

Machine Learning from Data: Homework 3 - Probabilities

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Question 1

Given a random sample $\{x_1, x_2, \dots, x_n\}$, derive the maximum likelihood estimator \hat{p} of the Binomial distribution.

$$B(x, p) = \binom{n}{x} p^x (1 - p)^{n-x}$$

We first want to calculate the likelihood:

$$L = P(x_1, \dots, x_n \mid p) = \prod_{i=1}^n P(x_i \mid p)$$

$$= \prod_{i=1}^n \binom{n}{x_i} p^{x_i} (1 - p)^{1-x_i}$$

$$= \prod_{i=1}^n p^{x_i} (1 - p)^{1-x_i} \prod_{i=1}^n \binom{n}{x_i}$$

$$= p^{\sum_{i=1}^n x_i} (1 - p)^{n - \sum_{i=1}^n x_i} \prod_{i=1}^n \binom{n}{x_i}$$

From the likelihood we calculate the log-likelihood:

$$\begin{aligned}
\ln(L) &= \ln(p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i} \prod_{i=1}^n \binom{n}{x_i}) \\
&= \ln(p^{\sum_{i=1}^n x_i}) + \ln((1-p)^{n-\sum_{i=1}^n x_i}) + \ln(\prod_{i=1}^n \binom{n}{x_i}) \\
&= \ln(p) \sum_{i=1}^n x_i + \ln(1-p)(n - \sum_{i=1}^n x_i) + \sum_{i=1}^n \ln\left(\binom{n}{x_i}\right)
\end{aligned}$$

We will take the derivative in respect to p our given value:

$$\begin{aligned}
\frac{\partial[\ln(L)]}{\partial p} &= \frac{\partial[\ln(p) \sum_{i=1}^n x_i]}{\partial p} + \frac{\partial[\ln(1-p)(n - \sum_{i=1}^n x_i)]}{\partial p} \\
&= \frac{\sum_{i=1}^n x_i}{p} - \frac{(n - \sum_{i=1}^n x_i)}{1-p}
\end{aligned}$$

To find the a maximum we set the derivative to 0 obtaining:

$$\frac{\sum_{i=1}^n x_i}{\hat{p}} - \frac{n - \sum_{i=1}^n x_i}{1 - \hat{p}} = 0$$

$$(1 - \hat{p}) \sum_{i=1}^n x_i - \hat{p}(n - \sum_{i=1}^n x_i) = 0$$

$$\sum_{i=1}^n x_i - \hat{p} \sum_{i=1}^n x_i - \hat{p}n + \hat{p} \sum_{i=1}^n x_i = 0$$

$$\sum_{i=1}^n x_i - \hat{p}n = 0$$

$$\hat{p}n = \sum_{i=1}^n x_i$$

Thus we obtain:

$$\hat{p} = \frac{\sum_{i=1}^n x_i}{n}$$

Question 2

A student wants to know her chances to pass and fail an exam if she studies and if she doesn't study. From last year's results, she sees that $P(Pass) = 60\%$. She also found out that $P(Studied | Pass) = 95\%$, $P(Studied | Failed) = 60\%$. You can assume that every student either studied or didn't study, and either passed or failed.

$$P(Failed) = 1 - P(Pass) = 1 - 0.6 = 0.4$$

$$\begin{aligned} P(Studied) &= P(Studied | Pass)P(Pass) + P(Studied | Failed)P(Failed) \\ &= 0.95 \cdot 0.6 + 0.6 \cdot 0.4 = 0.81 \end{aligned}$$

$$P(\overline{Studied}) = 1 - P(Studied) = 1 - 0.81 = 0.19$$

$$P(\overline{Studied} | Pass) = 1 - P(Studied | Pass) = 1 - 0.95 = 0.05,$$

a.

What is her probability of passing the exam if she studies?

$$P(Pass | Studied) = \frac{P(Studied | Pass)P(Pass)}{P(Studied)} = \frac{0.95 \cdot 0.6}{0.81} = 0.7037$$

b.

What is her probability of passing if she doesn't study?

$$P(Pass | \overline{Studied}) = \frac{P(\overline{Studied} | Pass)P(Pass)}{P(\overline{Studied})} = \frac{0.05 \cdot 0.6}{0.19} = 0.1578$$

Question 3

Find 3 random variables X, Y, C such that:

- a) $X \perp Y \mid C$
- b) X and Y are not independent
- c) X, Y are integers such that $3 \leq X, Y \leq 9$ and C is binary.
- d) The following conditions hold:
 - i. $P(1 \leq X, \leq 5) = 0.4$
 - ii. $P(1 \leq Y, \leq 5) = 0.4$
 - iii. $P(C = 0) = 0.3$

Z is a random variable with the following distribution:

$$A \sim B(1, \frac{6}{7})$$

$$B \sim B(1, \frac{4}{7})$$

$$X = 5 + CA$$

$$Y = 6 - CB$$

$$C \sim B(1, 0.7)$$

Question 4

The probability of Wolt arriving on time is 0.75.

a.

What is the probability of having 2 on-time meals in a week (7 days)?

$$X \sim B(7, 0.75)$$

$$P(X = 2) = \binom{7}{2} \cdot 0.75^2 \cdot (1 - 0.75)^{7-2} = \binom{7}{2} \cdot 0.75^2 \cdot 0.25^5 = 21 \cdot 0.5625 \cdot 0.000976 = 0.01154$$

b.

What is the probability of having at least 4 on-time meals in a week?

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.07056 = 0.92944$$

c.

A company of 100 employees recorded the number of on-time meals they had during a particular week and averaged their results. What do you expect the value of that average to be?

$$\text{Let } X_i \sim B(7, 0.75),$$

$$Y = \sum_{i=1}^{100} X_i \sim B(7 * 100, 0.75)$$

Now we take the average as follows:

$$E\left(\frac{Y}{100}\right) \stackrel{\text{linearity}}{=} \frac{E(Y)}{100} \stackrel{\text{binomial formula}}{=} \frac{7 * 100 * 0.75}{100} = 5.25$$