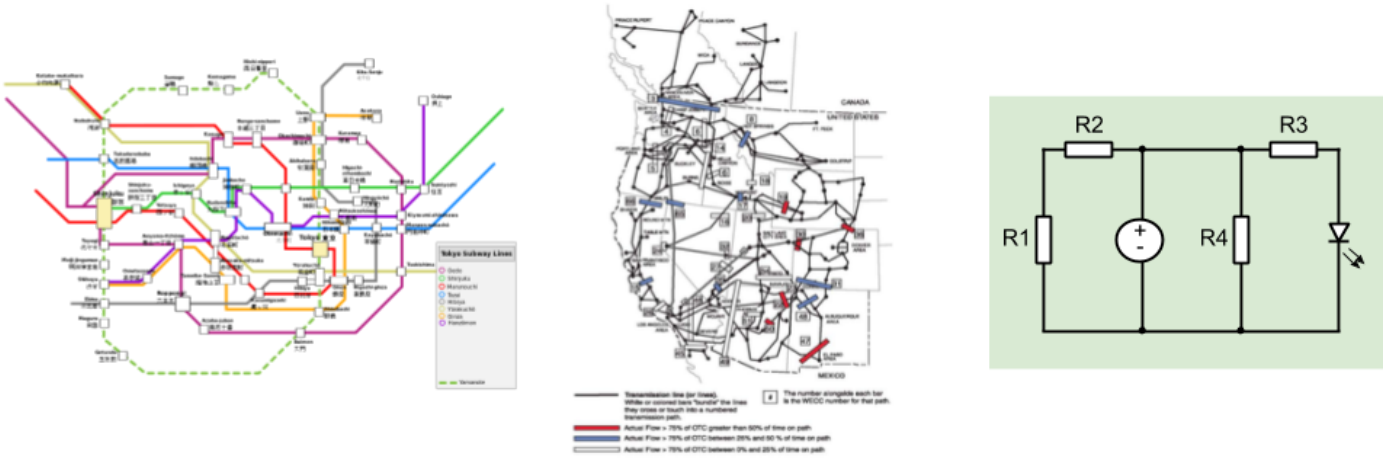
# 9.1 Graphs: Introduction

## Introduction to graphs



Many items in the world are connected, such as computers on a network connected by wires, cities connected by roads, or people connected by friendship.

Figure 9.1.1: Examples of connected items: Subway map, electrical power transmission, electrical circuit.




Source: Subway map (Comicinker (Own work) / CC-BY-SA-3.0 via Wikimedia Commons), Internet map (Department of Energy / Public domain via Wikimedia Commons), Electrical circuit (zyBooks)

A ***graph*** is a data structure for representing connections among items, and consists of vertices connected by edges.

- A ***vertex*** (or node) represents an item in a graph.
- An ***edge*** represents a connection between two vertices in a graph.

**PARTICIPATION ACTIVITY** 9.1.1: A graph represents connections among items, like among computers, or people.



## Animation captions:

1. Items in the world may have connections, like a computer network.
2. A graph's vertices represent items.
3. A graph's edges represent connections.

4. A graph can represent many different things, like friendships among people. Raj and Maya are friends, but Raj and Jen are not.

For a given graph, the number of vertices is commonly represented as V, and the number of edges as E.

**PARTICIPATION ACTIVITY** 9.1.2: Graph basics.

Refer to the above graphs.

1) The computer network graph has how many vertices?
   - ○ 5
   - ○ 6

2) The computer network graph has how many edges?
   - ○ 5
   - ○ 6

3) Are Maya and Thuy friends?
   - ○ Yes
   - ○ No

4) Can a vertex have more than one edge?
   - ○ Yes
   - ○ No

5) Can an edge connect more than two vertices?
   - ○ Yes
   - ○ No

6) Given 4 vertices A, B, C, and D and at most one edge between a vertex pair, what is the maximum number of edges?
   - ○ 6
   - ○ 16

## Adjacency and paths

In a graph:

- Two vertices are ***adjacent*** if connected by an edge.
- A ***path*** is a sequence of edges leading from a source (starting) vertex to a destination (ending) vertex. The ***path length*** is the number of edges in the path.
- The ***distance*** between two vertices is the number of edges on the shortest path between those vertices.

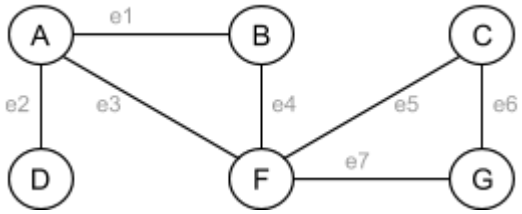
**PARTICIPATION ACTIVITY** 9.1.3: Graphs: adjacency, paths, and distance.

### Animation captions:

1. Two vertices are adjacent if connected by an edge. PC1 and Server1 are adjacent. PC1 anc Server3 are not adjacent.
2. A path is a sequence of edges from a source vertex to a destination vertex.
3. A path's length is the path's number of edges. Vertex distance is the length of the shortest path: Distance from PC1 to PC2 is 2.

**PARTICIPATION ACTIVITY** 9.1.4: Graph properties.

Refer to the following graph.




1) A and B are adjacent.

- ○ True
- ○ False

2) A and C are adjacent.

- ○ True
- ○ False

3) Which of the following is a path from B to G?

- ○

e3, e7

- e1, e3, e7
- No path from B to G.

4) What is the distance from D to C?

- 3
- 4
- 5

How was this section? Provide feedback

# 9.2 Applications of graphs

## Geographic maps and navigation

Graphs are often used to represent a geographic map, which can contain information about places and travel routes. Ex: Vertices in a graph can represent airports, with edges representing available flights. Edge weights in such graphs often represent the length of a travel route, either in total distance or expected time taken to navigate the route. Ex: A map service with access to real-time traffic information can assign travel times to road segments.

**PARTICIPATION ACTIVITY** 9.2.1: Driving directions use graphs and shortest path algorithms to determine the best route from start to destination.

### Animation content:

undefined

### Animation captions:

1. A road map can be represented by a graph. Each intersection of roads is a vertex. Destinations like the beach or a house are also vertices.
2. Roads between vertices are edges. A map service with realtime traffic information can assig travel times as edge weights.
3. A shortest path finding algorithm can be used to find the shortest path starting at the house and ending at the beach.

**PARTICIPATION ACTIVITY** 9.2.2: Using graphs for road navigation.

1) The longer a street is, the more vertices will be needed to represent that street.
   - ○ True
   - ○ False

2) Using the physical distance between vertices as edge weights will often suffice in contexts where the fastest route needs to be found.
   - ○ True
   - ○ False

3) Navigation software would have no need to place a vertex on a road in a location where the road does not intersect anything other roads.
   - ○ True
   - ○ False

4) If navigation software uses GPS to automatically determine the start location for a route, the vertex closest to the GPS coordinates can be used as the starting vertex.
   - ○ True
   - ○ False

**PARTICIPATION ACTIVITY** 9.2.3: Using graphs for flight navigation.

Suppose a graph is used to represent airline flights. Vertices represent airports and edge weights represent flight durations.

1) The weight of an edge connecting two airport vertices may change based on _____.
   - ○

- flight delays
- weather conditions
- flight cost

2) Edges in the graph could potentially be added or removed during a single day's worth of flights.
   - True
   - False

## Product recommendations

A graph can be used to represent relationships between products. Vertices in the graph corresponding to a customer's purchased products have adjacent vertices representing products that can be recommended to the customer.

**PARTICIPATION ACTIVITY** 9.2.4: A graph of product relationships can be used to produce recommendations based on purchase history.

### Animation content:

`undefined`

### Animation captions:

1. An online shopping service can represent relationships between products being sold using graph.
2. Relationships may be based on the products alone. A game console requires a TV and games, so the game console vertex connects to TV and game products.
3. Vertices representing common kitchen products, such as a blender, dishwashing soap, kitchen towels, and oven mitts, are also connected.
4. Connections might also represent common purchases that occur by chance. Maybe several customers who purchase a Blu-ray player also purchase a blender.
5. A customer's purchases can be linked to the product vertices in the graph.
6. Adjacent vertices can then be used to provide a list of recommendations to the customer.

**PARTICIPATION ACTIVITY** 9.2.5: Product recommendations.

1) If a customer buys only a Blu-ray player, which product is not likely to be

recommended?

- Television 1 or 2
- Blender
- Game console

2) Which single purchase would produce the largest number of recommendations for the customer?

- Tablet computer
- Blender
- Game console

3) If "secondary recommendations" included all products adjacent to recommended products, which products would be secondary recommendations after buying oven mitts?

- Blender, kitchen towels, and muffin pan
- Dishwashing soap, Blu-ray player, and muffin mix
- Game console and tablet computer case

## Social and professional networks

A graph may use a vertex to represent a person. An edge in such a graph represents a relationship between 2 people. In a graph representing a social network, an edge commonly represents friendship. In a graph representing a professional network, an edge commonly represents business conducted between 2 people.

**PARTICIPATION ACTIVITY** 9.2.6: Professional networks represented by graphs help people establish business connections.

### Animation content:

undefined

### Animation captions:

1. In a graph representing a professional network, vertices represent people and edges represent a business connection.
2. Tuyet conducts business with Wilford, adding a new edge in the graph. Similarly, Manuel conducts business with Keira.
3. The graph can help professionals find new connections. Shayla connects with Octavio through a mutual contact, Manuel.

**PARTICIPATION ACTIVITY** 9.2.7: Professional network.

Refer to the animation's graph representing the professional network.

1) Who has conducted business with Eusebio?
   - ○ Giovanna
   - ○ Manuel
   - ○ Keira

2) If Octavio wishes to connect with Wilford, who is the best person to introduce the 2 to each other?
   - ○ Shayla
   - ○ Manuel
   - ○ Rita

3) What is the length of the shortest path between Salvatore and Reva?
   - ○ 5
   - ○ 6
   - ○ 7

How was this section? **Provide feedback**

# 9.3 Graph representations: Adjacency lists

## Adjacency lists

Various approaches exist for representing a graph data structure. A common approach is an adjacency list. Recall that two vertices are ***adjacent*** if connected by an edge. In an ***adjacency list*** graph representation, each vertex has a list of adjacent vertices, each list item representing an edge.

