6.1 Binary trees

Binary tree basics

In a list, each node has up to one successor. In a ***binary tree***, each node has up to two children, known as a *left child* and a *right child*. "Binary" means two, referring to the two children. Some more definitions related to a binary tree:

- ***Leaf***: A tree node with no children.
- ***Internal node***: A node with at least one child.
- ***Parent***: A node with a child is said to be that child's parent. A node's ***ancestors*** include the node's parent, the parent's parent, etc., up to the tree's root.
- ***Root***: The one tree node with no parent (the "top" node).

Another section discusses binary tree usefulness; this section introduces definitions.

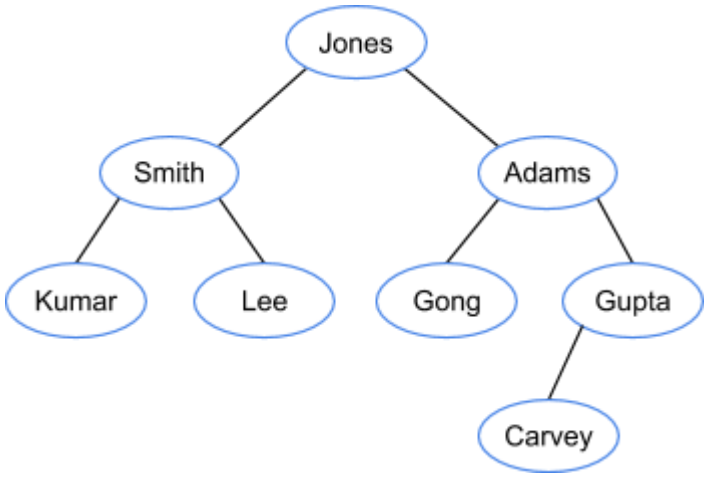
Below, each node is represented by just the node's key, as in B, although the node may have other data.

PARTICIPATION ACTIVITY 6.1.1: Binary tree basics.

Animation captions:

1. In a list, each node has up to one successor.
2. In a binary tree, each node has up to two children.
3. A tree is normally drawn vertically. Edge arrows are optional.
4. A node can have a left child and right child. A node with a child is called the child's parent.
5. First node: Root node. Node without child: Leaf node. Node with child: Internal nodes.

PARTICIPATION ACTIVITY 6.1.2: Binary tree basics.



1) Root node: _____.

Check Show answer

2) Smith's left child: ____

Check Show answer

3) The tree has four leaf nodes: Kumar, Lee, Gong, and _____.

Check Show answer

4) How many internal nodes does the tree have?

Check Show answer

5) The tree has an internal node, Gupta, with only one child rather than two. Is the tree still a binary tree? Type yes or no.

Check Show answer

Depth, level, and height

A few additional terms:

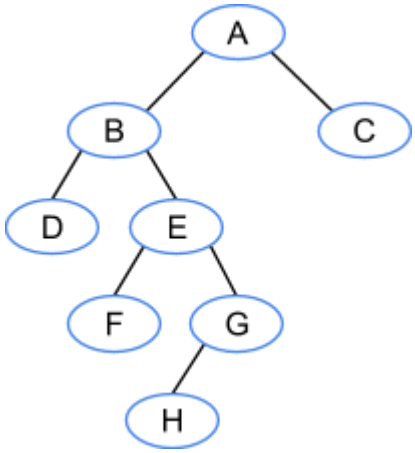
- The link from a node to a child is called an ***edge***.
- A node's ***depth*** is the number of edges on the path from the root to the node. The root node thus has depth 0.
- All nodes with the same depth form a tree ***level***.
- A tree's ***height*** is the largest depth of any node. A tree with just one node has height 0.

PARTICIPATION ACTIVITY 6.1.3: Binary tree terminology: height, depth, and level.

Animation captions:

1. The binary tree has edges A to B, A to C, and C to D.
2. A node's depth is the number of edges from the root to the node.
3. Nodes with the same depth form a level.
4. A tree's height is the largest depth of any node.

PARTICIPATION ACTIVITY 6.1.4: Binary tree height, depth, and level.



1) A's depth is 1.

- ○ True
- ○ False

2) E's depth is 2.

- ○ True
- ○ False

3) B and C form level 1.

- True
- False

4) D, F, and H form level 2.

- True
- False

5) The tree's height is 4.

- True
- False

6) A tree with just one node has height 0.

