Functions 13

Exact Differentials

We know that for a function of two variables $u(x,y)$ the total differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

and of course, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ will be, in general, functions of x and y.

Now let's consider the inverse problem!

Given $P(x,y)dx + Q(x,y)dy$
when is it the case that this is the total differential of some (as yet unknown) function $u(x,y)$? If it is such then $P(x,y) = \frac{\partial u}{\partial x}$ and $Q(x,y) = \partial u/\partial y$. for the function $u(x,y)$.

This implies and is implied by the condition of intergrability

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Example $\quad y^2dx + (x^2+2y)dy$

$P(x,y) = y^2 \qquad \partial P/\partial y = 2y$
$Q(x,y) = x^2+2y \qquad \partial Q/\partial x = 2x$

$\partial P/\partial y \neq \partial Q/\partial x$ not exact

Example $(2xy + \cos x \cos y)dx + (x^2 - \sin x \sin y)dy$

$\partial P/\partial y = 2x + \cos x \sin y$ $\quad \frac{\partial Q}{\partial x} = 2x - \cos x \sin y$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \underline{\text{exact}}$$

$$u(x,y) = x^2 y + \sin x \cos y + f(y)$$

Then either:

$$\frac{\partial u}{\partial y} = x^2 - \sin x \cos y + f'(y)$$
$$= x^2 - \sin x \cos y$$

$$\therefore f'(y) = 0 \quad f(y) = k$$

or: $\quad u(x,y) = x^2 y + \sin x \cos y + g(x)$

$$g'(x) = 0 \quad g(x) = k$$

Note:

$$\boxed{\frac{dy}{dx} = -\frac{P(x,y}{Q(x,y)}}$$

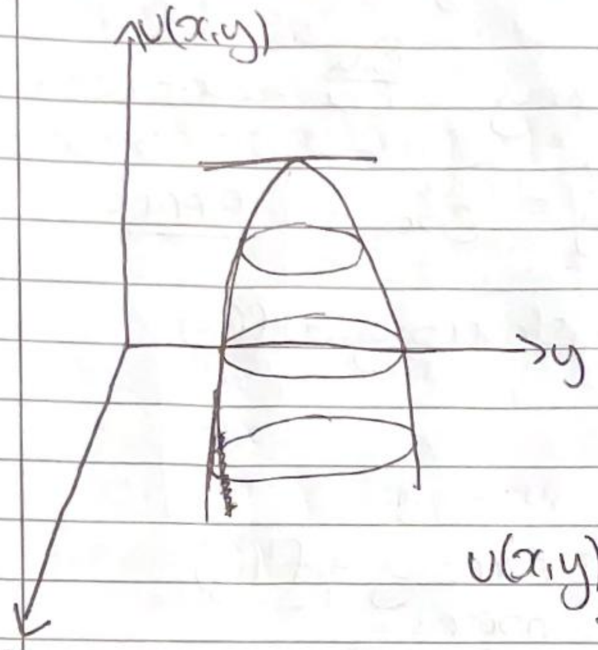
Stationary Points

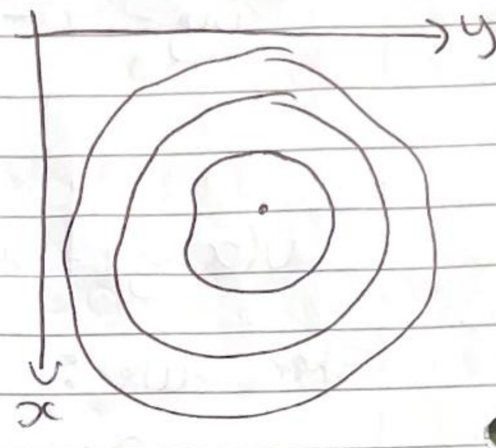
How do stationary points work when we have two variables?

There are 3 types of stationary points.

