3.1 Searching and algorithms

Algorithms

An ***algorithm*** is a sequence of steps for accomplishing a task. ***Linear search*** is a search algorithm that starts from the beginning of a list, and checks each element until the search key is found or the end of the list is reached.

PARTICIPATION ACTIVITY 3.1.1: Linear search algorithm checks each element until key is found.

Animation captions:

1. Linear search starts at first element and searches elements one-by-one.
2. Linear search will compare all elements if the search key is not present.

Figure 3.1.1: Linear search algorithm.

```
LinearSearch(numbers, int numbersSize, int key) {
   i = 0

   for (i = 0; i < numbersSize; ++i) {
      if (numbers[i] == key) {
         return i
      }
   }

   return -1 // not found
}

main() {
   numbers = {2, 4, 7, 10, 11, 32, 45, 87}
   NUMBERS_SIZE = 8
   i = 0
   key = 0
   keyIndex = 0

   print("NUMBERS: ")
   for (i = 0; i < NUMBERS_SIZE; ++i) {
      print(numbers[i] + " ")
   }
   printLine()

   print("Enter a value: ")
   key = getIntFromUser()

   keyIndex = LinearSearch(numbers, NUMBERS_SIZE, key)

   if (keyIndex == -1) {
      printLine(key + " was not found.")
   }
   else {
      printLine("Found " + key + " at index " + keyIndex + ".")
   }
}
```

```
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 10
Found 10 at index 3.
...
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 17
17 was not found.
```

PARTICIPATION ACTIVITY 3.1.2: Linear search algorithm execution.

Given list: (20, 4, 114, 23, 34, 25, 45, 66, 77, 89, 11).

1) How many list elements will be compared to find 77 using linear search?

Check Show answer

2) How many list elements will be checked to find the value 114 using linear search?

Check Show answer

3) How many list elements will be checked if the search key is not found using linear search?

Check Show answer

Algorithm runtime

An algorithm's ***runtime*** is the time the algorithm takes to execute. If each comparison takes 1 µs (1 microsecond), a linear search algorithm's runtime is up to 1 s to search a list with 1,000,000 elements, 10 s for 10,000,000 elements, and so on. Ex: Searching Amazon's online store, which has more than 200 million items, could require more than 3 minutes.

An algorithm typically uses a number of steps proportional to the size of the input. For a list with 32 elements, linear search requires at most 32 comparisons: 1 comparison if the search key is found at index 0, 2 if found at index 1, and so on, up to 32 comparisons if the the search key is not found. For a list with N elements, linear search thus requires at most N comparisons. The algorithm is said to require "on the order" of N comparisons.

PARTICIPATION ACTIVITY 3.1.3: Linear search runtime.

1) Given a list of 10,000 elements, and if each comparison takes 2 µs, what is the fastest possible runtime for linear search?

µs

Check Show answer

2) Given a list of 10,000 elements, and if each comparison takes 2 µs, what is

the longest possible runtime for linear search?

[ ] µs

Check Show answer

How was this section? Provide feedback

3.2 Binary search

Linear search vs. binary search

Linear search may require searching all list elements, which can lead to long runtimes. For example, searching for a contact on a smartphone one-by-one from first to last can be time consuming. Because a contact list is sorted, a faster search, known as binary search, checks the middle contact first. If the desired contact comes alphabetically before the middle contact, binary search will then search the first half and otherwise the last half. Each step reduces the contacts that need to be searched by half.

PARTICIPATION ACTIVITY 3.2.1: Using binary search to search contacts on your phone.

Animation captions:

1. A contact list stores contacts sorted by name. Searching for Pooja using a binary search starts by checking the middle contact.
2. The middle contact is Muhammad. Pooja is alphabetically after Muhammad, so the binary search only searches the contacts after Muhammad. Only half the contacts now need to be searched.
3. Binary search continues by checking the middle element between Muhammad and the last contact. Pooja is before Sharod, so the search continues with only those contacts between Muhammad and Sharod, which is one fourth of the contacts.
4. The middle element between Muhammad and Sharod is Pooja. Each step reduces the number of contacts to search by half.

PARTICIPATION ACTIVITY 3.2.2: Using binary search to search a contact list.

A contact list is searched for Bob.
Assume the following contact list: Amy, Bob, Chris, Holly, Ray, Sarah, Zoe

1) What is the first contact searched?

Check Show answer

2) What is the second contact searched?

Check Show answer

Binary search algorithm

Binary search is a faster algorithm for searching a list if the list's elements are sorted and directly accessible (such as an array). Binary search first checks the middle element of the list. If the search key is found, the algorithm returns the matching location. If the search key is not found, the algorithm repeats the search on the remaining left sublist (if the search key was less than the middle element) or the remaining right sublist (if the search key was greater than the middle element).

PARTICIPATION ACTIVITY 3.2.3: Binary search efficiently searches sorted list by reducing the search space by half each iteration.

Animation captions:

1. Elements with indices between low and high remain to be searched.
2. Search starts by checking the middle element.
3. If search key is greater than element, then only elements in right sublist need to be searched
4. Each iteration reduces search space by half. Search continues until key found or search space is empty.

