

Machine Learning from Data -IDC

HW5—Theory+ SVM

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Question 1)

a)

Let K, L be two kernels (operating on the same space V) and let α, β be two positive scalars

Prove that $\alpha K + \beta L$ is a kernel.

Let $\langle x, y \rangle_V = \alpha K(x, y) + \beta L(x, y)$

We will show that $\langle x, y \rangle_V$ is a inner product by proving the four properties required to be an inner product operation.

Symmetry:

$$\begin{aligned}\langle x, y \rangle_V &= \alpha K(x, y) + \beta L(x, y) \\ &= \alpha K(y, x) + \beta L(y, x) \text{ by symmetry of inner products of } K, L \\ &= \langle y, x \rangle_V\end{aligned}$$

Additivity:

$$\langle u_1 + u_2, v \rangle_V = \alpha K(u_1 + u_2, v) + \beta L(u_1 + u_2, v)$$

$$\begin{aligned}
&= \alpha(K(u_1, v) + K(u_2, v)) + \beta(L(u_1, v) + L(u_2, v)) \text{ as } K, L \text{ are inner products} \\
&= (\alpha K(u_1, v) + \beta L(u_1, v)) + (\alpha K(u_2, v) + \beta L(u_2, v)) \\
&= \langle u_1, v \rangle_V + \langle u_2, v \rangle_V
\end{aligned}$$

Homogeneity

$$\begin{aligned}
&\langle c \cdot u, v \rangle_V = \alpha K(c \cdot u, v) + \beta L(c \cdot u, v) \\
&= \alpha \cdot c \cdot K(u, v) + \beta \cdot c \cdot L(u, v) \text{ as } K, L \text{ are inner products} \\
&= c \cdot (\alpha K(u, v) + \beta L(u, v)) = c \cdot \langle u, v \rangle_V
\end{aligned}$$

Positivity

$$\langle u, u \rangle_1 = \alpha K(u, u) + \beta L(u, u)$$

As K, L are inner products, then $K(u, u) \geq 0$ and $L(u, u) \geq 0$

As $\alpha, \beta > 0$ by the given assumption, then we can conclude that $\langle u, u \rangle_1 \geq 0$ as the right side is a sum and product of positive numbers

Now to show that $\langle u, u \rangle_V = 0$ iff $u = 0$.

First assume that $\langle u, u \rangle_V = 0$.

Then, as $\alpha, \beta > 0$, the only way this can occur are if $K(u, u) = L(u, u) = 0$.

As K, L are inner products, this means that $u = 0$.

Now assume that $u = 0$.

Notice that $\langle u, u \rangle_V = \alpha K(u, u) + \beta L(u, u)$.

As K, L are inner products:

$$\langle u, u \rangle_V = \alpha \cdot 0 + \beta \cdot 0$$

$= 0$ which proves the other direction.

b)

Provide (two different) examples of non-zero kernels K, L (operating on the same space), so that:

i. $K - L$ is a kernel

Let $K(x, y) = \varphi_1(x) \cdot \varphi_1(y)$ and $L(x, y) = \varphi_2(x) \cdot \varphi_2(y)$

Define $\varphi_1(x) = (2x, -4x^2)$ and $\varphi_2(x) = (x, x^2)$ Notice that $K(x, y) - L(x, y) =$

$$\begin{aligned}
& \varphi_1(x) \cdot \varphi_1(y) - \varphi_2(x) \cdot \varphi_2(y) \\
&= (2x, -4x^2) \cdot (2y, -4y^2) - (x, x^2) \cdot (y, y^2) \\
&= 2x \cdot 2y + (-4x^2) \cdot (-4y^2) - x \cdot y - x^2 \cdot y^2 \\
&= 4(x \cdot y) + 16(x^2 \cdot y^2) - x \cdot y - x^2 \cdot y^2 \\
&= 3(x \cdot y) + 15(x^2 \cdot y^2) \\
&= 3x \cdot 3y + 15x^2 \cdot 15y^2
\end{aligned}$$

This is a kernel with the mapping $\varphi(x) = (3x, 15x)$.

ii. $K - L$ is not a kernel

Let $K(x, y) = \varphi_1(x) \cdot \varphi_1(y)$ and $L(x, y) = \varphi_2(x) \cdot \varphi_2(y)$

Define $\varphi_1(x) = (x^4, x^2)$ and $\varphi_2(x) = (2x^4, -4x^2)$

Notice that $K(x, y) - L(x, y) = \varphi_1(x) \cdot \varphi_1(y) - \varphi_2(x) \cdot \varphi_2(y)$

$$= (x^4, x^2) \cdot (y^4, y^2) - (2x^4, -4x^2) \cdot (2y^4, -4y^2)$$

$$\begin{aligned}
&= x^4 \cdot y^4 + x^2 \cdot y^2 - 2x^4 \cdot 2y^4 - (-4x^2) \cdot (-4y^2) \\
&= x^4 \cdot y^4 + x^2 \cdot y^2 - 2(x^4 \cdot y^4) - 4(x^2 \cdot y^2) \\
&= -(x^4 \cdot y^4) - 3(x^2 \cdot y^2)
\end{aligned}$$

Notice that this is not a kernel. Proof:

ATC that this is a kernel. That means there exists a mapping $\varphi(x)$

s.t. $-(x^4 \cdot y^4) - 3(x^2 \cdot y^2) = \varphi(x) \cdot \varphi(y)$

Notice that the left side is always negative.

However, by the basic properties of an inner product, then $\varphi(x) \cdot \varphi(x) \geq 0$ for any x .

This means we have a norm that is negative, which is a contradiction!

Question 2)

Use Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraints:

Function: $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $g(x, y, z) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = 1$, where $\alpha > \beta > 0$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix} = 2 \begin{bmatrix} \frac{x}{\alpha^2} \\ \frac{y}{\beta^2} \\ \frac{z}{\beta^2} \end{bmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$\iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \lambda \begin{bmatrix} \frac{x}{\alpha^2} \\ \frac{y}{\beta^2} \\ \frac{z}{\beta^2} \end{bmatrix} = 0$$

$$\iff \begin{bmatrix} x(1 - \frac{\lambda}{\alpha^2}) \\ y(1 - \frac{\lambda}{\beta^2}) \\ z(1 - \frac{\lambda}{\beta^2}) \end{bmatrix} = 0$$

Notice that if $x \neq 0 \Rightarrow \lambda = \alpha^2$ and if $y \neq 0 \vee z \neq 0 \Rightarrow \lambda = \beta^2$. Therefore: $\alpha = \beta$ contradicting our constraint.

Consider $x \neq 0$:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = \frac{x^2}{\alpha^2} = 1 \iff x = \pm \alpha$$

Consider $x = 0$:

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = 1$$

$$y^2 + z^2 = \beta^2$$

Notice: $\alpha > \beta > 0 \Rightarrow \alpha^2 > \beta^2$

Therefore we can define the maxima $f(\pm\alpha, 0, 0) = \alpha^2$

And minima $f(0, x, z) = y^2 + z^2 = \beta^2$

Question 3)

Let $X = \mathbb{R}^2$. Let $C = H = \{h(a, b, c) = \{(x, y, z) \mid |x| \leq a, |y| \leq b, |z| \leq c\} \mid a, b, c \in \mathbb{R}^+\}$ the set of all origin centered boxes.

Describe the polynomial sample complexity algorithm L that learns C using H . State the time complexity and sample complexity of your suggested algorithm. Prove all your steps.