Functions 6

Main Principls (- Graph Drawing)

- examine behavair for $x \to 0, +\infty, -\infty$.
- look for symmetries (even, odd)
- if $y = P(x)/Q(x)$, with P, Q polynomials
  - zeros of P give x axis intercept.
  - zeros of Q give infinite discontinuities.
- stationary points / inflection points

Example $$y = \frac{x(x-2)}{x-3}$$

$y = 0$ when $x = 0, 2$

$y \sim \frac{2}{3}x$ when x is very small

near $x = 3$ we have $y \sim \frac{3}{x-3}$

SO, $y \to +\infty$ as $x \to 3^+$ (from right)
$y \to -\infty$ as $x \to 3^-$ (from left)

what happen when x is small?

$$y = \frac{2}{3}x \frac{(1-\frac{1}{2}x)}{(1-\frac{1}{3}x)} = \frac{2}{3}x\left(1-\frac{1}{2}x\right)\left(1+\frac{1}{3}x+\frac{1}{9}x^2+\dots\right)$$

binomial exp

$$= \frac{2}{3}x\left(1 - \frac{1}{6}x + \dots\right)$$ tends to $\frac{2x}{3}$ for small x.

what happens when $|x|$ is large?

$$y = \frac{x^2\left(1-\frac{2}{x}\right)}{x\left(1-\frac{3}{x}\right)}$$ binomial exp.

$$= x\left(1-\frac{2}{x}\right)\left(1+\frac{3}{x}+\frac{9}{x^2}+\dots\right)$$

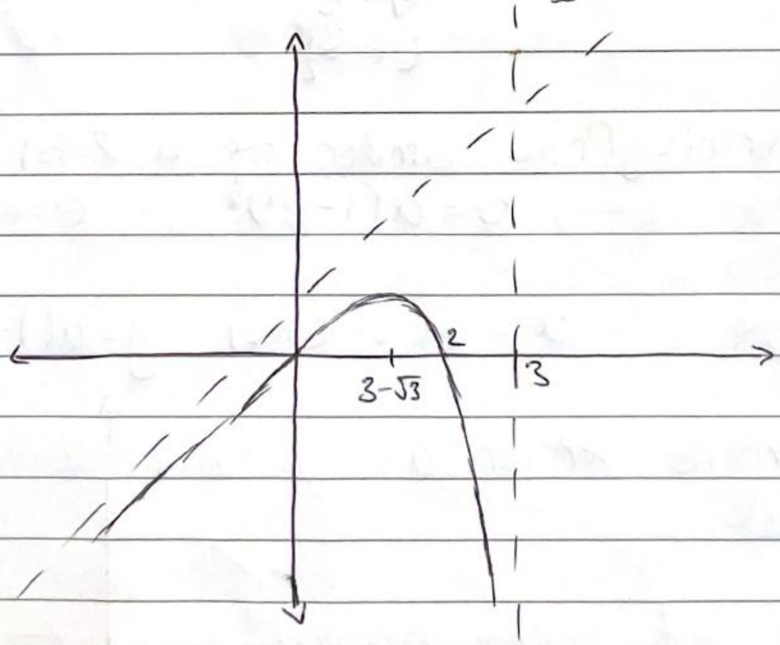
$$= x\left(1+\frac{1}{x}+\frac{3}{x^2}+\dots\right)$$

$$= \underbrace{x + 1}_{\text{have to include the } \underline{+1}} + \underbrace{\frac{3}{x} + \dots}_{\text{smaller terms ignore}}$$

So $y(x) \to$ straight line asymtote as $x \to +\infty$.

