Intro to Math Thinking Fall 2024: Assignment 10.1

1. Let $A = (a, b)$, $C = (c, d)$ where A, C are intervals.

$$\begin{aligned} A \cap C &= \{x|a < x < b\} \cap \{x|c < x < d\} \\ &= \{x|max(a, c) < x < min(b, d)\} \\ &= (max(a, c), min(b, d)) \end{aligned}$$

   Similarly for closed intervals and $\frac{1}{2}$-intervals. Hence, true.
   False for unions. Observe that $(0, 1) \cup (3, 4)$ is not an interval.

2. (a) $(-\infty, 1) \cup (3, +\infty)$
   (b) $(-\infty, 1] \cup (7, +\infty)$
   (c) $(-\infty, 5] \cup (8, +\infty)$
   (d) $(3, 8]$
   (e) $[3, +\infty)$
   (f) $(-\infty, \pi) \cup (\pi, +\infty)$
   (g) $[4]$
   (h) $\emptyset$
   (i) $(-\infty, 7] \cup [8, +\infty)$
   (j) $(5, 7]$

3. Let A be a set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ that has an upper bound $c \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. We can find another upper bound $c + 1 > c$. Since this can be done with any upper bound, there are infinitely many different upper bounds. ■

4. Let A be a set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ that has a least upper bound c such that for any upper bound b of the set, $c \le b$. Suppose there exists 2 lubs, c_1, c_2. Since they are lubs, they are also upper bounds such that $(c_1 \le c_2) \wedge (c_2 \le c_1) \Leftrightarrow c_1 = c_2$. Hence, the lub is unique. ■

5. Let A be a set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ for the following

   (a) ($\Rightarrow$) Let b be the least upper bound of A. By definition of lub, $(\forall a \in A)(a \le b)$ and for any other upper bound c, $b \le c$.
   ($\Leftarrow$) $(\forall a \in A)(a \le b)$ and for any other upper bound c, $b \le c$. If A has a least upper bound, then by definition b is the lub of A.
   Thus, the statement is true. ■

   (b) ($\Rightarrow$) Let b be the least upper bound of A. If $\exists c, c < b$, then c is not an upper bound of A. Thus, $\exists a \in \mathbb{A}$ such that $a > c$.
   ($\Leftarrow$) $\exists c, c < b$ where c is not an upper bound of A, and $\exists a \in \mathbb{A}$ such that $a > c$. Since **(a)** is also true, then b is the lub of A.
   Hence, the statement is true. ■

6. (a) ($\Rightarrow$) Let $lub(A) = b$. Then by definition of least upper bound, $(\forall a \in A)(a \leq b)$. Hence, true.
   ($\Leftarrow$) $(\forall a \in A)(a \leq b)$. By definition of least upper bound, we can write $lub(A) = b$. Hence, true.
   Thus, the statement is true. $\blacksquare$

   (b) ($\Rightarrow$) Let $lub(A) = b$. Let $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$. $\exists a \in A$ such that $a > b - \epsilon$. By definition of lub, $a \leq b$. Rearranging $a > b - \epsilon \Leftrightarrow \epsilon > b - a$. As a tends to b, $\epsilon > b - a \Leftrightarrow \epsilon > 0$. Hence, $(\forall \epsilon > 0)(\exists a \in A)(a > b - \epsilon)$ is true.
   ($\Leftarrow$) Suppose $(\forall \epsilon > 0)(\exists a \in A)(a > b - \epsilon)$. Since $\epsilon > 0$ and $a \in A$, it is also true that $a \leq b$. By definition of lub, we can write $lub(A) = b$. Hence, true.
   Thus, the statement is true. $\blacksquare$

7. $\mathbb{N}$

8. Let A be a finite set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ such that $\exists b(\forall a \in A)(a \leq b)$. If $b \leq c$ where c is any other element in $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ that is an upper bound, then b is the least upper bound of A.

9. (a) $lub(a, b) = b$
   (b) $lub[a, b] = b$
   (c) $max(a, b) =$ undefined
   (d) $max[a, b] = b$

10. Let $A = \{|x - y| \mid x, y \in (a, b)\}$. The largest element of A can be constructed by letting $x = b - \epsilon, y = a + \epsilon$ such that $|x - y| = |(b - \epsilon) - (a + \epsilon)| = |(b - a) - 2\epsilon| = (b - a) - 2\epsilon$ where ϵ tends to 0 from above. Since $b - a \geq b - a - 2\epsilon \Leftrightarrow \epsilon \geq 0$. This is a least upper bound since any element x such that $0 < x < b - a$ will be inside A.

11. Let A be a set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$. Then $(\forall a \in A)(a \geq b)$, where b is a lower bound of A.

12. Let A be a set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$. A has a greatest lower bound b if b is a lower bound, and for any other lower bound $c, c \leq b$.

13. Let A be a set of $\mathbb{Z}/\mathbb{Q}/\mathbb{R}$ for the following

    (a) ($\Rightarrow$) Let b be the greatest lower bound of A. By definition of glb, $(\forall a \in A)(a \geq b)$ and for any other lower bound c, $b \geq c$.
    ($\Leftarrow$) $(\forall a \in A)(a \geq b)$ and for any other lower bound c, $b \geq c$. If A has a greatest lower bound, then by definition b is the glb of A.
    Thus, the statement is true. $\blacksquare$

    (b) ($\Rightarrow$) Let b be the greatest lower bound of A. If $\exists c, c > b$, then c is not a lower bound of A. Thus, $\exists a \in \mathbb{A}$ such that $a < c$.
    ($\Leftarrow$) $\exists c, c > b$ where c is not a lower bound of A, and $\exists a \in \mathbb{A}$ such that $a < c$. Since

(a) is also true, then b is the glb of A.
Hence, the statement is true. ■

14. (a) ($\Rightarrow$) Let $glb(A) = b$. Then by definition of greatest lower bound, $(\forall a \in A)(a \geq b)$. Hence, true.
($\Leftarrow$) $(\forall a \in A)(a \geq b)$. By definition of greatest lower bound, we can write $glb(A) = b$. Hence, true.
Thus, the statement is true. ■

    (b) ($\Rightarrow$) Let $glb(A) = b$. Let $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$. $\exists a \in A$ such that $a < b + \epsilon$. By definition of glb, $a \geq b$. Rearranging $a < b + \epsilon \Leftrightarrow \epsilon > a - b$. As a tends to b, $\epsilon > a - b \Leftrightarrow \epsilon > 0$. Hence, $(\forall \epsilon > 0)(\exists a \in A)(a < b + \epsilon)$ is true.
($\Leftarrow$) Suppose $(\forall \epsilon > 0)(\exists a \in A)(a < b + \epsilon)$. Since $\epsilon > 0$ and $a \in A$, it is also true that $a \geq b$. By definition of glb, we can write $glb(A) = b$. Hence, true.
Thus, the statement is true. ■

15. Let A be a nonempty set of $\mathbb{R}$ that has a lower bound. $\exists b(\forall a \in A)(a \geq b)$ where any lower bound $c, c \leq b$. By definition of greatest lower bound, we can write $glb(A) = b$, and thus $\mathbb{R}$ is complete. ■

16. Completeness Property for $\mathbb{Z}$ states every non-empty set of $\mathbb{Z}$ that contains a lower/upper bound contains a greatest/least lower/upper bound.