Figure 3.2.1: Binary search algorithm.

```
BinarySearch(numbers, numbersSize, key) {
   mid = 0
   low = 0
   high = numbersSize - 1

   while (high >= low) {
      mid = (high + low) / 2
      if (numbers[mid] < key) {
         low = mid + 1
      }
      else if (numbers[mid] > key) {
         high = mid - 1
      }
      else {
         return mid
      }
   }

   return -1 // not found
}

main() {
   numbers = { 2, 4, 7, 10, 11, 32, 45, 87 }
   NUMBERS_SIZE = 8
   i = 0
   key = 0
   keyIndex = 0

   print("NUMBERS: ")
   for (i = 0; i < NUMBERS_SIZE; ++i) {
      print(numbers[i])
   }
   printLine()

   print("Enter a value: ")
   key = getIntFromUser()

   keyIndex = BinarySearch(numbers, NUMBERS_SIZE, key)

   if (keyIndex == -1) {
      printLine(key + " was not found.")
   }
   else {
      printLine("Found " + key + " at index " + keyIndex + ".")
   }
}
```

```
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 10
Found 10 at index 3.
...
NUMBERS: 2 4 7 10 11 32 45 87
Enter a value: 17
17 was not found.
```

PARTICIPATION ACTIVITY 3.2.4: Binary search algorithm execution.

Given list: (4, 11, 17, 18, 25, 45, 63, 77, 89, 114).

1) How many list elements will be checked to find the value 77 using binary search?

Check Show answer

2) How many list elements will be checked to find the value 17 using binary search?

Check Show answer

3) Given an array with 32 elements, how many list elements will be checked if the key is less than all elements in the list, using binary search?

Check Show answer

Binary search efficiency

Binary search is incredibly efficient in finding an element within a sorted list. During each iteration or step of the algorithm, binary search reduces the search space (i.e., the remaining elements to search within) by half. The search terminates when the element is found or the search space is empty (element not found). For a 32 element list, if the search key is not found, the search space is halved to have 16 elements, then 8, 4, 2, 1, and finally none, requiring only 6 steps. For an N element list, the maximum number of steps required to reduce the search space to an empty sublist is $\lfloor log_2 N \rfloor + 1$. Ex: $\lfloor log_2 32 \rfloor + 1 = 6$.

PARTICIPATION ACTIVITY 3.2.5: Speed of linear search versus binary search to find a number within a sorted list.

Animation captions:

1. A binary search begins with the middle element of the list. Each subsequent search reduces the search space by half. Using binary search, a match was found with only 3 comparisons.
2. Using linear search, a match was found after 6 comparisons. Compared to a linear search, binary search is incredibly efficient in finding an element within a sorted list.

If each comparison takes 1 µs (1 microsecond), a binary search algorithm's runtime is at most 20 µs to search a list with 1,000,000 elements, 21 µs to search 2,000,000 elements, 22 µs to search 4,000,000 elements, and so on. Ex: Searching Amazon's online store, which has more than 200 million items, requires less than 28 µs; up to 7,000,000 times faster than linear search.

PARTICIPATION ACTIVITY 3.2.6: Linear and binary search runtime.

Answer the following questions assuming each comparison takes 1 µs.

1) Given a list of 1024 elements, what is the runtime for linear search if the search key is less than all elements in the list?

µs

Check Show answer

2) Given a list of 1024 elements, what is the runtime for binary search if the search key is less than all elements in the list?

µs

Check Show answer

How was this section? Provide feedback

3.3 Constant time operations

Constant time operations

In practice, designing an efficient algorithm aims to lower the amount of time that an algorithm runs. However, a single algorithm can always execute more quickly on a faster processor. Therefore, the theoretical analysis of an algorithm describes runtime in terms of number of

constant time operations, not nanoseconds. A ***constant time operation*** is an operation that, for a given processor, always operates in the same amount of time, regardless of input values.

PARTICIPATION ACTIVITY 3.3.1: Constant time vs. non-constant time operations.

Animation content:

undefined

Animation captions:

1. x = 10, y = 20, a = 1000, and b = 2000 are assigning variables with values, and are constan time operations.
2. A CPU multiplies values 10 and 20 at the same speed as 1000 and 2000. Multiplication is a constant time operation.
3. A loop that iterates x times, adding y to a sum each iteration, will take longer if x is larger. Th loop is not constant time.
4. String concatenation is another common operation that is not constant time, because more characters need to be copied for larger strings.

PARTICIPATION ACTIVITY 3.3.2: Constant time operations.

1) The statement below that assigns x with y is a constant time operation.

```
y = 10
x = y
```

- ○ True
- ○ False

2) A loop is never a constant time operation.

- ○ True
- ○ False

3) The 3 constant time operations in the code below can collectively be considered 1 constant time operation.

```
x = 26.5
y = 15.5
z = x + y
```

- ○ True
- ○

False

Identifying constant time operations

The programming language being used, as well as the hardware running the code, both affect what is and what is not a constant time operation. Ex: Most modern processors perform arithmetic operations on integers and floating point values at a fixed rate that is unaffected by operand values. Part of the reason for this is that the floating point and integer values have a fixed size. The table below summarizes operations that are generally considered constant time operations.

