Optimal Multiplicative partitions: Number vs Individual size

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Abstract

This article introduces a function $P_\pi(n)$ defined on positive integers, representing the maximum number k of factors in a multiplicative partition of n where each factor d_i must satisfy $d_i \geq k$. We analyze the local behavior of $P_\pi(n)$ by defining three sequences based on whether $P_\pi(m) > P_\pi(m+1)$ (sequence of decrease, d_j), $P_\pi(m) = P_\pi(m+1)$ (sequence of equality, e_j), or $P_\pi(m) < P_\pi(m+1)$ (sequence of increase, i_j). We conjecture that these sequences possess asymptotic densities, denoted $C^{(-)}$, $C^{(0)}$, and $C^{(+)}$, respectively. This relies on the observation that the largest sequence terms s_M up to a large integer M are very close to M. Empirical evidence for M up to 9×10^5 suggests that these densities converge, with $C^{(-)} \approx C^{(+)} \approx 0.373$ and $C^{(0)} \approx 0.252$. Furthermore these three local behaviors comprehensively describe the transitions of $P_\pi(n)$ across the integers. The study provides statistical support for these conjectures and discusses properties of the sequences and the function $P_\pi(n)$.

Contents

1 Principal Definitions

Multiplicative partitions, which are ways of expressing an integer as a product of integer factors, constitute a classical area of study in number theory, see [2] and [1]. This paper focuses on a particular type of constrained multiplicative partition, characterized by a function denoted $P_\pi(n)$.

The function $P_\pi(n)$ is defined as the largest possible number of factors in a multiplicative partition of n under the specific constraint that every factor must be at least as large as the total count of factors in that partition. As noted, "this function is related to the multiplicative structure of n, and the growth of $P_\pi(n)$ is irregular and influenced by the density of its divisors."

1.1 Partition Function

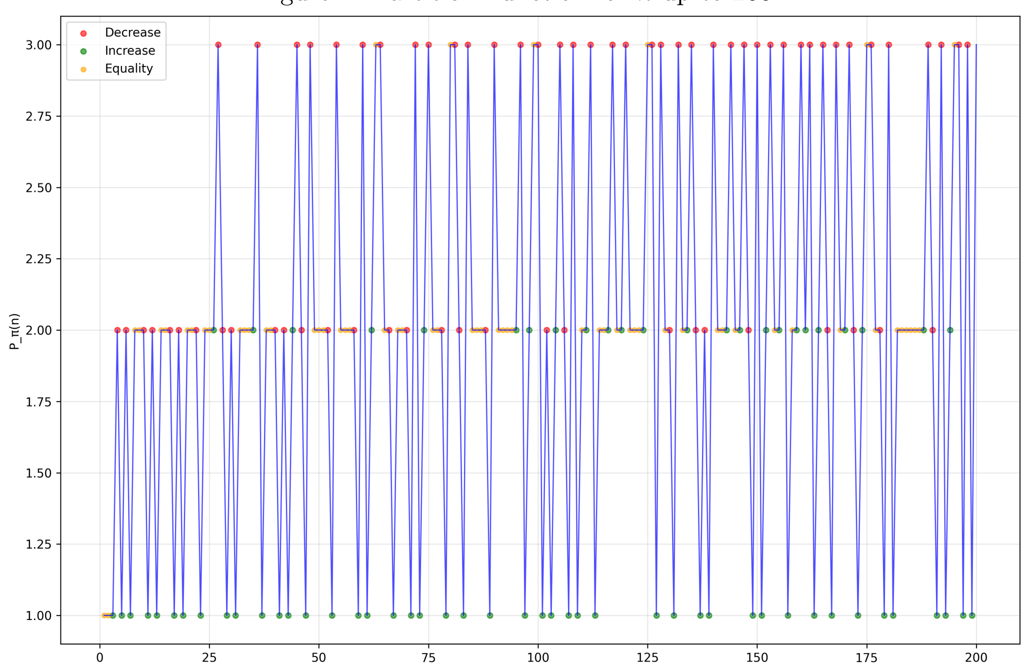
Definition 1.1 (The function $P_\pi(n)$)**.** Let n be a positive integer. The function $P_\pi(n)$ is defined as the largest positive integer k such that n can be written as a product of k integer factors $d_1, d_2, \ldots, d_k$, i.e., $n = d_1 \cdot d_2 \cdot \ldots \cdot d_k$, where each factor d_i satisfies the condition $d_i \geq k$.

