Machine Learning from Data -IDC
HW5–Theory+ SVM

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1.

(a) Let K, L be two kernels (operating on the same space) and let α, β be two positive scalars
Prove that $\alpha K + \beta L$ is a kernel.
As $K(x,y) = \langle \varphi_1(x), \varphi_1(y) \rangle$, $L(x,y) = \langle \varphi_2(x)\, \varphi_2(y) \rangle$ (inner product,)
then $\alpha K(x,y) + \beta L(x,y) = \alpha \cdot \langle \varphi_1(x), \varphi_1(y) \rangle + \beta \cdot \langle \varphi_2(x\, \varphi_2(y) \rangle$
$= \langle \alpha \cdot \varphi_1(x), \varphi_1(y) \rangle + \langle \beta \cdot \varphi_2(x), \varphi_2(y) \rangle$ by linearity of inner product

(b) Provide (two different) examples of non-zero kernels K, L (operating on the same space), so that:

i. $K - L$ is a kernel
ii. $K - L$ is not a kernel

Prove your answers.
Suppose we are working over the space $\mathbb{R}$.

i. Let $K(x,y) = \varphi_1(x) \cdot \varphi_1(y)$ and $K(x,y) = \varphi_2(x) \cdot \varphi_2(y)$
Define $\varphi_1(x) = (2x, -4x^2)$ and $\varphi_2(x) = (x, x^2)$ Notice that

$K(x, y) - L(x, y) = \varphi_1(x) \cdot \varphi_1(y) - \varphi_2(x) \cdot \varphi_2(y)$
$= (2x, -4x^2) \cdot (2y, -4y^2) - (x, x^2) \cdot (y, y^2)$
$= 2x \cdot 2y + (-4x^2) \cdot (-4y^2) - x \cdot y - x^2 \cdot y^2$
$= 4(x \cdot y) + 16(x^2 \cdot y^2) - x \cdot y - x^2 \cdot y^2$
$= 3(x \cdot y) + 15(x^2 \cdot y^2)$
$= 3x \cdot 3y + 15x^2 \cdot 15y^2$
This is a kernel with the mapping $\varphi(x) = (3x, 15x)$.

ii. Let $K(x, y) = \varphi_1(x) \cdot \varphi_1(y)$ and $K(x, y) = \varphi_2(x) \cdot \varphi_2(y)$
Define $\varphi_1(x) = (x^4, x^2)$ and $\varphi_2(x) = (2x^4, -4x^2)$ Notice that
$K(x, y) - L(x, y) = \varphi_1(x) \cdot \varphi_1(y) - \varphi_2(x) \cdot \varphi_2(y)$
$= (x^4, x^2) \cdot (y^4, y^2) - (2x^4, -4x^2) \cdot (2y^4, -4y^2)$
$= x^4 \cdot y^4 + x^2 \cdot y^2 - 2x^4 \cdot 2y^4 - (-4x^2) \cdot (-4y^2)$
$= x^4 \cdot y^4 + x^2 \cdot y^2 - 2(x^4 \cdot y^4) - 4(x^2 \cdot y^2)$
$= -(x^4 \cdot y^4) - 3(x^2 \cdot y^2)$
Notice that this is not a kernel. Proof:
ATC that this is a kernel. That means there exists a mapping
$\varphi(x)$
s.t. $-(x^4 \cdot y^4) - 3(x^2 \cdot y^2) = \varphi(x) \cdot \varphi(y)$
Notice that the left side is always negative.
However, by the basic properties of an inner product, then the right side must ≥ 0.
This means we have a inner product (that is always positive) equal a negative number, which is a contradiction!

2. Use Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraints:
Function: $f(x, y, z,) = x^2 + y^2 + z^2$. Constraint: $g(x, y, z) = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\beta^2} = 1$
where $\alpha > \beta > 0$

3. Let $X = \mathbb{R}^2$. Let $C = H + \{h(a,b,c) = (x,y,z) s.t. |x| \leq a, |y| \leq b, |z| \leq c s.t. a,b,c \in \mathbb{R}_+\}$ the set of all origin centered boxes. Describe the polynomial sample complexity algorithm L that learns C using H. State the time complexity and sample complexity of your suggested algorithm. Prove all your steps.