Table 3.3.1: Common constant time operations.

Operation	Example

Addition, subtraction, multiplication, and division of fixed size integer or floating point values.	`w = 10.4`
	`x = 3.4`
	`y = 2.0`
	`z = (w - x) / y`

Assignment of a reference, pointer, or other fixed size data value.	`x = 1000`
	`y = x`
	`a = true`
	`b = a`

Comparison of two fixed size data values.	`a = 100`
	`b = 200`
	`if (b > a) {`
	`  ...`
	`}`

Read or write an array element at a particular index.	`x = arr[index]` `arr[index + 1] = x + 1`

PARTICIPATION ACTIVITY 3.3.3: Identifying constant time operations.

1) In the code below, suppose str1 is a pointer or reference to a string. The code only executes in constant time if the assignment copies the pointer/reference, and not all the characters in the string.

```
str2 = str1
```

- True
- False

2) Certain hardware may execute division more slowly than multiplication, but both may still be constant time operations.
   - True
   - False

3) The hardware running the code is the only thing that affects what is and what is not a constant time operation.
   - True
   - False

How was this section? Provide feedback

3.4 Growth of functions and complexity

Upper and lower bounds

An algorithm with runtime complexity T(N) has a lower bound and an upper bound.

- ***Lower bound***: A function f(N) that is ≤ the best case T(N), for all values of N ≥ 1.
- ***Upper bound***: A function f(N) that is ≥ the worst case T(N), for all values of N ≥ 1.

Ex: An algorithm with best case runtime $T(N) = 7N + 36$ and worst case runtime $T(N) = 3N^2 + 10N + 17$, has a lower bound $f(N) = 7N$ and an upper bound $f(N) = 30N^2$. These lower and upper bounds provide a general picture of the runtime, while using simpler functions than the exact runtime.

Upper and lower bounds in the context of runtime complexity

In strict mathematical terms, upper and lower bounds do not relate to best or worst case runtime complexities. In the context of algorithm complexity analysis, the terms are often

used to collectively bound all runtime complexities for an algorithm. Therefore, the terms are used in this section such that the upper bound is ≥ the worst case T(N) and the lower bound is ≤ the best case T(N).

PARTICIPATION ACTIVITY 3.4.1: Upper and lower bounds.

Animation content:

undefined

Animation captions:

1. An algorithm's worst and best case runtimes are represented by the blue and purple curves respectively.
2. $2N^2$, shown in yellow, is a lower bound. The lower bound is less than or equal to both runtime functions for all N ≥ 1.
3. $30N^2$, shown in orange, is an upper bound. The upper bound is greater than or equal to both runtime functions for all N ≥ 1.
4. Together, the upper and lower bounds enclose all possible runtimes for this algorithm.

PARTICIPATION ACTIVITY 3.4.2: Upper and lower bounds.

Suppose an algorithm's best case runtime complexity is $T(N) = 3N + 6$, and the algorithm's worst case runtime is $T(N) = 5N^2 + 7N$.

1) Which function is a lower bound for the algorithm?
   - $5N^2$
   - $5N$
   - $3N$

2) Which function is an upper bound for the algorithm?
   - $12N^2$
   - $5N^2$
   - $7N$

3) $5N^2 + 7N$ is an upper bound for the algorithm.

- ○ True
- ○ False

4) $3N + 6$ is a lower bound for the algorithm.

- ○ True
- ○ False

Growth rates and asymptotic notations

An additional simplification can factor out the constant from a bounding function, leaving a function that categorizes the algorithm's growth rate. Ex: Instead of saying that an algorithm's runtime function has an upper bound of $30N^2$, the algorithm could be described as having a worst case growth rate of N^2. ***Asymptotic notation*** is the classification of runtime complexity that uses functions that indicate only the growth rate of a bounding function. Three asymptotic notations are commonly used in complexity analysis:

- ***O notation*** provides a growth rate for an algorithm's upper bound.
- ***Ω notation*** provides a growth rate for an algorithm's lower bound.
- ***Θ notation*** provides a growth rate that is both an upper and lower bound.

Table 3.4.1: Notations for algorithm complexity analysis.

Notation	General form	Meaning
O	$T(N) = O(f(N))$	A positive constant c exists such that, for all N ≥ 1, $T(N) \leq c * f(N)$.
Ω	$T(N) = \Omega(f(N))$	A positive constant c exists such that, for all N ≥ 1, $T(N) \geq c * f(N)$.
Θ	$T(N) = \Theta(f(N))$	$T(N) = O(f(N))$ and $T(N) = \Omega(f(N))$.

PARTICIPATION ACTIVITY 3.4.3: Asymptotic notations.

Suppose $T(N) = 2N^2 + N + 9$.

1) $T(N) = O(N^2)$

- ○ True
- ○ False

2) $T(N) = \Omega(N^2)$

- True
- False

3) $T(N) = \Theta(N^2)$

- True
- False

4) $T(N) = O(N^3)$

- True
- False

5) $T(N) = \Omega(N^3)$

- True
- False

How was this section? Provide feedback

3.5 O notation

Big O notation

Big O notation is a mathematical way of describing how a function (running time of an algorithm) generally behaves in relation to the input size. In Big O notation, all functions that have the same growth rate (as determined by the highest order term of the function) are characterized using the same Big O notation. In essence, all functions that have the same growth rate are considered equivalent in Big O notation.