Equivalently, $P_\pi(n)$ is the maximum size k of a multi-set of integers $A = \{d_1, d_2, \ldots, d_k\}$ such that:

1. The product of the elements of A is n: $\prod_{i=1}^{k} d_i = n$.
2. Each element $d_i \in A$ is greater than or equal to the size of the multi-set A: $d_i \geq k$ for all $i = 1, \ldots, k$.

For $n = 1$, the only multi-set is $A = \{1\}$. The size of this multi-set is $k = 1$. The single factor $d_1 = 1$ satisfies the condition $d_1 \geq k$ (since $1 \geq 1$). Thus, $P_\pi(1) = 1$.

Figure 1: Partition Function for n up to 200



Example 1.1. The following examples illustrate the calculation of $P_\pi(n)$:

- For $n = 10$: If $k = 1$, the partition is $\{10\}$. Since $10 \geq 1$, this is valid. If $k = 2$, we need factors d_1, d_2 such that $d_1 d_2 = 10$ and $d_1, d_2 \geq 2$. The partition $\{2, 5\}$ satisfies these conditions ($2 \geq 2$ and $5 \geq 2$). So $k = 2$ is possible. If $k = 3$, we need factors d_1, d_2, d_3 such that $d_1 d_2 d_3 = 10$ and $d_1, d_2, d_3 \geq 3$. The smallest possible product of three integers, each at least 3, is $3 \cdot 3 \cdot 3 = 27$. Since $27 > 10$, $k = 3$ is not possible for $n = 10$. Thus, the maximum k is 2, so $P_\pi(10) = 2$.
- For $n = 11$: If $k = 1$, the partition is $\{11\}$. Since $11 \geq 1$, this is valid. If $k = 2$, we need factors d_1, d_2 such that $d_1 d_2 = 11$ and $d_1, d_2 \geq 2$. As 11 is prime, its only positive integer factors are 1 and 11. The partition $\{1, 11\}$ has $d_1 = 1$, which does not satisfy $d_1 \geq 2$. Thus, $k = 2$ is not possible. Therefore, $P_\pi(11) = 1$.
- For $n = 63$: The problem states that $P_\pi(63) = 3$. This is supported by the partition $\{3, 3, 7\}$. Here, $k = 3$, and all factors $(3, 3, 7)$ are greater than or equal to $k = 3$. To confirm k cannot be 4, we would require four factors d_1, d_2, d_3, d_4 each at least 4. Their product would be at least $4^4 = 256$, which is greater than 63. So $P_\pi(63) = 3$.
- For $n = 64$: The problem states that $P_\pi(64) = 3$. This is supported by the partition $\{4, 4, 4\}$. Here, $k = 3$, and all factors $(4, 4, 4)$ are greater than or equal to $k = 3$. Similarly, $k = 4$ would require factors ≥ 4, leading to a product of at least $4^4 = 256 > 64$. So $P_\pi(64) = 3$.

1.2 Local Behavior Sequences

Using the function $P_\pi(n)$, we define three sequences based on the relationship between $P_\pi(m)$ and $P_\pi(m + 1)$.

Definition 1.2 (Sequence of Decrease (d_j))**.** The sequence $(d_j)_{j \geq 1}$ consists of all positive integers m, listed in increasing order, such that $P_\pi(m) > P_\pi(m + 1)$.

