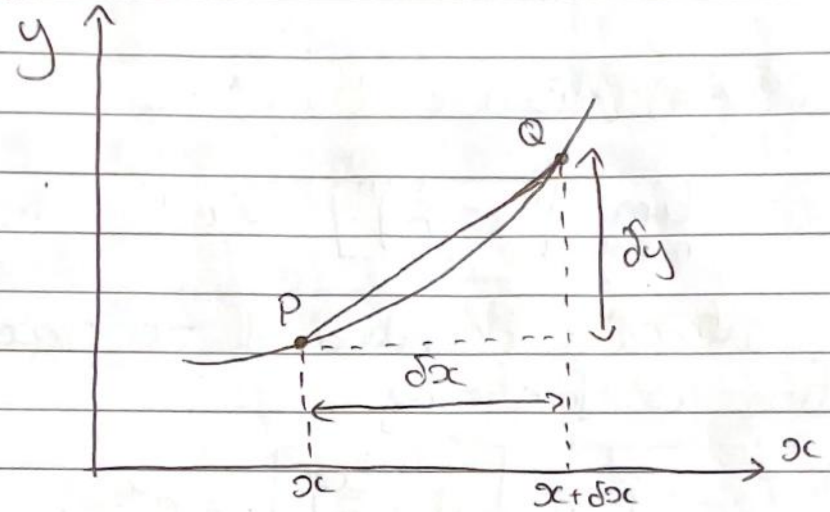
Functions 5

Differentiation from First Principles

Consider the tangent to the curve at point P as the limit of the chord PQ as Q approaches P.



gradient of PQ $= \dfrac{f(x+\delta x) - f(x)}{\delta x} = \dfrac{\delta y}{\delta x}$

If P is fixed (x is fixed), let $\delta x \to 0$, and note that (in the above) $\delta y \to 0$.

If the limit exists, we define

$$\lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} \right] = \frac{dy}{dx}$$

Example $\quad y = x^2$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left[\frac{(x+\delta x)^2 - x^2}{\delta x} \right] =$$

$$= \lim_{\delta x \to 0} \frac{2\delta x\, x + \delta x^2}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x)$$

$$= \underline{\underline{2}}$$

$$\boxed{\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

Example $y = \sin x$

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left[\frac{\sin(x+\delta x) - \sin x}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right]$$

$$= \underbrace{\lim_{\delta x \to 0}\left[\cos\left(x + \frac{\delta x}{2}\right)\right]}_{=\cos x} \cdot \underbrace{\lim_{\delta x \to 0}\left[\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x/2}\right]}_{=1}$$

$$\frac{dy}{dx} = \cos x$$

We need to know how this technique works, but normally 'apply' a stock set of derivates in pratice - formula sheet.

Product, Quotient, Chain Rule

Product

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

We note that:

$$\frac{d^2}{dx^2}\left[f(x)g(x)\right] = \frac{d^2f(x)}{dx^2}g(x) + 2\frac{df(x)}{dx}\frac{dg(x)}{dx} + f(x)\frac{dg(x)}{dx}$$

From Leibniz (1684), we get the general result:

$$\frac{d^n}{dx^n}(fg) = \binom{n}{0}\frac{d^nf}{dx^n}g + \binom{n}{1}\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \binom{n}{2}\frac{d^{n-2}f}{dx^{n-2}}\frac{d^2g}{dx^2} + \dots + \binom{n}{n-1}\frac{df}{dx}\frac{d^{n-1}g}{dx^{n-1}} + \binom{n}{n}f\frac{d^ng}{dx^n}$$

← Binomial

can be proved via induction.

<u>Quotient</u>

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

<u>Chain</u>

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg} \times \frac{dg}{dx}$$

<u>Example</u> $y = \ln(\cos x)$ $\qquad f = \ln(g) \quad g = \cos x$

$$\frac{df}{dg} = \frac{1}{g} \qquad \frac{dg}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{1}{\cos x} \times -\sin x = -\tan x$$

<u>Parametric Differentiation</u>

Suppose $y = y(t)$ and $x = x(t)$, which ~~height~~ might be the coordinates of a moving point.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$$

Similarly, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{dt}{dx} \cdot \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)$$

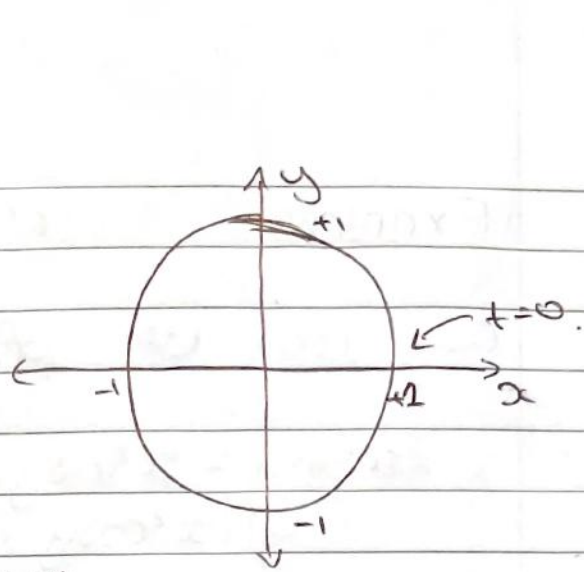
$$= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^3}$$

<u>Example</u>

$x = \cos t$

$y = \sin t$

$\dot{x} = -\sin t$
$\dot{y} = \cos t$



$\ddot{x} = -\cos t$
$\ddot{y} = -\sin t$

$$\frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot(t)$$

$$\frac{d^2y}{dx^2} = \frac{\sin^2 t + \cos^2 t}{-\sin^3 t} = \frac{-1}{y^3}$$

Inverse Function

$y = \arcsin x \iff x = \sin y$ implicit

$$1 = \cos(y)\frac{dy}{dx}$$

$$\frac{dx}{dy} = \frac{1}{\cos y}$$

$$y' = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$y' = 1/\sqrt{1-x^2}$$

take care with principle values. we don't need ± here as we only have +grad

$y = \arctan x \qquad x = \tan y$

$$\frac{dx}{dy} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Implicit Functions

Sometimes the form $y=f(x)$ is not avaible, the function may be similer to

$$f(x,y) = 0.$$

we could write this as a function of y.

