Lecture 2 11/10/21

Recap

$$z = x + iy \qquad z^* = x - iy$$

$$(z \cdot z')^* = z^* \cdot z'^*$$

$$(z \cdot z)^* = (z^*)^2 \qquad (z^n)^* = (z^*)^n$$

$$f(z^*) = [f(z)]^*$$

$$z \cdot z^* = (x+iy)(x-iy)$$
$$= x^2 + \cancel{ixy} - \cancel{ixy} + y^2$$
$$= x^2 + y^2$$
$$= |z|^2$$

called the absolute value of a complex no.

$|z|$ = radius = r

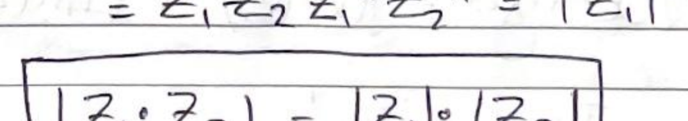
$$|z| = \sqrt{x^2+y^2}$$

properties of $|z|$:

$$|z_1 \cdot z_2|^2 = z_1 \cdot z_2 \cdot (z_1 \cdot z_2)^*$$
$$= z_1 z_2 z_1^* z_2^* = |z_1|^2 \cdot |z_2|^2$$

$$\boxed{|z_1 \cdot z_2| = |z_1| \cdot |z_2|}$$

in natural numbers, inequalities are easy to understand.



does this make sense for complex numbers?

for example, which is larger:

$3-2i$ or $5+i$

Suggestion:

$z_1 > z_2$ if $|z_1| > |z_2|$

DOESN'T WORK!! apply to negative numbers!

!! we do not have any way to establish which complex number is bigger.

Inequalities Involving $|z|$:

$$-|z| \leq Re(z) \leq |z|$$

$$-|z| \leq Im(z) \leq |z|$$

$$|z| = \sqrt{x^2+y^2} = \sqrt{(Re(z))^2 + (Im(z))^2}$$

Triangle Inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof:

$$\begin{aligned}|z_1+z_2|^2 &= (z_1+z_2)(z_1^* + z_2^*) \\ &= z_1 z_1^* + z_2 z_2^* + z_1 z_2^* + z_1^* z_2 \\ &= |z_1|^2 + |z_2|^2 + z_1 z_2^* + (z_1 z_2^*)^* \\ &= |z_1|^2 + |z_2|^2 + 2Re(z_1 z_2^*)\end{aligned}$$

$$Re(z_1 z_2^*) \leq |z_1 z_2^*|$$

$$|z_1 z_2^*| = |z_1| \cdot |z_2^*| = |z_1||z_2|$$

how?

$$|z| = \sqrt{x^2 + y^2}$$

$$|z^*| = \sqrt{x^2 + (-y)^2} = |z|$$

$$Re(z_1 z_2^*) \leq |z_1||z_2|$$

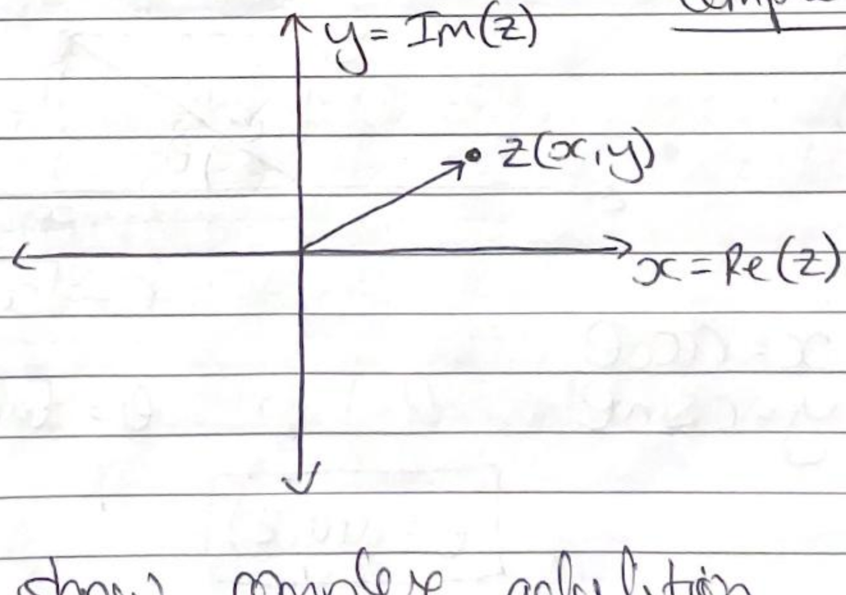
$$\ldots \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

