AND → ∩ (min)
OR → ∪ (max)

- For the Systems with little complexity, hence little uncertainty, & has mathematical Expressions, provide precise description of the system.
- For the System with little complexity, but has ~~S~~ significant data, with model free methods, such as ANN, provides a powerful & robust means to reduce some uncertainty through learning, based on patterns in the available data.
- For most complex System, where few numerical data exist & where ambiguous info. is available, fuzzy reasoning provides a ~~S~~ way to understand the ~~of~~ System. ~~Behavior~~

6.36e − 10

(orange) → fruit? or color?

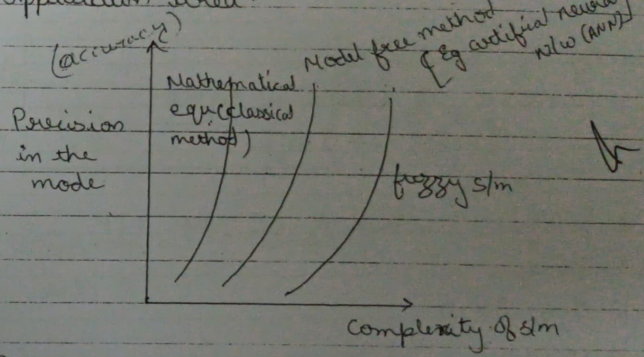
accuracy

L.M.Sani

Introduction to Fuzzy logic

complexity in world ~~..~~, arises from uncertainty in the form of ambiguity.

* Application area:



Precision − Accuracy

uncertainity − no certain Eg. It may rain today

Ambiguity − Eg cake is little hot.

uncertain, ambiguous information → Fuzzy s/m → Certain information

Not having one meaning or vague

* conclusion

1. Fuzzy s/ms are not very precise
2. Fuzzy s/ms find application where complexity is high

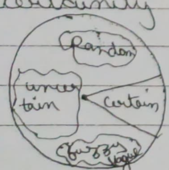
* Eg of simple s/m
  Control of intensity of lamp
  1 I/P → 1 O/P

* Eg of complex s/m
  Control of speed of motor
  20 I/P → 1 O/P
  random

* Speed control of DC motor I/P.
1 Torque
2 I/P vtg
3 current
4. Size of motor
5 Reference speed
6 Temp of motor
7 Temp of surrounding
8 Max & Min speed.

* Limitation of fuzzy logic
  Not suitable for shooting, missile targetting

* Uncertainity in information world

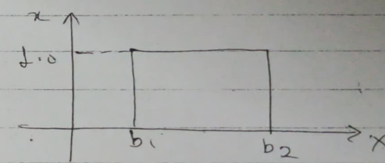


ambigious → vague

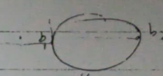
* Classical sets
  If X = {set of real no}.
  if A is a no
  $x = 1$ if $A \in X$
  $x = 0$ if $A \notin X$.

  $x \rightarrow$ membership value. It is binary (either 0 or 1)

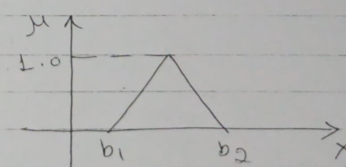
* Shape of membership function



$b_1, b_2 \rightarrow$ boundaries of the set

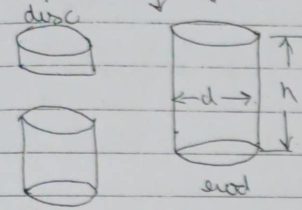


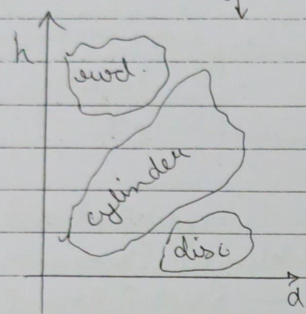
* Fuzzy sets
  If μ is membership fun
  $0 \leq \mu \leq 1$



* Mathematical of terms   Fuzzy terms

1 disc





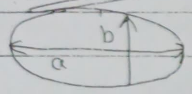
disc → $d >> h$

cyl → $d \approx h$

rod → $h >> d$

2. Ellipse :- when it becomes circle?

$\Rightarrow a/b = 1$



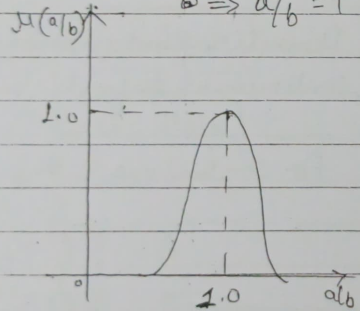
a = base

b = height

Geometric shape

$a >> b$ ——

$b >> a$ |



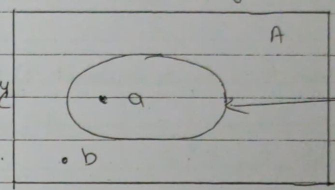
Membership fun of a circle

$\mu(a/b) = e^{-3(a/b-1)^2}$

* Classical sets are also known as crisp sets b/oz boundaries are well defn.

* Crisp set boundary.

X (universe of discourse)



NO uncertainty in boundries.

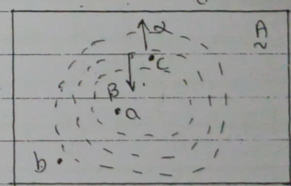
boundary of set A

universal set → universal discourse

set $A \in U$.

* Fuzzy set boundary.

X (universe of discourse)



vague & ambiguous boundries.

$\mu(c)$ is ambiguous.

Pt c may or may not belong to set A

If pt c moved towards α → μ decreases.

If pt c moves towards β → μ increases

* classical set.

A = set of real nos from 1 to 10
$= \{1, 2, \ldots 10\}$

If B is contained in A, B is subset of A

$B \in A$ Eg: $B = \{2, 4, 5\}$.

Membership fun : $\chi \Rightarrow$

$\left.\begin{array}{l}\chi(2) = 1 \\ \chi(3) = 0\end{array}\right\}$ $\chi = 1$ or $\chi = 0$

* Operations of classical sets
union (∪), intersection (∩), complement ... ($\bar{\ }$)

$A \cup B$ $A \cap B$ $\bar{A}$ & $\bar{B}$

* Fuzzy sets
1 notation of fuzzy set = $\underset{\sim}{A}$

Simplify $\Rightarrow$ A.

2 universe of discourse $\rightarrow$ U or X
Fuzzy set is contained in U or X
If U or X is discrete & finite
Fuzzy set $\underset{\sim}{A} = A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots + \frac{\mu_A(x_n)}{x_n} \right\}$

Membership degree

$$A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots\ldots + \frac{\mu_A(x_n)}{x_n} \right\}$$

universe of discourse

$$= \left\{ \sum_i \frac{\mu_A(x_i)}{x_i} \right\}$$

$+ \rightarrow$ Seperation
$- \rightarrow$ delimiter

NOTE 1. + symbol doesn't represent addition but a delimiter (seperator)
2. — symbol doesn't represent division but a delimiter
3 Σ symbol doesn't represent summation,
4 $\int$ symbol doesn't represent integration but delimiter

* Descrete & finite universe $\rightarrow$ Eg. Set of even nos from 1 to 20
* Continuous & Infinite discourse $\rightarrow$ Eg set of real nos

For Continuous & Infinite discourse,

$$A = \left\{ \int \frac{\mu_A(x)}{x} \right\}$$

$$= \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots\ldots \right\}$$

* Operations on Fuzzy sets.
If A & B are the Fuzzy sets defn on U.
If U is finite, $U = \{1, 2, 3, 4, 5, 6\}$.

