Exercise 2

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Question 1

Let ℓ_1 be the line that passes through $p_1 = (2, 9, 8)$ and $p_2 = (1, 9, 9)$ and let ℓ_2 be the line that passes through $p_3 = (1, 1, 1)$ and $p_4 = (2, 5, 4)$

1. Find out if the two lines intersect and if so, find the intersection point of ℓ_1 and ℓ_2

   $u_1 = p_2 - p_1 = \begin{bmatrix} 1-2 \\ 9-9 \\ 9-8 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

   $\ell_1(\lambda) = p_1 + \lambda u_1 = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

   $u_2 = p_4 - p_3 = \begin{bmatrix} 2-1 \\ 5-1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

   $\ell_2(\lambda) = p_3 + \lambda u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

   $\frac{u_1 \cdot u_2}{\|u_1\|\|u_2\|} = \frac{-1+3}{\sqrt{2}\sqrt{1+16+9}} \neq 1, -1 \implies \ell_1$ is not parallel to ℓ_2

   $\begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

   $\iff \lambda_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix}$

   $\iff \lambda_2 = -\lambda_1 = 2$

   Now we plug in!

   $p_0 = \ell_2(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = (1 + 2(1), 1 + 2(4), 1 + 2(3)) = (3, 9, 7) \square$

2. Let S be the sphere whose center is the intersection of ℓ_1 and ℓ_2 and whose radius is $r = 4$. Write the implicit representation of the sphere.

   $(x-3)^2 + (y-9)^2 + (z-7)^2 = 4^2$

3. Find the implicit representation of the two planes Π_1 and Π_2 that are tangent to the sphere S at the points of intersections of ℓ_1 towards p_2 and ℓ_2 towards p_4, respectively.

   $(1 - \lambda - 3)^2 + (9-9)^2 + (9 + \lambda - 7)^2 = 4^2$

   $(-2-\lambda)^2 + (2+\lambda)^2 = 16$

   $(2+\lambda)^2 = 8$

   $2 + \lambda = \pm\sqrt{8}$

   $\lambda = \pm\sqrt{8} - 2$

We get $(3 \mp \sqrt{8}, 9, 7 \pm \sqrt{8})$, using euclidian distance, we get $(3-\sqrt{8}, 9, 7+\sqrt{8})$

Now since the line are passing through the center then they are normal to the sphere at their point of intersection

We do $(3-\sqrt{8}, 9, 7+\sqrt{8}) \cdot (-1, 0, 1) = 7 - 3 + 2\sqrt{8} = 4 + 2\sqrt{8}$

We get the plane equation $\Pi_1 = (-1, 0, 1) \cdot (x, y, z) - 4 - 2\sqrt{8} = 0$

Since we get a negative value when we plug in p_2 we negate the normal and obtain:

$\Pi_1 = (1, 0, -1) \cdot (x, y, z) + 4 + 2\sqrt{8} = 0$

$(1 + \lambda - 3)^2 + (1 + 4\lambda - 9)^2 + (1 + 3\lambda - 7)^2 = 4^2$

$(\lambda - 2)^2 + (-8 + 4\lambda)^2 + (-6 + 3\lambda)^2 = 16$

$4 - 4\lambda + \lambda^2 + 64 - 64\lambda + 16\lambda^2 + 36 - 36\lambda + 9\lambda^2 = 16$

$26\lambda^2 - 104\lambda + 88 = 0$

Solving the equation: $\lambda = 2.7845, 1.2155$

We repeat the procedure as above:

$(3.7845, 12.138, 9.3535)$, $(2.2155, 4.862, 3.6465)$ and take $(2.2155, 4.862, 3.6465)$ as its distance is the closest to p_4

$(2.2155, 4.862, 3.6465) \cdot (1, 4, 3) = 32.603$

We get the plane equation $\Pi_1 = (1, 4, 3) \cdot (x, y, z) - 32.6 = 0$

If we plug in p_4 we get a positive value so our surface is in the right direction $\square$

Question 2

1. Let $\ell = (2, 0, 3) + \lambda(1, 5, 2)$ be a line. Find the projection of $p = (3, -1, 4)$ on ℓ

   Clearly $p_0 = \ell(0) = (2, 0, 3)$ is a point on ℓ

   We will project $p - p_0$ onto ℓ

   Let $\hat{u} = \dfrac{\begin{bmatrix}1\\5\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\5\\2\end{bmatrix}\right\|} = \frac{1}{\sqrt{1^2+5^2+2^2}} \begin{bmatrix}1\\5\\2\end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix}1\\5\\2\end{bmatrix}$

   $proj_{\hat{u}}(p - p_0) = (\hat{u} \cdot (p - p_0))\hat{u}$

   $= \frac{1}{30}\left(\begin{bmatrix}1\\5\\2\end{bmatrix} \cdot \begin{bmatrix}3-2\\-1-0\\4-3\end{bmatrix}\right)\begin{bmatrix}1\\5\\2\end{bmatrix}$

