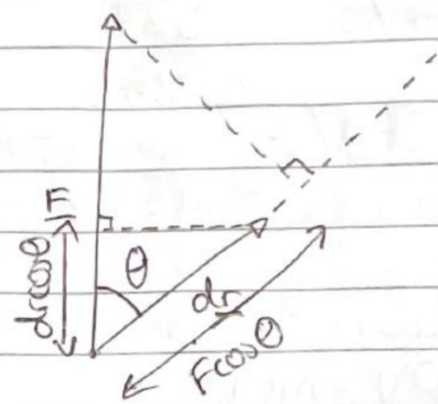
Classical Mechanics 12

Work-Energy Theorem in 3D

$$dW = \underline{F} \cdot \underline{dr} = F\cos\theta \times dr = F \times dr\cos\theta$$

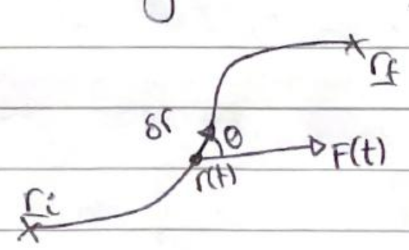


Suppose a bodies moves from $\underline{r_i}$ to $\underline{r_f}$ along a path $\underline{r(t)}$ subject to external force $\underline{F(t)}$. At every point on the journey:

$$\underline{F} = m\frac{d\underline{v}}{dt}$$

The force may depend on: position, time, velocity, ... So $\underline{F(t)}$ may differ from journey to journey.

The route $\underline{r(t)}$ may also change journey to journey.



$$K_f - K_i = \sum \delta K$$

$$\begin{aligned}
\delta K &= \tfrac{1}{2}m|\underline{v}+\delta\underline{v}|^2 - \tfrac{1}{2}m|\underline{v}|^2 \\
&= \tfrac{1}{2}m(\underline{v}+\delta\underline{v})(\underline{v}+\delta\underline{v}) - \tfrac{1}{2}m|\underline{v}|^2 \\
&= \tfrac{1}{2}m(|\underline{v}|^2 + 2\underline{v}\cdot\delta\underline{v} + |\delta\underline{v}|^2) - \tfrac{1}{2}m|\underline{v}|^2 \\
&= m\underline{v}\cdot\delta\underline{v} + O(|\delta\underline{v}|^2)
\end{aligned}$$

For small enough δt, we get

$$K_f - K_i \approx \sum m\underline{v} \cdot \delta\underline{v}$$

Since δt is arbitrarily small, we can also assume that the acceleration is constant through δt.

$$\delta\underline{v} = \underline{a}\delta t = \frac{\underline{F}}{m}\delta t$$

$$K_f - K_i = \sum m\underline{v} \cdot \frac{\underline{F}}{m}\delta t = \sum \underline{F} \cdot \underline{v}\delta t = \sum \underline{F} \cdot \delta\underline{r}$$

Now take limit as $\delta t \to 0$. This defines the path intergral.

$$K_f - K_i = \lim_{\delta t \to 0} \sum \underline{F} \cdot \delta\underline{r} = \int \underline{F} \cdot d\underline{r}$$

Potentials in 3D

In 1D, any position-dependent force field $F(x)$ is conservative: the work done by $F(x)$ in moving a body from x_0 to x is a function of x_0 and x only independent of the details of the journey between them.

This is not true in 3D. Even when the forces is a function of position only, the work done moving from $\underline{r_0}$ to $\underline{r}$ may depend upon the path taken.

This means we cannot use our 1D definition.

$$\cancel{U(\underline{r}) = -\int_{r_0}^{r} F(\underline{r}')\, d\underline{r}'}$$

Fortunately, almost all forces in physics are central, and position-dependent central force fields <u>can</u> be described using potentials.

Central Force

not a vector: <u>Scalar</u>

$$\underline{F}(r) = -\frac{GMm}{r^2}\hat{r} = F(r)\hat{r}$$

A central force is always parallel or anti-parallel to the vector between the two bodies involved.

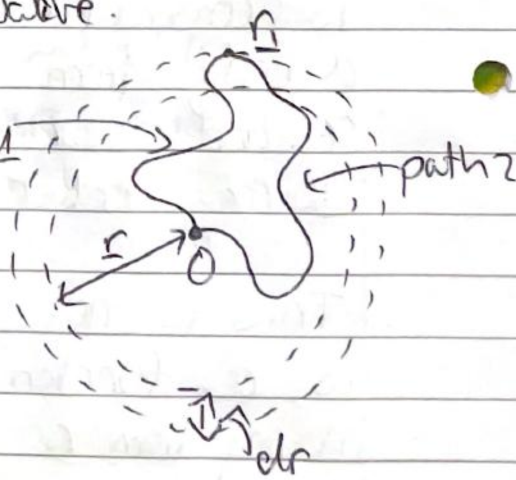
A Note on Reduced Mass

N2 for moon:

- if we treat the earth as infinitley massive $m\ddot{\underline{r}} = F(r)\hat{\underline{r}}$
- more accurately $\mu\ddot{\underline{r}} = F(r)\hat{\underline{r}}$

In the world of central forces (ours), any central force field is conservative.

∴ Our equations for force/potential can apply!



$$U(r) = -\int_{r_0}^{r} F(r')\,dr'$$

$$F(r) = -dU/dr$$