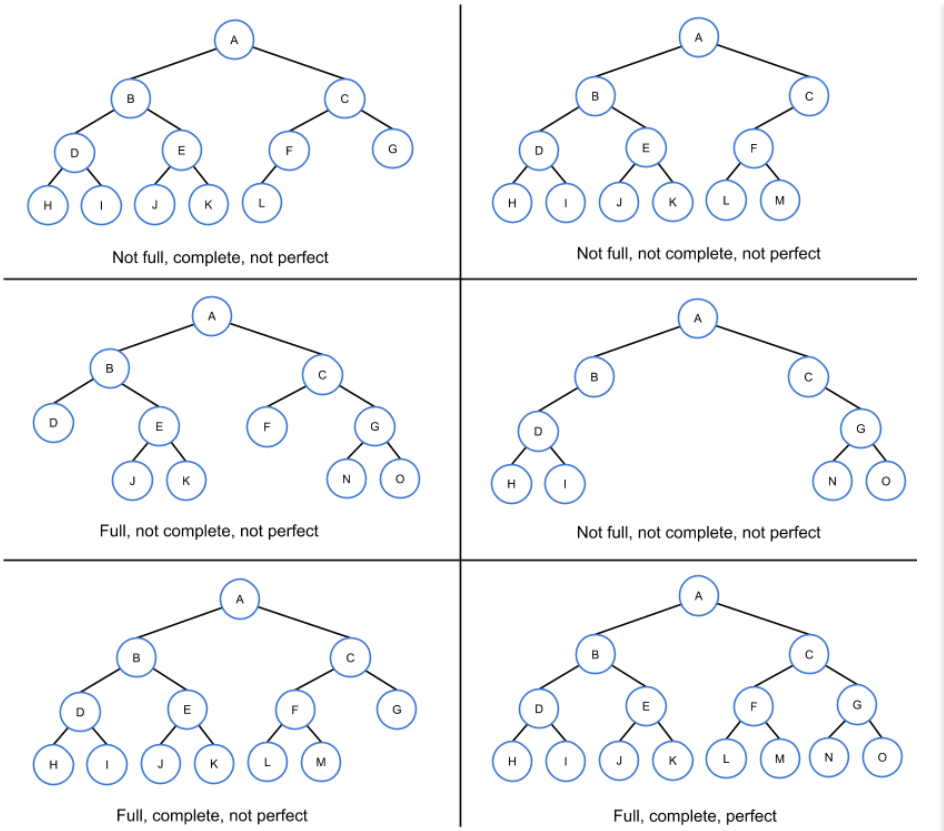
- True
- False

Special types of binary trees

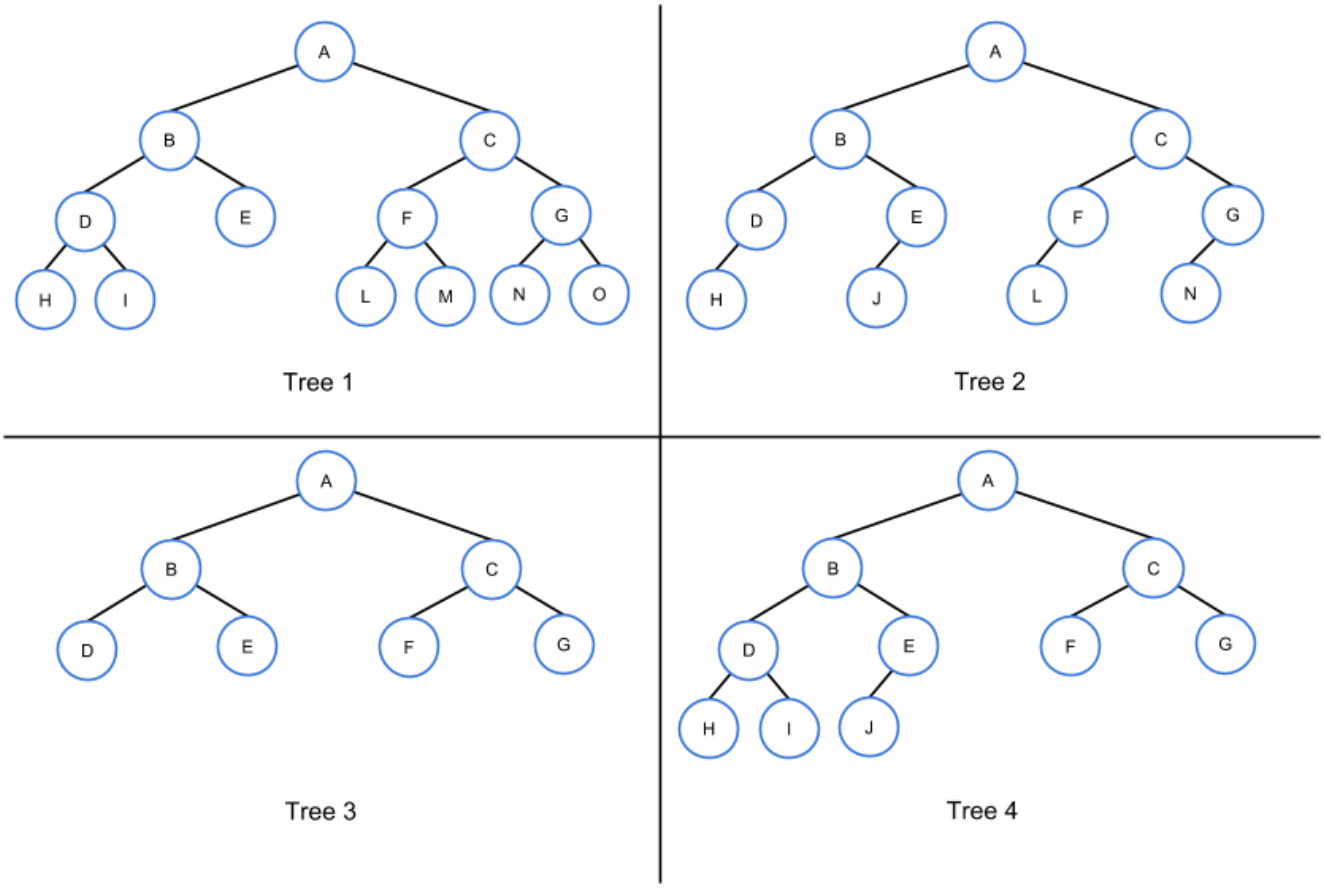
Certain binary tree structures can affect the speed of operations on the tree. The following describe special types of binary trees:

- A binary tree is ***full*** if every node contains 0 or 2 children.
- A binary tree is ***complete*** if all levels, except possibly the last level, are completely full and all nodes in the last level are as far left as possible.
- A binary tree is ***perfect***, if all internal nodes have 2 children and all leaf nodes are at the same level.

Figure 6.1.1: Special types of binary trees: full, complete, perfect.



PARTICIPATION ACTIVITY 6.1.5: Identifying special types of binary trees.



Full, not complete, not perfect

Not full, complete, not perfect

Full, complete, perfect

Not full, not complete, not perfect

Tree 1

Tree 2

Tree 3

Tree 4

Reset

How was this section? **Provide feedback**

6.2 Applications of trees

File systems

Trees are commonly used to represent hierarchical data. A tree can represent files and directories in a file system, since a file system is a hierarchy.

PARTICIPATION ACTIVITY 6.2.1: A file system is a hierarchy that can be represented by a tree.

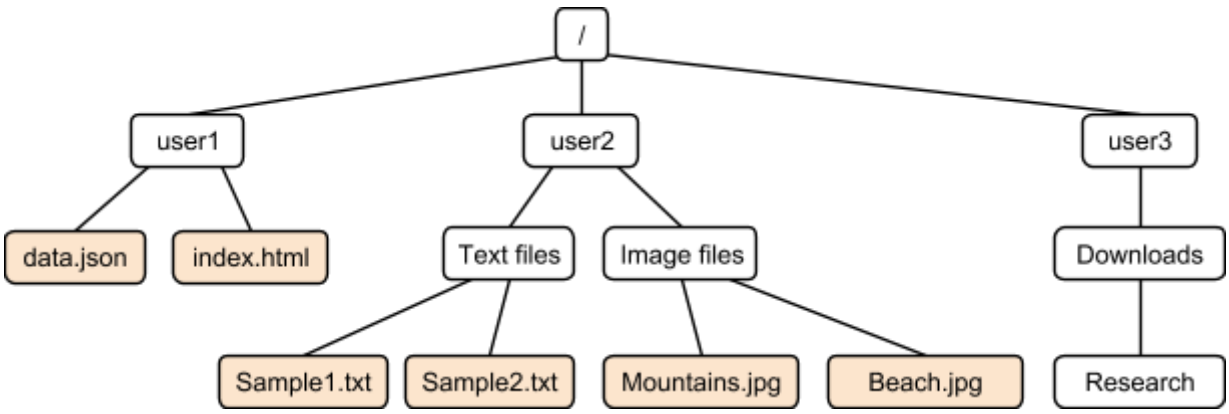
Animation content:

undefined

Animation captions:

1. A tree representing a file system has the filesystem's root directory ("/"), represented by the root node.
2. The root contains 2 configuration text files and 2 additional directories: user1 and user2.
3. Directories contain additional entries. Only empty directories will be leaf nodes. All files are leaf nodes.

PARTICIPATION ACTIVITY 6.2.2: Analyzing a file system tree.



1) What is the depth of the "Mountains.jpg" file node?

- 3
- 4

2) What is the height of this tree?

- 3
- 4
- 14

3) What is the parent of the "Text files" node?

- The "user2" directory node
- The "Image files" directory node
- The "Sample1.txt" file node.

4) Which operation would increase the height of the tree?

- Adding a new file into the user1 directory
- Adding a new directory into the "Image files" directory
- Adding a new directory into the "Research" directory

PARTICIPATION ACTIVITY 6.2.3: File system trees.

1) A file in a file system tree is always a leaf node.