Given a function that describes the running time of an algorithm, the Big O notation for that function can be determined using the following rules:

1. If f(N) is a sum of several terms, the highest order term (the one with the fastest growth rate) is kept and others are discarded.
2. If f(N) has a term that is a product of several factors, all constants (those that are not in terms of N) are omitted.

PARTICIPATION

ACTIVITY 3.5.1: Determining Big O notation of a function.

Animation content:

undefined

Animation captions:

1. Determine a function that describes the running time of the algorithm, and then compute the Big O notation of that function.
2. Apply rules to obtain the Big O notation of the function.
3. All functions with the same growth rate are considered equivalent in Big O notation.

PARTICIPATION ACTIVITY 3.5.2: Big O notation.

1) Which of the following Big O notations is equivalent to O(N+9999)?
   - O(1)
   - O(N)
   - O(9999)

2) Which of the following Big O notations is equivalent to O(734·N)?
   - O(N)
   - O(734)
   - $O(734 \cdot N^2)$

3) Which of the following Big O notations is equivalent to $O(12 \cdot N + 6 \cdot N^3 + 1000)$?
   - O(1000)
   - O(N)
   - $O(N^3)$

Big O notation of composite functions

The following rules are used to determine the Big O notation of composite functions: c denotes a constant

Figure 3.5.1: Rules for determining Big O notation of composite functions.

Composite function	Big O notation
c · O(f(N))	O(f(N))
c + O(f(N))	O(f(N))
g(N) · O(f(N))	O(g(N) · f(N))
g(N) + O(f(N))	O(g(N) + f(N))

PARTICIPATION ACTIVITY 3.5.3: Big O notation for composite functions.

Determine the simplified Big O notation.

1) $10 \cdot O(N^2)$
   - O(10)
   - $O(N^2)$
   - $O(10 \cdot N^2)$

2) $10 + O(N^2)$
   - O(10)
   - $O(N^2)$
   - $O(10 + N^2)$

3) $3 \cdot N \cdot O(N^2)$
   - $O(N^2)$
   - $O(3 \cdot N^2)$
   - $O(N^3)$

4) $2 \cdot N^3 + O(N^2)$
   - $O(N^2)$
   - $O(N^3)$
   - $O(N^2 + N^3)$

5) $log_2 N$
   - $O(log_2 N)$

- O($logN$)
- O(N)

Runtime growth rate

One consideration in evaluating algorithms is that the efficiency of the algorithm is most critical for large input sizes. Small inputs are likely to result in fast running times because N is small, so efficiency is less of a concern. The table below shows the runtime to perform f(N) instructions for different functions f and different values of N. For large N, the difference in computation time varies greatly with the rate of growth of the function f. The data assumes that a single instruction takes 1 μs to execute.

Table 3.5.1: Growth rates for different input sizes.

Function	N = 10	N = 50	N = 100	N = 1000	N = 10000	N = 100000
log_2N	3.3 μs	5.65 μs	6.6 μs	9.9 μs	13.3 μs	16.6 μs
N	10 μs	50 μs	100 μs	1000 μs	10 ms	100 ms
$Nlog_2N$	.03 ms	.28 ms	.66 ms	.099 s	.132 s	1.66 s
N^2	.1 ms	2.5 ms	10 ms	1 s	100 s	2.7 hours
N^3	1 ms	.125 s	1 s	16.7 min	11.57 days	31.7 years
2^N	.001 s	35.7 years	> 1000 years			

The interactive tool below illustrates graphically the growth rate of commonly encountered functions.

PARTICIPATION ACTIVITY 3.5.4: Computational complexity graphing tool.

Common Big O complexities

Many commonly used algorithms have running time functions that belong to one of a handful of growth functions. These common Big O notations are summarized in the following table. The table shows the Big O notation, the common word used to describe algorithms that belong to that notation, and an example with source code. Clearly, the best algorithm is one that has constant time complexity. Unfortunately, not all problems can be solved using constant complexity algorithms. In fact, in many cases, computer scientists have proven that certain types of problems can only be solved using quadratic or exponential algorithms.

Figure 3.5.2: Runtime complexities for various code examples.

