Lecture 4

Functions of complex variables

→ extend the idea of x^a to complex numbers z^a where a can be any real number.
when $a \in \mathbb{Z}$, leads to polynomials

$$\sum_{i=0}^{n} a_i x \rightarrow \sum_{i=0}^{n} a_i z$$

→ going from exponents of reals numbers to exponents of complex numbers
→ trig of $\mathbb{C}$, $\sin(z)$ $\cos(z)$
→ logs of $\mathbb{C}$, $\ln(z)$

→ differentiation of functions of complex no.

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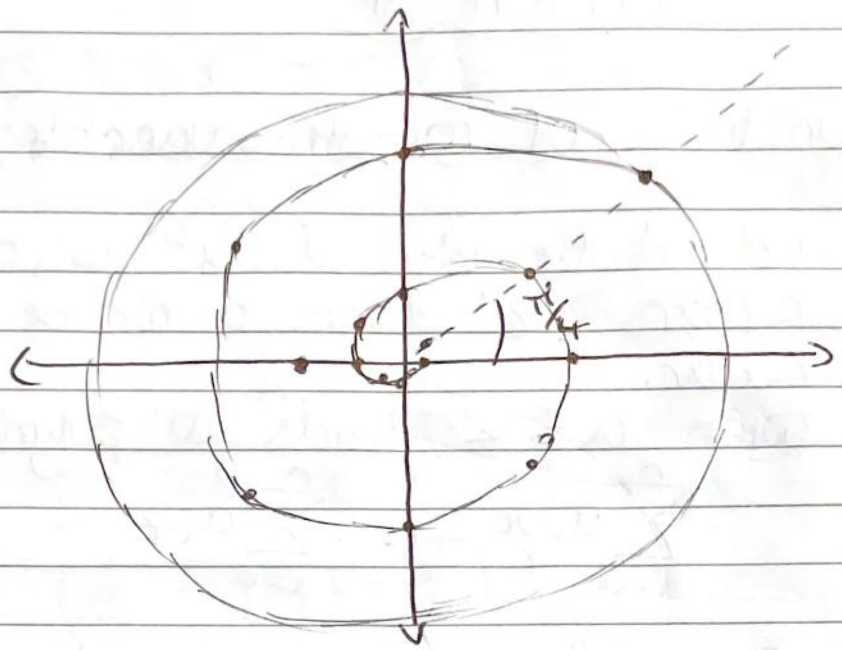
lets begin with z^a $\quad a \in \mathbb{Z}$

from Euler $z = re^{i\theta}$
then $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Example $\left(\frac{1+i}{2}\right) = \frac{1}{2} + \frac{i}{2}$ $\qquad r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$

$z = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}}$ $\qquad \theta = \tan^{-1}(1) = \frac{\pi}{4}$

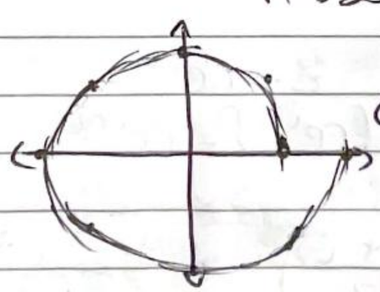
$z^n = \left(\frac{1}{\sqrt{2}}\right)^n e^{i\frac{n\pi}{4}}$



$\lim_{n\to\infty} \left(\frac{1}{\sqrt{2}}\right)^n = 0.$ $\theta_{new} = \theta_{old} + \frac{\pi}{4}.$

we see a spiraling in of the complex no. every time we increase n.

when $r>1$, then $\lim_{n\to\infty} r^n = \infty$.



There is, of course, an intermediate case when $r=1$, when the complex number stays on the unit circle.

$z^n = e^{i\theta}$ $|z^n|^2 = zz^* = e^{in\theta}e^{-in\theta} = \underline{1}.$

Next, lets look at the rational powers, this leads to an obvious question.

$$z^{\frac{1}{n}} = \sqrt[n]{z} = ? \quad (= z')$$

How do we do this?

We know that for z' to exist it must follow

$$\boxed{(z')^n = z}$$

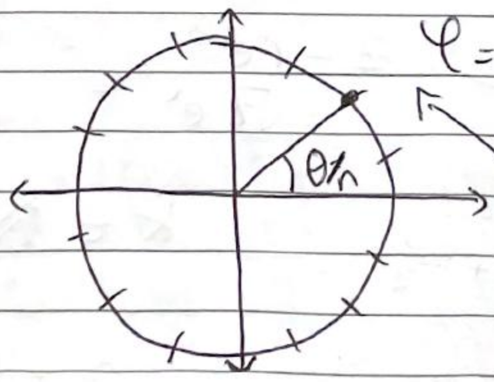
$z = re^{i\theta}$
$z' = \rho e^{i\varphi}$

$$(\rho e^{i\varphi})^n = re^{i\theta}$$

$$\rho^n e^{in\varphi} = re^{i\theta} \implies \rho^n = r \qquad n\varphi = \theta + 2\pi k$$

$$\rho = \sqrt[n]{r} \qquad \varphi = \frac{\theta + 2\pi k}{n} = \frac{\theta}{n} + \underbrace{\frac{2\pi k}{n}}_{\text{we can ignore this.}}$$