- local maximum
- local minimum
- saddle point

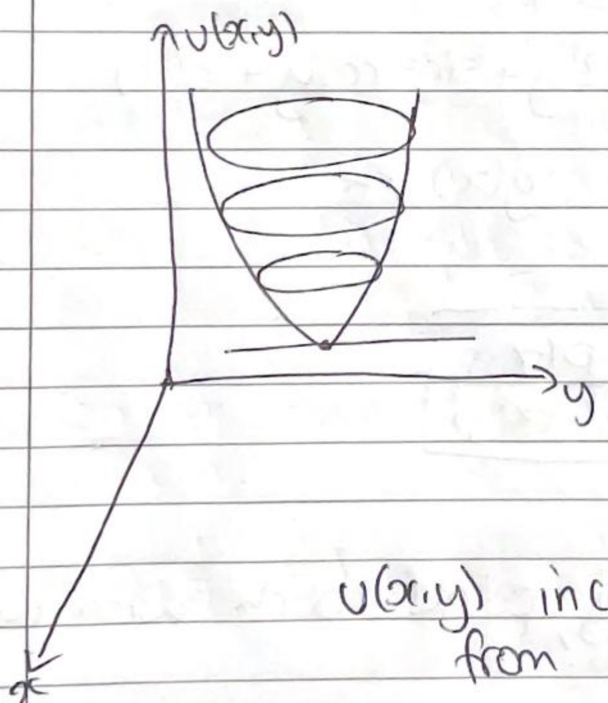
Local Maximum

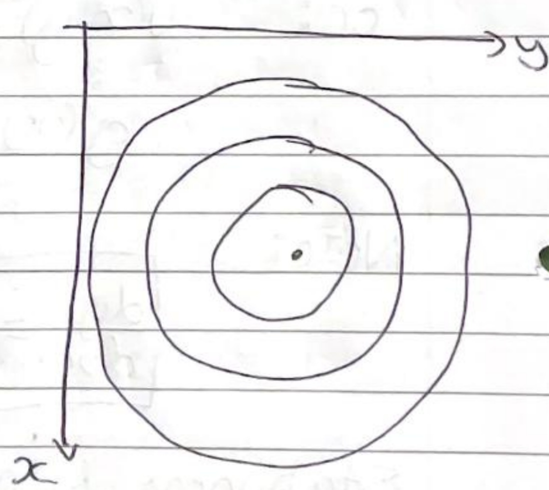




u(x,y) decreasing away from the point

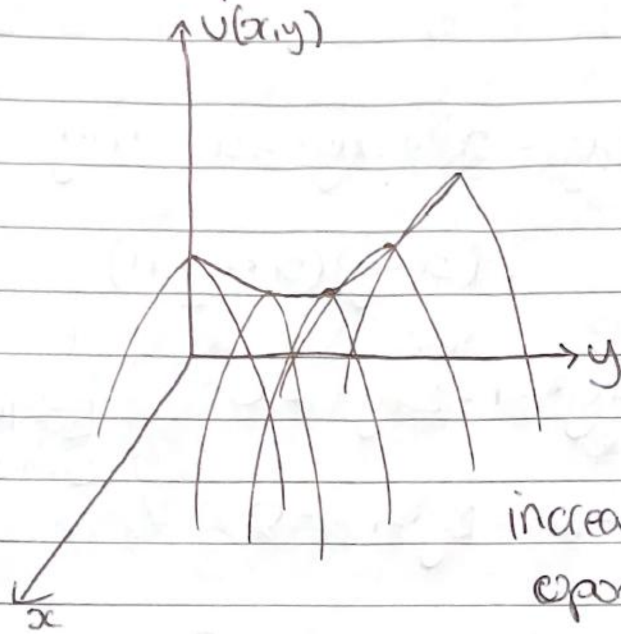
Local Maximum

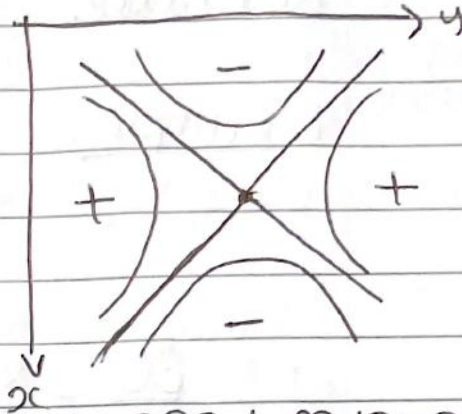




u(x,y) increases away from the point

Saddle Point





OPEN CONTOURS

increasing/decreasing depending upon direction

For functions of two variables $u(x,y)$ Stationary points are located at simultaneous solutions of the equations:

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0.$$

Which produces several (x_0, y_0) points.

Each point (x_0,y_0) has character determined by

$$E_0 \equiv \left[\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) \right]_{(x_0,y_0)} \equiv B^2 - AC$$

if $E_0 > 0$, we have a saddle point

if $E_0 < 0$: $\left(\frac{\partial^2 u}{\partial x^2}\right)^{\leftarrow A} < 0 \Rightarrow$ maximum

$\left(\frac{\partial^2 u}{\partial x^2}\right)^{\leftarrow A} > 0 \Rightarrow$ minimum if $E_0 = 0$, you need to look at higher order derivatives.

