Exercise 2

227367455

Question 1

Let ℓ_1 be the line that passes through $p_1 = (2, 9, 8)$ and $p_2 = (1, 9, 9)$ and let ℓ_2 be the line that passes through $p_3 = (1, 1, 1)$ and $p_4 = (2, 5, 4)$

1. Find out if the two lines intersect and if so, find the intersection point of ℓ_1 and ℓ_2

   $u_1 = p_2 - p_1 = \begin{bmatrix} 1-2 \\ 9-9 \\ 9-8 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

   $\ell_1(\lambda) = p_1 + \lambda u_1 = \begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

   $u_2 = p_4 - p_3 = \begin{bmatrix} 2-1 \\ 5-1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

   $\ell_2(\lambda) = p_3 + \lambda u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

   $\frac{u_1 \cdot u_2}{\|u_1\|\|u_2\|} = \frac{-1+3}{\sqrt{2}\sqrt{1+16+9}} \neq 1, -1 \implies \ell_1$ is not parallel to ℓ_2

   $\begin{bmatrix} 1 \\ 9 \\ 9 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

   $\iff \lambda_1 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix}$

   $\iff \lambda_1 = -\lambda_2 = 2$

   Now we plug in!

   $p_0 = \ell_2(-2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = (1 - 2(1), 1 - 2(4), 1 - 2(3)) = (-1, -7, -5)$

2. Let S be the sphere whose center is the intersection of ℓ_1 and ℓ_2 and whose radius is $r = 4$. Write the implicit representation of the sphere.

   $(x + 1)^2 + (y + 7)^2 + (z + 5)^2 = 4^2$

3. Find the implicit representation of the two planes Π_1 and Π_2 that are tangent to the sphere S at the points of intersections of ℓ_1 towards p_2 and ℓ_2 towards p_4, respectively.

   We could solve the quadratic equation but we already know that we need can the point of intersection in $\ell_i(\lambda_i \pm 4)$

   That is we want to solve $p' = argmin_p\big\{\|p - p_1\|\big|p = \ell_1(\lambda_1 + r), r \in \{4, -4\}\big\}$

   and $p'' = argmin_p\big\{\|p - p_1\|\big|p = \ell_2(\lambda_2 + r), r \in \{4, -4\}\big\}$

$$\nabla S = \nabla((x+1)^2 + (y+7)^2 + (z+5)^2) = \begin{bmatrix} 2(x+1) \\ 2(y+7) \\ 2(z+5) \end{bmatrix}$$

$n_1 = \nabla S(p'), n_2 = \nabla S(p'')$

We get $\Pi_1 = n_1 \cdot (p_0 - p')$ and $\Pi_2 = n_2 \cdot (p_0 - p'')$

Question 2

1. Let $\ell = (2,0,3) + \lambda(1,5,2)$ be a line. Find the projection of $p = (3,-1,4)$ on ℓ

   Clearly $p_0 = \ell(0) = (2,0,3)$ is a point on ℓ

   We will project $p - p_0$ onto ℓ

   Let $\hat{u} = \dfrac{\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \right\|} = \frac{1}{\sqrt{1^2+5^2+2^2}} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   $proj_{\hat{u}}(p - p_0) = (\hat{u} \cdot (p - p_0))\hat{u}$

   $= \frac{1}{30} \left(\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3-2 \\ -1-0 \\ 4-3 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   $= \frac{1}{30} \left(\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   $= \frac{1}{30}(1 - 5 + 2) \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   $= \frac{1}{30}(-2) \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   $= \frac{-1}{15} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   Now to find the projection of the point p we do:

   $p_0 + proj_{\hat{u}}(p - p_0) = (2,0,3) + \frac{-1}{15} \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$

   $= (1\frac{14}{15}, -\frac{1}{3}, 2\frac{13}{15})$

2. Find an implicit and a parametric representation for the plane that contains both p and ℓ

   $v = (p - p_0) = \begin{bmatrix} 3-2 \\ -1-0 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

