Machine Learning from Data: Homework 3 - Probabilities

227367455 and 323081950

May 19, 2021

Question 1

Given a random sample $\{x_1, x_2, ..., x_n\}$, derive the maximum likelihood estimator $\hat{p}$ of the Binomial distribution.

$$B(x,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

We first want to calculate the likelihood:

$L = P(x_1, ...x_n \mid p) = \prod_{i=1}^{n} P(x_i \mid p)$

$= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{1-x_i}$

$= \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} \prod_{i=1}^{n} \binom{n}{x_i}$

$= p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \binom{n}{x_i}$

From the likelihood we calculate the log-likelihood:

$ln(L) = ln(p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \binom{n}{x_i})$

$= ln(p^{\sum_{i=1}^{n} x_i}) + ln((1-p)^{n-\sum_{i=1}^{n} x_i}) + ln(\prod_{i=1}^{n} \binom{n}{x_i})$

$= ln(p)\sum_{i=1}^{n} x_i + ln(1-p)(n - \sum_{i=1}^{n} x_i) + \sum_{i=1}^{n} ln(\binom{n}{x_i})$

We will take the derivative in respect to p our given value:

$$\frac{\partial[ln(L)]}{\partial p} = \frac{\partial[ln(p)\sum_{i=1}^{n} x_i]}{\partial p} + \frac{\partial[ln(1-p)(n - \sum_{i=1}^{n} x_i)]}{\partial p}$$

$$= \frac{\sum_{i=1}^{n} x_i}{p} - \frac{(n - \sum_{i=1}^{n} x_i)}{1-p}$$

To find the a maximum we set the derivative to 0 obtaining:

$$\frac{\sum_{i=1}^{n} x_i}{\hat{p}} - \frac{n - \sum_{i=1}^{n} x_i}{1-\hat{p}} = 0$$

$(1-\hat{p})\sum_{i=1}^{n} x_i - \hat{p}(n - \sum_{i=1}^{n} x_i) = 0$

$\sum_{i=1}^{n} x_i - \hat{p}\sum_{i=1}^{n} x_i - \hat{p}n + \hat{p}\sum_{i=1}^{n} x_i) = 0$

$\sum_{i=1}^{n} x_i - \hat{p}n + = 0$

$\hat{p}n = \sum_{i=1}^{n} x_i$

Thus we obtain:

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Question 2

A student wants to know her chances to pass and fail an exam if she studies and if she doesn't study. From last year's results, she sees that $P(Pass) = 60\%$. She also found out that $P(Studied \mid Pass) = 95\%$, $P(Studied \mid Failed) = 60\%$. You can assume that every student either studied or didn't study, and either passed or failed.

$P(Failed) = 1 - P(Pass) = 1 - 0.6 = 0.4$

$P(Studied) = P(Studied \mid Pass)P(Pass) + P(Studied \mid Failed)P(Failed)$
$= 0.95 \cdot 0.6 + 0.6 \cdot 0.4 = 0.81$

$P(\overline{Studied}) = 1 - P(Studied) = 1 - 0.81 = 0.19$

$P(\overline{Studied} \mid Pass) = 1 - P(Studied \mid Pass) = 1 - 0.95 = 0.05,$

a.

What is her probability of passing the exam if she studies?

$$P(Pass|Studied) = \frac{P(Studied \mid Pass)P(Pass)}{P(Studied)} = \frac{0.95 \cdot 0.6}{0.81} = 0.7037$$

b.

What is her probability of passing if she doesn't study?

$$P(Pass \mid \overline{Studied}) = \frac{P(\overline{Studied} \mid Pass)P(Pass)}{P(\overline{Studied})} = \frac{0.05 \cdot 0.6}{0.19} = 0.1578$$

Question 3

Find 3 random variables X, Y, C such that:

a) $X \perp Y \mid C$

b) X and Y are not independent

c) X, Y are integers such that $3 \leq X, Y \leq 9$ and C is binary.

d) The following conditions hold:

i. $P(1 \leq X, \leq 5) = 0.4$

ii. $P(1 \leq Y, \leq 5) = 0.4$

iii. $P(C = 0) = 0.3$

You need to specify the value of $P(X = x, Y = y, C = c)$. How many relevant values exist?