Since, $\quad y' = \dfrac{(x^2 - 6x + 6^2)}{(x-3)^2}$

we will have stationary points at $x = 3 \pm \sqrt{3}$ ✓



Parametric Rep of Curves

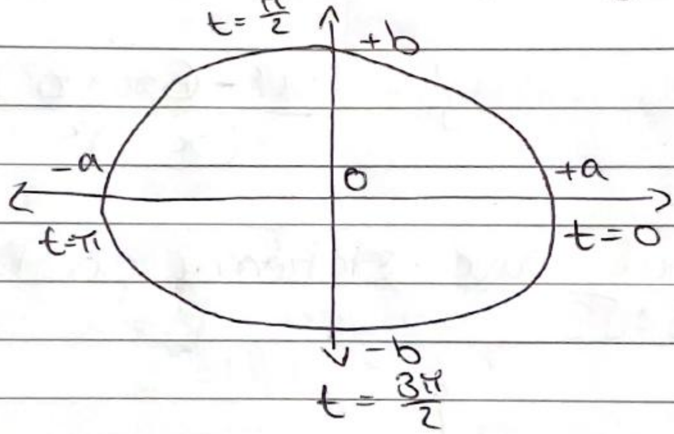
given $x = x(t)$ and $y = y(t)$

Example $x = a\cos t$ ~~b~~ $y = b\sin t$

We can eliminate t here:

$\cos t = \frac{x}{a}$ $\cos^2 t = \frac{x^2}{a^2}$ $\sin t = \frac{y}{b}$ $\sin^2 t = \frac{y^2}{b}$

$\sin^2 t + \cos^2 t = 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



ELLIPSE

$(a > b)$

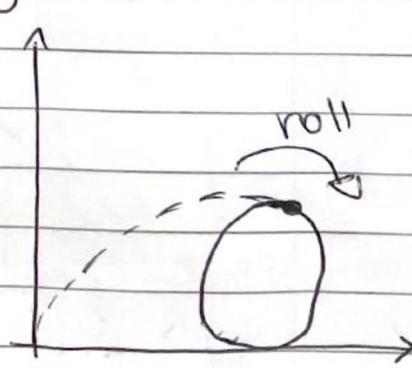
Eccentricity (how similar are a & b) found by

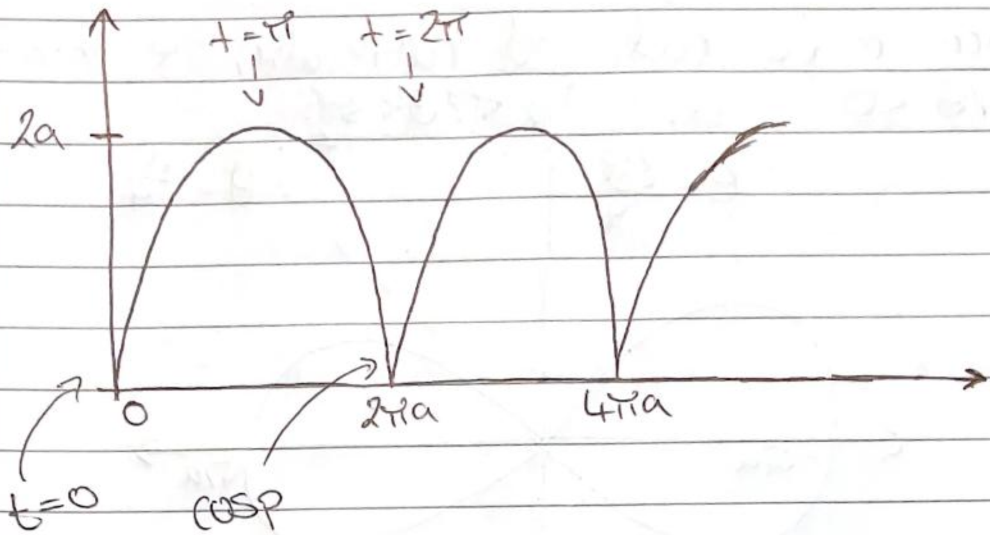
$b = a(1-e^2)^{\frac{1}{2}}$ e = eccentricity

Example $x = a(t - \sin t)$ $y = a(1-\cos t)$

v. famous - known as cycloid.

locus of point on a rolling circle.

