Intro to Math Thinking Fall 2024: Assignment 7

1. False. Counterexample. A penguin is a bird that cannot fly.
2. False. Counterexample. Let $x = y = 1$. Then $(x-y)^2 = (1-1)^2 = 0 \not> 0$.
3. True. Let $x, y \in \mathbb{Q}$ where $x < y$. Let $x = \frac{p}{q}$ and $y = \frac{r}{s}$ where $p, q, r, s \in \mathbb{Z}$ and $q \neq 0 \wedge s \neq 0$. Then

$$\begin{aligned} \frac{x+y}{2} &= \frac{\frac{p}{q} + \frac{r}{s}}{2} \text{ (substitute)} \\ &= \frac{ps+qr}{2qs} \text{ (simplify)} \end{aligned}$$

   Since $\frac{ps+qr}{2qs} \in \mathbb{Q}$, and $x < \frac{x+y}{2} < y$, there exists a third rational between any 2 unequal rationals.

4. Constructing a truth table for $[(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)] \Rightarrow [\phi \Leftrightarrow \psi]$ would result in a tautology.
5. $[(\neg\phi) \Rightarrow (\neg\psi)] \Leftrightarrow [\psi \Rightarrow \phi]$ is the contrapositive. Then from **4.**, $[(\phi \Rightarrow \psi) \wedge (\neg\phi \Rightarrow \neg\psi)] \Rightarrow [\phi \Leftrightarrow \psi]$ would result in a tautology.
6. Suppose \$2M is split evenly among 5 investors. \$2M/5 = \$400000, which is the most each investor will receive if split evenly. Thus, at least 1 investor receives at least \$400000.
7. Suppose $\sqrt{3}$ is rational. Then there exists $p, q \in \mathbb{N}$ such that p, q have no common factors.

$$\begin{aligned} \sqrt{3} &= \frac{p}{q} \\ 3 &= \frac{p^2}{q^2} \text{ (square)} \\ 3q^2 &= p^2 \text{ (rearrange)} \end{aligned}$$

   p^2 is a multiple of 3, and thus p is a multiple of 3. Let $r \in \mathbb{N}$ such that $p = 3r$.

$$\begin{aligned} 3q^2 &= (3r)^2 \text{ (substitute)} \\ 3q^2 &= 9r^2 \text{ (simplify)} \\ q^2 &= 3r^2 \text{ (divide by 3)} \end{aligned}$$

   q^2 is a multiple of 3, and thus q is a multiple of 3. p, q both are a multiple of 3, however p, q have no common factors, which is a contradiction. Thus, $\sqrt{3}$ is irrational. ■

8. (a) If the Yuan rises, the Dollar will fall.
   (b) $(\forall x, y \in \mathbb{R})[(-y < -x) \Rightarrow (x < y)]$
   (c) If 2 triangles have the same area, they are congruent.

(d) If $b^2 \geq 4ac$, then $ax^2 + bx + c = 0$ (where $a, b, c, x \in \mathbb{R}$ and $a \neq 0$).

(e) If the opposite angles of quadrilateral $ABCD$ are pairwise equal, then the opposite sides are pairwise equal.

(f) If all 4 angles of quadrilateral $ABCD$ are equal, then all 4 sides are equal.

(g) $(\forall n \in \mathbb{N})[3|(n^2 + 5) \Rightarrow \neg(3|n)]$

9. (b) og: True. $(\forall x, y \in \mathbb{R})[x < y \Rightarrow -y < -x]$
   conv: True. $(\forall x, y \in \mathbb{R})[-y < -x \Rightarrow x < y]$

   $(\forall x, y \in \mathbb{R})[x < y \Leftrightarrow 0 < y - x \Leftrightarrow -y < -x]$. Thus, equivalent.

   (c) og: True. Congruent triangles are the same size and shape, thus will have the same area.
   conv: False. The triangles could have the same area but not be congruent.

   Not equivalent.

   (d) Suppose $(\forall b, c \in \mathbb{R} \wedge \forall a \in \mathbb{R} \backslash \{0\})$ and $\exists x \in \mathbb{R}$ such that $ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Leftrightarrow (x + \frac{b}{2a})^2 + \frac{c}{a} = \frac{b^2}{4a^2} \Leftrightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \Leftrightarrow x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a} \Leftrightarrow x = -\frac{b \pm \sqrt{b^2-4ac}}{2a}$. Since $b^2 - 4ac \geq 0 \Leftrightarrow b^2 \geq 4ac$. Thus, a solution exists if $b^2 \geq 4ac$, and the converse is also true. Thus, equivalent.

   (e) If opposite sides of quadrilateral $ABCD$ are pairwise equal, it is a parallelogram. Then the opposite angles are pairwise equal. The converse is also true. Thus, equivalent.

   (f) og: False. Counterexample. $ABCD$ is a rhombus.
   conv: False. $ABCD$ is a rectangle.

   Not equivalent.

   (g) If $(n \in \mathbb{N})$ and $3 \nmid n$, then $n = (3k + 1) \vee (3k + 2)$ where $k \in \mathbb{Z}$. Then $n^2 + 5 = [3(3k^2 + 2k + 2) \vee 3(3k^2 + 4k + 3)]$. Thus, $3 \mid (n^2 + 5)$. If $3 \mid (n^2 + 5)$, then $n^2 + 5 = (3k + 1) \vee (3k + 2)$. Consider the case where $n^2 + 5 = 3k + 2$. Then $n^2 = 3k - 3 = 3(k - 1)$. Therefore, $3 \mid n^2$, and by Euclid's lemma, $3 \mid n$. Thus, the converse is false, and the statement and its converse are not equivalent.