**PARTICIPATION ACTIVITY** 9.3.1: Adjacency list graph representation.

### Animation captions:

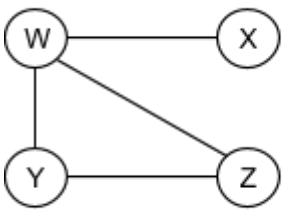
1. Each vertex has a list of adjacent vertices for edges. The edge connecting A and B appears in A's list and also in B's list.
2. Each edge appears in the lists of the edge's two vertices.

## Advantages of adjacency lists

A key advantage of an adjacency list graph representation is a size of O(V + E), because each vertex appears once, and each edge appears twice. V refers to the number of vertices, E the number of edges.

However, a disadvantage is that determining whether two vertices are adjacent is O(V), because one vertex's adjacency list must be traversed looking for the other vertex, and that list could have V items. However, in most applications, a vertex is only adjacent to a small fraction of the other vertices, yielding a sparse graph. A ***sparse graph*** has far fewer edges than the maximum possible. Many graphs are sparse, like those representing a computer network, flights between cities, or friendships among people (every person isn't friends with every other person). Thus, the adjacency list graph representation is very common.

**PARTICIPATION ACTIVITY** 9.3.2: Adjacency lists.




| Vertices | Adjacent vertices (edges) |
|---|---|
| W | X (a) Z |
| X | W |
| Y | (b) |
| Z | (c) |

1) (a)

- Z
- Y

2) (b)

- W, X, Z
- W, X
- W, Z

3) (c)

- W
- W, X
- W, Y

**PARTICIPATION ACTIVITY** 9.3.3: Adjacency lists: Bus routes.

The following adjacency list represents bus routes. Ex: A commuter can take the bus from Belmont to Sonoma.

| Vertices | Adjacent vertices (edges) | | | |
|---|---|---|---|---|
| Belmont | Marin City | Sonoma | | |
| Hillsborough | Livermore | Marin City | Sonoma | |
| Livermore | Hillsborough | Sonoma | | |
| Marin City | Belmont | Hillsborough | Sonoma | |
| Sonoma | Belmont | Hillsborough | Livermore | Marin City |

1) A direct bus route exists from Belmont to Sonoma or Belmont to _____.

Check Show answer

2) How many buses are needed to go from Livermore to Hillsborough?

Check Show answer

3) What is the minimum number of buses needed to go from Belmont to Hillsborough?

Check Show answer

4) Is the following a path from Livermore to Marin City? Type: Yes or No
Livermore, Hillsborough, Sonoma, Marin City

Check Show answer

How was this section? Provide feedback

# 9.4 Graph representations: Adjacency matrices

## Adjacency matrices

Various approaches exist for representing a graph data structure. One approach is an adjacency matrix. Recall that two vertices are ***adjacent*** if connected by an edge. In an ***adjacency matrix*** graph representation, each vertex is assigned to a matrix row and column, and a matrix element is 1 if the corresponding two vertices have an edge or is 0 otherwise.

**PARTICIPATION ACTIVITY** 9.4.1: Adjacency matrix representation.

### Animation captions:

1. Each vertex is assigned to a row and column.
2. An edge connecting A and B is a 1 in A's row and B's column.
3. Similarly, that same edge is a 1 in B's row and A's column. (The matrix will thus be symmetric.)
4. Each edge similarly has two 1's in the matrix. Other matrix elements are 0's (not shown).

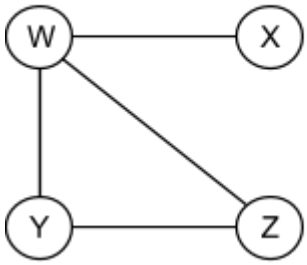
## Analysis of adjacency matrices

Assuming the common implementation as a two-dimensional array whose elements are accessible in O(1), then an adjacency matrix's key benefit is O(1) determination of whether two vertices are adjacent: The corresponding element is just checked for 0 or 1.

A key drawback is $O(V^2)$ size. Ex: A graph with 1000 vertices would require a 1000 x 1000 matrix, meaning 1,000,000 elements. An adjacency matrix's large size is inefficient for a sparse graph, in which most elements would be 0's.

An adjacency matrix only represents edges among vertices; if each vertex has data, like a person's name and address, then a separate list of vertices is needed.

**PARTICIPATION ACTIVITY** 9.4.2: Adjacency matrix.




| | W | X | Y | Z |
|---|---|---|---|---|
| W | 0 | (a) | (b) | 1 |
| X | 1 | 0 | 0 | (c) |
| Y | (d) | 0 | 0 | (e) |
| Z | 1 | 0 | (f) | 0 |

1) (a)
- 0
- 1

2) (b)
- 0
- 1

3) (c)
- 0
- 1

4) (d)
- 0
- 1

5) (e)
- 0
- 1

6) (f)

- 0
- 1

**PARTICIPATION ACTIVITY** 9.4.3: Adjacency matrix: Power grid map.

The following adjacency matrix represents the map of a city's electrical power grid. Ex: Sector A's power grid is connected to Sector B's power grid.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| A | 0 | 1 | 1 | 1 | 0 |
| B | 1 | 0 | 1 | 1 | 1 |
| C | 1 | 1 | 0 | 1 | 1 |
| D | 1 | 1 | 1 | 0 | 0 |
| E | 0 | 1 | 1 | 0 | 0 |

1) How many edges does D have?

Check Show answer

2) How many edges does the graph contain?

Check Show answer

3) Assume Sector D has a power failure. Can power from Sector A be diverted directly to Sector D? Type: Yes or No

Check Show answer

4) Assume Sector E has a power failure. Can power from Sector A be diverted directly to Sector E? Type: Yes or No

Check Show answer

5) Is the following a path from Sector A to E? Type: Yes or No
AD, DB, BE

Check Show answer

How was this section? Provide feedback

# 9.5 Graphs: Breadth-first search

## Graph traversal and breadth-first search

An algorithm commonly must visit every vertex in a graph in some order, known as a ***graph traversal***. A ***breadth-first search*** (BFS) is a traversal that visits a starting vertex, then all vertices of distance 1 from that vertex, then of distance 2, and so on, without revisiting a vertex.

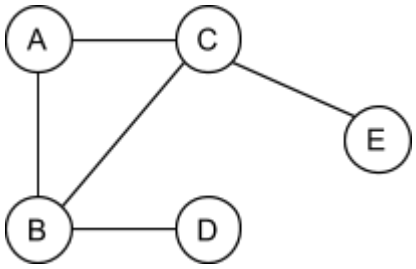
**PARTICIPATION ACTIVITY** 9.5.1: Breadth-first search.

### Animation captions:

1. Breadth-first search starting at A visits vertices based on distance from A. B and D are distance 1.
2. E and F are distance 2 from A. Note: A path of length 3 also exists to F, but distance uses shortest path.
3. C is distance 3 from A.
4. Breadth-first search from A visits A, then vertices of distance 1, then 2, then 3. Note: Visiting order of same-distance vertices doesn't matter.

**PARTICIPATION ACTIVITY** 9.5.2: Breadth-first search traversal.

Perform a breadth-first search of the graph below. Assume the starting vertex is E.




1) Which vertex is visited first?

Check Show answer

2) Which vertex is visited second?

Check Show answer

3) Which vertex is visited third?

Check Show answer

4) What is C's distance?

Check Show answer

5) What is D's distance?

Check Show answer

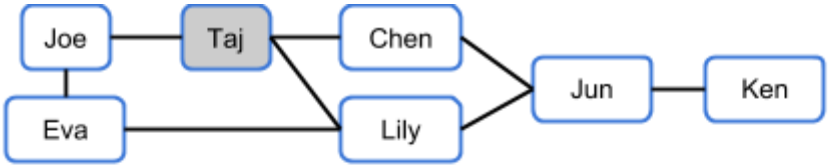
6) The BFS traversal of a graph is unique.
Type: Yes or No

Check Show answer

## Example: Social networking friend recommender

## Example 9.5.1: Social networking friend recommender using breadth-first search.

Social networking sites like Facebook, Google+, and LinkedIn use graphs to represent "friendship" among people. For a particular user, a site may wish to recommend new friends. One approach does a breadth-first search starting from the user, recommending new friends starting at distance 2 (distance 1 people are already friends with the user).




| Breadth-first traversal: | Taj | Joe | Chen | Lily | Eva | Jun | Ken |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 1 | 1 | 2 | 2 | 3 |

Hello Taj. People you may know:

*Eva*

*Jun*

*Ken*

**PARTICIPATION ACTIVITY** 9.5.3: BFS: Friend recommender.

Refer to the friend recommender example above.

1) A distance greater than 0 indicates people are not friends with the user.
   - True
   - False

2) People with a distance of 2 are recommended before people with a distance of 3.
   - True
   - False

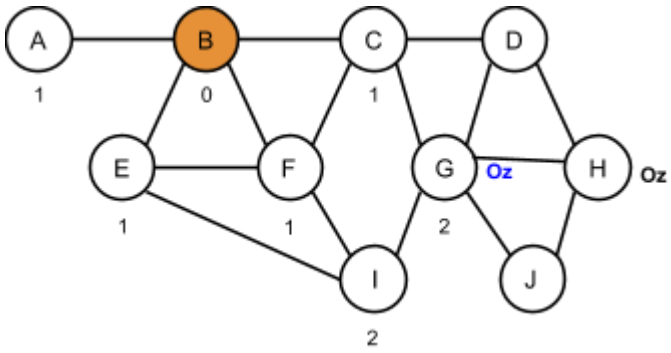
3) If Chen is the user, the system also recommends Eva, Jun, and Ken.