$A \cup B \Rightarrow \mu_A(x) \vee \mu_B(x)$ $\quad \vee \rightarrow$ union

$A \cap B \Rightarrow \mu_A(x) \wedge \mu_B(x)$ $\quad \wedge \rightarrow$ intersection

$\bar{A} \Rightarrow 1 - \mu_A(x)$ $\quad \bar{A} \rightarrow$ complement

* According to Zadeh's notation
(Pioneer of fuzzy logic, 1965)

$A \cup B \Rightarrow \max(\mu_A(x), \mu_B(x))$

$$\boxed{\mu_{(A\cup B)}(x) = \max(\mu_A(x), \mu_B(x))}$$

$$\mu_{(A\cap B)}(x) = \min(\mu_A(x), \mu_B(x))$$

$$\mu_{\bar{A}}(x) = (1 - \mu_A(x))$$

* If $A = \left\{\frac{1}{5} + \frac{0.8}{6} + \frac{0.9}{7} + \frac{0.8}{8} + \frac{1}{9}\right\}$

& $B = \left\{\frac{0.6}{5} + \frac{0.8}{6} + \frac{0.9}{7} + \frac{0.3}{8} + \frac{0.1}{9}\right\}$

Find $A \cup B$, $A \cap B$, $\bar{A}$ & $\bar{B}$

$U = \{5, 6, 7, 8, 9\}$

$\mu_A(5) = 1 \quad \mu_B(5) = 0.6$

$$A \cup B = \left\{\frac{1}{5} + \frac{0.8}{6} + \frac{0.9}{7} + \frac{0.8}{8} + \frac{1}{9}\right\}$$

$$A \cap B = \left\{\frac{0.6}{5} + \frac{0.8}{6} + \frac{0.9}{7} + \frac{0.3}{8} + \frac{0.1}{9}\right\}$$

$$\bar{A} = \left\{\frac{0}{5} + \frac{0.2}{6} + \frac{0.1}{7} + \frac{0.2}{8} + \frac{0}{9}\right\}$$

$$\bar{B} = \left\{\frac{0.4}{5} + \frac{0.2}{6} + \frac{0.1}{7} + \frac{0.7}{8} + \frac{0.9}{9}\right\}$$

$\Rightarrow$ * given $A = \left\{\frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\}$

& $B = \left\{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$ Find

(1) B/A (2) $A \cap B$ (3) $A \cup \bar{A}$ (4) $\bar{B} \cap B$ (5) $\overline{A \cap B}$

using Zadeh's notation
↳ * By default if not mentioned

Sol $B/A \rightarrow$ Difference Operator

$\frac{B}{A} = B \cap \bar{A}$ & $\frac{A}{B} = A \cap \bar{B}$

$\frac{B}{A} \Rightarrow \bar{A} = \left\{\frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5}\right\}$

$$\frac{B}{A} = \left\{ \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

(2) $A \cap B = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5} \right\}$

(3) $A \cup \bar{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\}$

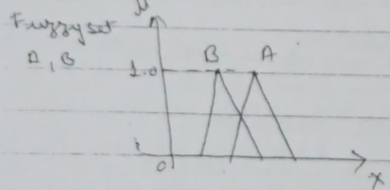
(4) $\bar{B} \cap B$.

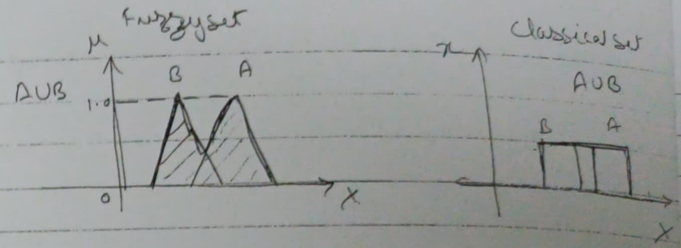
$$\bar{B} = \left\{ \frac{0.5}{2} + \frac{0.3}{2} + \frac{0.8}{4} + \frac{0.6}{5} \right\}$$

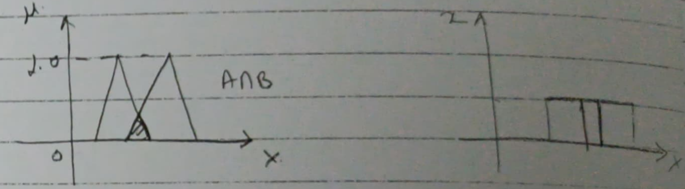
$$\bar{B} \cap B = \left\{ \frac{0.5}{2} + \frac{0.3}{2} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$$

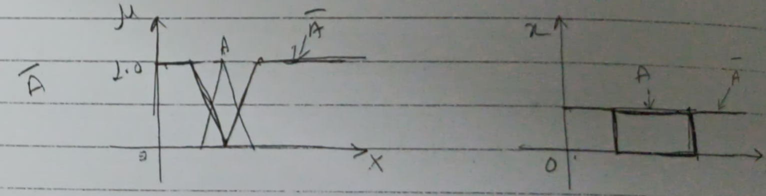
(5) $\overline{A \cap B} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5} \right\}$

* Fuzzy set operations with venn diagram







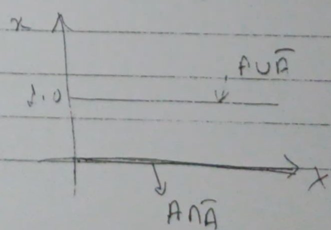


* Excluded middle laws.

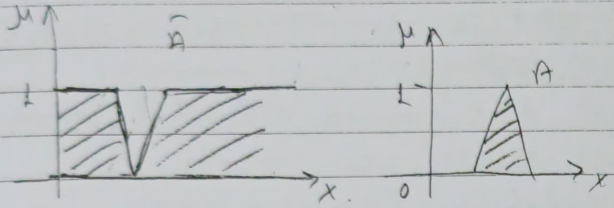
For classical set

$A \cup \bar{A} = X \Rightarrow$ universe of discourse

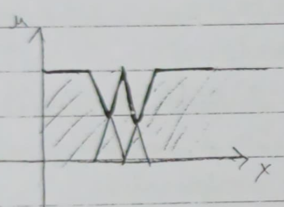
$A \cap \bar{A} = \phi \rightarrow$ null set



For Fuzzy sets → not followed.



A∪Ā



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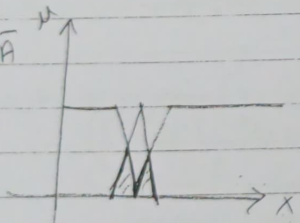


* Demorgan's law

$\overline{A \cap B} = \bar{A} \cup \bar{B}$ & $\overline{A \cup B} = \bar{A} \cap \bar{B}$

for classical sets

Followed by fuzzy sets.

