Classical Mechanics 6

From potentials to forces

$$F(x) = -\frac{\partial U}{\partial x}$$

- minimum of $U(x)$ $\Leftrightarrow$ stable equilibrium
- maximum of $U(x)$ $\Leftrightarrow$ unstable equalibruim

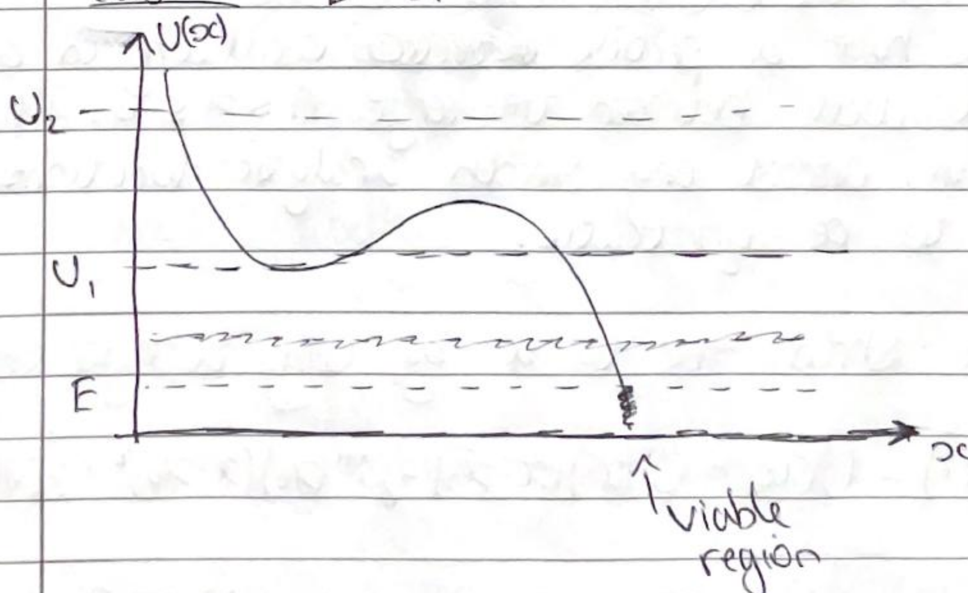
Energy Conservation

The force $F(x) = -dU(x)/dx$ corresponding to the potential $U(x)$ must be conservative, $\therefore$ total energy will be conserved:

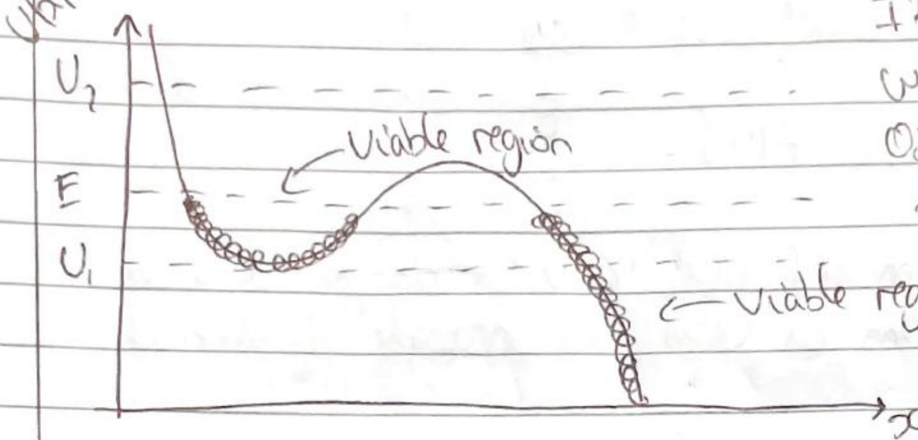
$$E = K + U = \text{const.}$$

As $K = \frac{1}{2}mv^2$ which will always be greater than zero, we can learn a lot from just the potential function.