$A \sim B(1, \frac{6}{7})$
$B \sim B(1, \frac{4}{7})$
$X = 5 + CA$
$Y = 6 - CB$
$C \sim B(1, 0.7)$

Relevant values for when $C = 1$:

$$
\begin{aligned}
&P(X = x, Y = y, C = 1) \\
&= P(5 + X = x \mid C = 1)P(6 - Y = y \mid C = 1)P(C = 1) \\
&= P(X = x - 5 \mid C = 1) \cdot P(Y = 6 - y \mid C = 1) \cdot 0.7 \\
&= \tbinom{1}{x-5} \tfrac{6}{7}^{x-5} \tfrac{1}{7}^{1-(x-5)} \tbinom{1}{6-y} \tfrac{4}{7}^{6-y} \tfrac{3}{7}^{1-(6-y)} \cdot 0.7 \\
&= \tfrac{6}{7}^{x-5} \tfrac{1}{7}^{1-x+5} \tfrac{4}{7}^{6-y} \tfrac{3}{7}^{1-6+y} \cdot 0.7 \\
&= \tfrac{6}{7}^{x-5} \tfrac{1}{7}^{6-x} \tfrac{4}{7}^{6-y} \tfrac{3}{7}^{y-5} \cdot 0.7
\end{aligned}
$$

$= \frac{6^{x-5}}{7} 4^{6-y} \frac{3^{y-5}}{7} \cdot \frac{7}{10}$
$= 6^{x-5} \cdot 4^{6-y} \cdot 3^{y-5} \cdot \frac{1}{70} = 2^{x-5+12-2y} \cdot 3^{x+y-10} \cdot \frac{1}{70}$
$= 2^{x-2y+7} \cdot 3^{x+y-10} \cdot \frac{1}{70}$
For the values of $x = 5, 6$ and $y = 5, 6$

When $C = 0$, the only relevant values are when $X = 5, Y = 6$, for which:
$P(X = 5, Y = 6, C = 0) = P(X = 5 \mid C = 0)P(Y = 6 \mid C = 0)P(C = 0)$
$= 1 \cdot 1 \cdot 0.3 = 0.3$

C = 1	**X = 5**	**X = 6**	**P(Y=k)**
Y = 5	0.057	0.343	0.4
Y = 6	0.043	0.257	0.3
P(X = k)	0.1	0.6	0.7

C = 0	**X = 5**	**X = 6**	**P(Y=k)**
Y = 5	0	0	0
Y = 6	0.3	0	0.3
P(X = k)	0.3	0	0.3

Question 4

The probability of Wolt arriving on time is 0.75.

a.

What is the probability of having 2 on-time meals in a week (7 days)?

$X \sim B(7, 0.75)$

$P(X = 2) = \binom{7}{2} \cdot 0.75^2 \cdot (1 - 0.75)^{7-2} = \binom{7}{2} \cdot 0.75^2 \cdot 0.25^5 = 21 \cdot 0.5625 \cdot 0.000976 = 0.01154$

b.

What is the probability of having at least 4 on-time meals in a week?

$P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.07056 = 0.92944$

c.

A company of 100 employees recorded the number of on-time meals they had during a particular week and averaged their results. What do you expect the value of that average to be?

Let $X_i \sim B(7, 0.75)$,

$Y = \sum_{i=1}^{100} X_i \sim B(7 * 100, 0.75)$

Now we take the average as follows:

$$E\left(\frac{Y}{100}\right) \overset{linearity}{=} \frac{E(Y)}{100} \overset{\text{binomial formula}}{=} \frac{7 * 100 * 0.75}{100} = 5.25$$