Exercise 4

227367455

Question 1

Let $R_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_2 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $R_3 = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. Closure: Let R_1, R_2 be two rotations around the origin, prove that $R_1 \circ R_2$ is also a rotation around the origin. $R_1 \circ R_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   $= \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha & -(\cos\theta\sin\alpha + \sin\theta\cos\alpha) & 0 \\ \cos\theta\sin\alpha + \sin\theta\cos\alpha & \cos\theta\cos\alpha - \sin\theta\sin\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   $= \begin{bmatrix} \cos(\alpha+\theta) & -\sin(\alpha+\theta) & 0 \\ \sin(\alpha+\theta) & \cos(\alpha+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   2 successive rotations under θ and α is therefore closed under one rotation with angle $\alpha + \theta$ $\square$

2. Associativity: Let R_1, R_2, R_3 be three rotations around the origin, prove that $(R_1 \circ R_2) \circ R_3$ $= R_1 \circ (R_2 \circ R_3)$

   $(R_1 \circ R_2) \circ R_3 = \left(\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \circ \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   $= \begin{bmatrix} \cos(\alpha+\theta) & -\sin(\alpha+\theta) & 0 \\ \sin(\alpha+\theta) & \cos(\alpha+\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by Closure

   $= \begin{bmatrix} \cos(\alpha+\theta+\beta) & -\sin(\alpha+\theta+\beta) & 0 \\ \sin(\alpha+\theta+\beta) & \cos(\alpha+\theta+\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ By Closure

   $R_1 \circ (R_2 \circ R_3) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \left(\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$

   $= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 0 \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ By Closure

   $= \begin{bmatrix} \cos(\alpha+\theta+\beta) & -\sin(\alpha+\theta+\beta) & 0 \\ \sin(\alpha+\theta+\beta) & \cos(\alpha+\theta+\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ By Closure

   $\implies (R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$ as needed $\square$

3. Identity element: Prove that there exists rotation around the origin R_{id} s.t. for all rotations around the origin R it holds that $R \circ R_{id} = R_{id} \circ R = R$

   There exist an Identity rotation

$\iff$ there exists an identity matrix representation R_{id}

$\iff R_{id} = I_3$

$\iff R_{id} = R_\theta$ st $\cos\theta = 1 \wedge \sin\theta = -\sin\theta = 0$

Clearly there exists such $\theta = 0$ and therefore all the above holds $\square$

4. Inverse element: Prove that for every rotation around the origin R there exists R' which is also a rotation around the origin s.t. $R \circ R' = R' \circ R = R_{id}$. Given R, what is R'?

   Let $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   $|R| = \cos^2\theta - (-\sin^2\theta) = 1$ so R is invertible

   $R' = R^{-1} = \frac{1}{|R|} Adj(R) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   Notice $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   This inverse rotation matrix can be represented with a rotation matrix with $\theta' = -\theta$

5. Now, show that all of the above are true for rotations around any single point (For example all the rotations around (1,1)) are a group.

   Let T be the translation matrix $T = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$ and its inverse is $T^{-1} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

   We will show closure of rotation around the point (t_x, t_y) as follows:

   Closure: $T \circ R_1 \circ T^{-1} \circ T \circ R_2 \circ T^{-1}$

   $= T \circ R_1 \circ R_2 \circ T^{-1}$ Since $R_1 \circ R_2$ is closed by 1a then we rotate (t_x, t_y) by $R_1 \circ R_2$ $\square$

   Associativity: $(T \circ R_1 \circ T^{-1} \circ T \circ R_2 \circ T^{-1}) \circ T \circ R_3 \circ T^{-1}$

   $= T \circ (R_1 \circ R_2) \circ T^{-1} \circ T \circ R_3 \circ T^{-1}$

   $= T \circ (R_1 \circ R_2) \circ R_3 \circ T^{-1}$

   $= T \circ R_1 \circ (R_2 \circ R_3) \circ T^{-1}$ using associativity of rotation matrices 1b

   $= T \circ R_1 \circ T \circ T^{-1} \circ (R_2 \circ R_3) \circ T^{-1}$

   $= T \circ R_1 \circ T \circ T^{-1} \circ (R_2 \circ T^{-1} \circ T \circ R_3) \circ T^{-1}$

   $= T \circ R_1 \circ T \circ (T^{-1} \circ R_2 \circ T^{-1} \circ T \circ R_3 \circ T^{-1})$ as needed $\square$

   Identity element:

   $T \circ R \circ T^{-1} = I_3$

   $\iff R \circ T^{-1} = I_3 \circ T^{-1}$

   $\iff R = I_3$ and we know such rotation matrix exists by 1c $\square$

   Inverse element: Since T is invertible and R is invertible by 1d

   $(T \circ R \circ T^{-1})^{-1} = T^{-1} \circ R^{-1} \circ T$ as needed $\square$

Question 2

Write the matrices representing the following transformations. For each transformation right its specific type (linear/rigid/similarity/affine/projective)

1. Rotate 30° counterclockwise around the line $\ell(t) = (1, -2, 1) + t(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ and uniformly scale by 1.5.

   $$R_x(\tfrac{\pi}{6}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $$T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $$S = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $v_1 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

   $v_2 = \frac{v_1 \times (0,0,1)}{|v_1 \times (0,0,1)|} = (0, 1, 0)$

   $v_3 = \frac{v_1 \times v_2}{|v_1 \times v_2|} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

   $$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $$F\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right) = T^{-1} \circ R^T \circ S \circ R_x(\tfrac{\pi}{6}) \circ R \circ T \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