10. $(\Rightarrow)$ If $n \in \mathbb{N}$ and $12 \mid n$, then $(\forall k \in \mathbb{Z})[n = 12k]$. Then $n^3 = 12^3 k^3$ that can be written as $12 \mid n^3$.
    $(\Leftarrow)$ False. Counterexample. Let $n = 6$ such that $n^3 = 216$. Since $216 = 12 * 18$, it can be written $12 \mid n^3$. However, $12 \nmid n$, which is a contradiction.
    Thus, the statement is false.

11. $1, 2, 5$ are irrational.

    For the following, let r, s be irrationals.

1. Let $r+3$ be rational such that $p, q \in \mathbb{Z}$ where p, q have no common factors. Then $r+3 = \frac{p}{q} \Leftrightarrow r = \frac{p}{q} - 3 \Leftrightarrow r = \frac{p-3q}{q} \in \mathbb{Q}$. But r was assumed to be irrational, which is a contradiction. Thus, $r+3$ must be irrational.
2. Let $5r$ be rational such that $p, q \in \mathbb{Q}$ where p, q have no common factors. Then $5r = \frac{p}{q} \Leftrightarrow r = \frac{p}{5q} \in \mathbb{Q}$. But r was assumed to be irrational, which is a contradiction. Thus, $5r$ is irrational.
3. Counterexample. Let $r = -s = \sqrt{2}$. Then $r + s = \sqrt{2} - \sqrt{2} = 0 \in \mathbb{Q}$. Thus, $r+s$ could be rational.
4. Counterexample. Let $r = s = \sqrt{2}$. Then $rs = \sqrt{2} * \sqrt{2} = 2 \in \mathbb{Q}$. Thus, rs could be rational.
5. Let $\sqrt{r}$ be rational such that $p, q \in \mathbb{Q}$ where p, q have no common factors. Then $\sqrt{r} = \frac{p}{q} \Leftrightarrow r = \frac{p^2}{q^2} \in \mathbb{Q}$. But r was assumed to be irrational, which is a contradiction. Thus, $\sqrt{r}$ is irrational.
6. Consider
   Case 1. If $\sqrt{2}^{\sqrt{2}}$ is rational, let $r = s = \sqrt{2}$.
   Case 2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, let $r = \sqrt{2}^{\sqrt{2}}$ and $s = \sqrt{2}$. Then $r^s = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$. Thus, r^s is rational.

12. For the following, let $m, n \in \mathbb{Z}$.

    (a) Let m, n be even such that $\forall q, r \in \mathbb{Z}, m = 2q$, and $n = 2r$. Then $m + n = 2q + 2r = 2(q+r)$, or $2 \mid (m+n)$. Thus, $m+n$ is even.

    (b) Let m, n be even such that $\forall q, r \in \mathbb{Z}, m = 2q$, and $n = 2r$. Then $mn = (2q)(2r) = 4qr$, or $4 \mid mn$. Thus, mn is divisible by 4.

    (c) Let m, n be odd such that $\forall q, r \in \mathbb{Z}, m = 2q+1$, and $n = 2r+1$. Then $m + n = (2q+1) + (2r+1) = 2q + 2r + 2 = 2(q+r+1)$, or $2 \mid (m+n)$. Thus, $m+n$ is even.

    (d) Let m be even and n be odd such that $\forall q, r \in \mathbb{Z}, m = 2q$, and $n = 2r+1$. Then $m + n = 2q + 1 + 2r = 2(q+r) + 1$, or $\neg 2 \mid (m+n)$. Thus, $m+n$ is odd.

    (e) Let m be even and n be odd such that $q, r \in \mathbb{Z}, m = 2q$, and $n = 2r+1$. Then $mn = (2q)(2r+1) = 2(2qr + q)$, or $2 \mid (m+n)$. Thus, mn is even.

1 OPTIONAL PROBLEM

(a) True. Let $x, y \in \mathbb{R}$ such that $x = 0$ and $(\forall y \in \mathbb{R}) \mid [x + y = y]$.

(b) True. Let $x, y \in \mathbb{R}$ and $\forall x \exists y (x + y = 0)$ where $x = -y$.

(c) False. Let $a, b, c \in \mathbb{Z}$ such that $a \mid bc$. Let $a = 4$ and $b = c = 2$. Then $a \mid bc \Leftrightarrow 4 \mid (2 * 2) \Leftrightarrow 4 \mid 4$, but $(a \nmid b \vee a \nmid c) \Leftrightarrow (4 \nmid 2 \vee 4 \nmid 2)$.

(d) True. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}\backslash\mathbb{Q}$. Let $x + y$ is rational such that $p, q \in \mathbb{Q}$ where p, q have no common factors. Then $x + y = \frac{p}{q} \Leftrightarrow y = \frac{p}{q} - x \Leftrightarrow y = \frac{p-qx}{q} \in \mathbb{Q}$. But y is irrational, which is a contradiction. Thus, $x + y$ is irrational.

(e) True. Case 1. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}\backslash\mathbb{Q}$. See **(d)**.
Case 2. Let $x, y \in \mathbb{R}$ such that $p, q \in \mathbb{Z}$ where p, q have no common factors. Then $x + y = \frac{p}{q} \Leftrightarrow x = \frac{p}{q} - y \Leftrightarrow x = \frac{p-qy}{q} \in \mathbb{Q}$. Thus, $x + y \in \mathbb{Q} \in \mathbb{R}$.
Case 3. See **(11.3)**.
Thus, at least 1 of x, y is irrational.

(f) False. Counterexample. Let $x, y \in \mathbb{R}\backslash\mathbb{Q}$ such that $x = -y = \sqrt{2}$. Then $x + y = \sqrt{2} - \sqrt{2} = 0 \in \mathbb{Q}$. Thus, $x + y$ could be rational without either $x, y \in \mathbb{Q}$.