   $\Pi(\lambda, \mu) = (2,0,3) + \lambda \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

   $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{vmatrix} = [(5)(1)-(2)(-1)]i - [(1)(1)-(1)(2)]j + [(1)(-1)-(5)(1)]k = 7i + j - 6k$

$$n = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}$$

$$n \cdot ((2,0,3) - (0,0,0)) = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 14 - 18 = -4$$

$$\Pi : \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -4$$

Question 3

Given the following points:
$P_0 = (-1,-1,3), P_1 = (-1,3,-1), P_2 = (3,3,5), P_3 = (3,5,3), P_4 = (\frac{-1}{2},3,3)$

1. Show that P_0, P_1, P_2, P_3 lie on the same plane, H, and find the implicit equation of H

$$P_1 - P_0 = \begin{bmatrix} -1-(-1) \\ 3-(-1) \\ -1-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}$$

$$P_1 - P_0 = \begin{bmatrix} 3-(-1) \\ 3-(-1) \\ 5-3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{vmatrix} i & j & k \\ 0 & 4 & -4 \\ 4 & 4 & 2 \end{vmatrix} = [(4)(2)-(-4)(4)]i-[(0)(2)-(-4)(4)]j+[(0)(4)-(4)(4)] = 24i-16j-16k = 3i-2j-2k$$

$$n = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = -3+2-6 = -7$$

Therefore we can build H out of these 3 points.

$$H : \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + 7 = 0$$

Now plug in P_3:

$$\begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} + 7 = 9 - 10 - 6 + 7 = 0 \text{ as needed}$$

Therefore since P_3 lies on H which was made with P_0, P_1 and P_2, then they all belong to the same plane H

2. b and c) Determine the outwards facing unit normal vector of each triangular face Calculate the implicit representation of the planes

In order to do so we will iteratively find the normal n and invert it if we get a positive value when plugging in the 2 remaining points

First we plug in P_4 in H

$$\begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{2} \\ 3 \\ 3 \end{bmatrix} + 7 < 7 - 12 < 0 \text{ as needed}$$

$$v_0 = 2(P_4 - P_0) = \begin{bmatrix}1\\8\\0\end{bmatrix},\ v_1 = 2(P_4 - P_1) = \begin{bmatrix}1\\0\\8\end{bmatrix},$$
$$v_2 = 2(P_4 - P_2) = \begin{bmatrix}-7\\0\\-4\end{bmatrix},\ v_3 = 2(P_4 - P_3) = \begin{bmatrix}-7\\-4\\0\end{bmatrix}$$

H_1 defined by P_0, P_2, P_4 or v_0, v_2, P_4:

$$\begin{vmatrix} i & j & k \\ 1 & 8 & 0 \\ -7 & 0 & -4 \end{vmatrix} = \begin{bmatrix}-32\\4\\56\end{bmatrix} = \begin{bmatrix}-8\\1\\14\end{bmatrix}$$

H_2 defined by P_0, P_1, P_4 or v_0, v_1, P_4:

$$\begin{vmatrix} i & j & k \\ 1 & 8 & 0 \\ 1 & 0 & 8 \end{vmatrix} = \begin{bmatrix}64\\-8\\-8\end{bmatrix} = \begin{bmatrix}16\\-2\\-2\end{bmatrix}$$

H_3 defined by P_1, P_3, P_4 or v_0, v_2, P_4:

$$\begin{vmatrix} i & j & k \\ 1 & 0 & 8 \\ -7 & -4 & 0 \end{vmatrix} = \begin{bmatrix}32\\-56\\-4\end{bmatrix} = \begin{bmatrix}8\\-14\\-1\end{bmatrix}$$

H_4 defined by P_2, P_3, P_4 or v_3, v_2, P_4:

$$\begin{vmatrix} i & j & k \\ -7 & 0 & -4 \\ -7 & -4 & 0 \end{vmatrix} = \begin{bmatrix}-16\\-28\\28\end{bmatrix} = \begin{bmatrix}-4\\-7\\4\end{bmatrix}$$

Question 4