- True
- False

2) A directory in a file system tree is always an internal node.

- True
- False

3) Using a tree data structure to implement a file system requires that each directory node support a variable number of children.

- True
- False

Binary space partitioning

Binary space partitioning (***BSP***) is a technique of repeatedly separating a region of space into 2 parts and cataloging objects contained within the regions. A ***BSP tree*** is a binary tree used to store information for binary space partitioning. Each node in a BSP tree contains information about a region of space and which objects are contained in the region.

In graphics applications, a BSP tree can be used to store all objects in a multidimensional world. The BSP tree can then be used to efficiently determine which objects must be rendered on screen. The viewer's position in space is used to perform a lookup within the BSP tree. The lookup quickly eliminates a large a number of objects that are not visible and therefore should not be rendered.

PARTICIPATION ACTIVITY 6.2.4: A BSP tree is used to quickly determine which objects do not need to be rendered.

Animation content:

`undefined`

Animation captions:

1. Data for a large, open 2-D world contains many objects. Only a few objects are visible on screen at any given moment.
2. Avoiding rendering off-screen objects is crucial for realtime graphics. But checking the intersection of all objects with the screen's rectangle is too time consuming.
3. A BSP tree represents partitioned space. The root represents the entire world and stores a li of all objects in the world, as well as the world's geometric boundary.
4. The root's left child represents the world's left half. The node stores information about the lef half's geometric boundary, and a list of all objects contained within.
5. The root's right child contains similar information for the right half.
6. Using the screen's position within the world as a lookup into the BSP tree quickly yields the right node's list of objects. A large number of objects are quickly eliminated from the list of potential objects on screen.
7. Further partitioning makes the tree even more useful.

PARTICIPATION ACTIVITY 6.2.5: Binary space partitioning.

1) When traversing down a BSP tree, half the objects are eliminated each level.

   - True
   -

False

2) A BSP implementation could choose to split regions in arbitrary locations, instead of right down the middle.

- True
- False

3) In the animation, if the parts of the screen were in 2 different regions, then all objects from the 2 regions would have to be analyzed when rendering.

- True
- False

4) BSP can be used in 3-D graphics as well as 2-D.

- True
- False

Using trees to store collections

Most of the tree data structures discussed in this book serve to store a collection of values. Numerous tree types exist to store data collections in a structured way that allows for fast searching, inserting, and removing of values.

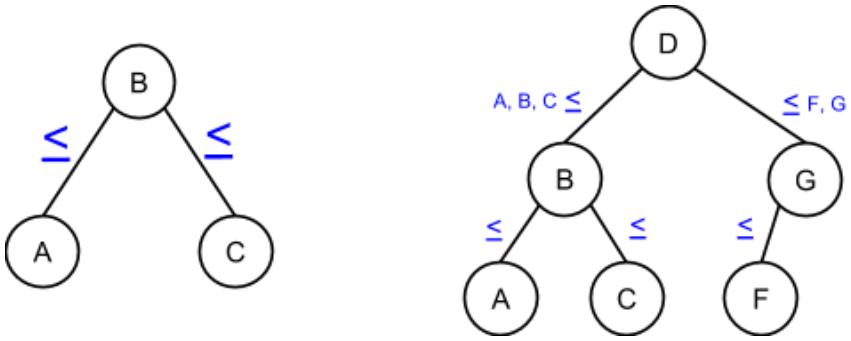
How was this section? **Provide feedback**

6.3 Binary search trees

Binary search trees

An especially useful form of binary tree is a ***binary search tree*** (BST), which has an ordering property that any node's left subtree keys ≤ the node's key, and the right subtree's keys ≥ the node's key. That property enables fast searching for an item, as will be shown later.

Figure 6.3.1: BST ordering property: For three nodes, left child is less-than-or-equal-to parent, parent is less-than-or-equal-to right child. For more nodes, all keys in subtrees must satisfy the property, for every node.



PARTICIPATION ACTIVITY 6.3.1: BST ordering properties.

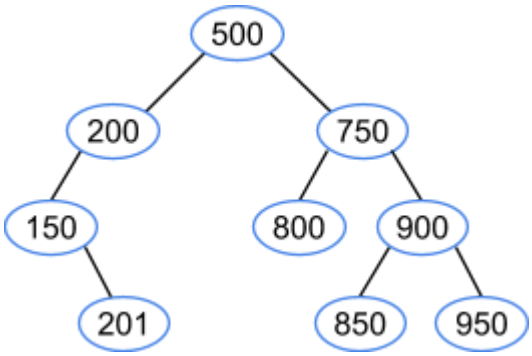
Animation content:

undefined

Animation captions:

1. BST ordering property: Left subtree's keys ≤ node's key, right subtree's keys ≥ node's key.
2. All keys in subtree must obey the ordering property. Not a BST.
3. All keys in subtree must obey the ordering property. Not a BST.
4. All keys in subtree must obey the ordering property. Valid BST.

PARTICIPATION ACTIVITY 6.3.2: Binary search tree: Basic ordering property.



1) Does node 900 and the node's subtrees obey the BST ordering property?

- Yes
- No

2) Does node 750 and the node's subtrees obey the BST ordering property?

- Yes
- No

3) Does node 150 and the node's subtrees obey the BST ordering property?

- Yes
- No

4) Does node 200 and the node's subtrees obey the BST ordering property?

- Yes
- No

5) Is the tree a binary search tree?

- Yes
- No

6) Is the tree a binary tree?

- Yes
- No

7) Would inserting 300 as the right child

of 200 obey the BST ordering property (considering only nodes 300, 200, and 500)?

- Yes
- No

Searching

To ***search*** nodes means to find a node with a desired key, if such a node exists. A BST may yield faster searches than a list. Searching a BST starts by visiting the root node (which is the first currentNode below):

Figure 6.3.2: Searching a BST.

```
if (currentNode->key == desiredKey) {
   return currentNode; // The desired node was found
}
else if (desiredKey < currentNode->key) {
   // Visit left child, repeat
}
else if (desiredKey > currentNode->key) {
   // Visit right child, repeat
}
```

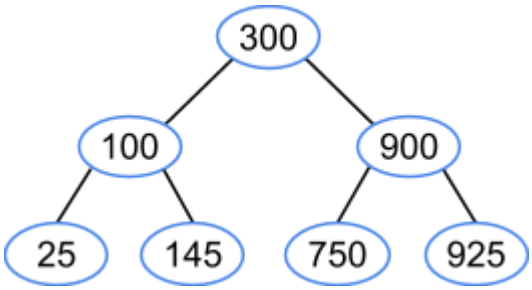
If a child to be visited doesn't exist, the desired node does not exist. With this approach, only a small fraction of nodes need be compared.

PARTICIPATION ACTIVITY 6.3.3: A BST may yield faster searches than a list.

Animation captions:

1. Searching a 7-node list may require up to 7 comparisons.
2. In a BST, if desired key equals current node's key, return found. If less, descend to left child. If greater, descend to right child.
3. Searching a BST may require fewer comparisons, in this case 3 vs. 7.

PARTICIPATION ACTIVITY 6.3.4: Searching a BST.



1) In searching for 145, what node is visited first?