   $= \frac{1}{30}\left(\begin{bmatrix}1\\5\\2\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix}\right)\begin{bmatrix}1\\5\\2\end{bmatrix}$

   $= \frac{1}{30}(1 - 5 + 2)\begin{bmatrix}1\\5\\2\end{bmatrix}$

   $= \frac{1}{30}(-2)\begin{bmatrix}1\\5\\2\end{bmatrix}$

   $= \frac{-1}{15}\begin{bmatrix}1\\5\\2\end{bmatrix}$

Now to find the projection of the point p we do:

$p_0 + proj_{\hat{u}}(p - p_0) = (2,0,3) + \frac{-1}{15}\begin{bmatrix}1\\5\\2\end{bmatrix}$

$= (1\frac{14}{15}, -\frac{1}{3}, 2\frac{13}{15})$ □

2. Find an implicit and a parametric representation for the plane that contains both p and ℓ

$v = (p - p_0) = \begin{bmatrix}3-2\\-1-0\\4-3\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

$\Pi(\lambda, \mu) = (2,0,3) + \lambda\begin{bmatrix}1\\5\\2\end{bmatrix} + \mu\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$\begin{bmatrix}1\\-1\\1\end{bmatrix} \times \begin{bmatrix}1\\5\\2\end{bmatrix} = \begin{vmatrix}i & j & k\\1 & 5 & 2\\1 & -1 & 1\end{vmatrix} = [(5)(1)-(2)(-1)]i-[(1)(1)-(1)(2)]j+[(1)(-1)-(5)(1)]k = 7i+j-6k$

$n = \begin{bmatrix}7\\1\\-6\end{bmatrix}$

$n \cdot ((2,0,3) - (0,0,0)) = \begin{bmatrix}7\\1\\-6\end{bmatrix} \cdot \begin{bmatrix}2\\0\\3\end{bmatrix} = 14 - 18 = -4$

$\Pi : \begin{bmatrix}7\\1\\-6\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = -4$ □

Question 3

Given the following points:
$P_0 = (-1,-1,3), P_1 = (-1,3,-1), P_2 = (3,3,5), P_3 = (3,5,3), P_4 = (\frac{-1}{2},3,3)$

1. Show that P_0, P_1, P_2, P_3 lie on the same plane, H, and find the implicit equation of H

$P_1 - P_0 = \begin{bmatrix}-1-(-1)\\3-(-1)\\-1-3\end{bmatrix} = \begin{bmatrix}0\\4\\-4\end{bmatrix}$

$P_1 - P_0 = \begin{bmatrix}3-(-1)\\3-(-1)\\5-3\end{bmatrix} = \begin{bmatrix}4\\4\\2\end{bmatrix}$

$\begin{vmatrix}i & j & k\\0 & 4 & -4\\4 & 4 & 2\end{vmatrix} = [(4)(2)-(-4)(4)]i-[(0)(2)-(-4)(4)]j+[(0)(4)-(4)(4)] = 24i-16j-16k = 3i-2j-2k$

$n = \begin{bmatrix}3\\-2\\-2\end{bmatrix}$

$\begin{bmatrix}3\\-2\\-2\end{bmatrix} \cdot \begin{bmatrix}-1\\-1\\3\end{bmatrix} = -3+2-6 = -7$

Therefore we can build H out of these 3 points.

$H : \begin{bmatrix}3\\-2\\-2\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} + 7 = 0$

Now plug in P_3:

$$\begin{bmatrix}3\\-2\\-2\end{bmatrix} \cdot \begin{bmatrix}3\\5\\3\end{bmatrix} + 7 = 9 - 10 - 6 + 7 = 0 \text{ as needed}$$

Therefore since P_3 lies on H which was made with P_0, P_1 and P_2, then they all belong to the same plane H

2. b) Determine the outwards facing unit normal vector of each triangular face Calculate the implicit representation of the planes

   In order to do so we can iteratively find the normal n and invert it if we get a positive value when plugging in the 2 remaining points. However since the picture is provided we will use the RHR.

   H_1 defined by P_0, P_2, P_4:

   $$v_0 = 2(P_4 - P_0) = \begin{bmatrix}1\\8\\0\end{bmatrix},\ v_1 = P_2 - P_0 = \begin{bmatrix}4\\4\\2\end{bmatrix},$$

   $$v_1 \times v_0 = \begin{vmatrix}i & j & k\\4 & 4 & 2\\1 & 8 & 0\end{vmatrix} = \begin{bmatrix}-16\\2\\28\end{bmatrix} = \begin{bmatrix}-8\\1\\14\end{bmatrix} = \begin{bmatrix}-\frac{8}{261}\\\frac{1}{261}\\\frac{14}{261}\end{bmatrix}$$