Notation	Name	Example pseudocode
O(1)	Constant	`FindMin(x, y) { if (x < y) { return x } else { return y } }`
O(log N)	Logarithmic	`BinarySearch(numbers, N, key) { mid = 0 low = 0 high = N - 1 while (high >= low) { mid = (high + low) / 2 if (numbers[mid] < key) { low = mid + 1 } else if (numbers[mid] > key) { high = mid - 1 } else { return mid } } return -1 // not found }`
O(N)	Linear	`LinearSearch(numbers, numbersSize, key) { for (i = 0; i < numbersSize; ++i) { if (numbers[i] == key) { return i } } return -1 // not found }`

O(N log N)	Linearithmic	`MergeSort(numbers, i, k) {`
		`  j = 0`
		`  if (i < k) {`
		`  j = (i + k) / 2  // Find midpoint`
		
		`  MergeSort(numbers, i, j)  // Sort left part`
		`  MergeSort(numbers, j + 1, k) // Sort right part`
		`  Merge(numbers, i, j, k)  // Merge parts`
		`  }`
		`}`

$O(N^2)$	Quadratic	`SelectionSort(numbers, numbersSize) {`
		`  for (i = 0; i < numbersSize; ++i) {`
		`  indexSmallest = i`
		`  for (j = i + 1; j < numbersSize; ++j) {`
		`  if (numbers[j] < numbers[indexSmallest]) {`
		`  indexSmallest = j`
		`  }`
		`  }`
		
		`  temp = numbers[i]`
		`  numbers[i] = numbers[indexSmallest]`
		`  numbers[indexSmallest] = temp`
		`}`

$O(c^N)$	Exponential	`Fibonacci(N) {`
		`  if ((1 == N) || (2 == N)) {`
		`  return 1`
		`  }`
		`  return Fibonacci(N-1) + Fibonacci(N-2)`
		`}`

PARTICIPATION ACTIVITY 3.5.5: Big O notation and growth rates.

1) O(5) has a _____ runtime complexity.
   - ○ constant
   - ○ linear
   - ○ exponential

2) O(N log N) has a _____ runtime complexity.
   - ○ constant
   - ○ linearithmic
   - ○ logarithmic

3) $O(N + N^2)$ has a _____ runtime

complexity.

- linear-quadratic
- exponential
- quadratic

4) A linear search has a _____ runtime complexity.

- O(log N)
- O(N)
- $O(N^2)$

5) A selection sort has a _____ runtime complexity.

- O(N)
- O(N log N)
- $O(N^2)$

How was this section? Provide feedback

3.6 Algorithm analysis

Worst-case algorithm analysis

To analyze how runtime of an algorithm scales as the input size increases, we first determine how many operations the algorithm executes for a specific input size, N. Then, the big-O notation for that function is determined. Algorithm runtime analysis often focuses on the worst-case runtime complexity. The ***worst-case runtime*** of an algorithm is the runtime complexity for an input that results in the longest execution. Other runtime analyses include best-case runtime and average-case runtime. Determining the average-case runtime requires knowledge of the statistical properties of the expected data inputs.

PARTICIPATION ACTIVITY 3.6.1: Runtime analysis: Finding the max value.

Animation content:

```
undefined
```

Animation captions:

1. Runtime analysis determines the total number of operations. Operations include assignment addition, comparison, etc.
2. The for loop iterates N times, but the for loop's initial expression i = 0 is executed once.
3. For each loop iteration, the increment and comparison expressions are each executed once In the worst-case, the if's expression is true, resulting in 2 operations.
4. One additional comparison is made before the loop ends.
5. The function f(N) specifies the number of operations executed for input size N. The big-O notation for the function is the algorithm's worst-case runtime complexity.

PARTICIPATION ACTIVITY 3.6.2: Worst-case runtime analysis.

1) Which function best represents the number of operations in the worst-case?

```
i = 0
sum = 0
while (i < N) {
   sum = sum + numbers[i]
   ++i
}
```

- f(N) = 3N + 2
- f(N) = 3N + 3
- f(N) = 2 + N (N + 1)

2) What is the big-O notation for the worst-case runtime?

```
negCount = 0
for(i = 0; i < N; ++i) {
   if (numbers[i] < 0 ) {
      ++negCount
   }
}
```

- f(N) = 2 + 4N + 1
- O(4N + 3)
- O(N)

3) What is the big-O notation for the worst-case runtime?

```
for (i = 0; i < N; ++i) {
   if ((i % 2) == 0) {
      outVal[i] = inVals[i] * i
   }
}
```

- O(1)
- $O(\frac{N}{2})$
- O(N)

4) What is the big-O notation for the worst-case runtime?

```
nVal = N
steps = 0
while (nVal > 0) {
   nVal = nVal / 2
   steps = steps + 1
}
```

- O(log N)
- $O(\frac{N}{2})$
- O(N)

5) What is the big-O notation for the **best-case** runtime?

```
i = 0
belowMinSum = 0.0
belowMinCount = 0
while (i < N && numbers[i] <= maxVal)
{
   belowMinCount = belowMinCount + 1
   belowMinSum = numbers[i]
   ++i
}
avgBelow = belowMinSum /
belowMinCount
```

- O(1)
- O(N)

Counting constant time operations

For algorithm analysis, the definition of a single operation does not need to be precise. An operation can be any statement (or constant number of statements) that has a constant runtime complexity, O(1). Since constants are omitted in big-O notation, any constant number of constant time operations is O(1). So, precisely counting the number of constant time

operations in a finite sequence is not needed. Ex: An algorithm with a single loop that execute 5 operations before the loop, 3 operations each loop iteration, and 6 operations after the loop would have a runtime of f(N) = 5 + 3N + 6, which can be written as O(1) + O(N) + O(1) = O(N). If the number of operations before the loop was 100, the big-O notation for those operation is still O(1).