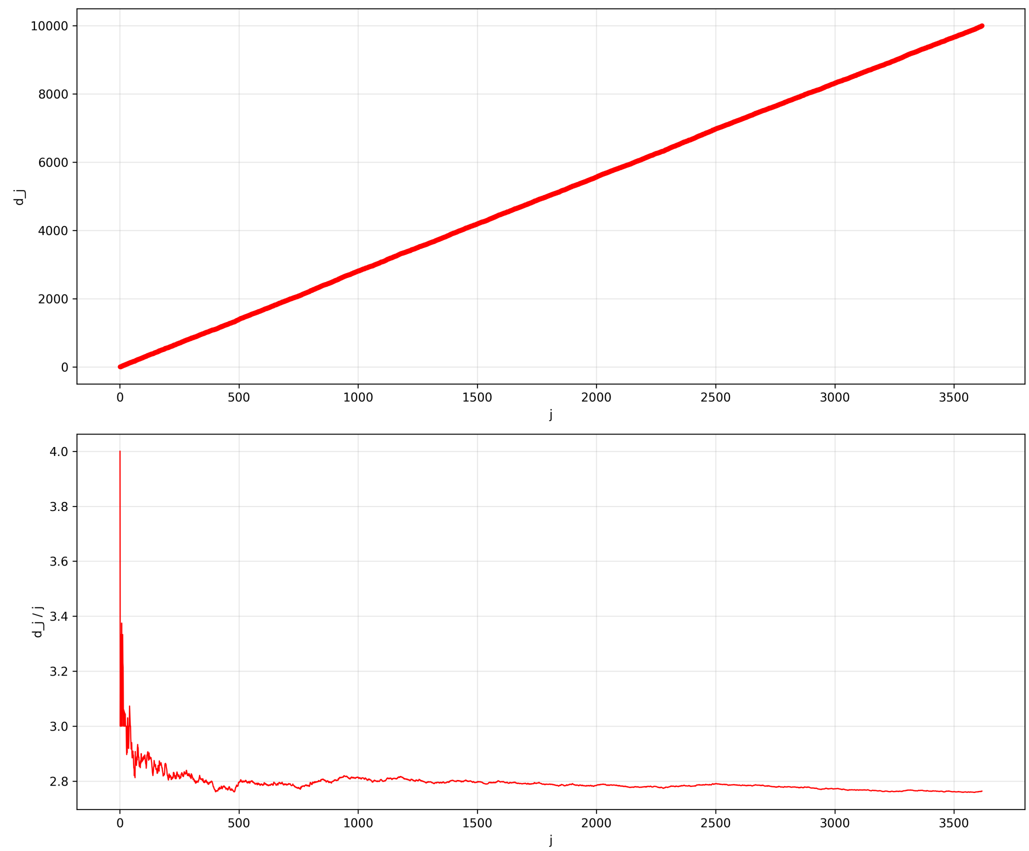
Definition 1.3 (Sequence of Equality (e_j))**.** The sequence $(e_j)_{j \geq 1}$ consists of all positive integers m, listed in increasing order, such that $P_\pi(m) = P_\pi(m + 1)$.

Definition 1.4 (Sequence of Increase (i_j))**.** The sequence $(i_j)_{j \geq 1}$ consists of all positive integers m, listed in increasing order, such that $P_\pi(m) < P_\pi(m + 1)$.

Example 1.2. The behavior of $P_\pi(n)$ at consecutive integers determines membership in these sequences:

- For $m = 10$: We have $P_\pi(10) = 2$ and $P_\pi(11) = 1$. Since $P_\pi(10) > P_\pi(11)$ (i.e., $2 > 1$), the integer 10 is a term in the sequence (d_j).
- For $m = 63$: We have $P_\pi(63) = 3$ and $P_\pi(64) = 3$. Since $P_\pi(63) = P_\pi(64)$ (i.e., $3 = 3$), the integer 63 is a term in the sequence (e_j).
- To illustrate the sequence (i_j), consider $m = 7$. $P_\pi(7) = 1$ (since 7 is prime, only partition is $\{7\}$, $k = 1$, $7 \geq 1$). For $P_\pi(8)$: If $k = 1$, $\{8\}$ is valid. If $k = 2$, factors $d_1, d_2 \geq 2$. $\{2, 4\}$ works $(2 \geq 2, 4 \geq 2)$. If $k = 3$, factors $d_1, d_2, d_3 \geq 3$. Smallest product $3^3 = 27 > 8$. So $P_\pi(8) = 2$. Since $P_\pi(7) < P_\pi(8)$ (i.e., $1 < 2$), the integer 7 is a term in the sequence (i_j).

Figure 2: Sequence of Decrease for d_j up to 10^5



1.3 Initial Terms of $P_\pi(n)$ and Derived Sequences

This section is reserved for listing the initial terms of the partition function $P_\pi(n)$ and the derived sequences (d_j), (e_j), and (i_j).

