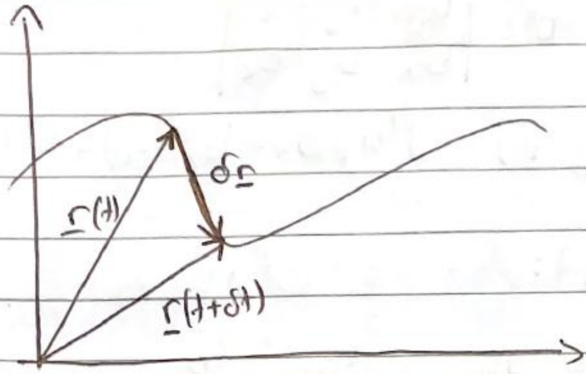
Vectors 5

Vector Derivatives



$$\underline{v} = \frac{d\underline{r}}{dt}$$

$$\underline{r} = r_x\hat{i} + r_y\hat{j}$$
$$d\underline{r} = \delta r_x\hat{i} + \delta r_y\hat{j}$$

$$\delta\underline{r} = \underline{r}(t+\delta t) - \underline{r}(t)$$

$$\frac{d\underline{r}}{dt} = \lim_{\delta t\to 0}\left(\frac{\delta\underline{r}}{\delta t}\right) = \lim_{\delta t\to 0}\frac{\underline{r}(t+\delta t)-\underline{r}(t)}{\delta t}$$

$$\frac{d\underline{r}}{dt} = \lim_{\delta t\to 0}\left(\frac{r_x(t+\delta t)-r_x(t)}{\delta t}, \frac{r_y(t+\delta t)-r_y(t)}{\delta t}\right)$$

$$\frac{d\underline{r}}{dt} = \left(\frac{dr_x}{dt}, \frac{dr_y}{dt}\right)$$

In general, in $\mathbb{R}^n$

$$\frac{d\underline{a}}{dt} = \left(\frac{da_1}{dt}, \frac{da_2}{dt}, \dots, \frac{da_n}{dt}\right)$$

Vector Derivative Rules

Scaler Product ~~Product Rule~~ in $\mathbb{R}^n$ $\quad f(s)$ = scaler $\underline{a}(s)$ = vector

$$\frac{d}{ds}\left[f(s)\underline{a}(s)\right] = \sum_{i=1}^{n}\frac{d}{ds}\left[f(s)a_i(s)\right] = \sum_{i=1}^{n} f(s)\frac{da_i}{ds} + \frac{df(s)}{ds}a_i$$

$$= f(s)\frac{d\underline{a}(s)}{ds} + \frac{df(s)}{ds}\underline{a}(s)$$

Dot Product

$$\frac{d}{ds}\left[\underline{a}(s)\cdot\underline{b}(s)\right] = \frac{d}{ds}\sum_{i=1}^{n}a_ib_i = \sum_{i=1}^{n}\frac{d}{ds}(a_ib_i)$$

$$= \sum_{i=1}^{n} a_i\frac{db_i}{ds} + \frac{da_i}{ds}b_i = \underline{a}\frac{d\underline{b}}{ds} + \frac{d\underline{a}}{ds}\underline{b}$$

Cross Product

$$\frac{d}{ds}\left[\underline{a}(s) \times \underline{b}(s)\right] = \frac{d}{ds}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \frac{d}{ds}\left[\hat{i}(a_y b_z - b_y a_z) + \hat{j}(a_z b_x - b_z a_x) + \hat{k}(a_x b_y - b_x a_y)\right]$$

consider $\hat{i}$ component:

$$\frac{d}{ds}(\underline{a} \times \underline{b})_{\hat{i}} = \hat{i}\left(\frac{da_y}{ds} b_z + a_y \frac{db_z}{ds} - \frac{db_y}{ds} a_z - b_y \frac{da_z}{ds}\right)$$

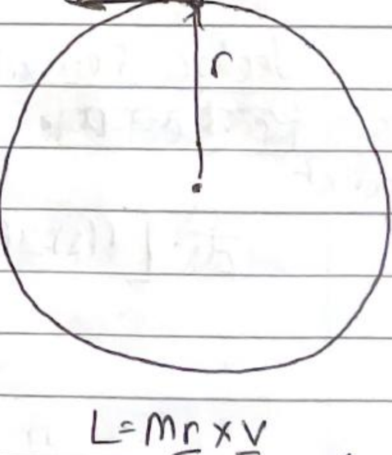
$$= \hat{i}\left(\frac{da_y}{ds} b_z - \frac{da_z}{ds} b_y + \frac{db_z}{ds} a_y - \frac{db_y}{ds} a_z\right)$$

without showing full working (need to inc. $\hat{j}$ + $\hat{k}$) we get the formula:

$$\frac{d}{ds}\left[\underline{a}(s) \times \underline{b}(s)\right] = \left(\frac{d\underline{a}}{ds} \times \underline{b}\right) + \left(\underline{a} \times \frac{d\underline{b}}{ds}\right)$$

Circular Motion

For circular motion with no torque, show that the radius is perpendicular to the velocity, $a = v^2/r$ and $|v|$ is constant.



circular so $|r|$ is const.

no torque so $|L| = 0$.