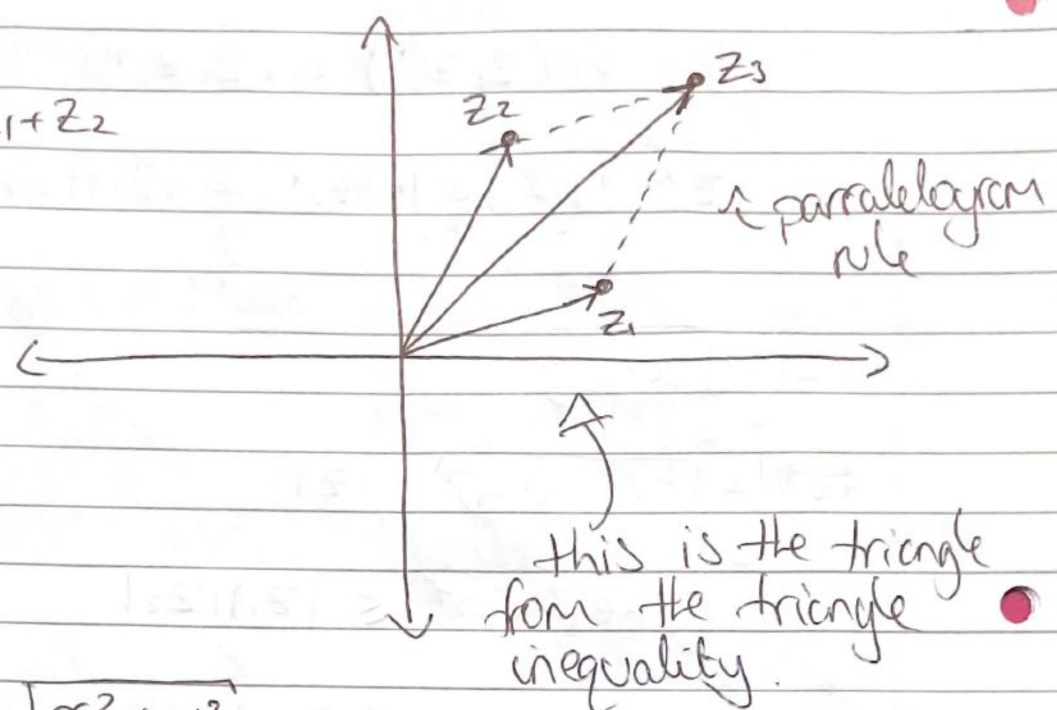
where is the triangle?

complex plane



lets show complex addition graphically.

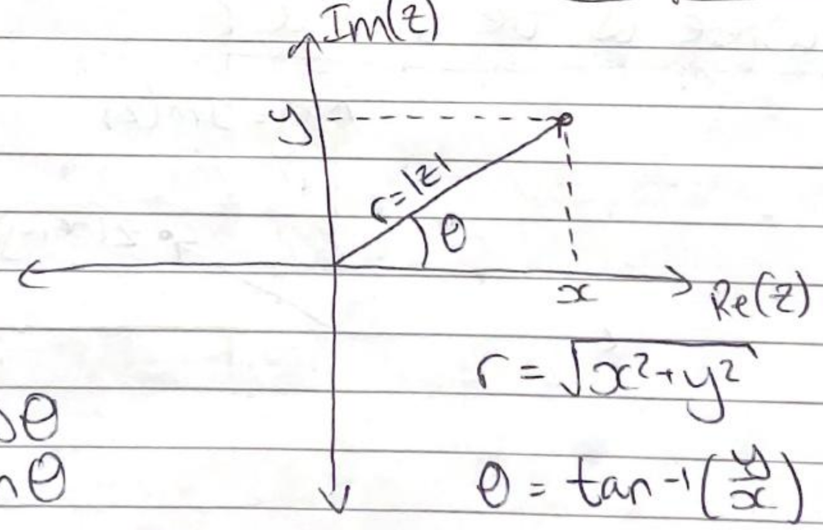
$z_3 = z_1 + z_2$



$|z| = \sqrt{x^2 + y^2}$

(length of vector from $(0,0)$ to (x,y))

how else can we show complex no.?



