Vectors 15

Special Matrices

Diagonal Matrix.

$A\underline{x} = \lambda \underline{x}$ has solutions λ_i (eigenvalues) and $\underline{x}_i$ (eigenvectors). Consider a matrix of eigenvectors

$$S = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$$

For now lets look at $\mathbb{R}^2$. $S = (\underline{x}_1, \underline{x}_2)$

$$AS = A(\underline{x}_1, \underline{x}_2) = (\lambda \underline{x}_1, \lambda \underline{x}_2)$$

$$= (\underline{x}_1, \underline{x}_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = S\Lambda$$

$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is a diagonal matrix (with eigenvalues) along the diagonals.

$$S^{-1}AS = S^{-1}S\Lambda = I\Lambda = \Lambda$$

$$\Lambda = S^{-1}AS^{-1}$$

This is called a similarity transformation.

Here $S^{-1}AS$ decomposes the matrix A into a basis corresponding to the eigenvalues of A.

N.B. since $\det(AB) = \det(A)\det(B)$

$$\det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S)$$
$$= \det(S^{-1})\det(S)\det A$$

$= \det(S^{-1}S)\det(A)$
$= \det(I)\det(A)$ $\quad \det(I) = 1$
$= \det(A)$

$$\det(\Lambda) = \det(A)$$

The determinant of a matrix to a similarity transformation.

Example $\quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

from earlier we found: $\lambda_1 = 1, \underline{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda_2 = 3, \underline{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then $S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$S^{-1}AS = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

N.B. $\det(A) = \det(\Lambda) = \prod_{i=1}^{n} \lambda_i$ $\quad$ product of eigenvalues.

Uses of diagonalisation

find A^{100}? $\quad$ lets consider $(\Lambda)^{100}$

$$(\Lambda)^{100} = (S^{-1}AS)^{100}$$
$$= (S^{-1}AS)(S^{-1}AS)\dots(S^{-1}AS)$$
$$= S^{-1}A(SS^{-1})A(SS^{-1})\dots(SS^{-1})AS$$
$$= S^{-1}AIAIA\dots IAS$$
$$= S^{-1}AA\dots AS$$
$$= S^{-1}A^{100}S$$

$$S\Lambda^{100}S^{-1} = SS^{-1}A^{100}SS^{-1}$$

$$A^{100} = S\Lambda^{100}S^{-1}$$

$$\begin{pmatrix}2&1\\1&2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1^{100}&0\\0&3^{100}\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\3^{100}&3^{100}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1+3^{100}&1+3^{100}\\-1+3^{100}&-1+3^{100}\end{pmatrix}$$

Trace of a Matrix

The trace of a matrix is defined as

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

N.B. $\mathrm{Tr}(A) = \mathrm{Tr}(A^T)$

N.B.2. $\underset{n\times(m=m)\times n}{\mathrm{Tr}(AB)} = \sum_{i}^{n} a_{ij}b_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ji}$

$$= \sum_{j=1}^{m}\sum_{i=1}^{n} b_{ji}a_{ij} = \mathrm{Tr}(BA)$$

$$\mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}(ASS^{-1}) = \mathrm{Tr}(AI) = \mathrm{Tr}(A)$$

$$\mathrm{Tr}(\Lambda) = \mathrm{Tr}(A)$$

A similarity transform does not change the trace.

eg. $A = \begin{pmatrix}2&1\\1&2\end{pmatrix}$ $\quad \Lambda = \begin{pmatrix}1&0\\0&3\end{pmatrix}$

$\mathrm{Tr}(A) = 4$ $\quad \mathrm{Tr}(\Lambda) = 4$

$$Tr(A) = \sum_{i=1}^{n} \lambda_i$$

You can use the trace and determinant to find eigenvalues.

Other Special Matrices

I) Symmetric Matrix

$$A = A^T \quad \text{(must be square)}$$

eg. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & -1 & 3 \end{pmatrix}$ The products of symmetric matrices commute.

$A, B \in$ symmetric matrix $\quad AB = BA$

Hermitian matrices are related: $A = A^{T*}$ where the complex conjugate of the transpose is the same matrix.

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

II) Skew-symmetric matrix

$$A = -A^T$$

$a_{ii} = -a_{ii} = 0 \qquad A = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix}$

III) Normal matrix

$$AA^T = AA^{T*}$$

only possible if eigenvectors of A form a normal basis. All diagonal, symmetric and skew symmetric matrices are normal.