PARTICIPATION ACTIVITY 3.6.3: Simplified runtime analysis: A constant number of constant time operations is O(1).

Animation captions:

1. Constants are omitted in big-O notation, so any constant number of constant time operations is O(1).
2. The for loop iterates N times. Big-O complexity can be written as a composite function and simplified.

PARTICIPATION ACTIVITY 3.6.4: Constant time operations.

1) A for loop of the form `for (i = 0; i < N; ++i) {}` that does not have nested loops or function calls, and does not modify i in the loop will always has a complexity of O(N).

   - True
   - False

2) The complexity of the algorithm below is O(1).

```
if (timeHour < 6) {
   tollAmount = 1.55
}
else if (timeHour < 10) {
   tollAmount = 4.65
}
else if (timeHour < 18) {
   tollAmount = 2.35
}
else {
   tollAmount = 1.55
}
```

   - True
   - False

3) The complexity of the algorithm below

is O(1).

```
for (i = 0; i < 24; ++i) {
   if (timeHour < 6) {
      tollSchedule[i] = 1.55
   }
   else if (timeHour < 10) {
      tollSchedule[i] = 4.65
   }
   else if (timeHour < 18) {
      tollSchedule[i] = 2.35
   }
   else {
      tollSchedule[i] = 1.55
   }
}
```

- ○ True
- ○ False

Runtime analysis of nested loops

Runtime analysis for nested loops requires summing the runtime of the inner loop over each outer loop iteration. The resulting summation can be simplified to determine the big-O notation.

PARTICIPATION ACTIVITY 3.6.5: Runtime analysis of nested loop: Selection sort algorithm.

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Animation captions:

1. For each iteration of the outer loop, the runtime of the inner loop is determined and added together to form a summation. For iteration i = 0, the inner loop executes N - 1 iterations.
2. For i = 1, the inner loop iterates N - 2 times: iterating from j = 2 to N - 1.
3. For i = N - 3, the inner loop iterates twice: iterating from j = N - 2 to N - 1. For i = N - 2, the inner loop iterates once: iterating from j = N - 1 to N - 1.
4. For i = N - 1, the inner loop iterates 0 times. The summation is the sum of a consecutive sequence of numbers from N - 1 to 0.
5. The sequence contains N / 2 pairs, each summing to N - 1, and can be simplified.
6. Each iteration of the loops requires a constant number of operations, which is defined as the constant c.
7. Additionally, each iteration of the outer loop requires a constant number of operations, which is defined as the constant d.
8. Big-O notation omits the constant values, and the runtime is equal to the summation of the total inner loop iterations.

Figure 3.6.1: Common summation: Summation of consecutive numbers.

$$(N-1)+(N-2)+\cdots+2+1=\frac{N(N-1)}{2}=O(N^2)$$

PARTICIPATION ACTIVITY 3.6.6: Nested loops.

Determine the big-O worst-case runtime for each algorithm.

1)
```
for (i = 0; i < N; i++)  {
   for (j = 0; j < N; j++) {
      if (numbers[i] < numbers[j]) {
         ++eqPerms
      }
      else {
         ++neqPerms
      }
   }
}
```

- O(N)
- O(N^2)

2)
```
for (i = 0; i < N; i++)  {
   for (j = 0; j < (N - 1); j++) {
      if (numbers[j + 1] <
numbers[j]) {
         temp = numbers[j]
         numbers[j] = numbers[j + 1]
         numbers[j + 1] = temp
      }
   }
}
```

- O(N)
- O(N^2)

3)
```
for (i = 0; i < N; i = i + 2)  {
   for (j = 0; j < N; j = j + 2) {
      cVals[i][j] = inVals[i] * j
   }
}
```

- O(N)
- O(N^2)

4)

```
for (i = 0; i < N; ++i)  {
   for (j = i; j < N - 1; ++j) {
      cVals[i][j] = inVals[i] * j
   }
}
```

- O(N^2)
- O(N^3)

5)
```
for (i = 0; i < N; ++i) {
   sum = 0
   for (j = 0; j < N; ++j) {
      for (k = 0; k < N; ++k) {
         sum = sum + aVals[i][k] *
bVals[k][j]
      }

      cVals[i][j] = sum
   }
}
```

- O(N)
- O(N^2)
- O(N^3)

How was this section? Provide feedback

3.7 Recursive definitions

Recursive algorithms

An ***algorithm*** is a sequence of steps, including at least 1 terminating step, for solving a problem. A ***recursive algorithm*** is an algorithm that breaks the problem into smaller subproblems and applies the algorithm itself to solve the smaller subproblems.

Because a problem cannot be endlessly divided into smaller subproblems, a recursive algorithm must have a ***base case***: A case where a recursive algorithm completes without applying itself to a smaller subproblem.

PARTICIPATION ACTIVITY 3.7.1: Recursive factorial algorithm.