Check Show answer

2) In searching for 145, what node is visited second?

Check Show answer

3) In searching for 145, what node is visited third?

Check Show answer

4) Which nodes would be visited when searching for 900? Write nodes in order visited, as: 5, 10

Check Show answer

5) Which nodes would be visited when searching for 800? Write nodes in order visited, as: 5, 10, 15

Check Show answer

6) What is the worst case (largest) number of nodes visited when

searching for a key?

Check Show answer

Best case BST search runtime

Searching a BST in the worst case requires H + 1 comparisons, meaning O(H) comparisons, where H is the tree height. Ex: A tree with a root node and one child has height 1; the worst case visits the root and the child: 1 + 1 = 2. A major BST benefit is that an N-node binary tree's height may be as small as O($logN$), yielding extremely fast searches. Ex: A 10,000 node list may require 10,000 comparisons, but a 10,000 node BST may require only 14 comparisons.

A binary tree's height can be minimized by keeping all levels full, except possibly the last level. Such an "all-but-last-level-full" binary tree's height is $H = \lfloor log_2 N \rfloor$.

Table 6.3.1: Minimum binary tree heights for N nodes are equivalent to $\lfloor log_2 N \rfloor$.

Nodes N	Height H	$log_2 N$	$\lfloor log_2 N \rfloor$	Nodes per level
1	0	0	0	1
2	1	1	1	1/1
3	1	1.6	1	1/2
4	2	2	2	1/2/1
5	2	2.3	2	1/2/2
6	2	2.6	2	1/2/3
7	2	2.8	2	1/2/4
8	3	3	3	1/2/4/1
9	3	3.2	3	1/2/4/2
...				
15	3	3.9	3	1/2/4/8
16	4	4	4	1/2/4/8/1

PARTICIPATION ACTIVITY 6.3.5: Searching a perfect BST with N nodes requires only O($logN$) comparisons.

Animation captions:

1. A perfect binary tree has height $\lfloor log_2 N \rfloor$.
2. A perfect binary tree search is O(H), so O($logN$).
3. Searching a BST may be faster than searching a list.

PARTICIPATION ACTIVITY 6.3.6: Searching BSTs with N nodes.

What is the worst case (largest) number of comparisons given a BST with N nodes?

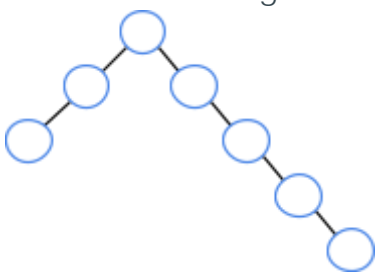
1) Perfect BST with N = 7

- $\lfloor log_2 N \rfloor$
- $\lfloor log_2 N \rfloor + 1$
- N

2) Perfect BST with N = 31

- 31
- 4
- 5

3) Given the following tree.



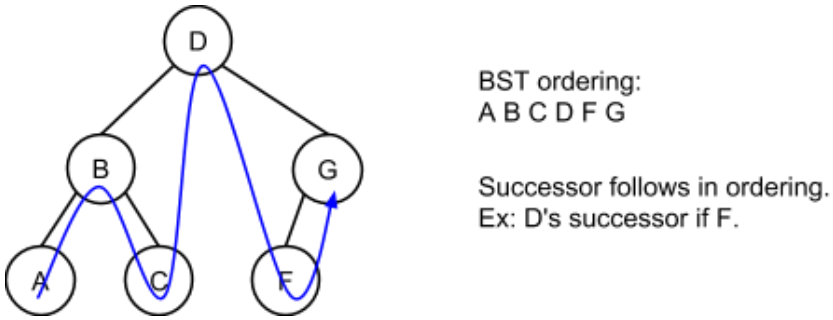
- 3
- 5

Successors and predecessors

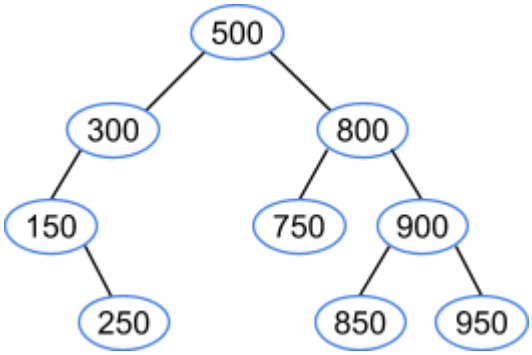
A BST defines an ordering among nodes, from smallest to largest. A BST node's ***successor*** is the node that comes after in the BST ordering, so in A B C, A's successor is B, and B's successor is C. A BST node's ***predecessor*** is the node that comes before in the BST ordering.

If a node has a right subtree, the node's successor is that right subtree's leftmost child: Starting from the right subtree's root, follow left children until reaching a node with no left child (may be that subtree's root itself). If a node doesn't have a right subtree, the node's successor is the first ancestor having this node in a left subtree. Another section provides an algorithm for printing a BST's nodes in order.

Figure 6.3.3: A BST defines an ordering among nodes.



PARTICIPATION ACTIVITY 6.3.7: Binary search tree: Defined ordering.



1) The first node in the BST ordering is 150.
   - True
   - False

2) 150's successor is 250.
   - True
   - False

3) 250's successor is 300.
   - True
   - False

4) 500's successor is 850.

- True
- False

5) 950's successor is 150.

- True
- False

6) 950's predecessor is 900.

- True
- False

How was this section? Provide feedback

6.4 BST search algorithm

Given a key, a ***search*** algorithm returns the first node found matching that key, or returns null if a matching node is not found. A simple BST search algorithm checks the current node (initially the tree's root), returning that node as a match, else assigning the current node with the left (if key is less) or right (if key is greater) child and repeating. If such a child is null, the algorithm returns null (matching node not found).

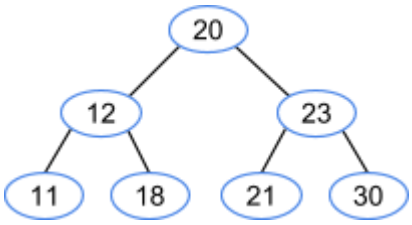
PARTICIPATION ACTIVITY 6.4.1: BST search algorithm.

Animation captions:

1. BST search algorithm checks current node, returning a match if found. Otherwise, assigns current node with left (if key is less) or right (if key is greater) child and continues search.
2. If the child to be visited does not exist, the algorithm returns null indicating no match found.

PARTICIPATION ACTIVITY 6.4.2: BST search algorithm.

Consider the following tree.



1) When searching for key 21, what node is visited first?
   - 20
   - 21

2) When searching for key 21, what node is visited second?
   - 12
   - 23

3) When searching for key 21, what node is visited third?
   - 21
   - 30

4) If the current node matches the key, when does the algorithm return the node?
   - Immediately
   - Upon exiting the loop

5) If the child to be visited is null, when does the algorithm return null?
   - Immediately
   - Upon exiting the loop

6) What is the maximum loop iterations for a perfect binary tree with 7 nodes, if a node matches?
   - 3
   - 7

7) What is the maximum loop iterations for a perfect binary tree with 7 nodes, if no node matches?
   -

3

- 7

8) What is the maximum loop iterations for a perfect binary tree with 255 nodes?