Case 1 $\quad E < U_1$

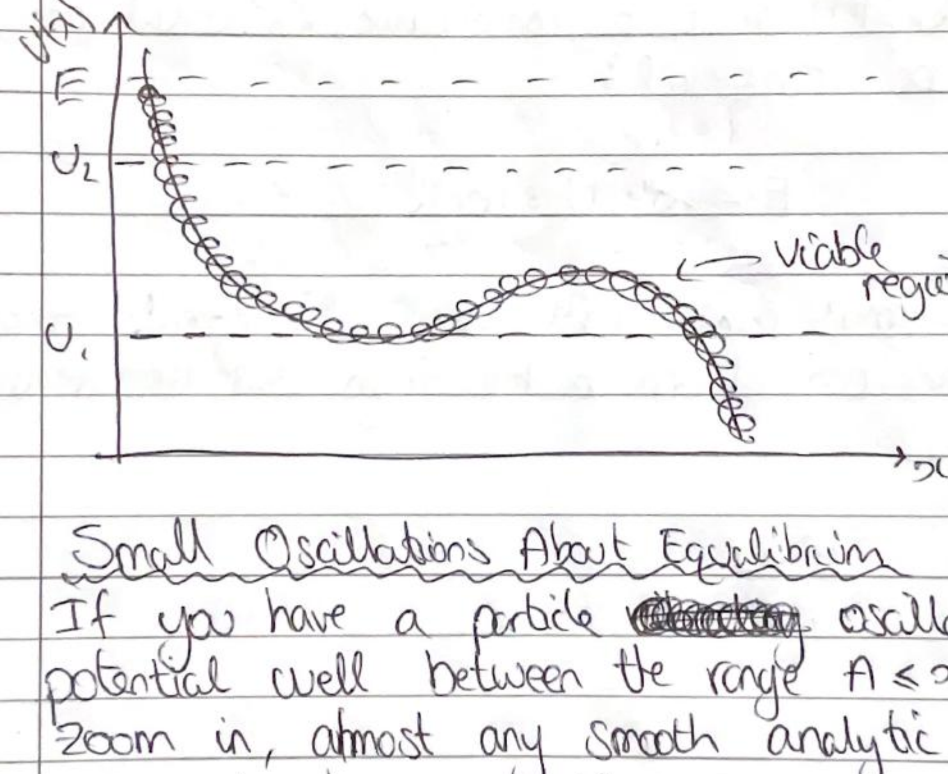


Case 2 $U_1 < E < U_2$



If in potential well would just oscillate back & forth.

Case 3 $U_1 < U_2 < E$



Small Oscillations About Equilibrium

If you have a particle ~~[illegible]~~ oscillating in a potential well between the range $A \le x \le B$. If we zoom in, almost any smooth analytic functions will appear to be quatratic.

We can show this to be by using a taylor expansion

$$U(x) = U(x_0) + U'(x_0)(x-x_0) + \frac{1}{2}U''(x_0)(x-x_0)^2 + \frac{1}{3!}U'''(x_0)(x-x_0)^3$$

The $U'(x)$ will be zero as we chose x_0 to be where the minimum of $U(x)$ is. Any terms larger than $(x-x_0)^3$ we can assume to be zero as they'll be too small to make any meaningful impact

$$U(x) \simeq U(x_0) + \frac{1}{2}U'(x)(x-x_0)^2$$
$$F(x) = -U''(x_0)(x-x_0)$$

This is just hooke's law!

N.B. There are functions for which this will not work, if they aren't (mostly) smooth & analytic.

Simple Harmonic Motion

The equation of motion is:

$$m\frac{d^2x}{dt^2} = -sx$$
$$\Downarrow$$
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\omega = \sqrt{\frac{s}{m}}$$

Method 1 (Guess)

We guess the solution:

$$x(t) = a\sin(\omega t + \phi)$$

Method 2 (better)

We start from conservation of energy:

$$\frac{1}{2}mv^2 + \frac{1}{2}sx^2 = E$$

$$v^2 + \omega^2 x^2 = \frac{2E}{m}$$

$$v^2 + \omega^2 x^2 = \omega^2 a^2$$

$$v^2 = \omega^2 a^2 - \omega^2 x^2$$

$$v = \omega\sqrt{a^2 - x^2}$$ ← Should be a ±

$$\frac{dx}{dt} = \omega\sqrt{a^2 - x^2}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \omega \int dt$$

$$\arcsin\left(\frac{x}{a}\right) = \omega t + \phi$$ ← constant of intergration

$$x = a\sin(\omega t + \phi)$$

Terminology

a = amplitude (max value)

ϕ = phase (translate graph).

ω = angular frequency $= 2\pi f$

T = time period $= \frac{1}{f} = \frac{2\pi}{\omega}$