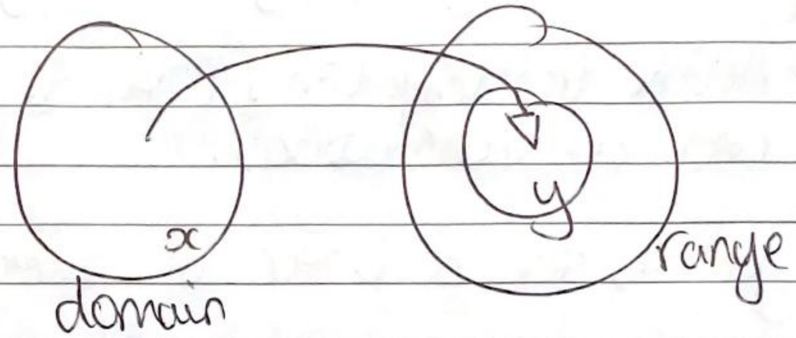
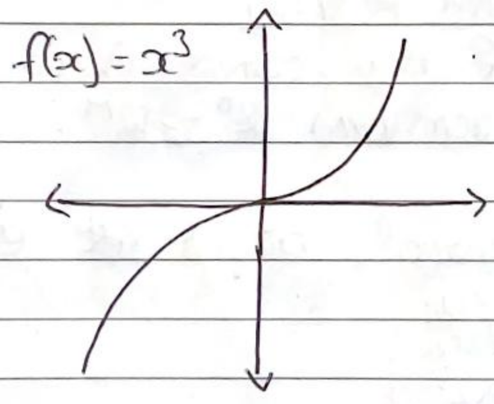
Vectors 12

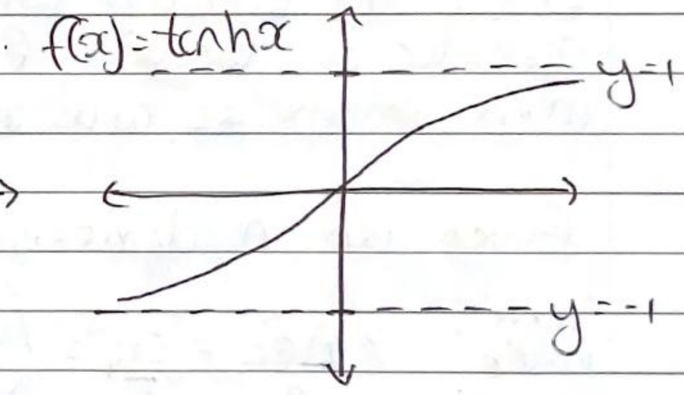
Linearity

Range & Domain

In general, a function maps one set onto another.







domain $\mathbb{R}$
range $\mathbb{R}$

domain $\mathbb{R}$
range $(-1, 1)$

Linear & Affine Transformations

By definition, a linear transformation obeys:

$$f(x + y) = f(x) + f(y)$$

and $$f(\lambda x) = \lambda f(x)$$

even $$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

Examples

$f(x) = x^2 \quad f(x+y) = (x+y)^2 \neq x^2 + y^2$ NOT LINEAR

$f(x) = 2x \quad f(x+y) = 2x + 2y = f(x) + f(y)$ LINEAR

$f(x) = 2x + 1 \quad f(x+y) = 2(x+y) + 1 \neq 2x + 2y + 1 + 1$ NOT LINEAR

Affine transformation (linear transformations and a translation).

In general, for a linear transformation $f(0) = 0$.

Linear Transformation with Matrices

consider a function $f = A\underline{x}$ where A is a $m \times n$ matrix. It will transform $\mathbb{R}^n \to \mathbb{R}^m$.

Given an n-dimensional basis set $\hat{e}_i$.

$$\underbrace{A}_{m \times n \text{ matrix}} \underbrace{\hat{e}_i}_{n \times 1 \text{ vector}} = \underbrace{\underline{a}_i}_{m \times 1 \text{ vector}} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

Now lets test for linearity:

$$f(\lambda \underline{x} + \mu \underline{y}) = A(\lambda \underline{x} + \mu \underline{y}) = \lambda A \underline{x} + \mu A \underline{y} = \lambda f(x) + \mu f(y)$$

now say $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + \dots + x_n \hat{e}_n$

$$\begin{aligned} f(\underline{x}) &= A(x_1 \hat{e}_1 + x_2 \hat{e}_2 + \dots + x_n \hat{e}_n) \\ &= x_1 A \hat{e}_1 + x_2 A \hat{e}_2 + \dots + x_n A \hat{e}_n \\ &= x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots x_n \underline{a}_n \end{aligned}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{A\underline{x}}$$