- True
- False

## Example: Find closest item in a peer-to-peer network

### Example 9.5.2: Application of BFS: Find closest item in a peer-to-peer network.

In a ***peer-to-peer network***, computers are connected via a network and may seek and download file copies (such as songs or movies) via intermediary computers or routers. For example, one computer may seek the movie "The Wizard of Oz", which may exist on 10 computers in a network consisting of 100,000 computers. Finding the closest computer (having the fewest intermediaries) yields a faster download. A BFS traversal of the network graph can find the closest computer with that movie. The BFS traversal can be set to immediately return if the item sought is found during a vertex visit. Below, visiting vertex G finds the movie; BFS terminates, and a download process can begin, involving a path length of 2 (so only 1 intermediary). Vertex H also has the movie, but is further from B so wasn't visited yet during BFS. (Note: Distances of vertices visited during the BFS from B are shown below for convenience).




| Visit order: | B | A | E | F | C | I | G |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | 1 | 1 | 1 | 2 | 2 |

Oz found; terminates

**PARTICIPATION ACTIVITY** 9.5.4: BFS application: Peer-to-peer search.

Consider the above peer-to-peer example.

1) In some BFS traversals starting from vertex B, vertex H may be visited before vertex G.
   - True
   - False

2) If vertex J sought the movie Oz, the download might occur from either G or H.

- True
- False

3) If vertex E sought the movie Oz, the download might occur from either G or H.

- True
- False

## Breadth-first search algorithm

An algorithm for breadth-first search pushes the starting vertex to a queue. While the queue is not empty, the algorithm pops a vertex from the queue and visits the popped vertex, pushes that vertex's adjacent vertices (if not already discovered), and repeats.

**PARTICIPATION ACTIVITY** 9.5.5: BFS algorithm.

### Animation captions:

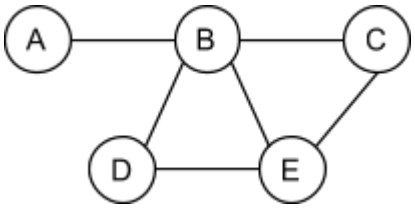
1. BFS pushes start vertex (in this case A) to frontierQueue, and adds to discoveredSet.
2. Pop vertex from frontierQueue, and visits popped vertex.
3. Pushes adjacent undiscovered vertices to frontierQueue, and adds to discoveredSet.
4. Vertex A's visit is complete. Proceed to next vertex in the frontierQueue (D). Then next (B).
5. Visit E. Then, even though B and F are adjacent to E, they are already in the discoveredSet so are not again added to frontierQueue or discoveredSet.
6. Continue until frontierQueue is empty. Note that discoveredSet above shows the visit order. Note that each vertex's distance from start is shown on the graph.

When the BFS algorithm first encounters a vertex, that vertex is said to have been ***discovered***. In the BFS algorithm, the vertices in the queue are called the ***frontier***, being vertices thus far discovered but not yet visited. Because each vertex is visited at most once, an already-discovered vertex is not pushed to the queue again.

A "visit" may mean to print the vertex, append the vertex to a list, compare vertex data to a value and return the vertex if found, etc.

**PARTICIPATION ACTIVITY** 9.5.6: BFS algorithm.

BFS is run on the following graph. Assume C is the starting vertex.




1) Which vertices are in the frontierQueue before the first iteration of the while loop?

- A
- C
- A, B, C, D, E

2) Which vertices are in discoveredSet after the first iteration of the while loop?

- C, B, E
- B, E
- C

3) In the second iteration, currentV = B. Which vertices are in discoveredSet after the second iteration of the while loop?

- C, B, E, A
- B, E, A, D
- C, B, E, A, D

4) In the second iteration, currentV = B. Which vertices are in frontierQueue after the second iteration of the while loop?

- C, B, E, A, D
- E, A, D
- frontierQueue is empty

5) BFS terminates after the second iteration, because all vertices are in the discoveredSet.

- True
- False

How was this section? Provide feedback

# 9.6 Graphs: Depth-first search

## Graph traversal and depth-first search

An algorithm commonly must visit every vertex in a graph in some order, known as a ***graph traversal***. A ***depth-first search*** (DFS) is a traversal that visits a starting vertex, then visits every vertex along each path starting from that vertex to the path's end before backtracking.

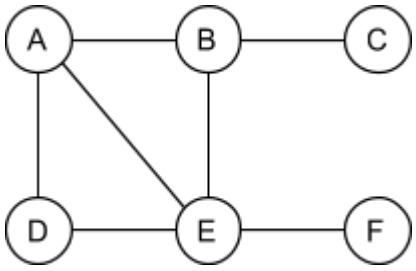
**PARTICIPATION ACTIVITY** 9.6.1: Depth-first search.

### Animation captions:

1. Depth-first search starting at A descends along a path to the path's end before backtracking
2. Reach path's end: Backtrack to F, visit F's other adjacent vertex (E). B already visited, backtrack again.
3. Backtracked all the way to A. Visit A's other adjacent vertex (D). No other adjacent vertices: Done.

**PARTICIPATION ACTIVITY** 9.6.2: Depth-first search traversal.

Perform a depth-first search of the graph below. Assume the starting vertex is E.




1) Which vertex is visited first?

Check Show answer

2) Assume DFS traverses the following verticies: E, A, B. Which vertex is visited next?

Check Show answer

3) Assume DFS traverses the following vertices: E, A, B, C. Which vertex is visited next?

Check Show answer

4) Is the following a valid DFS traversal?
Type: Yes or No
E, D, F, A, B, C

Check Show answer

5) Is the following a valid DFS traversal?
Type: Yes or No
E, D, A, B, C, F

Check Show answer

6) The DFS traversal of a graph is unique.
Type: Yes or No

Check Show answer

## Depth-first search algorithm

An algorithm for depth-first search pushes the starting vertex to a stack. While the stack is not empty, the algorithm pops the vertex from the top of the stack. If the vertex has not already been visited, the algorithm visits the vertex and pushes the adjacent vertices to the stack.

**PARTICIPATION**

ACTIVITY 9.6.3: Depth-first search traversal with starting vertex A.

## Animation content:

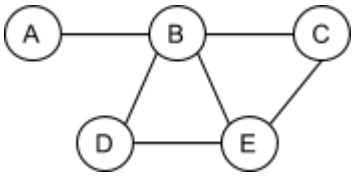
undefined

## Animation captions:

1. A depth-first search at vertex A first pushes vertex A onto the stack. The first loop iteration then pops vertex A off the stack and assigns currentV with that vertex.
2. Vertex A is visited and added to the visited set. Each vertex adjacent to A is pushed onto the stack.
3. Vertex B is popped off the stack and processed as the next currentV.
4. Vertices F and E are processed similarly.
5. Vertices F and B are in the visited set and are skipped after being popped off the stack.
6. Vertex C is popped off the stack and visited. All remaining vertices except D are in the visited set.
7. Vertex D is the last vertex visited.

PARTICIPATION ACTIVITY 9.6.4: DFS algorithm.

DFS is run on the following graph. Assume C is the starting vertex.




1) Which vertices are in the stack before the first iteration of the while loop?

- A
- C
- A, B, C, D, E

2) Which vertices are in the stack after the first iteration of the while loop?

- B, E, C
- B, E
- B

3) Which vertices are in visitedSet after the first iteration of the while loop?

- B, E, C
- C
- B

4) In the second iteration, currentV = B. Which vertices are in the stack after the second iteration of the while loop?

- C
- A, C, D, E
- A, C, D, E, E

5) Which vertices are in visitedSet after the second iteration of the while loop?

- C, B
- C
- B

6) DFS terminates once all vertices are added to visitedSet.

- True
- False

## Recursive DFS algorithm

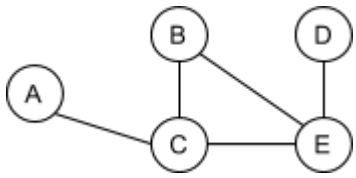
A recursive DFS can be implemented using the program stack instead of an explicit stack. The recursive DFS algorithm is first called with the starting vertex. If the vertex has not already been visited, the recursive algorithm visits the vertex and performs a recursive DFS call for each adjacent vertex.

Figure 9.6.1: Recursive depth-first search.

```
RecursiveDFS(currentV) {
   if ( currentV is not in visitedSet ) {
      Add currentV to visitedSet
      "Visit" currentV
      for each vertex adjV adjacent to currentV
         RecursiveDFS(adjV)
   }
}
```

**PARTICIPATION ACTIVITY** 9.6.5: Recursive DFS algorithm.

The recursive DFS algorithm is run on the following graph. Assume D is the starting vertex.




1) The recursive DFS algorithm uses a queue to determine which vertices to visit.
   - True
   - False

2) DFS begins with the function call RecursiveDFS(D).
   - True
   - False

3) If B is not yet visited, RecursiveDFS(B) will make subsequent calls to RecursiveDFS(C) and RecursiveDFS(E).
   - True
   - False

How was this section? **Provide feedback**

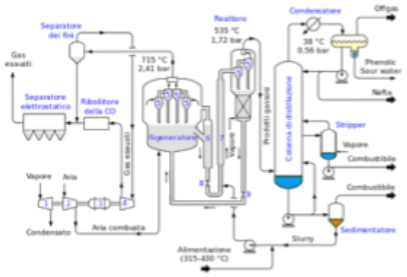
# 9.7 Directed graphs

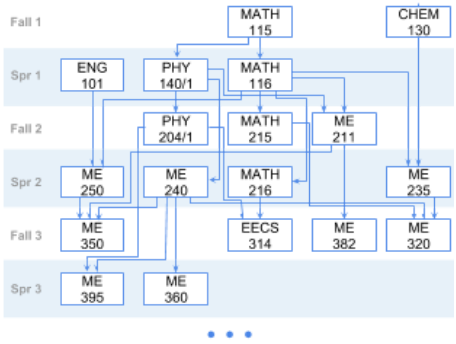
## Directed graphs

A ***directed graph***, or ***digraph***, consists of vertices connected by directed edges. A ***directed edge*** is a connection between a starting vertex and a terminating vertex. In a directed graph, a vertex Y is ***adjacent*** to a vertex X, if there is an edge from X to Y.

Many graphs are directed, like those representing links between web pages, maps for navigation, or college course prerequisites.

Figure 9.7.1: Directed graph examples: Process flow diagram, airline routes, and college course prerequisites.




Source: Fluid catalyic cracker (Mbeychok / Public Domain via Wikimedia Commons), Florida airline routes 1974 (Geez-oz (Own work) / CC-BY-SA-3.0 via Wikimedia Commons), college course prerequisites (zyBooks)

**PARTICIPATION ACTIVITY** 9.7.1: A directed graph represents connections among items, like links between web pages, or airline routes.