Eg $A = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5} \right\}$

$B = \left\{ \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5} \right\}$

* Excluded middle laws:

$A \cup \bar{A} \neq X$

$A \cap \bar{A} \neq \phi$

$\bar{A} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} \right\}$

$\bar{B} = \left\{ \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5} \right\}$

$\bar{A} \cup \bar{B} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5} \right\}$ — (1)

Demorgan's law
$\overline{A \cap B} = \bar{A} \cup \bar{B}$
$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$A \cap B = \left\{ \frac{.5}{2} + \frac{.5}{3} + \frac{.2}{4} + \frac{.2}{5} \right\}$

$\overline{A \cap B} = \{ .5, .5, .8, .8 \}$ → hence prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$

* Properties of fuzzy sets

1 Triangular norm or T-norm represents intersection operation

Axioms of T-norm ($\wedge$)

(1) $a \wedge b = b \wedge a$ — T_1

(2) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ — T_2

(3) $a \leq c$ & $b \leq d$ implies $a \wedge b \leq c \wedge d$ — T_3

(4) $a \wedge 1 = a$

with these norms $0 \wedge a = 0$.

Eg $\min(a, b)$, ab, $\max(0, a+b-1)$

2 S-norm or triangular conorm ($\vee$) represents union operation

Axioms of S-norm

(1) $a \vee b = b \vee a$ — S_1

(2) $(a \vee b) \vee c = a \vee (b \vee c)$ — S_2

(3) $a \leq c$ & $b \leq d$ implies $a \vee b \leq c \vee d$ — S_3

1 It follows all the first three ~~norm~~ axioms except 4^{th}.

(4) $a \vee 0 = a$

* General relationship b/w S-norm & T-norm

$a \wedge b = 1 - [(1-a) \vee (1-b)]$ This equ is followed S-norm & T-norm become conjugate of each other

3 Complement operation (c) [or c-norm]

Axioms of complement operation

~~(a) $c(0) = 1$~~

(a) $c(0) = 1$ — C_1

(b) $a < b$ implies $c(a) > c(b)$ — C_2

(c) $c(c(a)) = a$ — C_3

Zadeh

* Zadeh's implication

1 T-norm $\rightarrow \min(a, b)$

2 S-norm $\rightarrow \max(a, b)$

3 c-norm $\rightarrow (1-a)$ or $(1-b)$

* Yager family of implication

1 $Y_q(a, b) = 1 - \min\left[1, \left((1-a)^q + (1-b)^q\right)^{1/q}\right]$

where $q \geq 0$ — T-norm

* Frank family

$F_s(a, b) = \log_a\left(1 + \frac{(s^a - 1)\cdot(s^b - 1)}{s-1}\right)$

where $s > 0$ — T-norm

* Sugeno family

$S_\lambda(a, b) = \min(1, a + b + \lambda \cdot a \cdot b)$

where $\lambda \geq -1$ — S-norm

Dubois & Prade family

$\delta_\alpha(a,b) = \dfrac{a.b}{\max(a,b,\alpha)}$ for def. value of boundary α

— T-norm

let $A = \left\{ \frac{1}{2} + \frac{0.8}{3} + \frac{0.7}{4} + \frac{0.6}{5} \right\}$.

$B = \left\{ \frac{0.6}{2} + \frac{0.9}{3} + \frac{0.1}{4} + \frac{0.2}{5} \right\}$

cal $A \cap B$ or $A \wedge B$ using Yager family implication & Dubois & Prade family

1 Assume $q = 1$: (Yager family)

$Y_1(a,b) = 1 - \min\left[1, ((1-a)^1 + (1-b)^1)^1\right]$

$Y_1(1, 0.6) = 1 - \min\left[1, ((1-1)^1 + (1-0.6)^1)^1\right]$

$= 1 - \min[1, (0 + (0.4))]$

$= 1 - \min[1, 0.4]$

$Y_1(1, 0.6) = 1 - \cancel{\min[}\ 0.4$

$= 0.6$

$Y_1(0.8, 0.9) = 1 - \min[1, ((1-0.8) + (1-0.9))]$

$= 1 - \min[1, (0.2 + 0.1)]$

$= 1 - \min[1, 0.3]$

$= 1 - 0.3$

$Y_1(0.8, 0.9) = 0.7$

$Y_1(0.7, 0.1) = 1 - \min[1, ((1-0.7) + (1-0.1))]$

$= 1 - \min[1, (0.3 + 0.9)]$

$= 1 - \min[1, 1.2]$

$= 1 - 1$

$= 0$

$Y_1(0.6, 0.2) = 1 - \min[1, ((1-0.6) + (1-0.2))]$

$= 1 - \min[1, (0.4 + 0.8)]$

$= 1 - \min[1, 1.2]$

$= 1 - 1$

$= 0$

$Y_1(A, b) = \left\{ \frac{0.6}{2} + \frac{0.7}{3} + \frac{0}{4} + \frac{0}{5} \right\}$.

Dubois & Prade family. Assume $\alpha = 1$

$\delta_\alpha(a,b) = \dfrac{a.b}{\max(a,b,\alpha)}$

$\delta_1(1, 0.6) = \dfrac{1 \cdot 0.6}{\max(1, 0.6, 1)} = \dfrac{0.6}{1} = 0.6$

$$\sigma_1(0.8, 0.9) = \frac{0.8 \times 0.9}{\max(0.8, 0.9, 1)}$$
$$= \frac{0.72}{1} = 0.72.$$

$$\sigma_2(0.7, 0.1) = \frac{0.7 \times 0.1}{\max(0.7, 0.1, 1)}$$
$$= \frac{0.07}{1} = 0.07$$

$$\sigma_2(0.6, 0.2) = \frac{0.6 \cdot 0.2}{\max(0.6, 0.2, 1)}$$
$$= \frac{0.12}{1} = 0.12$$

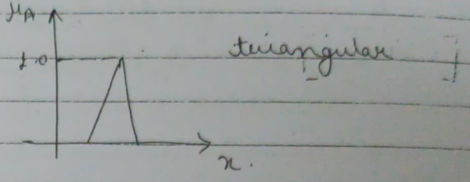
$$\sigma_2(A, B) = \left\{ \frac{0.6}{2} + \frac{0.72}{3} + \frac{0.07}{4} + \frac{0.12}{5} \right\}.$$

* Representation of fuzzy sets
1 using eqn for membership function
2 ~~using~~ Eg - $\mu_A(x) = \frac{1}{1+(x-6)^2}$

x is the element in the domain
A is the fuzzy set
μ_A is membership fun of fuzzy set A.

2 using graphical method
use various shapes.

