Machine Learning from Data: Homework 3 - Probabilities

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May 8, 2021

Question 1

Given a random sample $\{x_1, x_2, ..., x_n\}$, derive the maximum likelihood estimator $\hat{p}$ of the Binomial distribution.

$$B(x, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

We first want to calculate the likelihood:

$L = P(x_1, ...x_n \mid p) = \prod_{i=1}^{n} P(x_i \mid p)$

$= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$

$= \prod_{i=1}^{n} p^{x_i} (1-p)^{n-x_i} \prod_{i=1}^{n} \binom{n}{x_i}$

$= p^{\sum_{i=1}^{n} x_i} (1-p)^{n^2 - \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \binom{n}{x_i}$

From the likelihood we calculate the log-likelihood:

$$ln(L) = ln(p^{\sum_{i=1}^{n} x_i}(1-p)^{n^2-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \binom{n}{x_i})$$

$$= ln(p^{\sum_{i=1}^{n} x_i}) + ln((1-p)^{n^2-\sum_{i=1}^{n} x_i}) + ln(\prod_{i=1}^{n} \binom{n}{x_i})$$

$$= ln(p) \sum_{i=1}^{n} x_i + ln(1-p)(n^2 - \sum_{i=1}^{n} x_i) + \sum_{i=1}^{n} ln(\binom{n}{x_i})$$

We will take the derivative in respect to p our given value:

$$\frac{\partial[ln(L)]}{\partial p} = \frac{\partial[ln(p) \sum_{i=1}^{n} x_i]}{\partial p} + \frac{\partial[ln(1-p)(n^2 - \sum_{i=1}^{n} x_i)]}{\partial p}$$

$$= \frac{\sum_{i=1}^{n} x_i}{p} - \frac{(n^2 - \sum_{i=1}^{n} x_i)}{1-p}$$

To find the a maximum we set the derivative to 0 obtaining:

$$\frac{\sum_{i=1}^{n} x_i}{p} - \frac{n^2 - \sum_{i=1}^{n} x_i}{1-p} = 0$$

$$(1-p) \sum_{i=1}^{n} x_i - p(n^2 - \sum_{i=1}^{n} x_i) = 0$$

$$\sum_{i=1}^{n} x_i - p \sum_{i=1}^{n} x_i - pn^2 + p \sum_{i=1}^{n} x_i) = 0$$

$$\sum_{i=1}^{n} x_i - pn^2 + = 0$$

$$pn^2 = \sum_{i=1}^{n} x_i$$

Thus we obtain:

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n^2}$$

Question 2

A student wants to know her chances to pass and fail an exam if she studies and if she doesn't study. From last year's results, she sees that $P(Pass) = 60\%$. She also found out that $P(Studied \mid Pass) = 95\%$, $P(Studied \mid Failed) = 60\%$. You can assume that every student either studied or didn't study, and either passed or failed.

$P(Failed) = 1 - P(Pass) = 1 - 0.6 = 0.4$

$P(Studied) = P(Studied \mid Pass)P(Pass) + P(Studied \mid Failed)P(Failed)$
$= 0.95 \cdot 0.6 + 0.6 \cdot 0.4 = 0.81$

$P(\overline{Studied}) = 1 - P(Studied) = 1 - 0.81 = 0.19$

$P(\overline{Studied} \mid Pass) = 1 - P(Studied \mid Pass) = 1 - 0.95 = 0.05,$

a.

What is her probability of passing the exam if she studies?

$$P(Pass|Studied) = \frac{P(Studied \mid Pass)P(Pass)}{P(Studied)} = \frac{0.95 \cdot 0.6}{0.81} = 0.7037$$

b.

What is her probability of passing if she doesn't study?

$$P(Pass \mid \overline{Studied}) = \frac{P(\overline{Studied} \mid Pass)P(Pass)}{P(\overline{Studied})} = \frac{0.05 \cdot 0.6}{0.19} = 0.1578$$

Question 3

Find 3 random variables X, Y, C such that:

a) $X \perp Y \mid C$

b) X and Y are not independent

c) X, Y are integers such that $3 \leq X, Y \leq 9$ and C is binary.

d) The following conditions hold:

i. $P(1 \leq X, \leq 5) = 0.4$

ii. $P(1 \leq Y, \leq 5) = 0.4$

iii. $P(C = 0) = 0.3$

$C \sim B(1, 0.7)$

Question 4

The probability of Wolt arriving on time is 0.75.

a.

What is the probability of having 2 on-time meals in a week (7 days)?

$X \sim B(7, 0.75)$

$P(X = 2) =$

b.

What is the probability of having at least 4 on-time meals in a week?

$P(X \geq 4) = 0.5$ using the symmetry of binomial distribution

c.

A company of 100 employees recorded the number of on-time meals they had during a particular week and averaged their results. What do you expect the value of that average to be?

Let $X_i \sim B(7, 0.75)$,

$Y = \sum_{i=1}^{100} X_i \sim B(7 * 100, 0.75)$

Now we take the average as follows:

$$E(\frac{Y}{100}) \overset{linearity}{=} \frac{E(Y)}{100} \overset{\text{binomial formula}}{=} \frac{7 * 100 * 0.75}{100} = 5.25$$