## Animation captions:

1. Items in the world may have directed connections, like links on a website.
2. A directed graph's vertices represent items.
3. Vertices are connected by directed edges.
4. A directed edge represent a connection from a starting vertex to a terminating vertex; the terminating vertex is adjacent to the starting vertex. B is adjacent to A, but A is not adjacent to B.
5. A directed graph can represent many things, like airline connections between cities. A flight is available from Los Angeles to Tucson, but not Tucson to Los Angeles.

**PARTICIPATION ACTIVITY** 9.7.2: Directed graph basics.

Refer to the above graphs.

1) E is a _____ in the directed graph.

- vertex
- directed edge

2) A directed edge connects vertices A and D. D is the _____ vertex.

- starting

- terminating

3) The airline routes graph is a digraph.

- True
- False

4) Tucson is adjacent to _____.

- San Francisco
- Los Angeles
- Dallas

## Paths and cycles

In a directed graph:

- A ***path*** is a sequence of directed edges leading from a source (starting) vertex to a destination (ending) vertex.
- A ***cycle*** is path that starts and ends at the same vertex. A directed graph is ***cyclic*** if the graph contains a cycle, and ***acyclic*** if the graph does not contain a cycle.

**PARTICIPATION ACTIVITY** 9.7.3: Directed graph: Paths and cycles.

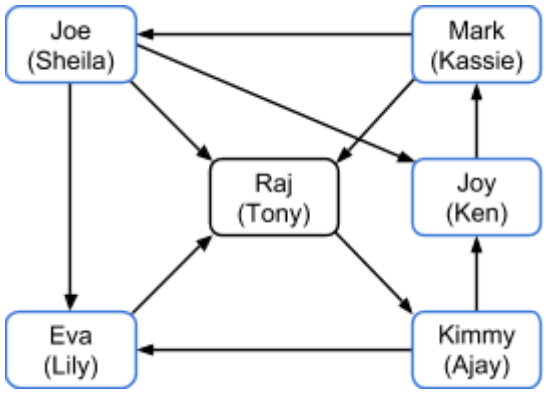
### Animation captions:

1. A path is a sequence of directed edges leading from a source (starting) vertex to a destination (ending) vertex.
2. A cycle is a path that starts and ends at the same vertex. A graph can have more than one cycle.
3. A cyclic graph contains a cycle. An acyclic graph contains no cycles.

## Example 9.7.1: Cycles in directed graphs: Kidney transplants.

A patient needing a kidney transplant may have a family member willing to donate a kidney but is incompatible. That family member is willing to donate a kidney to someone else, as long as their family member also receives a kidney donation. Suppose Gregory needs a kidney. Gregory's wife, Eleanor, is willing to donate a kidney but is not compatible with Gregory. However, Eleanor is compatible with another patient Joanna, and Joanna's husband Darrell is compatible with Gregory. So, Eleanor donates a kidney to Joanna, and Darrell donates a kidney to Gregory, which is an example of a 2-way kidney transplant. In 2015, a 9-way kidney transplant involving 18

patients was performed within 36 hours (Source: SF Gate). Multiple-patient kidney transplants can be represented as cycles within a directed graph.




3-way transplant: Eva (Lily), Raj (Tony), Kimmy (Ajay)

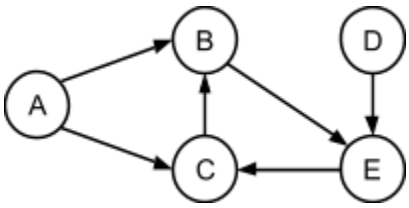
4-way transplant: Raj (Tony), Kimmy (Ajay), Joy (Ken), Mark (Kassie)

In this graph, vertices represent patients, and edges represent compatibility between a patient's family member (shown in parentheses) and another patient. An N-way kidney transplant is represented as a cycle with N edges. Due the complexity of coordinating multiple simultaneous surgeries, hospitals and doctors typically try to find the shortest possible cycle.

**PARTICIPATION ACTIVITY** 9.7.4: Directed graphs: Cyclic and acyclic.

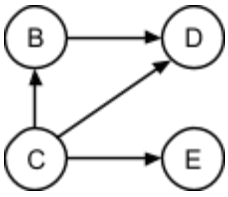
Determine if each of the following graphs is cyclic or acyclic.

1)



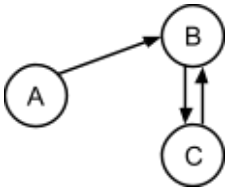

- Cyclic
- Acyclic

2)




- Cyclic
- Acyclic

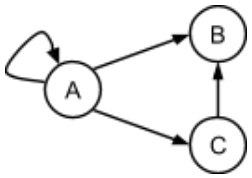
3)




- Cyclic

Acyclic

4)




Cyclic

Acyclic

How was this section? Provide feedback

# 9.8 Weighted graphs

## Weighted graphs

A ***weighted graph*** associates a weight with each edge. A graph edge's ***weight***, or ***cost***, represents some numerical value between vertex items, such as flight cost between airports, connection speed between computers, or travel time between cities. A weighted graph may be directed or undirected.

**PARTICIPATION ACTIVITY** 9.8.1: Weighted graphs associate weight with each edge.

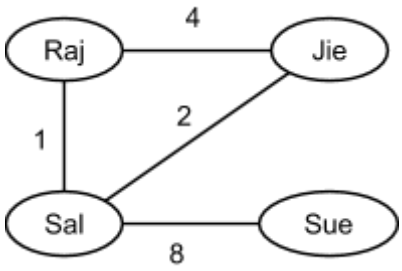
### Animation captions:

1. A weighted graph associates a numerical weight, or cost, with each edge. Ex: Edge weights may indicate connection speed (Mbps) between computers.
2. Weighted graphs can be directed. Ex: Edge weights may indicate travel time (hours) between cities; travel times may vary by direction.

**PARTICIPATION ACTIVITY** 9.8.2: Weighted graphs.

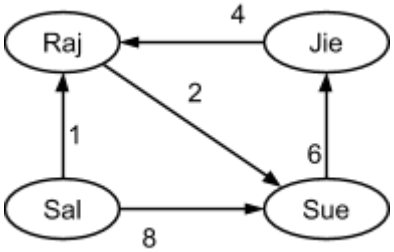
1) This graph's weights indicate the number of times coworkers have collaborated on projects. How many

times have Raj and Jie teamed up?



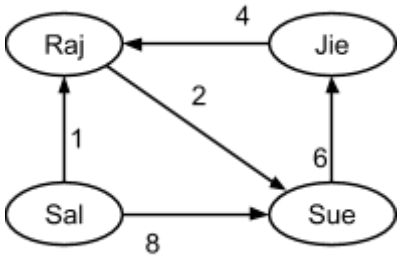

- 1
- 4

2) This graph indicates the number of times a worker has nominated a coworker for an award. How many times has Sal nominated Sue?




- 0
- 8

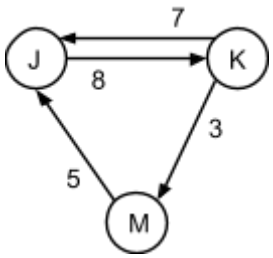
3) This graph indicates the number of times a worker has nominated a coworker for an award. How many times has Raj nominated Jie?




- 4
- 1
- 0

4) The weight of the edge from K to J is

_____.




- 7
- 8

## Path length in weighted graphs

In a weighted graph, the ***path length*** is the sum of the edge weights in the path.

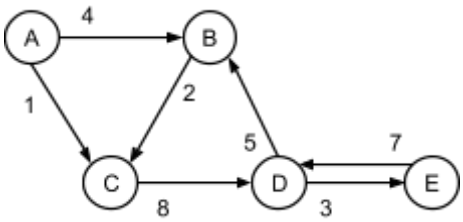
**PARTICIPATION ACTIVITY** 9.8.3: Path length is the sum of edge weights.

### Animation captions:

1. A path is a sequence of edges from a source vertex to a destination vertex.
2. The path length is the sum of the edge weights in the path.
3. The shortest path is the path yielding the lowest sum of edge weights. Ex: The shortest path from Paris to Marseille is 6.

**PARTICIPATION ACTIVITY** 9.8.4: Path length and shortest path.

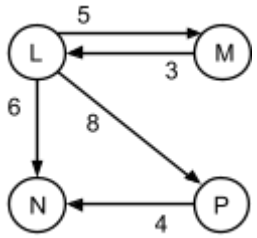
1) Given a path A, C, D, the path length is _____.




Check Show answer

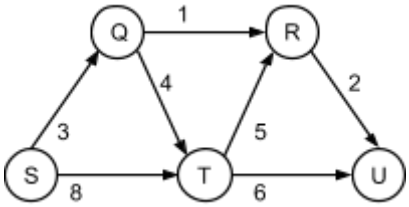
2) The shortest path from M to N has a

length of _____.



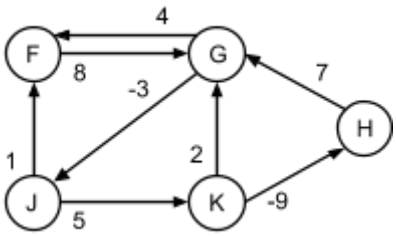

Check Show answer

3) The shortest path from S to U has a length of _____.




Check Show answer

4) Given a path H, G, J, F, the path length is _____.

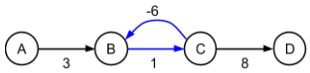



Check Show answer

## Negative edge weight cycles

The ***cycle length*** is the sum of the edge weights in a cycle. A ***negative edge weight cycle*** has a cycle length less than 0. A shortest path does not exist in a graph with a negative edge weight cycle, because each loop around the negative edge weight cycle further decreases the cycle length, so no minimum exists.

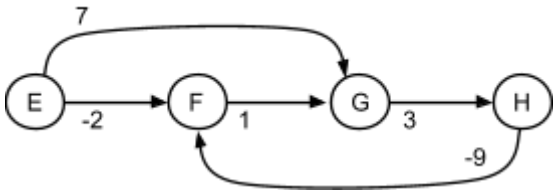
Figure 9.8.1: A shortest path from A to D does not exist, because the cycle B, C can be repeatedly taken to further reduce the cycle length.




