

# Machine Learning from Data -IDC

## HW5—Theory+ SVM

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### Question 1)

**a)**

Let  $K, L$  be two kernels (operating on the same space) and let  $\alpha, \beta$  be two positive scalars

Prove that  $\alpha K + \beta L$  is a kernel.

As  $K(x, y) = \langle \varphi_1(x), \varphi_1(y) \rangle$ ,  $L(x, y) = \langle \varphi_2(x), \varphi_2(y) \rangle$  (inner product,) then  $\alpha K(x, y) + \beta L(x, y) = \alpha \cdot \langle \varphi_1(x), \varphi_1(y) \rangle + \beta \cdot \langle \varphi_2(x), \varphi_2(y) \rangle$   
 $= \langle \alpha \cdot \varphi_1(x), \varphi_1(y) \rangle + \langle \beta \cdot \varphi_2(x), \varphi_2(y) \rangle$  by linearity of inner product

**b)**

Provide (two different) examples of non-zero kernels  $K, L$  (operating on the same space), so that:

**i.**  $K - L$  is a kernel

ii.  $K - L$  is not a kernel

Suppose we are working over the space  $\mathbb{R}$ .

1. Let  $K(x, y) = \varphi_1(x) \cdot \varphi_1(y)$  and  $K(x, y) = \varphi_2(x) \cdot \varphi_2(y)$

Define  $\varphi_1(x) = (2x, -4x^2)$  and  $\varphi_2(x) = (x, x^2)$  Notice that  $K(x, y) -$

$$\begin{aligned} L(x, y) &= \varphi_1(x) \cdot \varphi_1(y) - \varphi_2(x) \cdot \varphi_2(y) \\ &= (2x, -4x^2) \cdot (2y, -4y^2) - (x, x^2) \cdot (y, y^2) \\ &= 2x \cdot 2y + (-4x^2) \cdot (-4y^2) - x \cdot y - x^2 \cdot y^2 \\ &= 4(x \cdot y) + 16(x^2 \cdot y^2) - x \cdot y - x^2 \cdot y^2 \\ &= 3(x \cdot y) + 15(x^2 \cdot y^2) \\ &= 3x \cdot 3y + 15x^2 \cdot 15y^2 \end{aligned}$$

This is a kernel with the mapping  $\varphi(x) = (3x, 15x)$ .

2. Let  $K(x, y) = \varphi_1(x) \cdot \varphi_1(y)$  and  $K(x, y) = \varphi_2(x) \cdot \varphi_2(y)$

Define  $\varphi_1(x) = (x^4, x^2)$  and  $\varphi_2(x) = (2x^4, -4x^2)$  Notice that  $K(x, y) -$

$$\begin{aligned} L(x, y) &= \varphi_1(x) \cdot \varphi_1(y) - \varphi_2(x) \cdot \varphi_2(y) \\ &= (x^4, x^2) \cdot (y^4, y^2) - (2x^4, -4x^2) \cdot (2y^4, -4y^2) \\ &= x^4 \cdot y^4 + x^2 \cdot y^2 - 2x^4 \cdot 2y^4 - (-4x^2) \cdot (-4y^2) \\ &= x^4 \cdot y^4 + x^2 \cdot y^2 - 2(x^4 \cdot y^4) - 4(x^2 \cdot y^2) \\ &= -(x^4 \cdot y^4) - 3(x^2 \cdot y^2) \end{aligned}$$

Notice that this is not a kernel. Proof:

ATC that this is a kernel. That means there exists a mapping  $\varphi(x)$

$$\text{s.t. } -(x^4 \cdot y^4) - 3(x^2 \cdot y^2) = \varphi(x) \cdot \varphi(y)$$

Notice that the left side is always negative.

However, by the basic properties of an inner product, then the right side must  $\geq 0$ .

This means we have a inner product (that is always positive) equal a negative number, which is a contradiction!

## Question 2)

Use Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraints:

Function:  $f(x, y, z) = x^2 + y^2 + z^2$ .

Constraint:  $g(x, y, z) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = 1$ , where  $\alpha > \beta > 0$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{bmatrix} = 2 \begin{bmatrix} \frac{x}{\alpha^2} \\ \frac{y}{\beta^2} \\ \frac{z}{\beta^2} \end{bmatrix}$$

$$\nabla f = \lambda \nabla g$$

$$\iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \lambda \begin{bmatrix} \frac{x}{\alpha^2} \\ \frac{y}{\beta^2} \\ \frac{z}{\beta^2} \end{bmatrix} = 0$$

$$\iff \begin{bmatrix} x(1 - \frac{\lambda}{\alpha^2}) \\ y(1 - \frac{\lambda}{\beta^2}) \\ z(1 - \frac{\lambda}{\beta^2}) \end{bmatrix} = 0$$

Notice that if  $x \neq 0 \Rightarrow \lambda = \alpha^2$  and if  $y \neq 0 \vee z \neq 0 \Rightarrow \lambda = \beta^2$ . Therefore:  $\alpha = \beta$  contradicting our constraint.

Consider  $x \neq 0$  :

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = \frac{x^2}{\alpha^2} = 1 \iff x = \pm \alpha$$

Consider  $x = 0$  :

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = 1$$

$$y^2 + z^2 = \beta^2$$

Notice:  $\alpha > \beta > 0 \Rightarrow \alpha^2 > \beta^2$

Therefore we can define the maxima  $f(\pm\alpha, 0, 0) = \alpha^2$

And minima  $f(0, x, z) = y^2 + z^2 = \beta^2$

### Question 3)

Let  $X = \mathbb{R}^2$ . Let  $C = H = \{h(a, b, c) = \{(x, y, z) \mid |x| \leq a, |y| \leq b, |z| \leq c\} \mid a, b, c \in \mathbb{R}^+\}$  the set of all origin centered boxes.

Describe the polynomial sample complexity algorithm  $L$  that learns  $C$  using  $H$ . State the time complexity and sample complexity of your suggested algorithm. Prove all your steps.