Eg



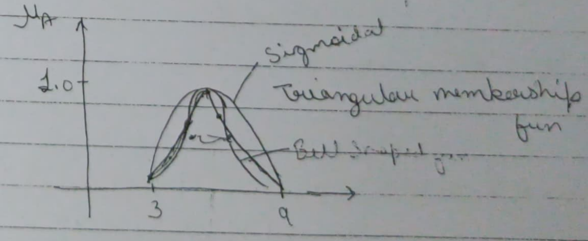
Eg → Let us define fuzzy set of numbers (real) close to 6

Sol Membership fun may be defn as

$$\mu_A = \frac{1}{1+(x-6)^2}$$

The universe of discourse if given as
$U = \{3, 4, 5, 6, 7, 8, 9\}$.
The fuzzy set $6 = \left\{ \frac{0.1}{3} + \frac{0.2}{4} + \frac{0.5}{5} + \frac{1}{6} + \frac{0.5}{7} + \frac{0.2}{8} + \frac{0.1}{9} \right\}$

Graphical representation



Fuzzy set of old people

Let us assume the domain of age as

$u = [0 \text{ to } 120]$ or $u = [0, 120]$

notation 2

$u = \{0 \text{ to } 120\}$

notation 1

Defn the membership fun.

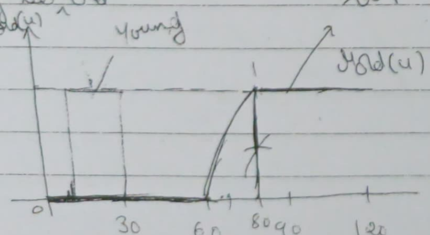
$$\mu_{old}(u) = 0 \quad \text{for } 0 \le u \le 60$$
$$= \frac{u-60}{20} \quad \text{for } 60 < u \le 80$$
$$= 1 \quad \text{for } 80 \le u \le 120.$$

$$\text{fuzzy set old} = \int_0^{120} \frac{\mu_{old}(u)}{u}$$

OR

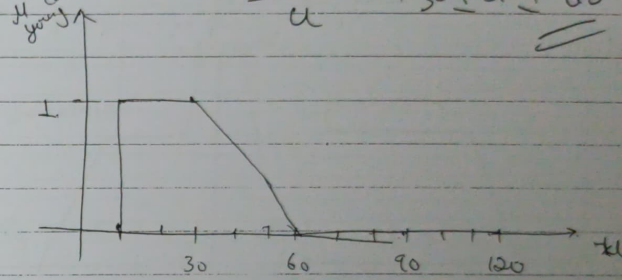
$$= \sum_{u=0}^{120} \frac{\mu_{old}(u)}{u}$$

fuzzy set old



Eg Fuzzy set young.

$$\mu_{young}(u) = 1 \quad \text{to} \quad 10 \le u \le 30$$
$$= 0 \quad u \ge 60$$
$$= \frac{60-u}{u} \quad 30 < u < 60$$

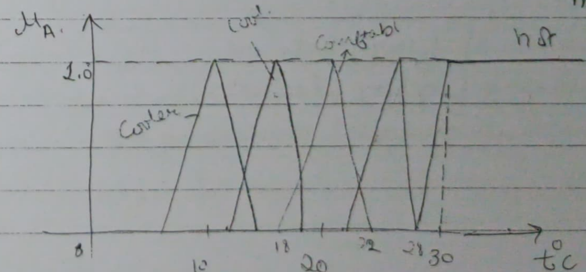


Eg Airconditioner

Domain of room temp. $T = \{0 \text{ to } 30^\circ\}$

form fuzzy sets of hot, warm, comfortable, cool.

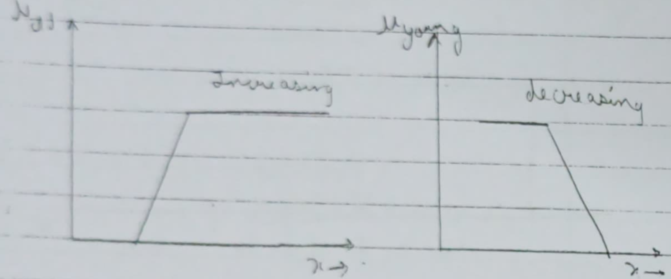
(hot, warm, comf, cool are called linguistic notions)

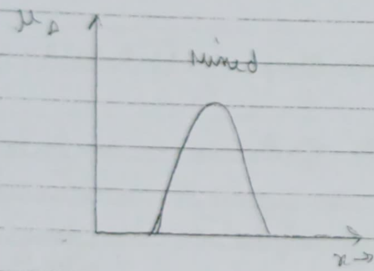


Observation

1 overlapping (desired nature)

* Fuzzy set Types
- increasing – Eg, old, hot, high etc
- decreasing – Eg young, cold, low etc.
- mixed

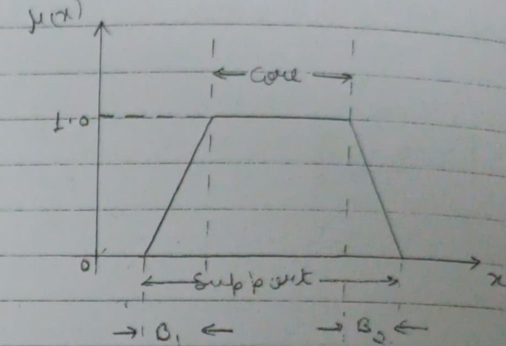




* Properties of fuzzy set

Membership function (Features)

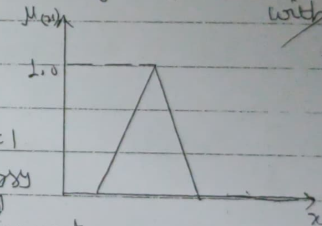
Assume trapezoidal membership fun



B_1 & B_2 → boundaries of fuzzy set

normal fuzzy set

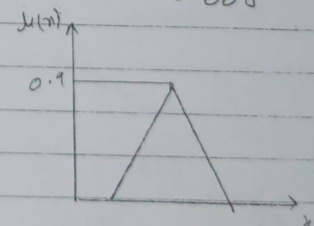
If height of fuzzy set is equal to 1 ie hgt(A) = 1 then that fuzzy set is called normal fuzzy set



~~with atleast one value of $\mu(x) = 1$~~

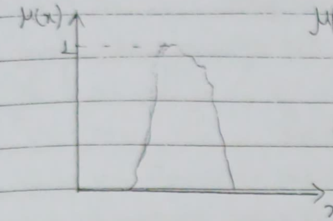
commonly used & easy to operate upon

Subnormal fuzzy set



If hgt(A) < 1 then the fuzzy set is called subnormal fuzzy s convex sets are normally used to represent fuzzy numbers, & are also useful for the rep of linguistic notions.