Path 1: A, B, C, B, C, D — Length = 3 + 1 + -6 + 1 + 8 = 7
Path 2: A, B, C, B, C, B, C, D — Length = 3 + 1 + -6 + 1 + -6 + 1 + 8 = 2
Path 3: A, B, C, B, C, B, C, B, C, D — Length = 3 + 1 + -6 + 1 + -6 + 1 + -6 + 1 + 8 = -3
*and so on...*

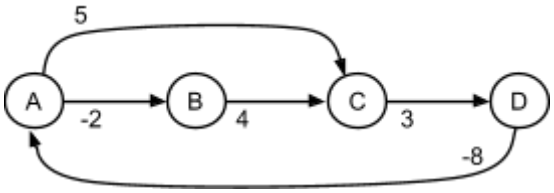
**PARTICIPATION ACTIVITY** 9.8.5: Negative edge weight cycles.

1) The cycle length for F, G, H, F is _____.



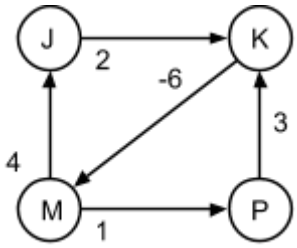

Check Show answer

2) Is A, C, D, A a negative edge weight cycle? Type Yes or No.




Check Show answer

3) The graph contains _____ negative edge weight cycles.




Check Show answer

How was this section? Provide feedback

# 9.9 Algorithm: Dijkstra's shortest path

## Dijkstra's shortest path algorithm

Finding the shortest path between vertices in a graph has many applications. Ex: Finding the shortest driving route between two intersections can be solved by finding the shortest path in a directed graph where vertices are intersections and edge weights are distances. If edge weights instead are expected travel times (possibly based on real-time traffic data), finding the shortest path will provide the fastest driving route.

***Dijkstra's shortest path algorithm***, created by Edsger Dijkstra, determines the shortest path from a start vertex to each vertex in a graph. For each vertex, Dijkstra's algorithm determines the vertex's distance and predecessor pointer. A vertex's ***distance*** is the shortest path distance from the start vertex. A vertex's ***predecessor pointer*** points to the previous vertex along the shortest path from the start vertex.

Dijkstra's algorithm initializes all vertices' distances to infinity (∞), initializes all vertices' predecessors to 0, and pushes all vertices to a queue of unvisited vertices. The algorithm then assigns the start vertex's distance with 0. While the queue is not empty, the algorithm pops the vertex with the shortest distance from the queue. For each adjacent vertex, the algorithm computes the distance of the path from the start vertex to the current vertex and continuing on to the adjacent vertex. If that path's distance is shorter than the adjacent vertex's current distance, a shorter path has been found. The adjacent vertex's current distance is updated to the distance of the newly found shorter path's distance, and vertex's predecessor pointer is pointed to the current vertex.

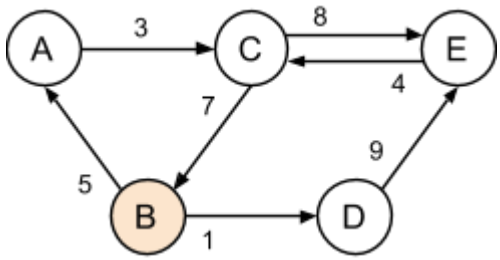
**PARTICIPATION ACTIVITY** 9.9.1: Dijkstra's algorithm finds the shortest path from a start vertex to each vertex in a graph.

## Animation captions:

1. Dijsktra's shortest path algorithm initializes each vertex's distance to Infinity, initializes each vertex's predecessor to 0, and pushes each vertex into the unvisitedQueue.
2. The start vertex's distance is 0. The algorithm visits the start vertex first.
3. For each adjacent vertex, if a shorter path from the start vertex to the adjacent vertex is found, the vertex's distance and predecessor pointer are updated.
4. B has the shortest path distance, and is popped from the queue. The path through B to C is not shorter, so no update is done. The path through B to D is shorter, so D's distance and predecessor pointer are updated.
5. D is then popped from the queue. The path through D to C is shorter, so C's distance and predecessor pointer are updated.
6. C is then popped from the queue. The path through C to D is not shorter, so no update is made.
7. The algorithm terminates when all vertices are visited. Each vertex's distance is the shortest path distance from the start vertex. The vertex's predecessor pointer points to the previous vertex in the shortest path.

**PARTICIPATION ACTIVITY** 9.9.2: Dijkstra's shortest path traversal.

Perform Dijkstra's shortest path algorithm on the graph below with starting vertex B.




1) Which vertex is visited first?

Check Show answer

2) A's distance after the algorithm's first while loop iteration is _____.
Type inf for infinity.

Check Show answer

Check

3) D's distance after the first while loop iteration is _____.
Type inf for infinity.

Check Show answer

4) C's distance after the first iteration of the while loop is _____.
Type inf for infinity.

Check Show answer

5) Which vertex is visited second?

Check Show answer

6) E's distance after the second while loop iteration is _____.

Check Show answer

## Finding shortest path from start vertex to destination vertex

After running Dijkstra's algorithm, the shortest path from the start vertex to a destination vertex can be determined using the vertices' predecessor pointers. If the destination vertex's predecessor pointer is not 0, the shortest path is traversed in reverse by following the predecessor pointers until the start vertex is reached. If the destination vertex's predecessor pointer is 0, then a path from the start vertex to the destination vertex does not exist.

**PARTICIPATION ACTIVITY** 9.9.3: Determining the shortest path from Dijkstra's algorithm.

## Animation captions:

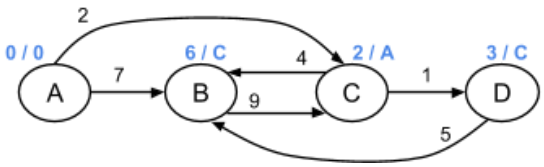
1. The vertex's predecessor pointer points to the previous vertex in the shortest path.

2. Starting with the destination vertex, the predecessor pointer is followed until the start vertex reached.
3. The vertex's distance is the shortest path distance from the start vertex.

**PARTICIPATION ACTIVITY** 9.9.4: Shortest path based on vertex predecessor.

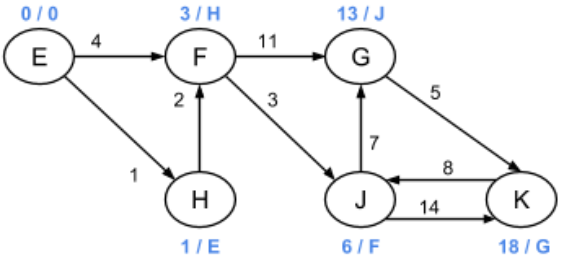
Type path as: A, B, C
If path does not exist, type: None

1) After executing DijkstraShortestPath(A), what is the shortest path from A to B?



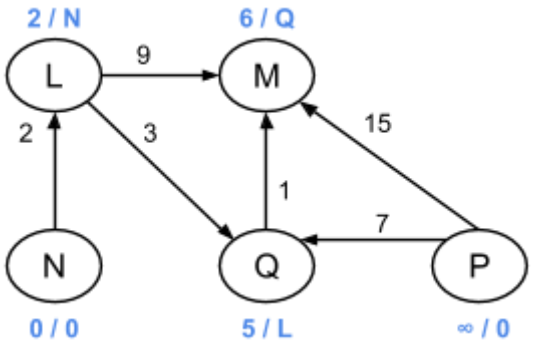

Check Show answer

2) After executing DijkstraShortestPath(E), what is the shortest path from E to G?




Check Show answer

3) After executing DijkstraShortestPath(N), what is the shortest path from N to P?




Check Show answer

## Algorithm efficiency

If the unvisited vertex queue is implemented using a list, the runtime for Dijkstra's shortest path algorithm is $O(V^2)$. The outer loop executes V times to visit all vertices. In each outer loop execution, popping the vertex from the queue requires searching all vertices in the list, which has a runtime of O(V). For each vertex, the algorithm follows the subset of edges to adjacent vertices; following a total of E edges across all loop executions. Given $E < V^2$, the runtime is $O(V*V + E) = O(V^2 + E) = O(V^2)$. Implementing the unvisited vertex queue using a standard binary heap reduces the runtime to O((E + V) log V), and using a Fibonacci heap data structure (not discussed in this material) reduces the runtime to O(E + V log V).

## Negative edge weights

Dijkstra's shortest path algorithm can be used for unweighted graphs (using a uniform edge weight of 1) and weighted graphs with non-negative edges weights. For a directed graph with negative edge weights, Dijkstra's algorithm may not find the shortest path for some vertices, so the algorithm should not be used if a negative edge weight exists.

**PARTICIPATION ACTIVITY** 9.9.5: Dijkstra's algorithm may not find the shortest path for a graph with negative edge weights.

### Animation captions:

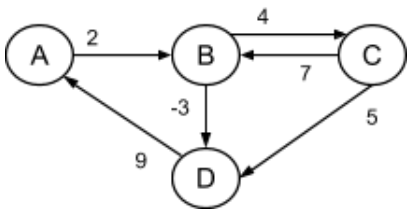
1. A is the start vertex. Adjacent vertices resulting in a shorter path are updated.
2. Path through B to D results in a shorter path. Vertex D is updated.
3. D has no adjacent vertices.

4. A path through C to B results in a shorter path. Vertex B is updated.
5. Vertex B has already been visited, and will not be visited again. So, D's distance and predecessor are not for the shortest path from A to D.

**PARTICIPATION ACTIVITY** 9.9.6: Dijkstra's shortest path algorithm: Supported graph types.

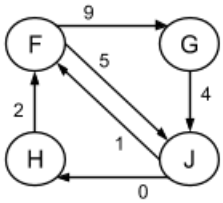
Indicate if Dijkstra's algorithm will find the shortest path for the following graphs.

