Lecture 10

Ordinary DEs of 1st Order

$$F(x, y, y') = 0 \iff \frac{dy}{dx} = f(x, y)$$

Ways of solution?

It doesn't matter how we find them. But we've got a system to solve them via classification!

Separable

This is the first type of DE we'll look at.

$$\frac{dy}{dx} = f(y)\, g(x)$$

Any function that can take this form can be solved the following way.

$$\frac{dy}{f(y)} = g(x)\, dx$$

N.B. this is not mathematically great, but okay for now.

$$\int \frac{dy}{f(y)} = \int g(x)\, dx$$

Example $\quad \dfrac{df}{dt} = -rf(t)$

$$\int \frac{df}{f} = -r\,dt \implies \ln f = -rt + c$$

$$f = Ae^{-rt} \qquad (e^c = A)$$

$$f = f_0 e^{-rt}$$

Useful in exponential decay

Linear homogeneous

$$\frac{dy}{dx} + P(x)y = 0$$

* Linear means both $\left(\frac{dy}{dx}\right)^1$ and $(y)^1$ are taken to the power of 1.

* homogenous means the right hand side is equal to zero.

These 'types' of equations are very important but are really just separable equations.

$$\frac{dy}{dx} = -P(x)y$$

$$\int \frac{dy}{y} = -\int P(x)dx$$

$$\ln y = -\int P(x)dx + c$$

$$y = e^{-\int P(x)dx + c}$$

$$y = Ae^{-\int P(x)dx}$$

Linear inhomogeneous ← r.h.s $\neq 0$.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

We cannot use separation to solve this.

to solve this we can use our answer to our homogenous case

$$y = e^{-\int P(x)dx}\,\tilde{y} \qquad \left(\tilde{y} = 1 \text{ when } Q(x) = 0\right)$$

$$y' = e^{-\int P(x)dx}\tilde{y}' - P(x)e^{-\int P(x)dx}\tilde{y}$$

Subbing this into our differential equation

$$\left(e^{-\int P(x)dx}\tilde{y}' - \cancel{P(x)e^{-\int P(x)dx}\tilde{y}}\right) + \cancel{P(x)e^{-\int P(x)dx}\tilde{y}} = Q(x)$$

$$e^{-\int P(x)dx}\tilde{y}' = Q(x)$$

$$\tilde{y}' = Q(x)e^{\int P(x)dx}$$

$$\tilde{y} = \int Q(x)e^{\int P(x)dx}\,dx$$

$$y = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)dx + C\right]$$

We use this so often we call

$$e^{\int P(x)dx}$$

the integration factor.

<u>Example</u> $\quad x^2y' + 3xy = 1$

$$y' + \frac{3}{x}y = \frac{1}{x^2} \qquad P(x) = \frac{3}{x} \quad Q(x) = \frac{1}{x^2}$$

$$\int P(x)dx = 3\int\frac{1}{x}dx = 3\ln x = \ln x^3$$

$$e^{\int P(x)dx} = x^3 \qquad e^{-\int P(x)dx} = x^{-3}$$

$$y = \frac{1}{x^3}\left[\int x^3\cdot\frac{1}{x^2}dx + c\right]$$

$$y = \frac{1}{x^3}\left[\frac{1}{2}x^2 + c\right]$$

$$\underline{\underline{y = \frac{1}{2x} + \frac{c}{x^3}}}$$

Second Order ODE

-> linear -> homogeneous

-> constant coeffecients.

$$ay'' + by' + cy = 0$$