- 8
- 255

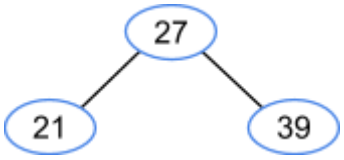
9) Suppose node 23 was instead 21, meaning two 21 nodes exist (which is allowed in a BST). When searching for 21, which node will be returned?

- Leaf
- Internal

PARTICIPATION ACTIVITY 6.4.3: BST search algorithm decisions.

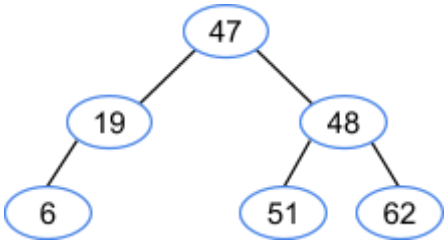
Determine cur's next assignment given the key and current node.

1) key = 40, cur = 27



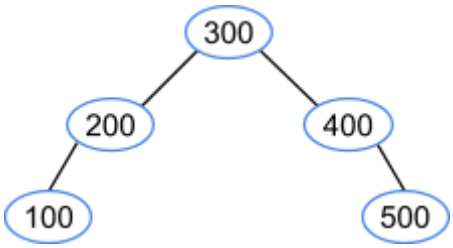
- 27
- 21
- 39

2) key = 6, cur = 47



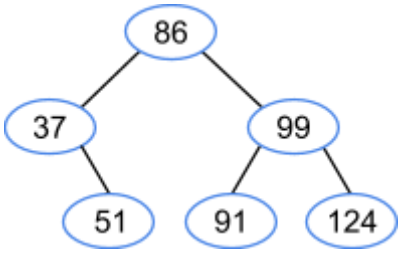
- 6
- 19
- 48

3) key 350, cur = 400



- Search terminates and returns 0.
- 400
- 500

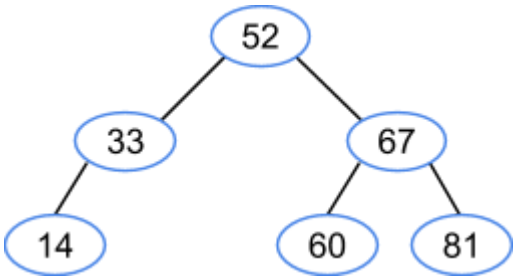
4) key 91, cur = 99



- 86
- 91
- 124

PARTICIPATION ACTIVITY 6.4.4: Tracing a BST search.

Consider the following tree. If node does not exist, enter null.



1) When searching for key 45, what node is visited first?

Check Show answer

2) When searching for key 45, what node is visited second?

Check Show answer

3) When searching for key 45, what node is visited third?

Check Show answer

How was this section? Provide feedback

6.5 BST insert algorithm

Given a new node, a BST ***insert*** operation inserts the new node in a proper location obeying the BST ordering property. A simple BST insert algorithm compares the new node with the current node (initially the root).

- *Insert as left child*: If the new node's key is less than the current node, and the current node's left child is null, the algorithm assigns that node's left child with the new node.
- *Insert as right child*: If the new node's key is greater than the current node, and the current node's right child is null, the algorithm assigns the node's right child with the new node.
- *Search for insert location*: If the left (or right) child is not null, the algorithm assigns the current node with that child and continues searching for a proper insert location.

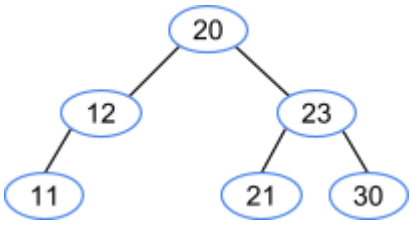
PARTICIPATION ACTIVITY 6.5.1: Binary search tree insertions.

Animation captions:

1. A node inserted into an empty tree will become the tree's root.
2. The BST is searched to find a suitable location to insert the new node as a leaf node.

PARTICIPATION ACTIVITY 6.5.2: BST insert algorithm.

Consider the following tree.



1) Where will a new node 18 be inserted?

- 12's right child
- 11's right child

2) Where will a new node 11 be inserted? (So two nodes of 11 will exist).

- 11's left child
- 11's right child

3) Assume a full 7-node BST. How many algorithm loop iterations will occur for an insert?

- 3
- 7

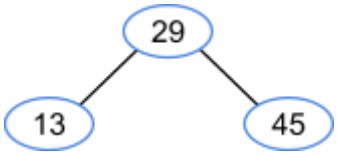
4) Assume a full 255-node BST. How many algorithm loop iterations will occur for an insert?

- 8
- 255

PARTICIPATION ACTIVITY 6.5.3: BST insert algorithm decisions.

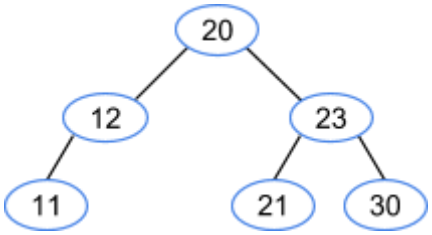
Determine the insertion algorithm's next step given the new node's key and the current node.

1) key = 7, cur = 29



- cur->left = node
- cur = cur->right
- cur = cur->left

2) key = 18, cur = 12

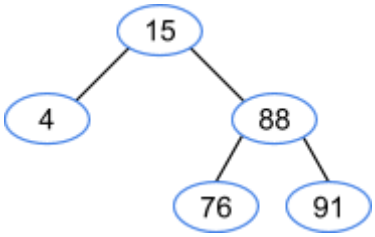


- cur->left = node
- cur->right = node
- cur = cur->right

3) key = 87, cur = null, tree->root = null (empty tree)

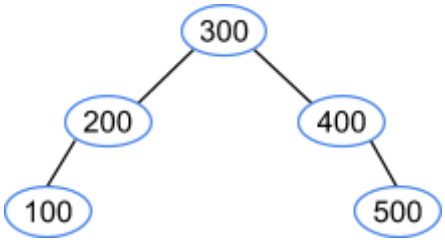
- tree->root = node
- cur->right = node
- cur->left = node

4) key = 53, cur = 76



- cur->left = node
- cur->right = node
- tree->root = node

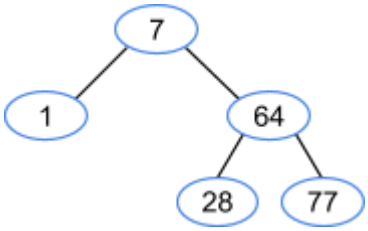
5) key = 600, cur = 400



- cur->left = node
- cur = cur->right
- cur->right = node

PARTICIPATION ACTIVITY 6.5.4: Tracing BST insertions.

Consider the following tree.



1) When inserting a new node with key 35, what node is visited first?

Check Show answer

2) When inserting a new node with key 35, what node is visited second?

Check Show answer

3) When inserting a new node with key 35, what node is visited third?