1)



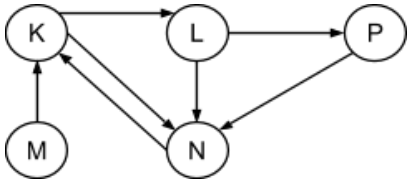

- Yes
- No

2)




- Yes
- No

3)




- Yes
- No

How was this section? Provide feedback

# 9.10 Algorithm: Bellman-Ford's shortest path

## Bellman-Ford shortest path algorithm

The ***Bellman-Ford shortest path algorithm***, created by Richard Bellman and Lester Ford, Jr., determines the shortest path from a start vertex to each vertex in a graph. For each vertex, the Bellman-Ford algorithm determines the vertex's distance and predecessor pointer. A vertex's ***distance*** is the shortest path distance from the start vertex. A vertex's ***predecessor pointer*** points to the previous vertex along the shortest path from the start vertex.

The Bellman-Ford algorithm initializes all vertices' current distances to infinity (∞) and predecessors to 0, and assigns the start vertex with a distance of 0. The algorithm performs V-1 main iterations, visiting all vertices in the graph during each iteration. Each time a vertex is visited, the algorithm follows all edges to adjacent vertices. For each adjacent vertex, the algorithm computes the distance of the path from the start vertex to the current vertex and continuing on to the adjacent vertex. If that path's distance is shorter than the adjacent vertex's current distance, a shorter path has been found. The adjacent vertex's current distance is updated to the newly found shorter path's distance, and the vertex's predecessor pointer is pointed to the current vertex.

The Bellman-Ford algorithm does not require a specific order for visiting vertices during each main iteration. So after each iteration, a vertex's current distance and predecessor may not yet be the shortest path from the start vertex. The shortest path may propagate to only one vertex each iteration, requiring V-1 iterations to propagate from the start vertex to all other vertices.

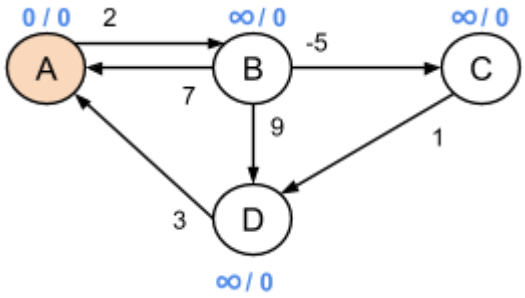
**PARTICIPATION ACTIVITY** 9.10.1: The Bellman-Ford algorithm finds the shortest path from a source vertex to all other vertices in a graph.

### Animation captions:

1. The Bellman-Ford algorithm initializes each vertex's distance to Infinity and each vertex's predecessor to 0, and assigns the start vertex's distance with 0.
2. For each vertex in the graph, if a shorter path from the currentV to the adjacent vertex is found, the adjacent vertex's distance and predecessor pointer are updated.
3. The path through B to D results in a shorter path to D than is currently known. So vertex D is updated.
4. The path through C to B results in a shorter path to B than is currently known. So vertex B is updated.
5. D has no adjacent vertices. After each iteration, a vertex's distance and predecessor may n yet be the shortest path from the start vertex.
6. In each main iteration, the algorithm visits all vertices. This time, the path through B to D results in an even shorter path. So vertex D is updated again.
7. During the third main iteration, no shorter paths are found, so no vertices are updated. V-1 iterations may be required to propagate the shortest path from the start vertex to all other vertices.
8. When done, each vertex's distance is the shortest path distance from the start vertex, and the vertex's predecessor pointer points to the previous vertex in the shortest path.

**PARTICIPATION ACTIVITY** 9.10.2: Bellman-Ford shortest path traversal.

The start vertex is A. In each main iteration, vertices in the graph are visited in the following order: A, B, C, D.



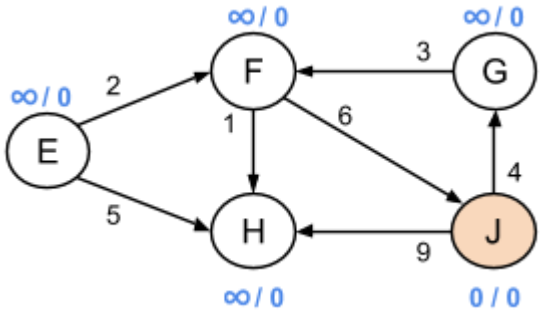

1) What are B's values after the first iteration?
   - 2 / A
   - ∞ / 0

2) What are C's values after the first iteration?
   - -5 / B
   - -3 / B
   - ∞ / 0

3) What are D's values after the first iteration?
   - 3 / A
   - -2 / C
   - ∞ / 0

**PARTICIPATION ACTIVITY** 9.10.3: Bellman-Ford: Distance and predecessor values.

The start vertex is J. Vertices in the graph are processed in the following order: E, F, G, H, J.




1) How many iterations are executed?

- 5
- 4
- 6

2) What are F's values after the first iteration?

- ∞ / 0
- 7 / G

3) What are G's values after the first iteration?

- 4 / J
- ∞ / 0

4) What are E's values after the final iteration?

- 9 / F
- ∞ / 0

## Algorithm Efficiency

The runtime for the Bellman-Ford shortest path algorithm is O(VE). The outer loop (the main iterations) executes V-1 times. In each outer loop execution, the algorithm visits each vertex and follows the subset of edges to adjacent vertices, following a total of E edges across all loop executions.

### Checking for negative edge weight cycles

The Bellman-Ford algorithm supports graphs with negative edge weights. However, if a negative edge weight cycle exists, a shortest path does not exist. After visiting all vertices V-1 times, the algorithm checks for negative edge weight cycles. If a negative edge weight cycle does not exist, the algorithm returns true (shortest path exists), otherwise returns false.

**PARTICIPATION ACTIVITY** 9.10.4: Bellman-Ford: Checking for negative edge weight cycles.

### Animation captions:

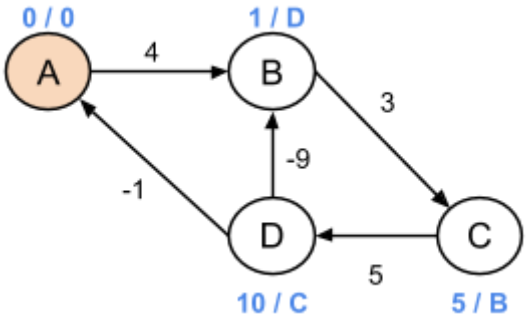
1. After visiting all vertices V-1 times, the Bellman-Ford algorithm checks for negative edge weight cycles.
2. For each vertex in the graph, adjacent vertices are checked for a shorter path. No such path exists through A to B.
3. But, a shorter path is still found through B to C, so a negative edge weight cycle exists.
4. The algorithm returns false, indicating a shortest path does not exist.

Figure 9.10.1: Bellman-Ford shortest path algorithm.

```
BellmanFord(startV) {
   for each vertex currentV in graph {
      currentV->distance = Infinity
      currentV->predV = 0
   }

   // startV has a distance of 0 from itself
   startV->distance = 0

   for i = 1 to number of vertices - 1 { // Main iterations
      for each vertex currentV in graph {
         for each vertex adjV adjacent to currentV {
            edgeWeight = weight of edge from currentV to adjV
            alternativePathDistance = currentV->distance + edgeWeight

            // If shorter path from startV to adjV is found,
            // update adjV's distance and predecessor
            if (alternativePathDistance < adjV->distance) {
               adjV->distance = alternativePathDistance
               adjV->predV = currentV
            }
         }
      }
   }

   // Check for a negative edge weight cycle
   for each vertex currentV in graph {
      for each vertex adjV adjacent to currentV {
         edgeWeight = weight of edge from currentV to adjV
         alternativePathDistance = currentV->distance + edgeWeight

         // If shorter path from startV to adjV is still found,
         // a negative edge weight cycle exists
         if (alternativePathDistance < adjV->distance) {
            return false
         }
      }
   }

   return true
}
```

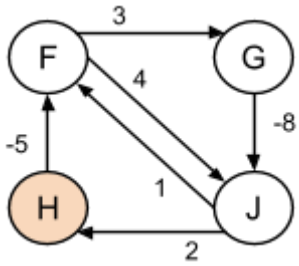
**PARTICIPATION ACTIVITY** 9.10.5: Bellman-Ford algorithm: Checking for negative edge weight cycles.

1) Given the following result from the Bellman-Ford algorithm for a start vertex of A, a negative edge weight cycle is found when checking adjacent vertices of vertex _____.




- B
- C
- D

2) What does the Bellman-Ford algorithm return for the following graph with a start vertex of H?




- True
- False

How was this section? **Provide feedback**

# 9.11 Topological sort

## Overview

A ***topological sort*** of a directed, acyclic graph produces a list of the graph's vertices such that for every edge from a vertex X to a vertex Y, X comes before Y in the list.

**PARTICIPATION ACTIVITY** 9.11.1: Topological sort.

## Animation captions:

1. Analysis of each edge in the graph determines if an ordering of vertices is a valid topologica sort.
2. If an edge from X to Y exists, X must appear before Y in a valid topological sort. C, D, A, F, E E is not valid because this requirements is violated for three edges.
3. Ordering D, A, F, E, C, B has 1 edge violating the requirement, so the ordering is not a valid topological sort.
4. For ordering D, A, F, E, B, C, the requirement holds for all edges, so the ordering is a valid topological sort.
5. A graph can have more than 1 valid topological sort. Another valid ordering is D, A, F, B, E, C.

**PARTICIPATION ACTIVITY** 9.11.2: Topological sort.

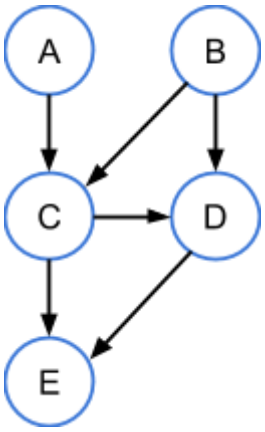
1) In the example above, D, A, B, F, E, C is a valid topological sort.
   - ○ True
   - ○ False

2) Which of the following is NOT a requirement of the graph for topological sorting?
   - ○ The graph must be acyclic.
   - ○ The graph must be directed.
   - ○ The graph must be weighted.

3) For a directed, acyclic graph, only one possible topological sort output exists.
   - ○ True
   - ○ False

**PARTICIPATION ACTIVITY** 9.11.3: Identifying valid topological sorts.

Indicate whether each vertex ordering is a valid topological sort of the graph below.