$r = \sqrt{x^2 + y^2}$

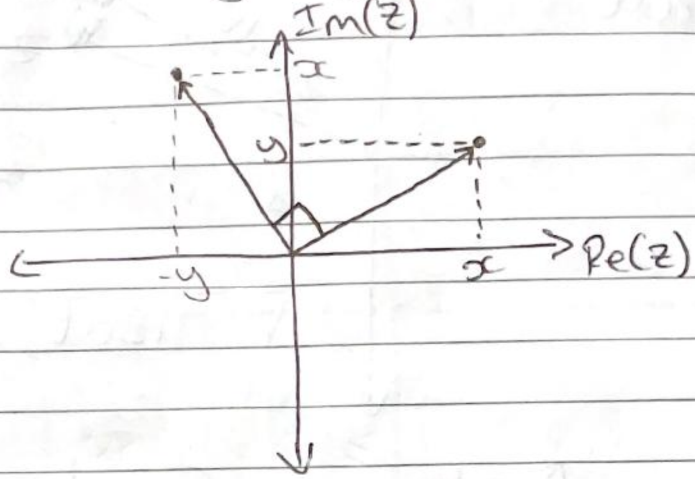
$x = r\cos\theta$
$y = r\sin\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\boxed{\theta = \arg(z)}$$

Lets do something we can't do with vectors, multiple them.

$$z = x + iy = (x, y)$$
$$iz = ix + i^2 y = -y + ix = (-y, x)$$



$$\theta = \arg\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$$
$$\theta = \arg(iz) = \tan^{-1}\left(\frac{x}{-y}\right) = \frac{\pi}{2} + \arg(z)$$

Complex powers?

What happens when we raise something to the power of a complex number.

$$a^z = a^{x+iy} = a^x \times \underbrace{a^{iy}}_{\to\ ?}$$

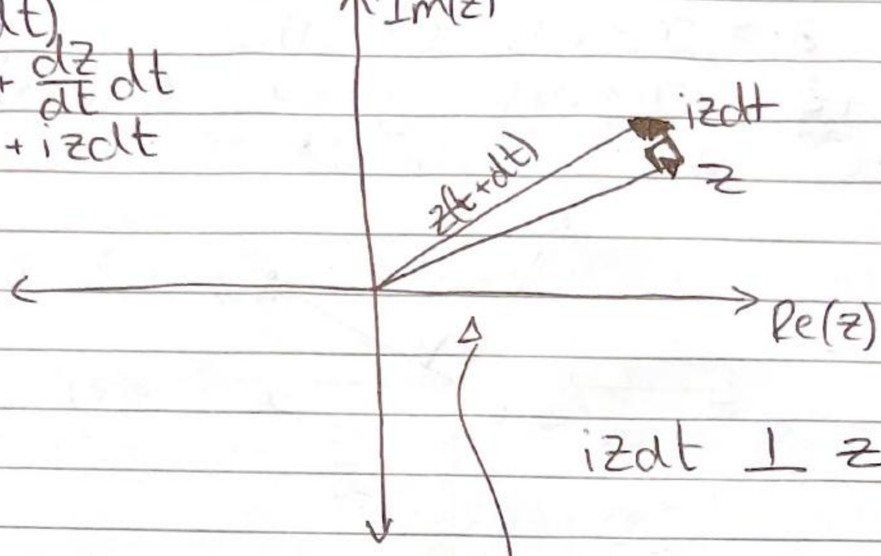
* use complex plain
* differential calculus.

$\frac{d}{dt}(a^t)$ $\qquad$ $\frac{d}{dt}(e^t) = e^t$

too difficult... $\qquad$ $\frac{d}{dt}(e^{it}) = ie^{it}$

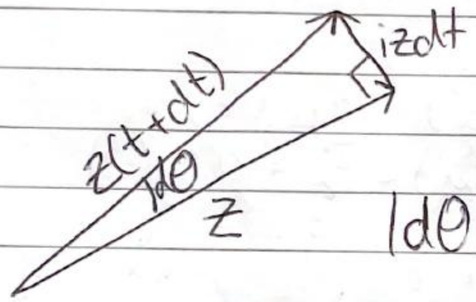
$z = e^{it}$ $\qquad$ $\frac{dz}{dt} = ie^{it} = iz$ $\qquad$ $\boxed{dz = iz\,dt}$

$$\begin{aligned} z(t+dt) &= z(t) + \frac{dz}{dt}dt \\ &= z(t) + iz\,dt \end{aligned}$$



$iz\,dt \perp z$

<u>Rotation!</u>



$$|d\theta| = \left|\frac{iz\,dt}{z}\right|$$

$$|d\theta| = |dt|$$

$\therefore$ t is the angle of the complex number.

^^^ this an improper discussion, proper proofs yet to come ^^^