Check Show answer

4) Where is the new node inserted? Type: left or right

Check Show answer

BST insert algorithm complexity

The BST insert algorithm traverses the tree from the root to a leaf node to find the insertion location. One node is visited per level. A BST with N nodes has at least log_2N levels and at most N levels. Therefore, the runtime complexity of insertion is best case $O(logN)$ and worst case $O(N)$.

The space complexity of insertion is $O(1)$ because only a single pointer is used to traverse the tree to find the insertion location.

Exploring further:

- Binary search tree visualization

How was this section? Provide feedback

6.6 BST remove algorithm

Given a key, a BST ***remove*** operation removes the first-found matching node, restructuring the tree to preserve the BST ordering property. The algorithm first searches for a matching node just like the search algorithm. If found (call this node X), the algorithm performs one of the following sub-algorithms:

- *Remove a leaf node:* If X has a parent (so X is not the root), the parent's left or right child (whichever points to X) is assigned with null. Else, if X was the root, the root pointer is assigned with null, and the BST is now empty.
- *Remove an internal node with single child:* If X has a parent (so X is not the root), the parent's left or right child (whichever points to X) is assigned with X's single child. Else, if X was the root, the root pointer is assigned with X's single child.
- *Remove an internal node with two children:* This case is the hardest. First, the algorithm locates X's successor (the leftmost child of X's right subtree), and copies the successor to X. Then, the algorithm recursively removes the successor from the right subtree.

PARTICIPATION ACTIVITY 6.6.1: BST remove: Removing a leaf, or an internal node with a single child.

Animation captions:

1. Removing a leaf node: The parent's right child is assigned with null.
2. Remove an internal node with a single child: The parent's right child is assigned with node's single child.

PARTICIPATION ACTIVITY 6.6.2: BST remove: Removing internal node with two children.

Animation captions:

1. Find successor: Leftmost child in node 25's right subtree is node 27.
2. Copy successor to current node.
3. Remove successor from right subtree.

Figure 6.6.1: BST remove algorithm.

```
BSTRemove(tree, key) {
   par = null
   cur = tree->root
   while (cur is not null) { // Search for node
      if (cur->key == key) { // Node found
         if (!cur->left && !cur->right) {        // Remove leaf
            if (!par) // Node is root
               tree->root = null
            else if (par->left == cur)
               par->left = null
            else
               par->right = null
         }
         else if (cur->left && !cur->right) {    // Remove node with only left child
            if (!par) // Node is root
               tree->root = cur->left
            else if (par->left == cur)
               par->left = cur->left
            else
               par->right = cur->left
         }
         else if (!cur->left && cur->right) {    // Remove node with only right child
            if (!par) // Node is root
               tree->root = cur->right
            else if (par->left == cur)
               par->left = cur->right
            else
               par->right = cur->right
         }
         else {                                  // Remove node with two children
            // Find successor (leftmost child of right subtree)
            suc = cur->right
            while (suc->left is not null)
               suc = suc->left
            successorData = Create copy of suc's data
            BSTRemove(tree, suc->key)     // Remove successor
            Assign cur's data with successorData
         }
         return // Node found and removed
      }
      else if (cur->key < key) { // Search right
         par = cur
         cur = cur->right
      }
      else {                     // Search left
         par = cur
         cur = cur->left
      }
   }
   return // Node not found
}
```

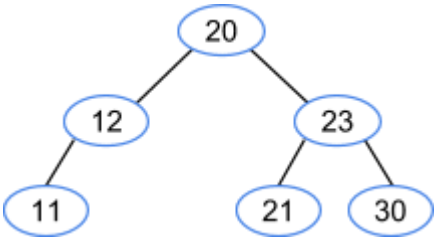
BST remove algorithm complexity

The BST remove algorithm traverses the tree from the root to find the node to remove. When the node being removed has 2 children, the node's successor is found and a recursive call is made. One node is visited per level, and in the worst case scenario the tree is traversed twice from the root to a leaf. A BST with N nodes has at least $log_2 N$ levels and at most N levels. Therefore, the runtime complexity of removal is best case $O(logN)$ and worst case $O(N)$.

Two pointers are used to traverse the tree during removal. When the node being removed has 2 children, a third pointer and a copy of one node's data are also used, and one recursive call is made. Thus, the space complexity of removal is always $O(1)$.

PARTICIPATION ACTIVITY 6.6.3: BST remove algorithm.

Consider the following tree. Each question starts from the original tree. Use this text notation for the tree: (20 (12 (11, -), 23 (21, 30))). The - means the child does not exist.



1) What is the tree after removing 21?
   - (20 (12 (11, -), 23 (-, 30)))
   - (20 (12 (11, -), 23))

2) What is the tree after removing 12?
   - (20 (- (11, -), 23 (21, 30)))
   - (20 (11, 23 (21, 30)))

3) What is the tree after removing 20?
   - (21 (12 (11, -), 23 (-, 30)))
   - (23 (12 (11, -), 30 (21, -)))

4) Removing a node from an N-node nearly-full BST has what computational complexity?

- O($logN$)
- O(N)

How was this section? Provide feedback

6.7 BST inorder traversal

A ***tree traversal*** algorithm visits all nodes in the tree once and performs an operation on each node. An ***inorder traversal*** visits all nodes in a BST from smallest to largest, which is useful for example to print the tree's nodes in sorted order. Starting from the root, the algorithm recursively prints the left subtree, the current node, and the right subtree.

Figure 6.7.1: BST inorder traversal algorithm.

```
BSTPrintInorder(node) {
  if (node is null)
      return

  BSTPrintInorder(node->left)
  Print node
  BSTPrintInorder(node->right)
}
```

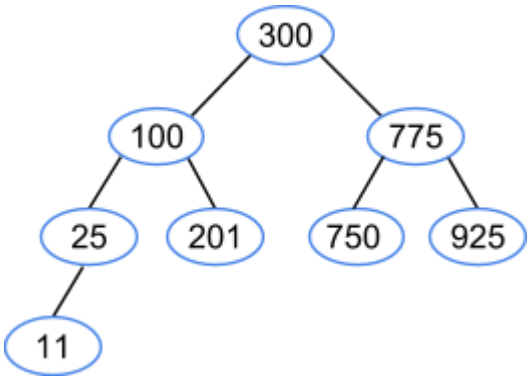
PARTICIPATION ACTIVITY 6.7.1: BST inorder print algorithm.

Animation captions:

1. An inorder traversals starts at the root. Recursive call descends into left subtree.
2. When left done, current is printed, then recursively descend into right subtree.
3. Return from recursive call causes ascending back up the tree; left is done, so do current an right.
4. Continues similarly.

PARTICIPATION ACTIVITY 6.7.2: Inorder traversal of a BST.

Consider the following tree.



1) What node is printed first?

Check Show answer

2) Complete the tree traversal after node 300's left subtree has been printed.

```
11 25 100 201
```

Check Show answer

3) How many nodes are visited?

Check Show answer

4) Using left, current, and right, what ordering will print the BST from largest to smallest? Ex: An inorder traversal uses left current right.

Check Show answer

How was this section? Provide feedback

6.8 BST height and insertion order

BST height and insertion order

Recall that a tree's ***height*** is the maximum edges from the root to any leaf. (Thus, a one-node tree has height 0.)