1) A, B, C, D, E

- Valid
- Invalid

2) E, D, C, B, A

- Valid
- Invalid

3) D, E, A, B, C

- Valid
- Invalid

4) B, A, C, D, E

- Valid
- Invalid

5) B, A, C, E, D

- Valid
- Invalid

## Example: course prerequisites

Graphs can be used to indicate a sequence of steps, where an edge from X to Y indicates that X must be done before Y. A topological sort of such a graph provides one possible ordering for performing the steps. Ex: Given a graph representing course prerequisites, a topological sort of the graph provides an ordering in which the courses can be taken.

**PARTICIPATION ACTIVITY** 9.11.4: Topological sorting can be used to order course prerequisites.

### Animation captions:

1. For a graph representing course prerequisites, the vertices represent courses, and the edge represent the prerequisites. CS 101 must be taken before CS 102, and CS 102 before CS 103.
2. CS 103 is "Robotics Programming" and has a physic course (Phys 101) as a prerequisite. Phys 101 also has a math prerequisite (Math 101).
3. The graph's valid topological sorts provide possible orders in which to take the courses.

**PARTICIPATION ACTIVITY** 9.11.5: Course prerequisites.

1) The "Math 101" and "CS 101" vertices have no incoming edges, and therefore one of these two vertices must be the first vertex in any topological sort.
   - ○ True
   - ○ False

2) Every topological sort ends with the "CS 103" vertex because this vertex has no outgoing edges.
   - ○ True
   - ○ False

## Topological sort algorithm

The topological sort algorithm uses three lists: a results list that will contain a topological sort of vertices, a no-incoming-edges list of vertices with no incoming edges, and a remaining-edges list. The result list starts as an empty list of vertices. The no-incoming-edges vertex list starts as a list of all vertices in the graph with no incoming edges. The remaining-edges list starts as a list of all edges in the graph.

The algorithm executes while the no-incoming-edges vertex list is not empty. For each iteration, a vertex is removed from the no-incoming-edges list and added to the result list. Next, a temporary list is built by removing all edges in the remaining-edges list that are outgoing from the removed vertex. For each edge currentE in the temporary list, the number of edges in the remaining-edges list that are incoming to currentE's terminating vertex are counted. If the incoming edge count is 0, then V is added to the no-incoming-edges vertex list.

Because each loop iteration can remove any vertex from the no-incoming-edges list, the algorithm's output is not guaranteed to be the graph's only possible topological sort.

**PARTICIPATION ACTIVITY** 9.11.6: Topological sort algorithm.

## Animation captions:

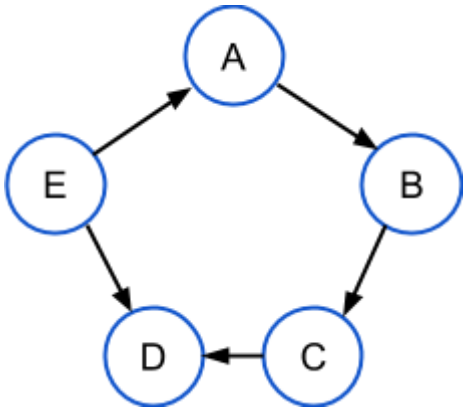
1. The topological sort algorithm begins by initializing an empty result list, a list of all vertices with no incoming edges, and a "remaining edges" list with all edges in the graph.
2. Vertex E is removed from the list of vertices with no incoming edges and added to resultList. Outgoing edges from E are removed from remainingEdges and added to outgoingEdges.
3. Edge EF goes to vertex F, which still has 2 incoming edges. Edge EG goes to vertex G, which still has 1 incoming edge.
4. Vertex A is removed and added to resultList. Outgoing edges from A are removed from remainingEdges. Vertices B and C are added to noIncoming.
5. Vertices C and B are processed, each with 1 outgoing edge.
6. Vertices B and F are processed, each also with 1 outgoing edge.
7. Vertex G is processed last. No outgoing edges remain. The final result is E, A, C, D, B, F, G.

Figure 9.11.1: GraphGetIncomingEdgeCount function.

```
GraphGetIncomingEdgeCount(edgeList, vertex) {
   count = 0
   for each edge currentE in edgeList {
      if (edge->toVertex == vertex)
         count = count + 1
   }
   return count
}
```

**PARTICIPATION ACTIVITY** 9.11.7: Topological sort algorithm.

Consider calling GraphTopologicalSort on the graph below.




1) In the first iteration of the while loop,

what is assigned to currentV?

- Vertex A
- Vertex E
- Undefined

2) In the first iteration of the while loop, what is the contents of outgoingEdges right before the for-each loop begins?

- Edge from E to A and edge from E to D
- Edge from A to B
- No edges

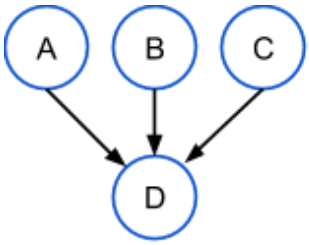
3) When currentV becomes vertex C, what is the contents of resultList?

- A, B
- E, A, B
- E, D

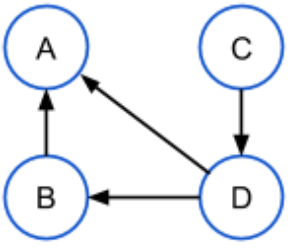
4) What is the final contents of resultList?

- A, B, C, D, E
- E, A, B, C, D
- E, A, B, D, C

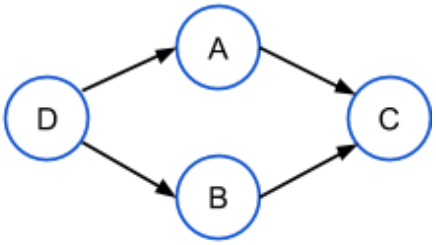
**PARTICIPATION ACTIVITY** 9.11.8: Topological sort matching.




1




2




3

Graph 2 Graph 3 Graph 1

D, B, A, C

C, D, B, A

B, C, A, D

Reset

**PARTICIPATION ACTIVITY** 9.11.9: Topological sort algorithm.

1) What does `GraphTopologicalSort` return?
   - A list of vertices.
   - A list of edges.
   - A list of indices.

2) GraphTopologicalSort will not work on a graph with a positive number of vertices but no edges.
   - True
   - False

3) If a graph implementation stores incoming and outgoing edge counts in each vertex, then the statement `GraphGetIncomingEdgeCount(remainingEdges, edge->to)` can be replaced with `currentE->toVertex->incomingEdgeCount`.
   - True
   - False

## Algorithm efficiency

The two vertex lists used in the topological sort algorithm will at most contain all the vertices in the graph. The remaining-edge list will at most contain all edges in the graph. Therefore, for a graph with a set of vertices V and a set of edges E, the space complexity

of topological sorting is $O(|V| + |E|)$. If a graph implementation allows for retrieval of a vertex's incoming and outgoing edges in constant time, then the time complexity of topological sorting is also $O(|V| + |E|)$.

How was this section? Provide feedback

# 9.12 Minimum spanning tree

## Overview

A graph's ***minimum spanning tree*** is a subset of the graph's edges that connect all vertices in the graph together with the minimum sum of edge weights. The graph must be weighted and connected. A ***connected*** graph contains a path between every pair of vertices.

**PARTICIPATION ACTIVITY** 9.12.1: Using the minimum spanning to minimize total length of power lines connecting cities.

### Animation captions:

1. A minimum spanning tree can be used to find the minimal amount of power lines needed to connect cities. Each vertex represents a city. Edges represent roads between cities. The city P has a power plant.
2. Power lines are along roads, such that each city is connected to a powered city. But power lines along every road would be excessive.
3. The minimum spanning tree, shown in red, is the set of edges that connect all cities to powe with minimal total power line length.
4. The resulting minimum spanning tree can be viewed as a tree with the power plant city as th root.

**PARTICIPATION ACTIVITY** 9.12.2: Minimum spanning tree.

1) If no path exists between 2 vertices in a weighted and undirected graph, then

no minimum spanning tree exists for the graph.

- True
- False

2) A minimum spanning tree is a set of vertices.

- True
- False

3) The "minimum" in "minimum spanning tree" refers to the sum of edge weights.

- True
- False

4) A minimum spanning tree can only be built for an undirected graph.

- True
- False

## Kruskal's minimum spanning tree algorithm

***Kruskal's minimum spanning tree algorithm*** determines subset of the graph's edges that connect all vertices in an undirected graph with the minimum sum of edge weights. Kruskal's minimum spanning tree algorithm uses 3 collections:

- An edge list initialized with all edges in the graph.
- A collection of vertex sets that represent the subsets of vertices connected by current set of edges in the minimum spanning tree. Initially, the vertex sets consists of one set for each vertex.
- A set of edges forming the resulting minimum spanning tree.

The algorithm executes while the collection of vertex sets has at least 2 sets and the edge list has at least 1 edge. In each iteration, the edge with the lowest weight is removed from the list of edges. If the removed edge connects two different vertex sets, then the edge is added to the resulting minimum spanning tree, and the two vertex sets are merged.

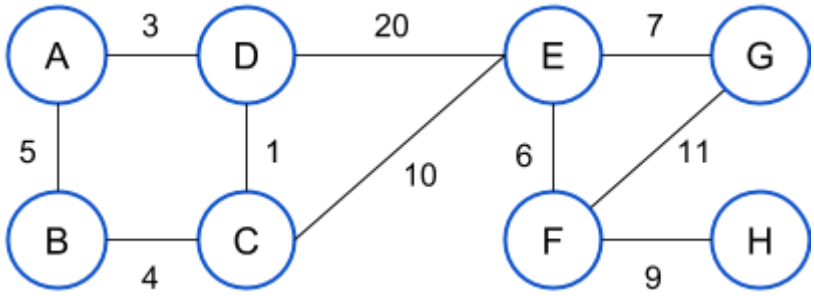
**PARTICIPATION ACTIVITY** 9.12.3: Minimum spanning tree algorithm.