Example $\quad v(x,y) = x^3 + xy^2 - x - x^2y - y^3 + y$

$$= (x-y)(x^2+y^2+1)$$

$$\frac{\partial v}{\partial x} = 3x^2 + y^2 - 1 - 2xy = 0$$

$$\frac{\partial v}{\partial y} = 2xy - x^2 - 3y^2 + 1 = 0$$

solve simultaneously

$+$

$$2x^2 - 2y^2 = 0$$

$$x^2 - y^2 = 0$$

$$y^2 = x^2$$

$$y = \pm x$$

plug back into $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ eq^n.

$+$

$$3x^2 + x^2 - 1 - 2x^2 = 0$$

$$2x^2 = 1$$

$$x = \pm 1/\sqrt{2}$$

$-$

$$-2x^2 - x^2 - 3x^2 + 1 = 0$$

$$6x^2 = 1$$

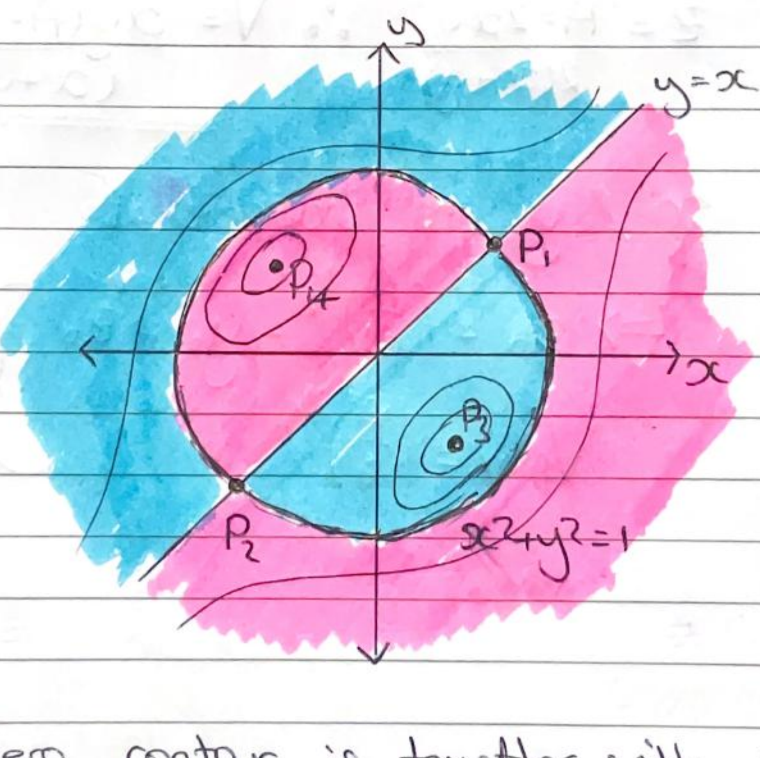
$$x = \pm 1/\sqrt{6}$$

So we have four stationary points for this function.

$$P_1\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), P_2\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), P_3\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), P_4\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

P_T	$A = \left(\frac{\partial^2 v}{\partial x^2}\right) = (6x-2y)$	$B = \left(\frac{\partial^2 v}{\partial x \partial y}\right) = 2x-2y$	$C = \left(\frac{\partial^2 v}{\partial y^2}\right) = 2x-6y$	$B^2-4AC = E_0$	V_0	Type
P_1	$4/\sqrt{2}$	0	$-4/\sqrt{2}$	8	0	Saddle
P_2	$-4/\sqrt{2}$	0	$4/\sqrt{2}$	8	0	Saddle
P_3	$8/\sqrt{6}$	$-4/\sqrt{6}$	$8/\sqrt{6}$	-8	$-\frac{2}{3}\sqrt{\frac{2}{3}}$	Minimum
P_4	$-8/\sqrt{6}$	$4/\sqrt{6}$	$-8/\sqrt{6}$	-8	$\frac{2}{3}\sqrt{\frac{2}{3}}$	Maximum

We can now use this information to plot a graph.



The zero contour is together with $x = y$ and $\underline{x^2+y^2=1}$. Note, link to factorisation!

Warning
when we are faced with a function of several variables and we need to find stationary points - we need to make sure our independent variables are indeed independent. ⟸ !!!!

Example to maximise volume, V, of a rectangular box given surface area A (fixed).

$$\max V = x \cdot y \cdot z \quad \text{given that } A = 2xy + 2xz + 2yz$$

however, x, y, z are not independent here!!

$$z = \frac{A - 2xy}{2(x+y)} \quad \therefore V = \frac{xy(A-2xy)}{(x+y)}$$

now, x & y are independent

$$\left.\begin{aligned} \Rightarrow x_0 &= \left(\frac{A}{6}\right)^{1/2} = y_0 = z_0 \\ V_{max} &= \left(\frac{A}{6}\right)^{3/2} \end{aligned}\right\} \text{eqn for a box.}$$