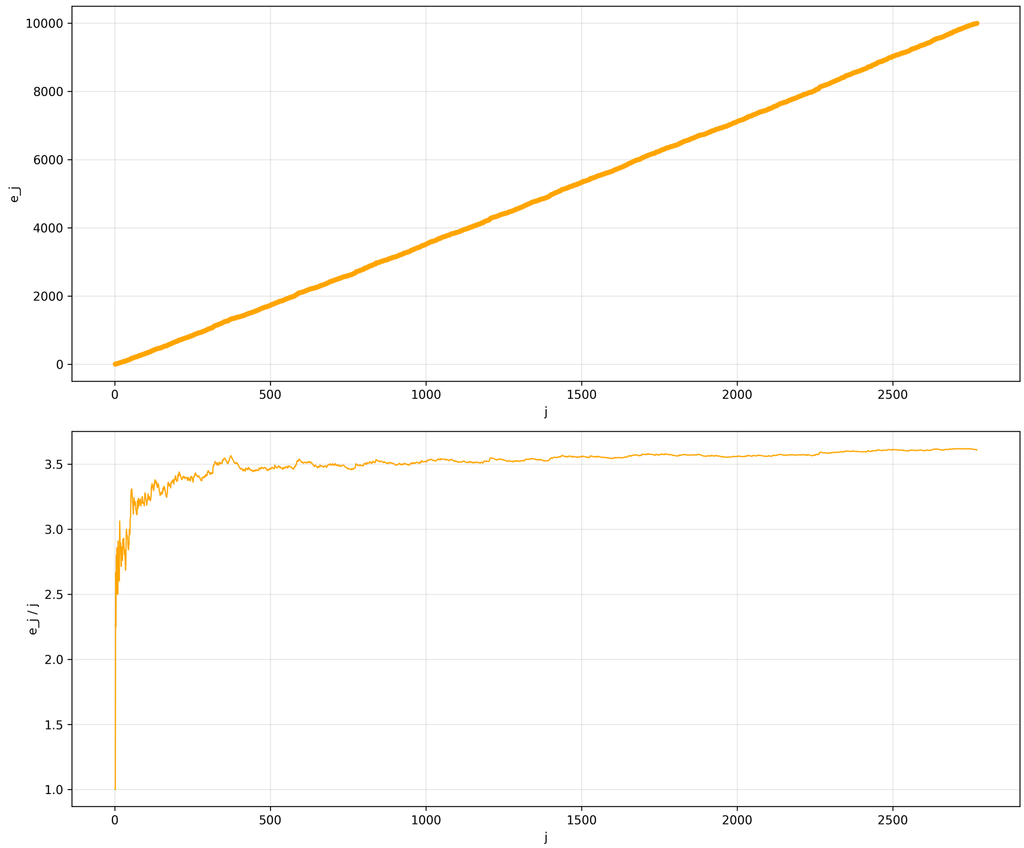
Initial values of $P_\pi(n)$: 1, 1, 1, 2, 1, 2, 1, 2, 2, 2, 1, 2, 1, 2, 2, 2, 1, 2, 1, 2, 2, 2, 1, 2, 2, 2, 3, 2, 1, 2, 1, 2, 2, 2, 2, 3, 1, 2, 2, 2, 1, 2, 1, 2, 3, 2, 1, 3, 2, 2, 2, 2, 1, 3, 2 . . .

Initial terms of Sequence of Decrease (d_j): 4, 6, 10, 12, 16, 18, 22, 27, 28, 30, 36, 40, 42, 45, 46, 48, 52, 54, 58, 60, 64, 66, 70, 72, 75, 78, 81, 82, 84, 88, 90, 96, 100, 102, 105, 106, 108, 112, 117, 120 . . .

Initial terms of Sequence of Equality (e_j): 1, 2, 8, 9, 14, 15, 20, 21, 24, 25, 32, 33, 34, 38, 39, 49, 50, 51, 55, 56, 57, 63, 65, 68, 69, 76, 77, 80, 85, 86, 87, 91, 92, 93, 94, 99, 110, 114, 115, 118, 121 . . .

Initial terms of Sequence of Increase (i_j): 3, 5, 7, 11, 13, 17, 19, 23, 26, 29, 31, 35, 37, 41, 43, 44, 47, 53, 59, 61, 62, 67, 71, 73, 74, 79, 83, 89, 95, 97, 98, 101, 103, 104, 107, 109, 111, 113, 116, 119 . . .