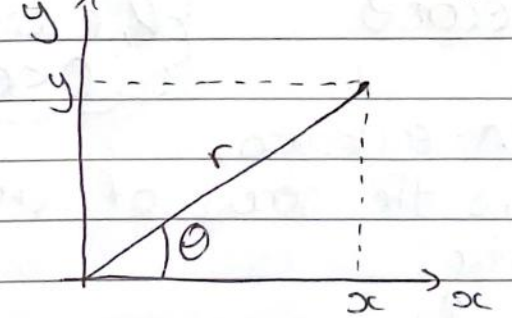




N.B. We cannot eliminate t completely to get $f(x)$.

Polar Coordinates

In 2 dimensions it is often useful to employ plane polar coordinates (r, θ) instead of cartesian (x, y).



r = radial distance to origin

θ = angle between point and x axis.

$r = \sqrt{x^2 + y^2}$ (always +ve)

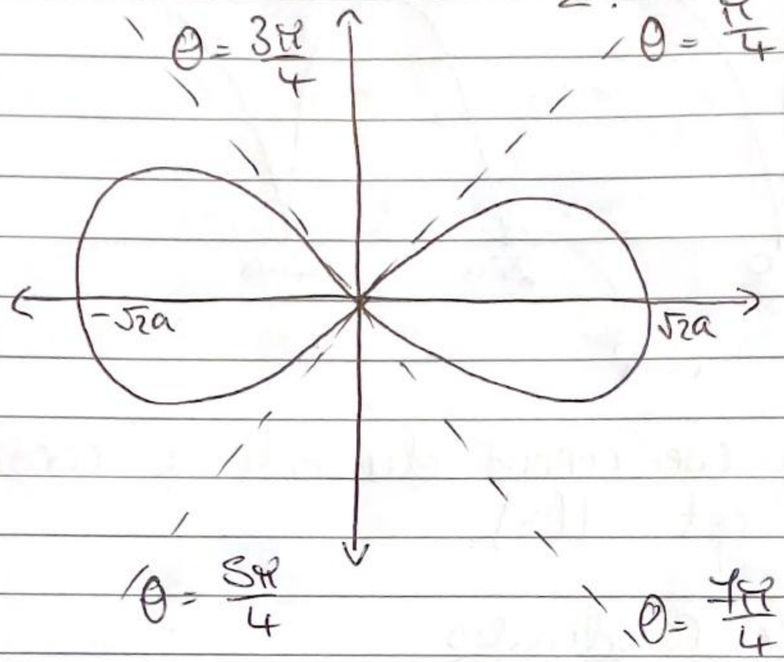
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (defined as $0 < \theta < 2\pi$)

$$x = r\cos\theta \qquad y = r\sin\theta$$

Lemniscate Function

$$r^2 = 2a^2\cos 2\theta \iff (x^2+y^2)^2 = 2a^2(x^2-y^2)$$

Since r is real and non-negative, we cannot have $\cos 2\theta < 0$ i.e. $\pi/2 < 2\theta < \frac{3\pi}{2}$.