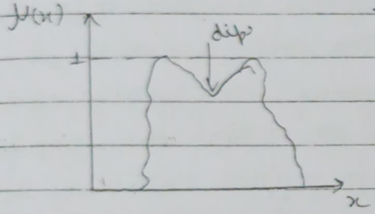
Convex fuzzy set



~~$\mu(x)$ increases continuously or decreases continuously~~

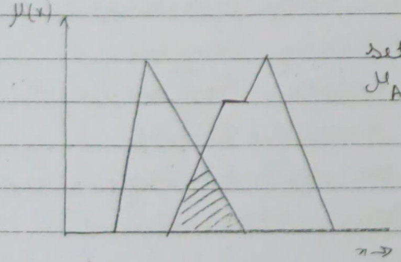
A fuzzy set is called convex if its membership fun does not contain 'dips'. This mean ie the $\mu(x)$ is, fuzzy inc, dec or Bell shaped.

non convex fuzzy set



$\mu(x)$ increases, decrease & again increases & decreases

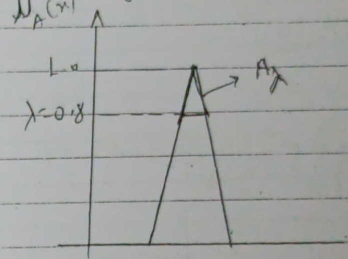
NOTE - Intersection of two convex fuzzy set is always a convex fuzzy set.



A fuzzy set is called non-convex fuzzy set ~~is a set~~ if its $\mu_A(x)$ contain dips

* λcut or α cut

If A is a fuzzy set, lambda cut of fuzzy set is A_λ for which $\lambda \leq \mu_A(x) \leq 1$



$A_\lambda \simeq A_\alpha$

Eg $A = \left\{ \frac{0.2}{2} + \frac{0.3}{3} + \frac{0.5}{4} + \frac{0.3}{5} \right\}$

cal A_α where $\alpha = 0.3$.

$A_\alpha = \left\{ \frac{0.3}{3} + \frac{0.5}{4} + \frac{0.3}{5} \right\}$.

* Linguistic variables

Any linguistic notion.

Eg temp, speed, ht, ct etc

linguistic terms → low, medium, high, very high etc

Linguistic variable with linguistic terms are known as linguistic prepositions

Linguistic hedges are similar to linguistic terms
↓
adjectives associated with linguistic variables

If α is a linguistic variable

$$\alpha = \int_0^n \frac{\mu_\alpha(y)}{y}$$

Linguistic hedges

1 very $\alpha = \int_0^n \frac{[\mu_\alpha(y)]^2}{y}$

2 very very $\alpha = \int_0^n \frac{[\mu_\alpha(y)]^4}{y}$

3 Plus $\alpha = \int_0^n \frac{[\mu_\alpha(y)]^{1.25}}{y}$

4 slightly $\alpha = \int_0^n \frac{[\mu_\alpha(y)]^{0.5}}{y}$

5 Minus $\alpha = \int_0^n \frac{[\mu_\alpha(y)]^{0.75}}{y}$

1 to 3 are known as concentration
4 & 5 are known as Dilations

Eg → Fuzzy set of temp is given as $\mu_{cold}(x)$

$$A = \left\{ \frac{0.2}{30} + \frac{0.4}{25} + \frac{0.6}{20} + \frac{0.8}{15} + \frac{1.0}{10} + \frac{0.8}{5} \right\}$$

Find fuzzy set of very cold, very very cold, slightly cold & plus cold

they fuzzy set of very cold $= \left\{ \frac{0.04}{30} + \frac{0.16}{25} + \frac{0.64}{15} + \frac{1}{10} \right.$

* Linguistic variable

By a linguistic variable we mean a vari whose value are words or sentences in a nat or artifical language.

Eg Age is a linguistic variable & its values linguistic rather than numerical i.e young, not young, very young, quite you old, very old etc rather than 20, 2, 22, 2

It is usual in approx reasoning to have the folln framework associated with the notion of a linguistic variable.

(X, LX, x, Mx). Here X denotes the symbolic name of a linguistic variable

very, not very, very very etc are the linguistic hedges. These are adjectives ass -ed with linguistic variable. Linguistic hed have the effect of modifying the member fun for a basic atomic term.

* crisp relation (classical relations)

Let X be universe of discourse of men.

X = {Adam, charles, Rajeev}

Let Y be universe of discourse of women

Y = { Eva, Diana, Sonia }

A relation "married to" = X × Y

X × Y = { (Adam, Eva), (Charles, Diana), (Rajeev, Sonia) }

* Fuzzy relations

If universe of discourse is U = {1, 2, 3} then 'approx equal' is a binary relation.

(fuzzy) → can be represented as

$$\left\{ \frac{1}{(1,1)} + \frac{1}{(2,2)} + \frac{1}{(3,3)} + \frac{0.8}{(1,2)} + \frac{0.8}{(2,3)} + \frac{0.8}{(2,1)} + \frac{0.8}{(3,2)} + \frac{0.3}{(1,3)} + \frac{0.3}{(3,1)} \right\}$$

Fuzzy membership function μ_R

$$\mu_R(x,y) = \begin{cases} 1 & \text{when } x = y \\ 0.8 & \text{when } |x-y| = 1 \\ 0.3 & \text{when } |x-y| = 2 \end{cases}$$

In matrix notation "approx. equal" may be written as

$$\begin{array}{cc} & \quad Y \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ X \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.8 \\ 0.3 & 0.8 & 1 \end{bmatrix} \end{array}$$

X × Y → cartesian product

Eg. Let $X = \{1, 2, 3\}$ & $Y = \{a, b, c\}$

The relation X × Y :

$$\begin{array}{cc} & \quad Y \\ & \begin{array}{ccc} a & b & c \end{array} \\ X \times Y = X \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix} \end{array}$$

⇒ Let A be the fuzzy set defn on X, Let B be the fuzzy set defn on Y

$$A = \left\{ \frac{0.8}{1} + \frac{1}{2} + \frac{0.8}{3} \right\}$$

$$B = \left\{ \frac{0.5}{a} + \frac{0.8}{b} + \frac{0.4}{c} \right\}$$

Find A×B using cartesian product

Ans Zadeh rule ⇒ $\mu_{(A\times B)}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$

= {

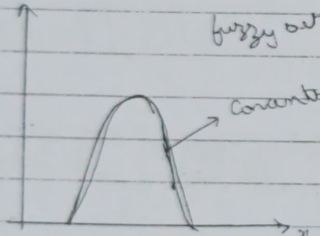
$$A \times B = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 0.5 & 0.8 & 0.4 \\ 2 & 0.5 & 0.8 & 0.4 \\ 3 & 0.5 & 0.8 & 0.4 \end{array}$$

$\mu_{(A \times B)}(1, a) = \min(0.8, 0.5)$

$= 0.5$

Binary relationship

* Fuzzy concentration (graphical representation)

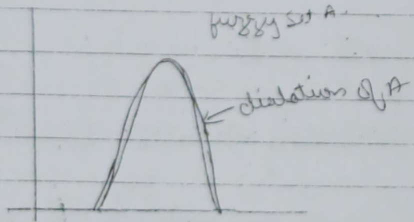


μ_α^2 or μ_α^4.

if $\mu_y = 0.4$

$\mu_\alpha^2 = 0.64$.