$$\varphi = \frac{\theta}{n}$$

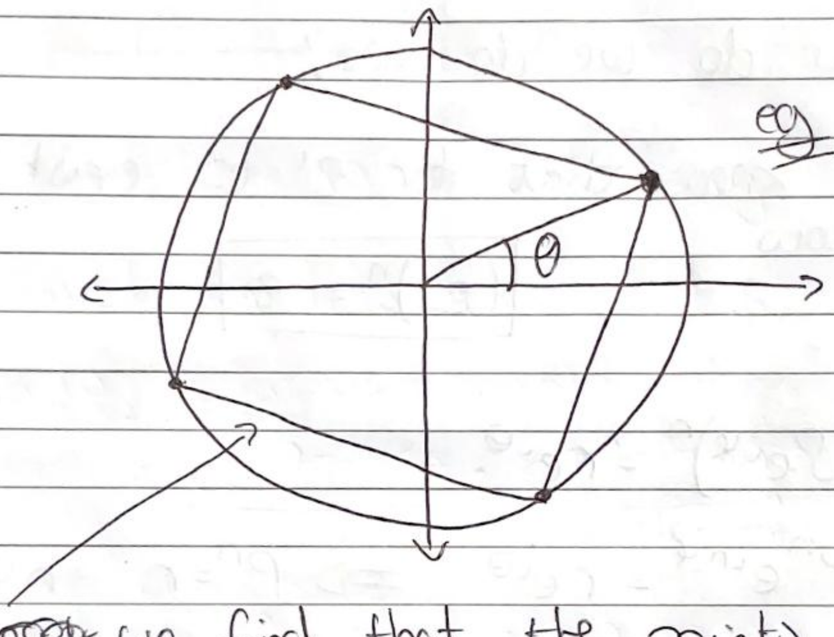


eventualy returns to same position

$$z' = \sqrt[n]{z} = r^{\frac{1}{n}} e^{i\frac{\theta + 2\pi k}{n}}$$
$$= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

Let look at this graphically:

What do we know:
1) Same $|z'|$, $\therefore$ lie on a circle.
2) there are n points.
3) their angles are equidistant. $\left(\frac{\theta + 2\pi k}{n}\right)$



eg $n=4$

we find that the points form a regular polygon. when $n=4$, we get a square.

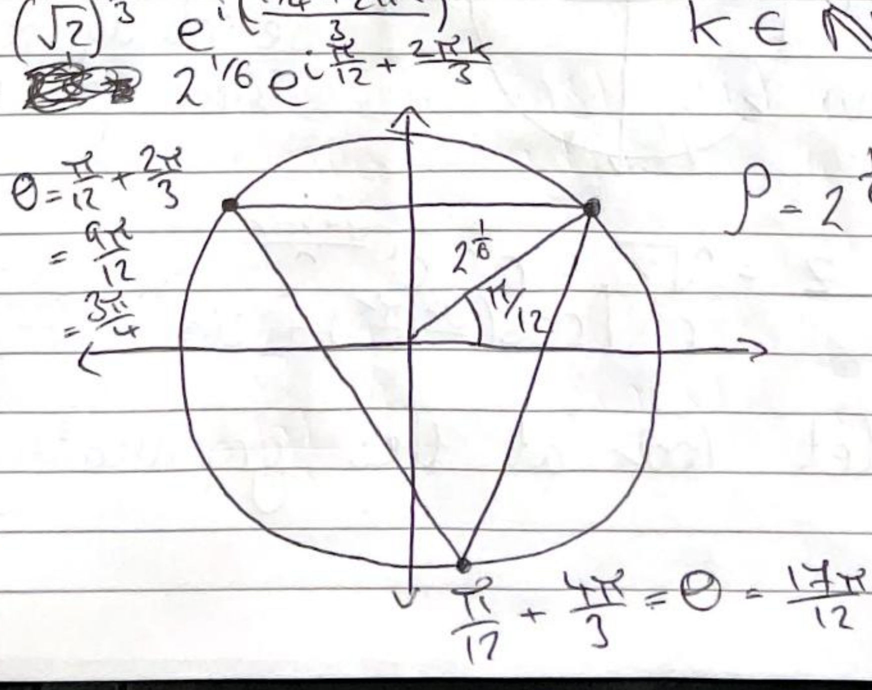
Example

$z = 1+i$
$\sqrt[3]{z} = z'$

$\Rightarrow$ $z = re^{i\theta}$
$= \sqrt{2}\, e^{i\pi/4}$ ← havn't shown working

$$z' = (\sqrt{2})^{\frac{1}{3}}\, e^{i\left(\frac{\pi/4 + 2\pi k}{3}\right)}$$
$$= 2^{1/6}\, e^{i\frac{\pi}{12} + \frac{2\pi k}{3}}$$

$k \in \mathbb{N}$



$\rho = 2^{\frac{1}{6}}$

$z^{\frac{p}{q}} = (z^{\frac{1}{q}})^p$ ⇐ ∴ all rational numbers have now been covered

Irrational $\quad a \neq \frac{p}{q}$

$z = re^{i\theta}$

$z^a = r^a (e^{i\theta})^a = r^a \cdot e^{i\theta a + i2\pi k a}$

$= r^a \cdot (e^{i(\theta + 2\pi k)})^a$

$$z^a = r^a \left[\cos(\theta + 2\pi k a) + i\sin(\theta + 2\pi k a) \right]$$

Eg

$i^{\sqrt{2}} = ?$ $\quad \boxed{i = 1e^{i\frac{\pi}{2}}}$

$(e^{i\frac{\pi}{2}})^{\sqrt{2}} = (e^{i\frac{\pi}{2} + 2\pi k})^{\sqrt{2}} = (e^{i\frac{\pi}{2}\sqrt{2} + i2\sqrt{2}\pi k})$

k now changes the arg(z), not by some fraction of 2π, we ∴ miss unity. We can no longer limit k to n. ∴ infinate number of values.

$= \cos(\frac{\pi}{2}\sqrt{2} + 2\sqrt{2}\pi k) + i\sin(\frac{\pi}{2}\sqrt{2} + 2\sqrt{2}\pi k)$

obviously $1^n = 1$, ∴ $1^{\sqrt{2}} = 1$.