   $d = ()$

   H_2 defined by P_0, P_1, P_4:

   $$v_0 = P_1 - P_0 = \begin{bmatrix}0\\4\\-4\end{bmatrix},\ v_1 = 2(P_4 - P_0) = \begin{bmatrix}1\\8\\0\end{bmatrix},$$

   $$v_0 \times v_1 = \begin{vmatrix}i & j & k\\1 & 8 & 0\\0 & 4 & -4\end{vmatrix} = \begin{bmatrix}-32\\4\\4\end{bmatrix} = \begin{bmatrix}-8\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{-8}{66}\\\frac{1}{66}\\\frac{1}{66}\end{bmatrix}$$

   H_3 defined by P_1, P_3, P_4 :

   $$v_0 = P_3 - P_1 = \begin{bmatrix}4\\2\\4\end{bmatrix},\ v_1 = 2(P_4 - P_1) = \begin{bmatrix}1\\0\\8\end{bmatrix},$$

   $$v_1 \times v_0 = \begin{vmatrix}i & j & k\\1 & 0 & 8\\4 & 2 & 4\end{vmatrix} = \begin{bmatrix}-16\\28\\2\end{bmatrix} = \begin{bmatrix}-8\\14\\1\end{bmatrix} = \begin{bmatrix}\frac{-8}{261}\\\frac{14}{261}\\\frac{1}{261}\end{bmatrix}$$

   H_4 defined by P_2, P_3, P_4:

   $$v_0 = P_2 - P_3 = \begin{bmatrix}0\\2\\-2\end{bmatrix},\ v_1 = 2(P_4 - P_3) = \begin{bmatrix}-7\\-4\\0\end{bmatrix},$$

   $$v_1 \times v_0 = \begin{vmatrix}i & j & k\\-7 & -4 & 0\\0 & -2 & 2\end{vmatrix} = \begin{bmatrix}-8\\14\\14\end{bmatrix} = \begin{bmatrix}-4\\7\\7\end{bmatrix} = \begin{bmatrix}\frac{-4}{114}\\\frac{7}{114}\\\frac{7}{114}\end{bmatrix}$$

3. c)

   $(-8, 1, 14) \cdot (\frac{-1}{2}, 3, 3) = 4 + 3 + 42 = 49$

   $H_1 : -8x + y + 14z = 49$

   $(-8, 1, 1) \cdot (\frac{-1}{2}, 3, 3) = 4 + 3 + 3 = 10$

$H_2 : -8x + y + z = 10$

$(-8, 14, 1) \cdot (\frac{-1}{2}, 3, 3) = 4 + 42 + 3 = 49$

$H_3 : -8x + 14y + 1 = 49$

$(-4, 7, 7) \cdot (\frac{-1}{2}, 3, 3) = 4 + 21 + 21 = 46$

$H_4 : -4x + 7y + 7z = 46$

And we already calculated H

4. d) We plug in in all the shapes and it must be negative for all shapes except H i) Is inside because : $H(-1/2, 1, 2) > 0$ and $H1(-1/2, 1, 2), H2(-1/2, 1, 2), H3(-1/2, 1, 2), H4(-1/2, 1, 2) < 0$ ii) Outside $H(1, 0, 1) < 0$ iii) Outside $H(3, 2, 4) < 0$

Question 4

A 2D light ray is sent from point P=(1,-1). It is reflected off a surface (represented by a line) at $R = (6, 11)$, and reaches a receiver point at $Q = (25, 13\frac{2}{17})$. Note that light rays hitting a surface reflect in a direction which is symmetric according to the normal.

1. Find the implicit representation of the surface s.t. its "up" is towards P (i.e. it faces the incoming ray). See the illustration below. The small black arrow is the direction the surface is facing. $RP = P - R = (5, 12), RQ = Q - R = (19, \frac{36}{17})$

   Normalizing we get $\hat{RP} = \frac{(5,12)}{\sqrt{5^2+12^2}} = \frac{(5,12)}{13}$

   Normalizing we get $\hat{RQ} = \frac{(19,\frac{36}{17})}{\sqrt{19^2+\frac{36^2}{17^2}}} = \frac{(19,\frac{36}{17})}{\sqrt{\frac{36^2+323^2}{17^2}}} = \frac{(19,\frac{36}{17})}{\frac{325}{17}}$

   Now we want to get $RQ - RP$ (Think of a diamond shaped figure, we could also get the average)

   $n = (\frac{17\cdot19}{325} - \frac{5}{13}, \frac{36}{325} - \frac{5}{13}) = (\frac{198}{325}, \frac{-264}{325}) = (3, -4)$

   $n \cdot (R - 0) = 6 \cdot 3 + 11(-4) = 18 - 44 = -26$

   We get $(3, -4) \cdot (x, y) + 26 = 0$

2. Find the angle between the ray and the surface.

   $cos(\alpha) = \frac{n\cdot RP}{\|n\|\cdot\|RP\|} = \frac{3(5)+(-4)12}{13\cdot\sqrt{3^2+4^2}} = \frac{-33}{13\cdot5} = \frac{-33}{65}$

   $\theta = \alpha - \frac{\pi}{2} = \arccos(\frac{-1}{13}) - \frac{\pi}{2} = 120.51^o - 90^o = 30.51^o$