* Fuzzy dialation

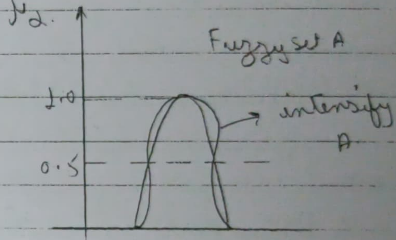


if $\mu_y = 0.8$

$\sqrt{\mu_y} = 0.89$

* Intensification

$$\text{"intensify"}\ \alpha = \begin{cases} 2\mu_\alpha^2(y) & \text{for } 0 \le \mu_\alpha(y) \\ 1 - 2[1 - \mu_\alpha(y)]^2 & 0.5 \le \mu_\alpha(y) \end{cases}$$



used to contrast the membership degrees & above 0.5

⇒* Let universe of discourse $U = \{1, 2, 3, 4, 5\}$.
foll^n fuzzy sets are defn on U.

$$\text{small} = \left\{ \frac{1.0}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.4}{4} + \frac{0.2}{5} \right\}$$

$$\text{large} = \left\{ \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1.0}{5} \right\}.$$

Find the fuzzy set of 'not very small' and 'not very very large'.

* Precedence for linguistic hedges & logical operations

Precedence	Operation
First	linguistic hedge, not.
second	And → intersection (min operator)
Third	or. → union (max operator)

logical operations - not, and, or.

linguistic hedge - very, slightly, very very, intensely ----.

very-small $= \left\{ \frac{1.0}{1} + \frac{0.64}{2} + \frac{0.36}{3} + \frac{0.16}{4} + \frac{0.04}{5} \right\}$
$[\alpha^2]$

very very large $= \left\{ \frac{1.6\times10^{-3}}{1} + \frac{0.0256}{2} + \frac{0.1296}{3} + \frac{0.4096}{4} + \frac{1.0}{5} \right\}$
$[\alpha^4]$

not very small $= \left\{ 1 - \alpha_{\text{very small}} \right\}$

$= \left\{ \frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.84}{4} + \frac{0.96}{5} \right\}$.

not very very large $= \left\{ \frac{0.9984}{1} + \frac{0.9744}{2} + \frac{0.8704}{3} \right.$
$\{1 - \alpha_{\text{very very large}}\}$

$\left. = + \frac{0.5904}{4} + \frac{0}{5} \right\}$

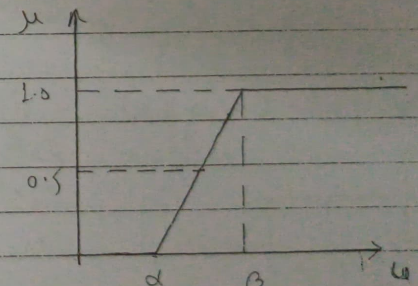
'not very small' and 'not very very large'
↳ intersection (min operator)

not very small $\wedge$ not very very large

$= \left\{ \frac{0}{1} + \frac{0.36}{2} + \frac{0.64}{3} + \frac{0.5904}{4} + \frac{0}{5} \right\}$.

* shape of membership fun

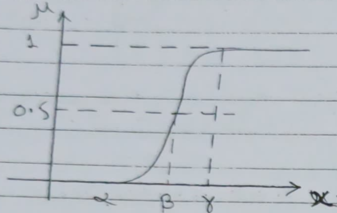
1. Γ function



Mathematical ~~ext~~ representation of membership fun

$$\Gamma(u:\alpha,\beta) = \begin{cases} 0 & \text{for } u<\alpha \\ (u-\alpha)/(\beta-\alpha) & \text{for } \alpha\le u\le\beta. \\ 1 & \text{for } u>\beta \end{cases}$$

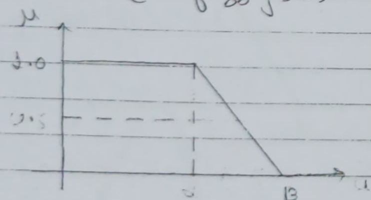
2 S function



Mathematical representation for $x \leq \gamma$

$$S(x;\alpha,\beta,\gamma) = \begin{cases} 0 & \text{for } x \leq \alpha \\ 2\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 & \text{for } \alpha < x \leq \beta \\ 1-2\left(\frac{x-\gamma}{\gamma-\alpha}\right)^2 & \text{for } \beta < x \leq \gamma \\ 1 & \text{for } x > \gamma \end{cases}$$

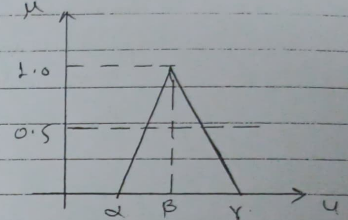
3 L function (L fuzzy set)



Mathematical representation

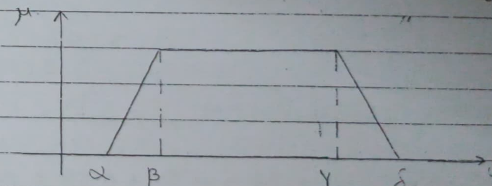
$$L(u;\alpha,\beta) = \begin{cases} 1 & \text{for } u < \alpha \\ \frac{(\beta-u)}{(\beta-\alpha)} & \text{for } \alpha \leq u \leq \beta \\ 0 & \text{for } u > \beta \end{cases}$$

4 Triangular fun (Λ fun)



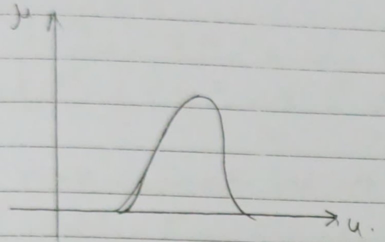
$$\Lambda(u;\alpha,\beta,\gamma) = \begin{cases} 0 & \text{for } u < \alpha \\ (u-\alpha)/(\beta-\alpha) & \text{for } \alpha \leq u \leq \beta \\ (\alpha-u)/(\beta-\alpha) & \text{for } \beta \leq u \leq \gamma \\ 0 & \text{for } u > \gamma \end{cases}$$

5 Π function (Pi fun or trapezodial fun)



$$\Pi(u;\alpha,\beta,\gamma,\delta) = \begin{cases} 0 & \text{for } u < \alpha \\ (u-\alpha)/(\beta-\alpha) & \text{for } \alpha \leq u < \beta \\ 1 & \text{for } \beta \leq u \leq \gamma \\ (\gamma-u)/(\delta-\gamma) & \text{for } \gamma < u \leq \delta \\ 0 & \text{for } u > \delta \end{cases}$$

6 Bell shaped fun.



* Fuzzy relations

1 Union & intersection operation on fuzzy relation

2 Zadeh's implication - similar to fuzzy set.

union → max

intersection → min

complement → negation $(1 - \mu(x))$

As per Sugeno, Yamakawa etc → the operator may differ from above

* Let R & S be a fuzzy relation

$$R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}$$

$$R = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.8 & 0.7 & 0.6 \\ x_2 & 1 & 0.6 & 0.8 \\ x_3 & 0.6 & 0.8 & 0.1 \end{array} \qquad S = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0.5 & 0.4 \\ x_2 & 0.6 & 0.8 & 0.6 \\ x_3 & 0.8 & 0.6 & 0.1 \end{array}$$

Cal $R \cup S$, $R \cap S$, $\bar{R}$

$$R \cup S \text{ (max)} = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0.7 & 0.6 \\ x_2 & 1 & 0.8 & 0.8 \\ x_3 & 0.8 & 0.8 & 0.1 \end{array}$$

$$R \cap S \text{ (min)} = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.8 & 0.5 & 0.4 \\ x_2 & 0.6 & 0.6 & 0.6 \\ x_3 & 0.6 & 0.6 & 0.1 \end{array}$$

$$\bar{R}\ (1-\mu(x)) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.3 & 0.4 \\ x_2 & 0 & 0.4 & 0.2 \\ x_3 & 0.4 & 0.2 & 0.9 \end{array}$$

* Operations of ~~classic~~ fuzzy set

* Projection & cylindrical extension

Projection operation brings a ternary relation back to binary relation or a binary relation to a fuzzy set or a fuzzy set to a single crisp value.

