Intro to Math Thinking Fall 2024: Assignment 8

1. Let $m, n \in \mathbb{Z}$ such that $m = 0$ and $n = 1$. Then $m^2 + mn + n^2 = 1 = 1^2$, which is a perfect square. Hence, the statement is true.

2. Suppose $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})[PerfectSquare(mn+1)]$. Let $n = m+2$ such that $mn+1 \Leftrightarrow m(m+2)+1 \Leftrightarrow m^2+2m+1 \Leftrightarrow (m+1)^2$. Since $m \in \mathbb{N}$, $n = m+2 \in \mathbb{N}$. Hence, the statement is true.

3. Suppose $(\forall n \in \mathbb{N})(\exists b, c \in \mathbb{N})[Composite(f(n) = n^2 + bn + c)]$. Let $n \in \mathbb{N}$ and $f(n) = (n+1)(n+2) = n^2 + 3n + 2$ where $b = 3$ and $c = 2$. Hence, the statement is true.

4. Suppose $(\forall n = 2k, k \in \mathbb{N}\backslash\{1\})[(\exists p_1, p_2 \in \mathbb{P})(n = p_1 + p_2)] \Rightarrow (\forall m = 2k+1, k \in \mathbb{N}\backslash\{1,2\})(\exists p_1, p_2, p_3 \in \mathbb{P})[m = p_1 + p_2 + p_3]$. If $n > 5$, then $n = 2k+3$ where $k > 1$. Since $2k > 2$ by Goldbach Conjecture, $2k = p + q$ where p, q are primes. Then $n = p + q + 3$, which is the sum of 3 primes. Hence, the statement is true.

5. **Prove** $S(n) = \sum_{i=1}^{n} 2i - 1 = n^2$
   Proof. Prove by induction. The base case is $S(1) = 2(1) - 1 = 1 = 1^2$, which is true. Assume for some k, $S(k)$ is true, and prove $P(k+1)$. For $S(k) = 1 + 3 + ... + (2k-1)$, the next term is obtained by adding $(2k+1)$. By the induction hypothesis, $S(k) = k^2$, and by adding the next term, we get $k^2 + 2k + 1 = (k+1)^2$, which is $S(k+1)$. Hence, by induction, we have proved $S(n) \forall n \in \mathbb{N}$. ■

6. **Prove** $(\forall n \in \mathbb{N}) : P(n) = \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
   Proof. Prove by induction. The base case is $P(1) = 1^2 = \frac{1}{6}(1)(1+1)(2(1)+1) \Leftrightarrow 1^2 = 1 \Leftrightarrow 1 = 1$, which is true. Assume for some k, $P(k)$ is true, and prove $P(k+1)$. For $P(k) = 1^2 + 2^2 + 3^2 + ... + k^2$, the next term is obtained by adding $(k+1)^2$. By the induction hypothesis, $S(k) = \frac{1}{6}k(k+1)(2k+1)$, and by adding the next term, we get

$$\begin{aligned}
\frac{1}{6}k(k+1)(2k+1) + (k+1)^2 &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \text{ (factor out } \frac{1}{6}(k+1)) \\
&= \frac{1}{6}(k+1)[2k^2 + 7k + 6] \text{ (simplify)} \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \text{ (rearrange)}
\end{aligned}$$

   which is $P(k+1)$. Hence, by the induction, we have proved $P(n) \forall n \in \mathbb{N}$. ■

1 OPTIONAL PROBLEMS

1. Let $n, N \in \mathbb{N}$ such that $1 + 2 + ... + n = N$. Note we can reverse the order of addition to get the same answer such that $n + (n-1) + ... + 1 = N$. Adding these 2 equations together

we get $(n+1)+(n+1)+...+(n+1)=2N$. Since there are n terms in the sum, we can write

$$\begin{aligned} 2N &= n(n+1) \\ N &= \frac{1}{2}n(n+1) \text{ (divide by 2)} \end{aligned}$$

Hence, $1+2+...+n=\frac{1}{2}n(n+1)$ is proven. ■

2. Prove by induction. Suppose $(\forall n \in \mathbb{N}\backslash\{1,2\})P(n)$, where $P(n)$ is *a collection of n points that are not all collinear in a plane where a triangle having 3 points as its vertices, which contains none of the other points in its interior, can be formed.* The base case is $P(3)$, which is true because 3 non-collinear points in a plane form a triangle with no interior points. For the induction hypothesis, we assume $P(k)$ is true for some k. We obtain the next term by adding a point on the plane such that there are 2 cases
Case 1. The added point is *not* within the triangle. Hence, $P(k+1)$ is true.
Case 2. The added point *is* within the triangle. We can form another triangle by connecting any 2 vertices of the previous triangle to the new point. This new triangle contains no points in its interior. Thus, $P(k+1)$ is true. Hence, by induction, we proved $P(n)\forall n \in \mathbb{N}\backslash\{1,2\}$. ■

3. Let $n \in \mathbb{N}$ for the following

   (a) Prove by induction. The base case is $P(n=1)$ such that $3|(4^1-1) \Leftrightarrow 3|3$, which is true. We assume for some k, $P(k)$ is true, $3|(4^k-1)$. The next is given by $4^{k+1}-1=4*4^k-1$. We can rewrite this as $4*4^k-4+3=4(4^k-1)+3$. We know the second term is divisible by 3 and the first term is divisible by 3 by the induction hypothesis. Hence, by induction, we proved $P(n)\forall n \in \mathbb{N}$.

   (b) Prove by induction. The base case is $P(n=5)$ such that $6!=720>2^8=256$, which is true. Assume for some k, $P(k)$ is true such that $(k+1)!>2^{k+3}$. To get the next term, multiply both sides by $(k+2)$ to obtain $(k+2)(k+1)!=(k+2)!>(k+2)*2^{k+3}=k*2^{k+3}+2*2^{k+3}=k*2^{k+3}+2^{k+4}>2^{(k+1)+3}$, which is $P(k+1)$. Hence, by induction, we proved $P(n)\forall n \in \mathbb{N}\backslash\{1,2,3,4\}$. ■

   (c) Prove by induction. The base case is $P(n=1)$ such that $1*1!=2!-1 \Leftrightarrow 1=1$, which is true. Obtain the next term by adding $(k+1)*(k+1)!$ to $P(k)$. We get $(k+1)!-1+(k+1)(k+1)!=(k+1)![1+(k+1)]-1=(k+2)!-1$, which is $P(k+1)$. Hence, by induction, we proved $P(n)\forall n \in \mathbb{N}$. ■