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```
undefined
```

Animation captions:

1. A recursive algorithm to compute N factorial has 2 parts: the base case and the non-base case.
2. N is assumed to be a positive integer. N = 1 is the base case, wherein a result of 1 is returned.
3. The non-base case computes the result by multiplying N by (N - 1) factorial.
4. The algorithm applying itself to a smaller subproblem is what makes the algorithm recursive.

PARTICIPATION ACTIVITY 3.7.2: Recursive algorithms.

1) A recursive algorithm applies itself to a smaller subproblem in all cases.
   - ○ True
   - ○ False

2) The base case is what ensures that a recursive algorithm eventually terminates.
   - ○ True
   - ○ False

3) The presence of a base case is what identifies an algorithm as being recursive.
   - ○ True
   - ○ False

Recursive functions

A ***recursive function*** is a function that calls itself. Recursive functions are commonly used to implement recursive algorithms.

Table 3.7.1: Sample recursive functions: Factorial, CumulativeSum, and ReverseList.

```
Factorial(N) {
   if (N == 1)
      return 1
   else
      return N * Factorial(N - 1)
}
```

```
CumulativeSum(N) {
   if (N == 0)
      return 0
   else
      return N + CumulativeSum(N - 1)
}
```

```
ReverseList(list, startIndex, endIndex) {
   if (startIndex >= endIndex)
      return
   else {
      Swap elements at startIndex and endIndex
      ReverseList(list, startIndex + 1, endIndex - 1)
   }
}
```

PARTICIPATION ACTIVITY 3.7.3: CumulativeSum recursive function.

1) What is the condition for the base case in the CumulativeSum function?
   - ○ N equals 0
   - ○ N does not equal 0

2) If Factorial(6) is called, how many additional calls are made to Factorial to compute the result of 720?
   - ○ 7
   - ○ 5
   - ○ 3

3) Suppose ReverseList is called on a list of size 3, a start index of 0, and an out-of-bounds end index of 3. The base case ensures that the function still properly sorts the list.
   - ○ True
   - ○ False

How was this section? **Provide feedback**

3.8 Recursive algorithms

Fibonacci numbers

The ***Fibonacci sequence*** is a numerical sequence where each term is the sum of the previous 2 terms in the sequence, except the first 2 terms, which are 0 and 1. A recursive function can be used to calculate a ***Fibonacci number***: A term in the Fibonacci sequence.

Figure 3.8.1: FibonacciNumber recursive function.

```
FibonacciNumber(termIndex) {
   if (termIndex == 0)
      return 0
   else if (termIndex == 1)
      return 1
   else
      return FibonacciNumber(termIndex - 1) + FibonacciNumber(termIndex - 2)
}
```

PARTICIPATION ACTIVITY 3.8.1: FibonacciNumber recursive function.

1) What does FibonacciNumber(2) return?

Check Show answer

2) What does FibonacciNumber(4) return?

Check Show answer

3) What does FibonacciNumber(8) return?

Check Show answer

Recursive binary search

Binary search is an algorithm that searches a sorted list for a key by first comparing the key to the middle element in the list and recursively searching half of the remaining list so long as the key is not found.

Binary search first checks the middle element of the list. If the search key is found, the algorithm returns the index. If the search key is not found, the algorithm recursively searches the remaining left sublist (if the search key was less than the middle element) or the remaining right sublist (if the search key was greater than the middle element).

Figure 3.8.2: BinarySearch recursive algorithm.

