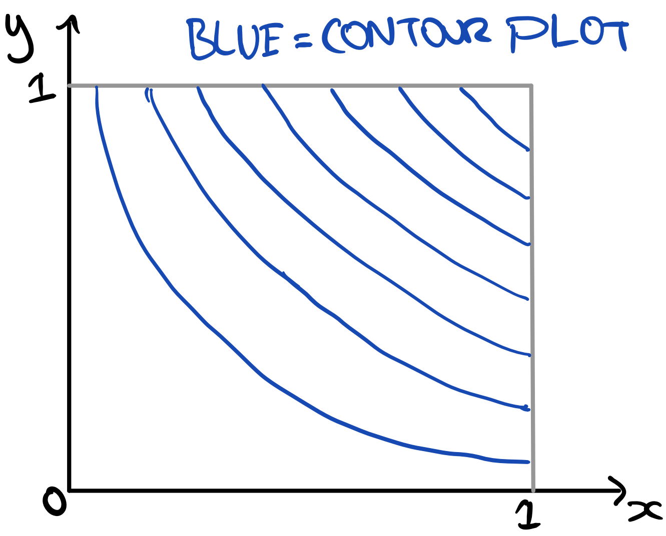
Example 1 find the flux into the surface for the region $0<x<1$ & $0<y<1$.

Surface: $z = f(x,y) = xy$

Vector field: $\underline{F} = x^2\hat{\imath} + y^2\hat{\jmath}$



$\underline{r} = x\hat{\imath} + y\hat{\jmath} + xy\hat{k}$

$\frac{\partial \underline{r}}{\partial x} = 1\hat{\imath} + 0\hat{\jmath} + y\hat{k}$

$\frac{\partial \underline{r}}{\partial y} = 0\hat{\imath} + 1\hat{\jmath} + x\hat{k}$

$\underline{N} = \frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix}$

$= -y\hat{\imath} - x\hat{\jmath} + \hat{k}$

$\therefore \; \underline{dS} = \underline{N}\,dx\,dy = (-y\hat{\imath} - x\hat{\jmath} + 1\hat{k})\,dy\,dx$

to get the flux into the surface, we want $-\underline{F}\cdot\underline{dS}$.

$-\underline{F}\cdot\underline{dS} = (x^2\hat{\imath} + y^2\hat{\jmath} + 0\hat{k})\cdot(y\hat{\imath} + x\hat{\jmath} - 1\hat{k})\,dx\,dy$

$= (x^2y + xy^2)\,dx\,dy$

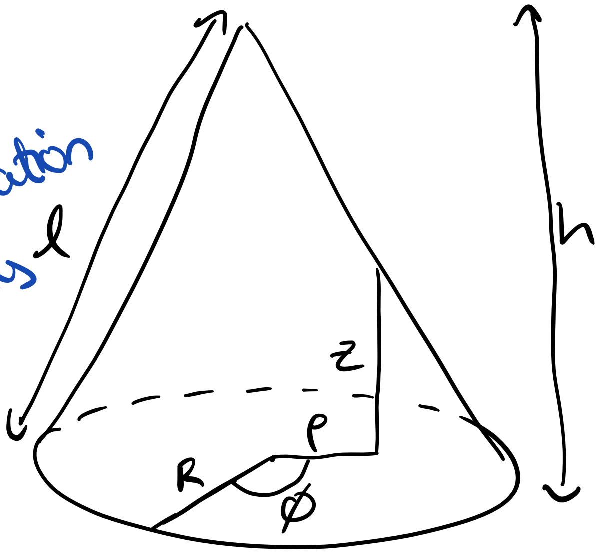
$\therefore$ total flux $= \int_{y=0}^{1}\int_{x=0}^{1} x^2y + xy^2 \,dx\,dy$

$= \int_{y=0}^{1} \left[\frac{1}{3}x^3y + \frac{1}{2}x^2y^2\right]_0^1 dy = \int_{y=0}^{1} \frac{1}{3}y + \frac{1}{2}y^2\,dy$

$$\int_{y=0}^{1} \frac{1}{3}y + \frac{1}{2}y^2 \, dy = \left[\frac{1}{6}y^2 + \frac{1}{6}y^3\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Example 2 Surface area of a cone

$$\rho = R\left(1 - \frac{z}{h}\right) \qquad z = \left(1 - \frac{\rho}{R}\right)h$$



$$\underline{r}(\rho, \phi, z) = \rho\hat{\rho}(\phi) + z\hat{z}$$

get equation in two variables

$$\underline{r}(\rho, \phi) = \rho\hat{\rho}(\phi) + \left(1 - \frac{\rho}{R}\right)h\hat{z}$$

$$\frac{\partial \underline{r}}{\partial \rho} = \hat{\rho}(\phi) - \frac{h}{R}\hat{z}$$

$$\left(\frac{\partial \hat{\rho}}{\partial \phi} = \hat{\phi}\right)$$

$$\frac{\partial \underline{r}}{\partial \phi} = \rho\hat{\phi}$$

$$\underline{N} = \frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ 1 & 0 & -h/R \\ 0 & \rho & 0 \end{vmatrix} = \frac{\rho h}{R}\hat{\rho} + \rho\hat{z}$$

$$\underline{dS} = \rho\left(\frac{h}{R}\hat{\rho} + \hat{z}\right)d\rho\, d\phi$$

$$|\underline{dS}| = \sqrt{\rho^2 + \frac{h^2\rho^2}{R^2}}\, d\rho\, d\phi = \rho\underbrace{\sqrt{1 + \frac{h^2}{R^2}}}_{= \frac{l}{R}}\, d\rho\, d\phi = \frac{\rho l}{R}\, d\rho\, d\phi$$

$$A = \iint_S |\underline{dS}| = \int_{\rho=0}^{R} \int_{\phi=0}^{2\pi} \frac{\rho l}{R}\, d\phi\, d\rho = \pi R l$$

(as expected)