Let, R be the fuzzy relation

R = 'x considerably larger than y'

$$R = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & \cancel{0.7}\ 1 & 0.7 & 0.8 \end{array}$$

then projection on x means.

→ x_1 assigned max value from the first row. x_2 from second row & x_3 from third row.

Cal projection of R on X. $X = \{x_1, x_2, x_3\}$

Sol

$Y = \{y_1\ y_2\ y_3\ y_4\}$.

Projection R on X $= \left\{ \frac{1}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3} \right\}$

~~Total proj = 1 (max of ↑) → single crisp value.~~

Proj R on Y $= \left\{ \frac{0.9}{y_1} + \frac{1}{y_2} + \frac{0.7}{y_3} + \frac{0.8}{y_4} \right\}$.

Total proj = 1 (max of ↑) → single crisp value

→ Cylindrical Extension

↳ This is more or less opp to proj

cylindrical extension (CE) converts single crisp value to fuzzy set.

fuzzy set $\xrightarrow{to}$ binary relation

Binary relation $\xrightarrow{to}$ ternary relation

Consider fuzzy set. proj of R on X

$$A = \left\{ \frac{1}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3} \right\}$$

Ce on the domain X × Y is given by

$$CE(A) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 1 & 1 & 1 & 1 \\ x_2 & 0.8 & 0.8 & 0.8 & 0.8 \\ x_3 & 1 & 1 & 1 & 1 \end{array}$$

≠ original binary relation

~~If proj of R on Y, B =~~

If proj of R on Y, $B = \left\{ \frac{0.9}{y_1} + \frac{1}{y_2} + \frac{0.7}{y_3} + \ldots \right.$

$$CE(B) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.9 & 1 & 0.7 & 0.8 \\ x_2 & 0.9 & 1 & 0.7 & 0.8 \\ x_3 & 0.9 & 1 & 0.8 & 0.8 \end{array}$$

* Find Ce of $C = 0.8$ on the domain $X = \{x_1\ x_2\ x_3\}$

$$Ce(C) = \left\{ \frac{0.8}{x_1} + \frac{0.8}{x_2} + \frac{0.8}{x_3} \right\}$$

* Consider relation R as 'x is approx equal to y'

$$R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{array}$$

$\mu_R(1,1) = 1$

$\mu_R(1,2) = 0.8$

$X = \{1, 2, 3\}$ $\quad Y = \{1, 2, 3\}$

Fuzzy set of 'x is small' is

$$A = \left\{ \frac{0.3}{x_1} + \frac{1}{x_2} + \frac{0.8}{x_3} \right\}$$

Now fuzzy set A is combined with R

Now combination 'x is approx equal to y' and 'x is small'

intersection

$$Ce(A) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.3 & 0.3 \\ x_2 & 1 & 1 & 1 \\ x_3 & 0.8 & 0.8 & 0.8 \end{array}$$

$R \cap Ce(A)$ (min) $=$

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.3 & 0.3 & 0.3 \\ x_2 & 0.8 & 1 & 0.8 \\ x_3 & 0.3 & 0.8 & 0.8 \end{array}$$

* Composition

Consider R = 'x considerably larger than y' fuzzy relation

$$= \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.8 & 1 & 0.1 & 0.7 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.9 & 1 & 0.7 & 0.8 \end{array}$$

& fuzzy set A = 'x is small'

$$= \left\{ \frac{0.3}{x_1} + \frac{1}{x_2} + \frac{0.8}{x_3} \right\}$$

Combination of R & A expressed as 'x considerably larger than y and x is small'

The extension of A on $(X \times Y)$

$$Ce(A) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.3 & 0.3 & 0.3 & 0.3 \\ x_2 & 1 & 1 & 1 & 1 \\ x_3 & 0.8 & 0.8 & 0.8 & 0.8 \end{array}$$

$$R \cap Ce(A) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 0.3 & 0.3 & 0.1 & 0.3 \\ x_2 & 0 & 0.8 & 0 & 0 \\ x_3 & 0.8 & 0.8 & 0.7 & 0.8 \end{array} = R_1$$

The combination of fuzzy sets & fuzzy relation with aid of CE & projection is called composition. Denoted by o(dot)

$\boxed{A \circ R = R_1}$

If a fuzzy set is defn on X & R is relation defn on X×Y then the composition of A& R result in a fuzzy set B defn on Y is given by,

$B = A \circ R = \text{Proj}[CE(A) \cap R]$ on Y.

Now if intersection is performed with min operation & projection with max operation

$\mu_B(y) = \max_x \min(\mu_A(x) ; \mu_R(x,y))$

↓

max-min composition.

If intersection is performed with product operation & projection with max operation

$\mu_B(y) = \max_x (\mu_A(x) \cdot \mu_R(x,y))$ (• represents multiplication)

↓

Max product composition

⟹ * You are faced with a prob of controlling a m/c is subjected to a variable load. Given

A = 'motor speed OK'

$= \left\{ \frac{0.6}{30} + \frac{0.8}{40} + \frac{1}{50} + \frac{0.7}{60} \right\}$

domain is speed (rps) = {30, 40, 50, 60}

B = 'motor Vtg nominal'

$= \left\{ \frac{0.3}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.7}{5} \right\}$

domain is Vtg = {2, 3, 4, 5}.

(Vtg × 10²).