Example $x^2 \sin y + xy = 1$

we can use product rule!

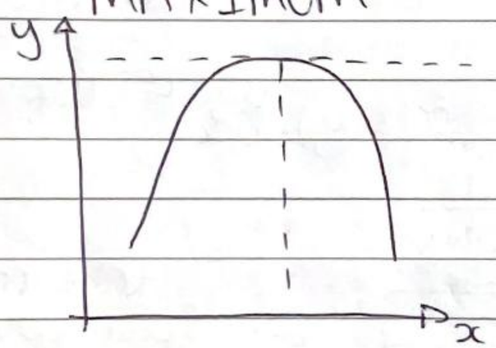
$$2x\sin y + x^2\cos y\, y' + xy' + y = 0$$
$$y'(x^2\cos y + x) = -2x\sin x - y$$

$$y' = \frac{-2x\sin x - y}{x^2\cos y + x}$$

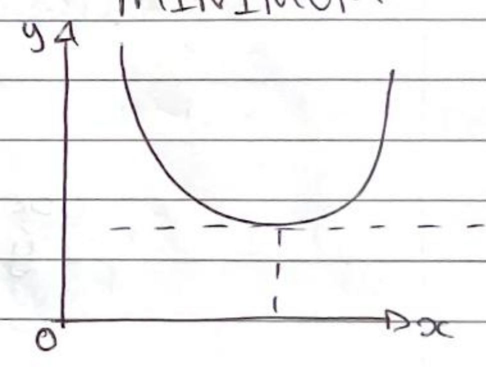
Stationary Points

At a stationary point, $\frac{dy}{dx} = 0$.

MAXIMUM



MINIMUM



$y' = 0$ and $y'' < 0$ $\qquad$ $y' = 0$ and $y'' > 0$

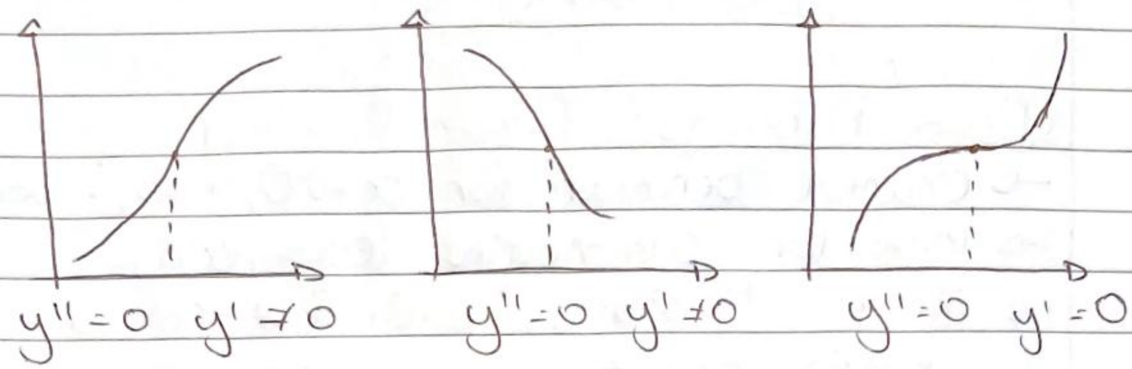
At a point of INFLECTION

$$\frac{d^2y}{dx^2} = 0.$$

but $\frac{dy}{dx}$ might not be zero.

y' also 0: stationary point of inflection

$y' \neq 0$: non-stationary point of inflection



Example $\quad y = x^2(x-1)$

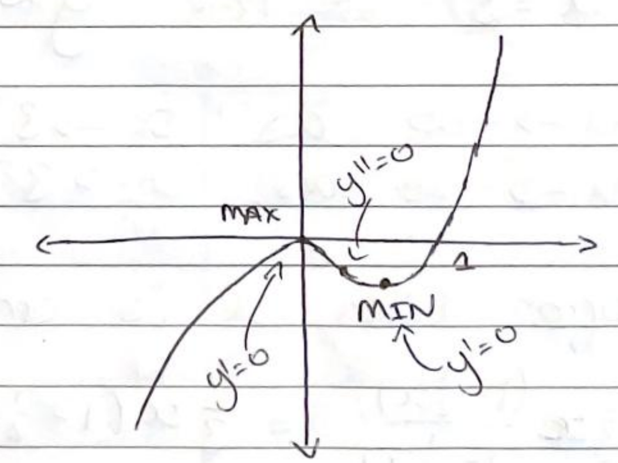
$$y' = 2x(x-1) + x^2 = 3x^2 - 2x$$
$$y'' = 6x - 2$$

lost an answer!

SP: $\quad 0 = 3x^2 - 2x \qquad 0 = 3x - 2 \qquad x = \frac{2}{3}, 0$

$y''|_{x=0} = -2 \qquad y''|_{x=\frac{2}{3}} = +2$

local maximum $\qquad$ local minimum



We know Point of Inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$,
But does $\frac{d^2y}{dx^2} \Rightarrow$ Point of Inflection?

NOT NECESSARILY!

We need to look at higher derivatives to determine nature of the point.