### Animation captions:

1. An edge list, a collection of vertex sets, and an empty result set are initialized. The edge list contains all edges from the graph.
2. Edge AD is removed from the edge list and added to resultList, which will contain the edges forming the minimum spanning tree.
3. The next 5 edges connect different vertex sets and are added to the result.
4. Edges AB and CE both connect 2 vertices that are in the same vertex set, and therefore are not added to the result.
5. Edge EF connects the 2 remaining vertex sets.
6. One vertex set remains, so the minimum spanning tree is complete.

**PARTICIPATION ACTIVITY** 9.12.4: Minimum spanning tree algorithm.

Consider executing Kruskal's minimum spanning tree algorithm on the following graph:




1) What is the first edge that will be added to the result?

- AD
- AB
- BC
- CD

2) What is the second edge that will be added to the result?

- AD
- AB
- BC
- CD

3) What is the first edge that will NOT be added to the result?

- BC
- AB

- FG
- DE

4) How many edges will be in the resulting minimum spanning tree?
   - 5
   - 7
   - 9
   - 10

**PARTICIPATION ACTIVITY** 9.12.5: Minimum spanning tree - critical thinking.

1) The edge with the lowest weight will always be in the minimum spanning tree.
   - True
   - False

2) The minimum spanning tree may contain all edges from the graph.
   - True
   - False

3) Only 1 minimum spanning tree exists for a graph that has no duplicate edge weights.
   - True
   - False

4) The edges from any minimum spanning tree can be used to create a path that goes through all vertices in the graph without ever encountering the same vertex twice.
   - True
   - False

## Algorithm efficiency

Algorithm efficiency

Kruskal's minimum spanning tree algorithm's use of the edge list, collection of vertex sets, and resulting edge list results in a space complexity of $O(|E| + |V|)$. If the edge list is sorted at the beginning, then the minimum edge can be removed in constant time within the loop. Combined with a mechanism to map a vertex to the containing vertex set in constant time, the minimum spanning tree algorithm has a runtime complexity of $O(|E|log|E|)$.

How was this section? Provide feedback

# 9.13 All pairs shortest path

## Overview and shortest paths matrix

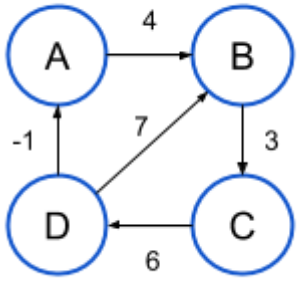
An ***all pairs shortest path*** algorithm determines the shortest path between all possible pairs of vertices in a graph. For a graph with vertices V, a |V| x |V| matrix represents the shortest path lengths between all vertex pairs in the graph. Each row corresponds to a start vertex, and each column in the matrix corresponds to a terminating vertex for each path. Ex: The matrix entry at row F and column T represents the shortest path length from vertex F to vertex T.

**PARTICIPATION ACTIVITY** 9.13.1: Shortest path lengths matrix.

### Animation captions:

1. For a graph with 4 vertices, the all-pairs-shortest-path matrix has 4 rows and 4 columns. An entry exists for every possible vertex pair.
2. An entry at row F and column T represents the shortest path length from vertex F to vertex T.
3. The shortest path from vertex A to vertex D has length 9.
4. The shortest path from B to A has length 2.
5. Only total path lengths are stored in the matrix, not the actual sequence of edges for the corresponding path.

**PARTICIPATION ACTIVITY** 9.13.2: Shortest path lengths matrix.




| | A | B | C | D |
|---|---|---|---|---|
| A | 0 | 4 | 7 | ? |
| B | 8 | 0 | 3 | 9 |
| C | 5 | 9 | ? | 6 |
| D | -1 | ? | 6 | 0 |

1) What is the shortest path length from A to D?

Check Show answer

2) What is the shortest path length from C to C?

Check Show answer

3) What is the shortest path length from D to B?

Check Show answer

**PARTICIPATION ACTIVITY** 9.13.3: Shortest path lengths matrix.

1) If a graph has 11 vertices and 7 edges, how many entries are in the all-pairs-shortest-path matrix?

- 11
- 7
- 77
- 121

2) An entry at row R and column C in the matrix represents the shortest path

length from vertex C to vertex R.

- ○ True
- ○ False

3) If a graph contains a negative edge weight, the matrix of shortest path lengths will contain at least 1 negative value.

- ○ True
- ○ False

4) For a matrix entry representing path length from A to B, when no such path exists, a special-case value must be used to indicate that no path exists.

- ○ True
- ○ False

## Floyd-Warshall algorithm

The ***Floyd-Warshall all-pairs shortest path algorithm*** generates a |V| x |V| matrix of values representing the shortest path lengths between all vertex pairs in a graph. Graphs with cycles and negative edge weights are supported, but the graph must not have any negative cycles. A ***negative cycle*** is a cycle with edge weights that sum to a negative value. Because a negative cycle could be traversed repeatedly, lowering the path length each time, determining a shortest path between 2 vertices in a negative cycle is not possible.

The Floyd-Warshall algorithm initializes the shortest path lengths matrix in 3 steps.

1. Every entry is assigned with infinity.
2. Each entry representing the path from a vertex to itself is assigned with 0.
3. For each edge from X to Y in the graph, the matrix entry for the path from X to Y is initialized with the edge's weight.

The algorithm then iterates through every vertex in the graph. For each vertex X, the shortest path lengths for all vertex pairs are recomputed by considering vertex X as an intermediate vertex. For each matrix entry representing A to B, existing matrix entries are used to compute the length of the path from A through X to B. If this path length is less than the current shortest path length, then the corresponding matrix entry is updated.

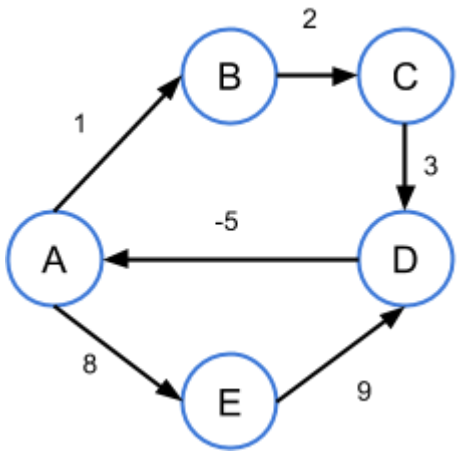
**PARTICIPATION ACTIVITY** 9.13.4: Floyd-Warshall algorithm.

## Animation captions:

1. All entries in the shortest path lengths matrix are first initialized with ∞. Vertex-to-same-verte values are then initialized with 0. For each edge, the corresponding entry for from X to Y is initialized with the edge's weight.
2. The k = 0 iteration corresponds to vertex A. Each vertex pair X, Y is analyzed to see if the path from X through A to Y yields a shorter path. Shorter paths from C to B and from D to B are found.
3. During the k = 1 iteration, 4 shorter paths that pass through vertex B are found from path from A to C, from A to D, from C to D, and from D to C.
4. During the k = 2 iteration, 1 shorter path that passes through vertex C is found from the path from B to A.
5. During the k=3 iteration, no entries are updated. The shortest path lengths matrix is complete.

**PARTICIPATION ACTIVITY** 9.13.5: Floyd-Warshall algorithm.

Consider executing the Floyd-Warshall algorithm on the graph below.




1) How many rows and columns are in the shortest path length matrix?

- 5 rows, 5 columns
- 5 rows, 6 columns
- 6 rows, 5 columns
- 6 rows, 6 columns

2) After matrix initialization, but before the k-loop starts, how many entries are set to non-infinite values?

- 5

- 11
- 14
- 25

3) After the algorithm completes, how many entries in the matrix are negative values?

- 0
- 3
- 5
- 7

4) After the algorithm completes, what is the shortest path length from vertex B to vertex E?

- -4
- 5
- 8
- Infinity (no path)

## Path reconstruction

Although only shortest path lengths are computed by the Floyd-Warshall algorithm, the matrix can be used to reconstruct the path sequence. Given the shortest path length from a start vertex to an end vertex is L. An edge from vertex X to the ending vertex exists such that the shortest path length from the starting vertex to X, plus the edge weight, equals L. Each such edge is found, and the path is reconstructed in reverse order.

**PARTICIPATION ACTIVITY** 9.13.6: Path reconstruction.

## Animation captions:

1. The matrix built by the Floyd-Warshall algorithm indicates that the shortest path from A to C of length 3.
2. The path from A to C is determined by backtracking from C.
3. Since A is the path's starting point, shortest distances from A are relevant at each vertex.
4. Traversing backwards along the only incoming edge to C, the expected path length at B is 3 - 5 = -2.
5. The computation at the edge from A to B doesn't hold, so the edge is not part of the path.

6. The computation on the D to B edge holds.
7. The computation on the A to D edge holds, and brings the path back to the starting vertex. The final path is A to D to B to C.

## Figure 9.13.1: FloydWarshallReconstructPath algorithm.

```
FloydWarshallReconstructPath(graph, startVertex, endVertex, distMatrix) {
   path = new, empty path

   // Backtrack from the ending vertex
   currentV = endVertex
   while (currentV != startVertex) {
      incomingEdges = all edges in the graph incoming to current vertex
      for each edge currentE in incomingEdges {
         expected = distMatrix[startVertex][currentV] - currentE.weight
         actual = distMatrix[startVertex][currentE.fromVertex]
         if (expected == actual) {
            currentV = currentE.fromVertex
            Prepend currentE to path
            break
         }
      }
   }

   return path
}
```

**PARTICIPATION ACTIVITY** 9.13.7: Path reconstruction.

1) Path reconstruction from vertex X to vertex Y is only possible if the matrix entry is non-infinite.
   - ○ True
   - ○ False

2) A path with positive length may include edges with negative weights.
   - ○ True
   - ○ False

3) More than 1 possible path sequence from vertex X to vertex Y may exists, even though the matrix will only store 1 path length from X to Y.
   - ○ True
   - ○ False

4) Path reconstruction is not possible if the graph has a cycle.

- ○ True
- ○ False

## Algorithm efficiency

The Floyd-Warshall algorithm builds a |V| x |V| matrix and therefore has a space complexity of $O(|V|^2)$. The matrix is constructed with a runtime complexity of $O(|V|^3)$.

How was this section? **Provide feedback**