   Similarity because of S

2. Scale by 5 in the direction of $(3, 1, -4)$ then shear by a factor of 0.2 in the Y direction (meaning after the shearing x'=x+0.2y, z'=z+0.2y

   $v_1 = \frac{(3,1,-4)}{|(3,1,-4)|} = (\frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{-4}{\sqrt{26}})$

   $v_2 = \frac{v_1 \times (0,0,1)}{|v_1 \times (0,0,1)|} = (\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}, 0)$

   $v_3 = \frac{v_1 \times v_2}{|v_1 \times v_2|} = (\frac{-6}{\sqrt{65}}, \frac{-4}{\sqrt{65}}, \frac{-5}{\sqrt{65}})$

   $$R = \begin{bmatrix} \frac{3}{\sqrt{26}} & \frac{1}{\sqrt{26}} & \frac{-4}{\sqrt{26}} & 0 \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} & 0 & 0 \\ \frac{-6}{\sqrt{65}} & \frac{-4}{\sqrt{65}} & \frac{-5}{\sqrt{65}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   $$\text{B} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right) = H \circ R^T \circ B \circ R \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Affine because of H

3. Reflect around the yz plane then translate in the (1,5,2) direction by a factor of 0.5

$$T = \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right) = T \circ R \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rigid because every properties are kept

4. Scale uniformly by a factor of 0.3 then project (cavalier projection) on the XY plain with an angle of 45°

$$U = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F\left(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\right) = P \circ U \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Linear because of P

Question 3

1. This is an orthographic projection $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. We see that the point (1,1,1) is projected to (1, 2, 0)

   $\phi = \frac{\pi}{2}$ special case where $x_p - x = 0 \wedge y_p - p \neq 0$

   $d = \sqrt{(1-1)^2 + (2-1)^2} = 1$

   $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. The point $(1, 0, 1)$ is projected to $(1.5, 0, 0)$

   $\phi = \tan^{-1} \frac{0}{0.5} = 0$

   $d = \sqrt{(1.5-1)^2 + (0-0)^2} = 0.5$

   $\begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. The point $(1, 1, 1)$ is projected to $(1.5, 1, 0)$

   $\phi = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}$

   $d = \sqrt{(2-1)^2 + (2-1)^2} = \sqrt{2}$

   $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. The point $(1, 1, 1)$ is projected to $(1.5, 1, 0)$

   $d = \sqrt{(1.42-1)^2 + (1.14-1)^2} = 0.4427$

   $\phi = \tan^{-1} \frac{0.14}{0.42} = 0.32175$ rad

   $\begin{bmatrix} 1 & 0 & 0.42 & 0 \\ 0 & 1 & 0.14 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Question 4

1. Project (perspective) a scene with C.O.P=(0,0,0) and a viewing plane z=-3. The line l=(14,2,-7)+t(1,3,0). What is the parametric representation of the line after the projection, l_p?

   $\ell_x(t) = \frac{-(14+t)(-3)}{7}, \ell_y(t) = \frac{-(2+3t)(-3)}{7}, \ell_z(t) = -3$

   $\ell(t) = (6, \frac{6}{7}, -3) + t(\frac{3}{7}, \frac{9}{7}, 0)$

2. We project (perspective) a scene with C.O.P = (0,0,0) and a viewing plane z=-2. We know that two intersecting lines in our scenes, l_1,l_2, have two different vanishing points, (1,1,-2),(2,4,-2) accordingly. What is the angle θ between l_1 and l_2?

   Since the vectors pass through the COP which is the origin we can consider the vanishing points as vectors and do the following:

   $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$

   $\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|} = \frac{2+4+4}{\sqrt{1+1+4}\sqrt{4+16+4}} = \frac{10}{\sqrt{6}\sqrt{24}} = \frac{10}{\sqrt{144}} = \frac{10}{12} = \frac{5}{6}$

   $\theta = 33.557^o$

3. Find the matrix representing the perspective projection where the COP=(1,-3,2) and projection plane with implicit representation -2x-y+3z=0.

   $f = \frac{(-2,-1,3)\cdot(1,-3,2)}{\|(-2,-1,3)\|} = \frac{7}{\sqrt{14}} = \sqrt{\frac{7}{2}}$

   $T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   We will create an orthonormal plane from which we will set our Z axis

   $v_1' = (-2, -1, 3) v_2 = \frac{v_1'}{\|v_1'\|} = (\frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}})$

   $v_2' = (-2, -1, 3) \times (1, 0, 0) = (0, 3, 1), v_2 = \frac{v_2'}{\|v_2'\|} = (0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$

   $v_3' = v_1' \times v_2' = (-10, 2, 6), v_2 = \frac{v_3'}{\|v_3'\|} = (\frac{-5}{\sqrt{35}}, \frac{1}{\sqrt{35}}, \frac{-3}{\sqrt{35}})$

   $R = \begin{bmatrix} 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ \frac{-5}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \frac{-3}{\sqrt{35}} & 0 \\ \frac{-2}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{3}{\sqrt{14}} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   Finally, we have the perspective projection matrix

   $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{-2}{7}} & 0 \end{bmatrix}$

   We therefore obtain the perspective projection matrix M by applying the following:

   $M = T^{-1} \circ R^T \circ F \circ R \circ T$