The *minimum* N-node binary tree height is $h = \lfloor log_2 N \rfloor$, achieved when each level is full except possibly the last. The *maximum* N-node binary tree height is N - 1 (the - 1 is because the root is at height 0).

Searching a BST is fast if the tree's height is near the minimum. Inserting items in random order naturally keeps a BST's height near the minimum. In contrast, inserting items in nearly-sorted order leads to a nearly-maximum tree height.

PARTICIPATION ACTIVITY 6.8.1: Inserting in random order keeps tree height near the minimum. Inserting in sorted order yields the maximum.

Animation captions:

1. Inserting in random order naturally keeps tree height near the minimum, in this case 3 (minimum: 2)
2. Inserting in sorted order yields the maximum height, in this case 6.
3. If nodes are given beforehand, randomizing the ordering before inserting keeps tree height near minimum.

PARTICIPATION ACTIVITY 6.8.2: BST height.

Draw a BST by hand, inserting nodes one at a time, to determine a BST's height.

1) A new BST is built by inserting nodes in this order:
   6 2 8

   What is the tree height? (Remember, the root is at height 0)

   Check Show answer

2) A new BST is built by inserting nodes

in this order:

```
20 12 23 18 30
```

What is the tree height?

Check Show answer

3) A new BST is built by inserting nodes in this order:

```
30 23 21 20 18
```

What is the tree height?

Check Show answer

4) A new BST is built by inserting nodes in this order:

```
30 11 23 21 20
```

What is the tree height?

Check Show answer

5) A new BST is built by inserting 255 nodes in sorted order. What is the tree height?

Check Show answer

6) A new BST is built by inserting 255 nodes in random order. What is the minimum possible tree height?

Check Show answer

BSTGetHeight algorithm

Given a node representing a BST subtree, the height can be computed as follows:

- If the node is null, return -1.
- Otherwise recursively compute the left and right child subtree heights, and return 1 plus the greater of the 2 child subtrees' heights.

PARTICIPATION ACTIVITY 6.8.3: BSTGetHeight algorithm.

Animation content:

```
undefined
```

Animation captions:

1. BSTGetHeight(tree->root) is called to get the height of the tree. The height of the root's left child is determined first using a recursive call.
2. BSTGetHeight for node 18 makes a recursive call on node 12. BSTGetHeight on node 12 makes a recursive call on the null left child, which returns -1.
3. Returning to the BSTGetHeight(node 12) call, a recursive call is now made on the right child Node 14 is a leaf, so both recursive calls return -1.
4. BSTGetHeight(node 14) returns 1 + max(-1, -1) = 1 + -1 = 0.
5. BSTGetHeight(node 12) has completed 2 recursive calls and returns 1 + max(-1, 0) = 1. BSTGetHeight(node 18) makes the recursive call on the null right child, which returns -1.
6. A recursive call is made for each node in the tree. BSTGetHeight(tree->root) returns 1 + max(2, 1) = 3, which is the tree's height.

PARTICIPATION ACTIVITY 6.8.4: BSTGetHeight algorithm.

1) BSTGetHeight returns 0 for a tree with a single node.
   - ○ True
   - ○ False

2) The base case for BSTGetHeight is when the node argument is null.
   - ○ True
   - ○ False

3) The worst-case time complexity for

BSTGetHeight is O(log N), where N is the number of nodes in the tree.

- ○ True
- ○ False

4) BSTGetHeight would also work if the recursive call on the right child was made before the recursive call on the left child.

- ○ True
- ○ False

How was this section? **Provide feedback**

6.9 BST parent node pointers

A BST implementation often includes a parent pointer inside each node. A balanced BST, such as an AVL tree or red-black tree, may utilize the parent pointer to traverse up the tree from a particular node to find a node's parent, grandparent, or siblings. The BST insertion and removal algorithms below insert or remove nodes in a BST with nodes containing parent pointers.

Figure 6.9.1: BSTInsert algorithm for BSTs with nodes containing parent pointers.

```
BSTInsert(tree, node) {
   if (tree->root == null) {
      tree->root = node
      node->parent = null
      return
   }

   cur = tree->root
   while (cur != null) {
      if (node->key < cur->key) {
         if (cur->left == null) {
            cur->left = node
            node->parent = cur
            cur = null
         }
         else
            cur = cur->left
      }
      else {
         if (cur->right == null) {
            cur->right = node
            node->parent = cur
            cur = null
         }
         else
            cur = cur->right
      }
   }
}
```

Figure 6.9.2: BSTReplaceChild algorithm.

```
BSTReplaceChild(parent, currentChild, newChild) {
   if (parent->left != currentChild &&
       parent->right != currentChild)
      return false

   if (parent->left == currentChild)
      parent->left = newChild
   else
      parent->right = newChild

   if (newChild != null)
      newChild->parent = parent
   return true
}
```

Figure 6.9.3: BSTRemoveKey and BSTRemoveNode algorithms for BSTs with nodes containing parent pointers.

```
BSTRemoveKey(tree, key) {
   node = BSTSearch(tree, key)
   BSTRemoveNode(tree, node)
}

BSTRemoveNode(tree, node) {
   if (node == null)
      return

   // Case 1: Internal node with 2 children
   if (node->left != null && node->right != null) {
      // Find successor
      succNode = node->right
      while (succNode->left)
         succNode = succNode->left

      // Copy value/data from succNode to node
      node = Copy succNode

      // Recursively remove succNode
      BSTRemoveNode(tree, succNode)
   }

   // Case 2: Root node (with 1 or 0 children)
   else if (node == tree->root) {
      if (node->left != null)
         tree->root = node->left
      else
         tree->root = node->right

      // Make sure the new root, if non-null, has a null parent
      if (tree->root != null)
         tree->root->parent = null
   }

   // Case 3: Internal with left child only
   else if (node->left != null)
      BSTReplaceChild(node->parent, node, node->left)

   // Case 4: Internal with right child only OR leaf
   else
      BSTReplaceChild(node->parent, node, node->right)
}
```

PARTICIPATION ACTIVITY 6.9.1: BST parent node pointers.

1) `BSTInsert` will not work if the tree's root is null.

- ○ True
- ○ False

2) `BSTReplaceChild` will not work if the parent pointer is null.

- True
- False

3) `BSTRemoveKey` will not work if the key is not in the tree.

- True
- False

4) `BSTRemoveNode` will not work to remove the last node in a tree.

- True
- False

5) `BSTRemoveKey` uses `BSTRemoveNode`.

- True
- False

6) `BSTRemoveNode` uses `BSTRemoveKey`.

- True
- False

7) `BSTRemoveNode` may use recursion.

- True
- False

8) `BSTRemoveKey` will not properly update parent pointers when a non-root node is being removed.

- True
- False

9) All calls to `BSTRemoveNode` to remove a non-root node will result in a call to `BSTReplaceChild`.

- True
- False

How was this section? Provide feedback

6.10 BST: Recursion

BST recursive search algorithm

BST search can be implemented using recursion. A single node and search key are passed as arguments to the recursive search function. Two base cases exist. The first base case is when the node is null, in which case null is returned. If the node is non-null, then the search key is compared to the node's key. The second base case is when the search key equals the node's key, in which case the node is returned. If the search key is less than the node's key, a recursive call is made on the node's left child. If the search key is greater than the node's key, a recursive call is made on the node's right child.