```
BinarySearch(numbers, low, high, key) {
   if (low > high)
      return -1

   mid = (low + high) / 2
   if (numbers[mid] < key) {
      return BinarySearch(numbers, mid + 1, high, key)
   }
   else if (numbers[mid] > key) {
      return BinarySearch(numbers, low, mid - 1, key)
   }
   return mid
}
```

PARTICIPATION ACTIVITY 3.8.2: Recursive binary search.

Suppose BinarySearch(numbers, 0, 7, 42) is used to search the list (14, 26, 42, 59, 71, 88, 92) for key 42.

1) What is the first middle element that is compared against 42?

- 42
- 59
- 71

2) What will the low and high argument values be for the first recursive call?

- low = 0
  high = 2
- low = 0
  high = 3

- low = 4
  high = 7

3) How many calls to BinarySearch will be made by the time 42 is found?

- 2
- 3
- 4

PARTICIPATION ACTIVITY 3.8.3: Recursive binary search base case.

1) Which does not describe a base case for BinarySearch?

- The low argument is greater than the high argument.
- The list element at index `mid` equals the key.
- The list element at index `mid` is less than the key.

How was this section? **Provide feedback**

3.9 Analyzing the time complexity of recursive alg

Recurrence relations

The runtime complexity T(N) of a recursive function will have function T on both sides of the equation. Ex: Binary search performs constant time operations, then a recursive call that operates on half of the input, making the runtime complexity T(N) = O(1) + T(N / 2). Such a function is known as a ***recurrence relation***: A function f(N) that is defined in terms of the same function operating on a value < N.

Using O-notation to express runtime complexity of a recursive function requires solving the recurrence relation. For simpler recursive functions such as binary search, runtime complexity can be determined by expressing the number of function calls as a function of N.

PARTICIPATION ACTIVITY 3.9.1: Worst case binary search runtime complexity.

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Animation captions:

1. In the non-base case, BinarySearch does some O(1) operations plus a recursive call on half the input list.
2. The maximum number of recursive calls can be computed for any known input size. For size 1, 1 recursive call is made.
3. Additional entries in the table can be filled. A list of size 32 is split in half 6 times before encountering the base case.
4. By analyzing the pattern, the total number of function calls can be expressed as a function of N.
5. The number of function calls corresponds to the runtime complexity.

PARTICIPATION ACTIVITY 3.9.2: Binary search and recurrence relations.

1) When the low and high arguments are equal, BinarySearch does not make a recursive call.
   - ○ True
   - ○ False

2) Suppose BinarySearch is used to search for a key within a list with 64 numbers. If the key is not found, how many recursive calls to BinarySearch are made?
   - ○ 1
   - ○ 6
   - ○ 64

3) Which function is a recurrence relation?
   - ○ $T(N) = N^2 + 6N + 2$
   - ○ $T(N) = 6N + T(N/4)$
   - ○

$T(N) = log_2 N$

Recursion trees

The runtime complexity of any recursive function can be split into 2 parts: operations done directly by the function and operations done by recursive calls made by the function. Ex: For binary search's T(N) = O(1) + T(N / 2), O(1) represents operations directly done by the function and T(N / 2) represents operations done by a recursive call. A useful tool for solving recurrences is a ***recursion tree***: A visual diagram of a operations done by a recursive function, that separates operations done directly by the function and operations done by recursive calls.

PARTICIPATION ACTIVITY 3.9.3: Recursion trees.

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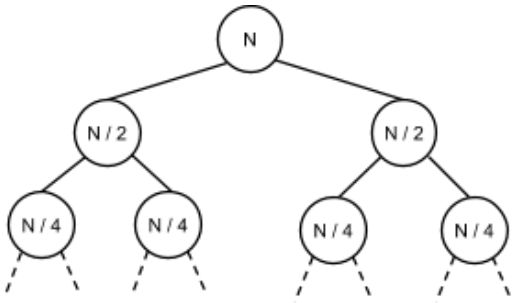
undefined

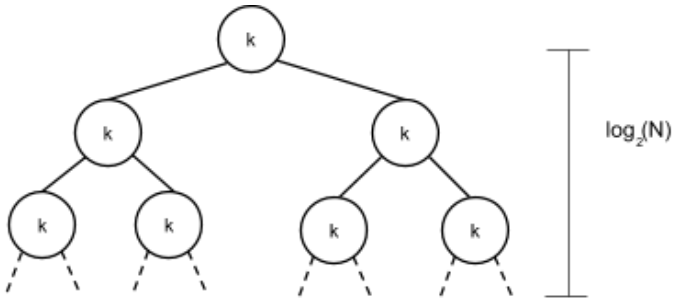
Animation captions:

1. An algorithm like binary search does a constant number of operations, k, followed by a recursive call on half the list.
2. The root node in the recursion tree represents k operations inside the first function call.
3. Recursive operations are represented below the node. The first recursive call also does k operations.
4. The number of nodes corresponds to the number of recursive calls. Splitting the input in half each time results in $log_2 N$ recursive calls. O($log_2 N$) = O($log N$).
5. Another algorithm may perform N operations then 2 recursive calls, each on N / 2 items. The root node represents N operations.
6. The initial call makes 2 recursive calls, each of which has a local N value of the initial N value / 2.
7. N operations are done per level.
8. The tree has $log_2 N$ levels. $N * log_2 N$ operations are done in total. O($N * log_2 N$) = O($N log N$).

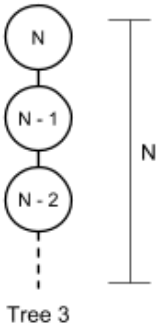
PARTICIPATION ACTIVITY 3.9.4: Matching recursion trees with runtime complexities.



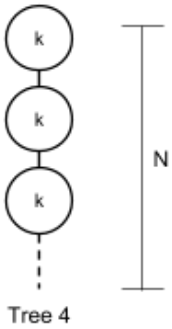
Tree 1



Tree 2



Tree 3



Tree 4

Tree 2 Tree 1 Tree 3 Tree 4

T(N) = k + T(N / 2) + T(N / 2)

T(N) = k + T(N - 1)

T(N) = N + T(N - 1)

T(N) = N + T(N / 2) + T(N / 2)

Reset

PARTICIPATION ACTIVITY 3.9.5: Recursion trees.

Suppose a recursive function's runtime is $T(N) = 7 + T(N - 1)$.

1) How many levels will the recursion tree have?

- 7
- $log_2 N$

- N

2) What is the runtime complexity of the function using O notation?

- O(1)
- O($logN$)
- O(N)

PARTICIPATION ACTIVITY 3.9.6: Recursion trees.

Suppose a recursive function's runtime is $T(N) = N + T(N - 1)$.

1) How many levels will the recursion tree have?

- $log_2 N$
- N
- N^2

2) The runtime can be expressed by the series N + (N - 1) + (N - 2) + ... + 3 + 2 + 1. Which expression is mathematically equivalent?

- $N * log_2 N$
- $(N/2) * (N + 1)$

3) What is the runtime complexity of the function using O notation?

- O(N)
- O(N^2)

How was this section? **Provide feedback**