Rule 1

Find out the relation R if motor speed is OK' then motor Vtg is nominal'

If new antecedent A' = 'motor speed little slow'

~~Rule 2 If motor speed is ...~~

$= \left\{ \frac{0.7}{30} + \frac{1}{40} + \frac{0.6}{50} + \frac{0.3}{60} \right\}$

If new antecedent A' is given, find out consequent $B' = A' \circ R$ using maxmin compo

Sol R = 'If motor speed OK then motor Vtg is nominal'

⟹ Cross product of A & B ⟹ A × B (Cartesian product) (use min operation

$$R = \begin{array}{c|cccc} & 2 & 3 & 4 & 5 \\ \hline 30 & 0.3 & 0.6 & 0.6 & 0.6 \\ 40 & 0.3 & 0.8 & 0.8 & 0.7 \\ 50 & 0.3 & 0.8 & 1 & 0.7 \\ 60 & 0.3 & 0.7 & 0.7 & 0.7 \end{array}$$

$$A' = \left\{ \frac{0.7}{30} + \frac{1}{40} + \frac{0.6}{50} + \frac{0.3}{60} \right\}$$

$B' = A' \circ R$ using max min composition

$$ce(A') = \begin{array}{c|cccc} & 2 & 3 & 4 & 5 \\ \hline 30 & 0.7 & 0.7 & 0.7 & 0.7 \\ 40 & 1 & 1 & 1 & 1 \\ 50 & 0.6 & 0.6 & 0.6 & 0.6 \\ 60 & 0.3 & 0.3 & 0.3 & 0.3 \end{array}$$

$$ce(A') \cap R = \begin{array}{c|cccc} 30 & 0.3 & 0.6 & 0.6 & 0.6 \\ 40 & 0.3 & 0.8 & 0.8 & 0.7 \\ 50 & 0.3 & 0.6 & 0.6 & 0.6 \\ 60 & 0.3 & 0.3 & 0.3 & 0.3 \end{array}$$

$B' = \text{Proj}\,[ce(A') \cap R]$ on $\{2, 3, 4, 5\}$

$$\text{Proj} = \left\{ \frac{0.3}{2} + \frac{0.8}{3} + \frac{0.8}{4} + \frac{0.7}{5} \right\} = B'$$

Max-Min Composition & Max-product composition

$\rightarrow \Rightarrow$ $A = \left\{ \frac{0.3}{30}, \frac{0.7}{60}, \frac{1.0}{100}, \frac{0.2}{120} \right\}$

$$B = \left\{ \frac{0.2}{20}, \frac{0.4}{40}, \frac{0.6}{60}, \frac{0.8}{80}, \frac{1.0}{100}, \frac{0.1}{120} \right\}$$

$$C = \left\{ \frac{0.33}{500}, \frac{0.67}{1000}, \frac{1.0}{1500}, \frac{0.15}{1800} \right\}.$$

B is mapped on Y & A is mapped on X, C is mapped on Z

SS

$$R = A \times B = \begin{array}{c|cccccc} & 20 & 40 & 60 & 80 & 100 & 120 \\ \hline 30 & 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.1 \\ 60 & 0.2 & 0.4 & 0.6 & 0.7 & 0.7 & 0.1 \\ 100 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 & 0.1 \\ 120 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.1 \end{array}$$

$= (X \times Y)$

$$S = B \times C = \begin{array}{c|cccc} & 500 & 1000 & 1500 & 1800 \\ \hline 20 & 0.2 & 0.2 & 0.2 & 0.15 \\ 40 & 0.33 & 0.4 & 0.4 & 0.15 \\ 60 & 0.33 & 0.6 & 0.6 & 0.15 \\ 80 & 0.33 & 0.67 & 0.8 & 0.15 \\ 100 & 0.33 & 0.67 & 1.0 & 0.15 \\ 120 & 0.1 & 0.1 & 0.1 & 0.1 \end{array}$$

$= (Y \times Z)$

x → row   Max product

| → column $\mu_T(30,500) = \max[(0.2)(0.2)...$

ROS using Max min composition

$$\boxed{T = R \circ S \Rightarrow \mu_T(x,z) = \bigvee_{y \in Y} (\mu_R(x,y) \wedge \mu_S(y,z))}$$

$\mu_T(30,500) = \max(\min(0.2,0.2),$
$\min(0.3,0.33),$
$\min(0.3,0.33), \min(0.3,0.33), \min(0.3,0.-)$
$\min(0.1,0.1))$

max min

$= \max(0.2, 0.3, 0.3, 0.3, 0.3, 0.1)$

$= \cancel{m}\ 0.3$

⇒ * Given $A = \{0.2, 0.5, 0.5\}$ $B = \{0.2, 0.5, 0.5\}$
Find ~~fuzzy~~ $A \cup B$, $A \cap B$ & $\bar{A}$ using Yager method

Sol

$C_w(a) = (1-a^q)^{1/q}$ where $q \in (0, \infty)$

Intersection

$1 - \min(1, [(1-a)^q + (1-b)^q]^{1/q})$

Union :

$\min[1, (a^q + b^q)^{1/q}]$

$A \cup B$ → assume $q = 0$.

$(A \cup B)_{q=0} = \min[1, (0.2 + 0.2)^{\infty}]$

$= \min[1, 2^{\infty}] = 1$

$= 1$

$(A \cap B)_{q=0} = 1 - \min[1, (1-0.2)^0 + (1+0.2)^0)^{1/0}]$
$= 0.$

$(\bar{A})_{q=0} = (1 - 0.2^0)^{1/0}$
$= 0.$

III^{ly} repeat for other elements.

* Cardinality of crisp sets
Suppose universe $X = \{a, b, c\}$.
The power set is, $P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,$
$\{a,c\}, \{b,c\}, \{a,b,c\}\}$

Number of combinations $= 8 = 2^3$
ie power set is possible no of combinat
of elements
$8 = 2^3$ ∴ Cardinal no is $3 = n_x =$ no of
elements in X.
Cardinality of power set $= n_{P(X)} = 2^{nx}$

* cardinality of fuzzy set.
cardinality of fuzzy sets & fuzzy relations $= \infty$.

extension principle

Let A be a fuzzy set on universe U.

$$A = \left\{ \frac{\mu_1}{u_1}, \frac{\mu_2}{u_2}, \frac{\mu_3}{u_3} ---- \frac{\mu_n}{u_n} \right\}.$$

For a fun f (mathematical fun) that performs one to one mapping given by

$$f(A) = f\left\{ \frac{\mu_1}{u_1}, \frac{\mu_2}{u_2}, \frac{\mu_3}{u_3} ---- \frac{\mu_n}{u_n} \right)$$

$$= \left\{ \frac{\mu_1}{f(u_1)}, \frac{\mu_2}{f(u_2)}, \frac{\mu_3}{f(u_3)} ---- \frac{\mu_n}{f(u_n)} \right\}.$$

mapping is said to be one to one.

Eg If $A = \left\{ \frac{0.6}{1}, \frac{1}{2}, \frac{0.8}{3} \right\}$ defn on $U = \{1, 2, 3\}$

we wish to map elements of this fuzzy set to another universe, V, under the fun $V = f(u) = 2u - 1$

$$V = \{1, 3, 5\}$$

$$f(A) = \left\{ \frac{0.6}{1} + \frac{1}{3} + \frac{0.8}{5} \right\} \rightarrow \text{new fuzzy set}$$

This is known as one to one mapping

⇒ consider two identical universes $U_1 = U_2$
$= \{1, 2, 3 ---- 10\}$

Fuzzy number A = 'approximately two'
$$= \left\{ \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right\}.$$

Fuzzy no → fuzzy set related to particular no or no's

Other than 1, 2 & 3 all the other members have membership degree equal to zero.

Let Fuzzy no B = 'approx six'
$$= \left\{ \frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right\}$$

The product of 'approx two' & 'approx six'
= 'approx twelve'
→ defn as domain V

$$V = \{5, 6, 18, 21 ---\}.$$

'approx twelve' can be evaluated based on extension principle

$$2 \times 6 = \left(\frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right) \times \left(\frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right)$$