PARTICIPATION ACTIVITY 6.10.1: BST recursive search algorithm.

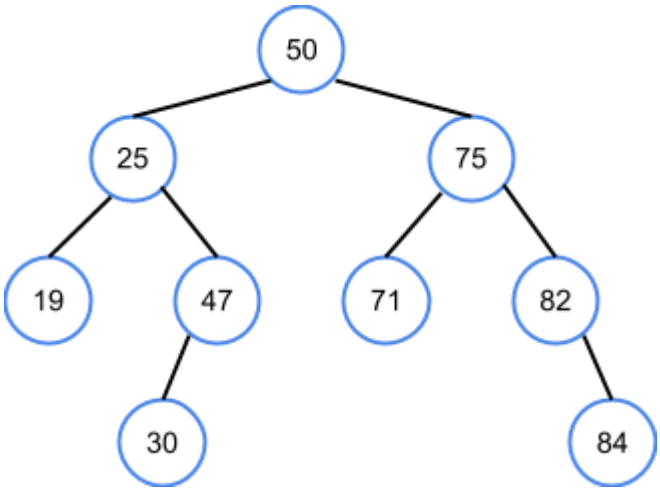
Animation content:

`undefined`

Animation captions:

1. A call to BSTSearch(tree, 40) calls the BSTSearchRecursive function with the tree's root as the node argument.
2. The search key 40 is less than 64, so a recursive call is made on the root node's left child.
3. An additional recursive call searches node 32's right child. The key 40 is found and node 40 is returned.
4. Each function returns the result of a recursive call, so BSTSearch(tree, 40) returns node 40.

PARTICIPATION ACTIVITY 6.10.2: BST recursive search algorithm.



1) How many calls to BSTSearchRecursive are made by calling BSTSearch(tree, 71)?

- 2
- 3
- 4

2) How many calls to BSTSearchRecursive are made by calling BSTSearch(tree, 49)?

- 3
- 4
- 5

3) What is the maximum possible number of calls to BSTSearchRecursive when searching the tree?

- 4
- 5

BST get parent algorithm

In a BST without parent pointers, a search for a node's parent can be implemented recursively. The algorithm recursively searches for a parent in a way similar to the normal BSTSearch algorithm. But instead of comparing a search key against a candidate node's key, the node is compared against a candidate parent's child pointers.

Figure 6.10.1: BST get parent algorithm.

```
BSTGetParent(tree, node) {
   return BSTGetParentRecursive(tree->root, node)
}

BSTGetParentRecursive(subtreeRoot, node) {
   if (subtreeRoot is null)
      return null

   if (subtreeRoot->left == node or
       subtreeRoot->right == node) {
      return subtreeRoot
   }

   if (node->key < subtreeRoot->key) {
      return BSTGetParentRecursive(subtreeRoot->left, node)
   }
   return BSTGetParentRecursive(subtreeRoot->right, node)
}
```

PARTICIPATION ACTIVITY 6.10.3: BST get parent algorithm.

1) BSTGetParent returns null when the node argument is the tree's root.
   - ○ True
   - ○ False

2) BSTGetParent always returns a non-null node when searching for a null node.
   - ○ True
   - ○ False

3) The base case for BSTGetParentRecursive is when subtreeRoot is null or is node's parent.
   - ○ True
   - ○ False

Recursive BST insertion and removal

BST insertion and removal can also be implemented using recursion. The insertion algorithm uses recursion to traverse down the tree until the insertion location is found. The removal algorithm uses the recursive search functions to find the node and the node's parent, then

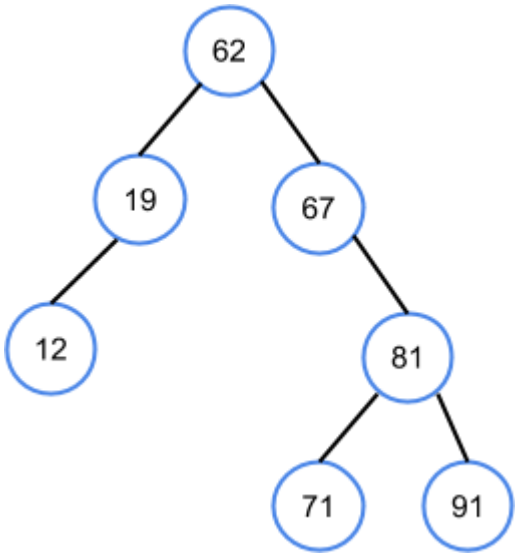
removes the node from the tree. If the node to remove is an internal node with 2 children, the node's successor is recursively removed.

Figure 6.10.2: Recursive BST insertion and removal.

```
BSTInsert(tree, node) {
   if (tree->root is null)
      tree->root = node
   else
      BSTInsertRecursive(tree->root, node)
}

BSTInsertRecursive(parent, nodeToInsert) {
   if (nodeToInsert->key < parent->key) {
      if (parent->left is null)
         parent->left = nodeToInsert
      else
         BSTInsertRecursive(parent->left, nodeToInsert)
   }
   else {
      if (parent->right is null)
         parent->right = nodeToInsert
      else
         BSTInsertRecursive(parent->right, nodeToInsert)
   }
}

BSTRemove(tree, key) {
   node = BSTSearch(tree, key)
   parent = BSTGetParent(tree, node)
   BSTRemoveNode(tree, parent, node)
}

BSTRemoveNode(tree, parent, node) {
   if (node == null)
      return false

   // Case 1: Internal node with 2 children
   if (node->left != null && node->right != null) {
      // Find successor and successor's parent
      succNode = node->right
      successorParent = node
      while (succNode->left != null) {
         successorParent = succNode
         succNode = succNode->left
      }

      // Copy the value from the successor node
      node = Copy succNode

      // Recursively remove successor
      BSTRemoveNode(tree, successorParent, succNode)
   }

   // Case 2: Root node (with 1 or 0 children)
   else if (node == tree->root) {
      if (node->left != null)
         tree->root = node->left
      else
         tree->root = node->right
   }

   // Case 3: Internal with left child only
   else if (node->left != null) {
```

```
            // Replace node with node's left child
            if (parent->left == node)
               parent->left = node->left
            else
               parent->right = node->left
         }

         // Case 4: Internal with right child only OR leaf
         else {
            // Replace node with node's right child
            if (parent->left == node)
               parent->left = node->right
            else
               parent->right = node->right
         }

         return true
      }
```

PARTICIPATION ACTIVITY 6.10.4: Recursive BST insertion and removal.



The following operations are executed on the above tree:
BSTInsert(tree, node 70)
BSTInsert(tree, node 56)
BSTRemove(tree, 67)

1) Where is node 70 inserted?

- Node 67's left child
- Node 71's left child
- Node 71's right child

2) How many times is BSTInsertRecursive called when inserting node 56?

- 2
- 3

- 5

3) How many times is BSTRemoveNode called when removing node 67?

- 1
- 2
- 3

4) What is the maximum number of calls to BSTRemoveNode when removing one of the tree's nodes?

- 2
- 4
- 5

How was this section? **Provide feedback**