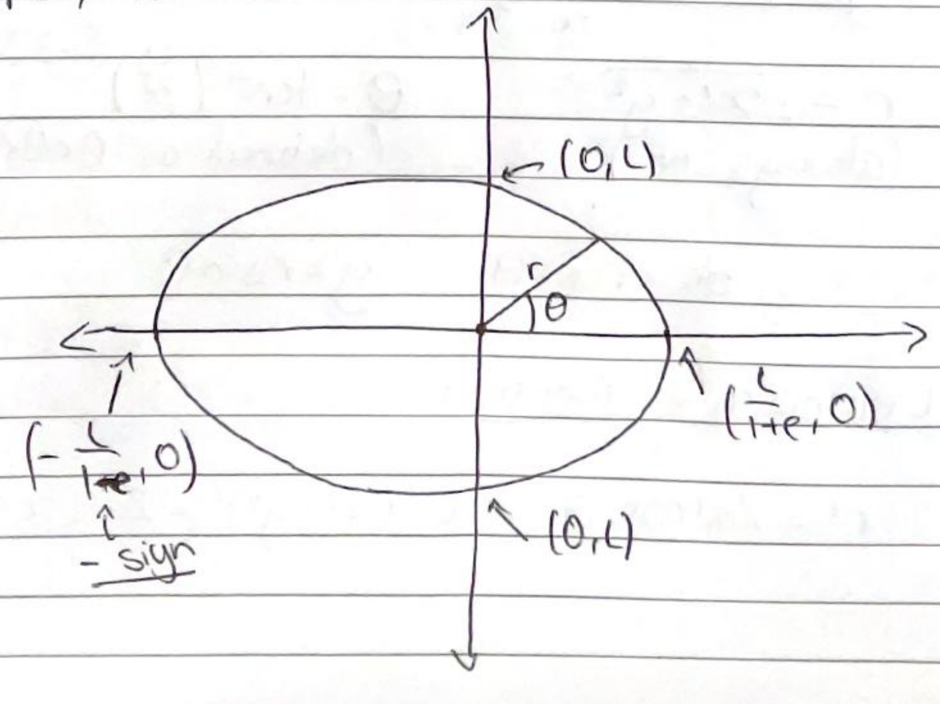


Ellipse (orbital mech.)

some constant $\frac{\ell}{r} = 1 + e\cos\theta$

ℓ, e are +ve
$0 < e < 1$

Unlike the cartesian equation, here the origin is the focus of the ellipse, not the centre.



e = 'eccentricity"
l = 'semi-latus rectum'

There are relationships between 'a' and 'l' and 'b', with 'b' only in cartesian form.

$$a = \frac{l}{1-e^2} \qquad a = \frac{l}{\sqrt{1-e^2}}$$

$$l = \frac{b^2}{a} \qquad e = \sqrt{1 - \frac{b^2}{a^2}}$$

Also note about how e changes:

$e = 0$ circle
$e = 1$ parabola
$e > 1$ hyperbola

Polynomial & Series Representation of Functions

So far we've look at the local behavoir of functions near specific points (eg. stationary points) and then expanded to curve sketching.

Here we note we can consider local approximations to a smooth function in the neighbourhood of a general point as a sequence of polynomials.

$$f(x) \simeq f(x_0) \quad \text{* best constant}$$
$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0) \quad \text{* best linear}$$
$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2}f''(x_0)(x - x_0)^2 \quad \text{* best quadratic}$$

At a stationary points, $f(x)$ is at least quadratic!

All of these polynomial representations are local - with a trade off to be expected between the accuracy over a domain $(x_0 - \epsilon, x_0 + \epsilon)$ and the degree of a polynomial.

If $f(x)$ has successive derivates at x_0, we can form a taylor series expansion around x_0.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \dots \dots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!}$$

For now we consider successive truncations to provide approximations to the local behaviour near $(x_0, f(x_0))$, with successive derivates providing more and more information.

We will properly explore this idea later.

Example $\quad f(x) = \sin x \qquad x_0 = 0$

Expanding, $f(x) = 0 + x + 0x^2 - \frac{x^3}{3!} + \dots$

These are local approximations about a point. We note in passing that we might alternatively seek a 'best' polynomial over an interval (a, b) e.g. linear regression.

A note of inequalities
It is import to realise that the stationary points are indicating local behaviour - a global behaviour may not follow.

In quadratic functions, the 'completing the square' approach did lead to global results - ∴ two inequalities are important.

* Cauchy-Schwarz
* AM/GM

With care, we can use our calculus of stationary points to arrive at global results and inequalities.

Example The AM/GM inequality is easy to generalise from 2 numbers (above) to 2^N numbers.

However, generalisation to a set not to a power of 2. is not so easy.

A Novel approach (due to Polya) is to consider the function

$$f(x) = e^x - (1+x)$$