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if $|r| = $ constant.

$$\frac{dr^2}{dt} = \frac{d(\underline{r} \cdot \underline{r})}{dt} = \underline{r} \cdot \frac{d\underline{r}}{dt} + \frac{d\underline{r}}{dt} \cdot \underline{r}$$

$$L = m\underline{r} \times \underline{v}$$

$$= 2\underline{r}\frac{d\underline{r}}{dt} = 2\underline{r}\underline{v} = 0 \quad \therefore \underline{r} \cdot \underline{v} = 0 \text{ (perp).} ✓$$

acceleration

this is just velocity.

this is zero because r^2 is const so $\frac{dr^2}{dt} = 0$.

(=0)

const. we can ignore.

$$\frac{dL}{dt} = \frac{d}{dt}(m\underline{r}\times\underline{v}) = \cancel{m}\frac{d}{dt}\left[\underline{r}\times\underline{v}\right] = \left(\frac{dr}{dt}\times\underline{v}\right)+\left(\underline{r}\times\frac{dv}{dt}\right)$$

$$= \underbrace{(v\times v)}_{=0} + (r\times a) = 0.$$

$\therefore\ r\times a = 0 \quad \therefore\ \underline{r}$ parralel to $\underline{a}$. ✓

$$\frac{d}{dt}\left[\underline{r}\cdot\frac{dr}{dt}\right] = \left(\frac{dr}{dt}\cdot\frac{dr}{dt}\right)+\left(\underline{r}\cdot\frac{d^2r}{dt^2}\right) = \underline{v}^2+\underline{r}\,\underline{a} = 0$$

$(r\perp v)$

$r\cdot a = -v^2 \implies a = -\frac{v^2}{r}$ ✓

$$\frac{d}{dt}\underline{v}^2 = \frac{d}{dt}(\underline{v}\cdot\underline{v}) = \underline{v}\frac{dv}{dt}+\frac{dv}{dt}\underline{v} = 2\underline{v}\cdot\underline{a}$$

$a \parallel r \quad v\perp r \quad \therefore\ v\perp a$

$\therefore\ v\cdot a = 0.$

$\therefore\ v^2 =$ constant

$|v| =$ constant. ✓

Keplers Laws

I) Planets orbit the sun in ellipses with the sun at one focus

II) The area traced out by a planet in a given time is constant.

III) The period of revolution of a planet is proportional to the semi-major axis to the power 3/2.

2nd Law Proof

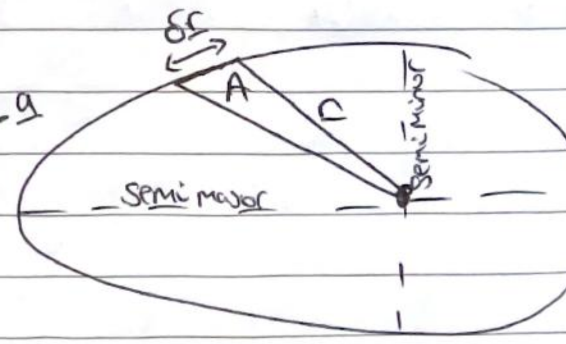
$v\times v = 0$ $\quad r\perp a$

$\delta\underline{r} = \frac{1}{2}(\underline{r}\times\delta\underline{r})$

$\dot{\underline{A}} = \frac{1}{2}\left(\underline{r}\times\frac{dr}{dt}\right)$

$\ddot{\underline{A}} = \frac{1}{2}\left(\frac{dr}{dt}\times\frac{dr}{dt}+\underline{r}\times\frac{dr^2}{dt^2}\right)$

$\ddot{A} = 0$



$\therefore\ \dot{\underline{A}} =$ const. $\implies$ area mapped out in equal time

Vector Fields (Intro)

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

This is know as 'nabla' or 'del'.

If we have a function $b = b(x, y, z)$

gradient

$$\text{grad}(b) = \nabla b = \frac{\partial b}{\partial x}\hat{i} + \frac{\partial b}{\partial y}\hat{j} + \frac{\partial b}{\partial z}\hat{k}$$

If we have a vector $\underline{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

divergence

$$\text{div}(a) = \nabla \cdot \underline{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$\text{curl}(a) = \nabla \times \underline{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{i} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{k}$$