example rotation in 3D

$$A_{\theta xy} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_{\theta yz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$A_{\theta xz} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

N.B. 3D rotation operations are not commutative, the order you do them is important.