Figure 3: Sequence of Equality for e_j up to 10^5



2 Convergence Conjecture

Conjecture 2.1 (Asymptotic Densities of Local Behaviors)**.** Let M be a fixed integer and let:

$$j_M^{(-)} = \max\{j : d_j \leq M\},$$

$$j_M^{(0)} = \max\{j : e_j \leq M\},$$

and

$$j_M^{(+)} = \max\{j : i_j \leq M\}.$$

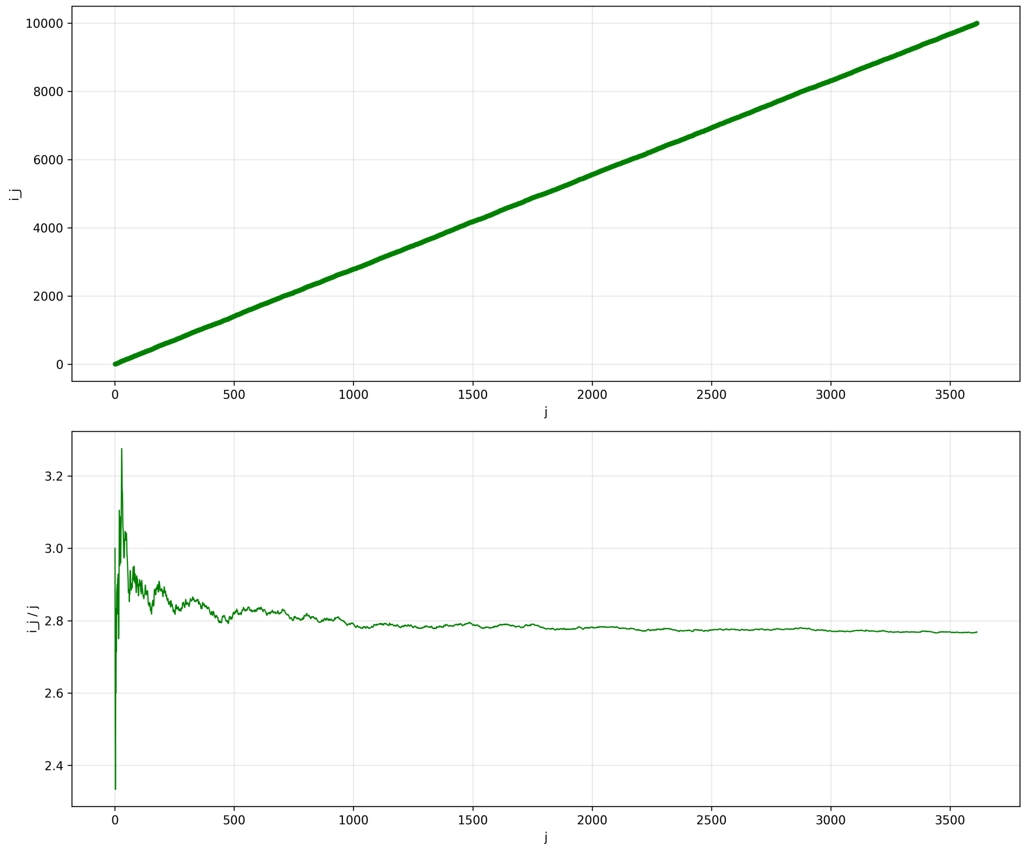
The elements of the three sequences are conjectured to satisfy

$$\forall s_M \in \{d_{j_M^{(-)}}, e_{j_M^{(0)}}, i_{j_M^{(+)}}\}, \quad |M - s_M| \ll M.$$

It is further conjectured the asymptotic densities of these sequences exist and are given by:

$$C^{(\pm)} = \lim_{M\to\infty} \frac{j_M^{(-)}}{d_{j_M^{(-)}}} = \lim_{M\to\infty} \frac{j_M^{(+)}}{i_{j_M^{(+)}}}, \quad C^{(0)} = \lim_{M\to\infty} \frac{j_M^{(0)}}{e_{j_M^{(0)}}},$$

Figure 4: Sequence of Increase for i_j up to 10^5

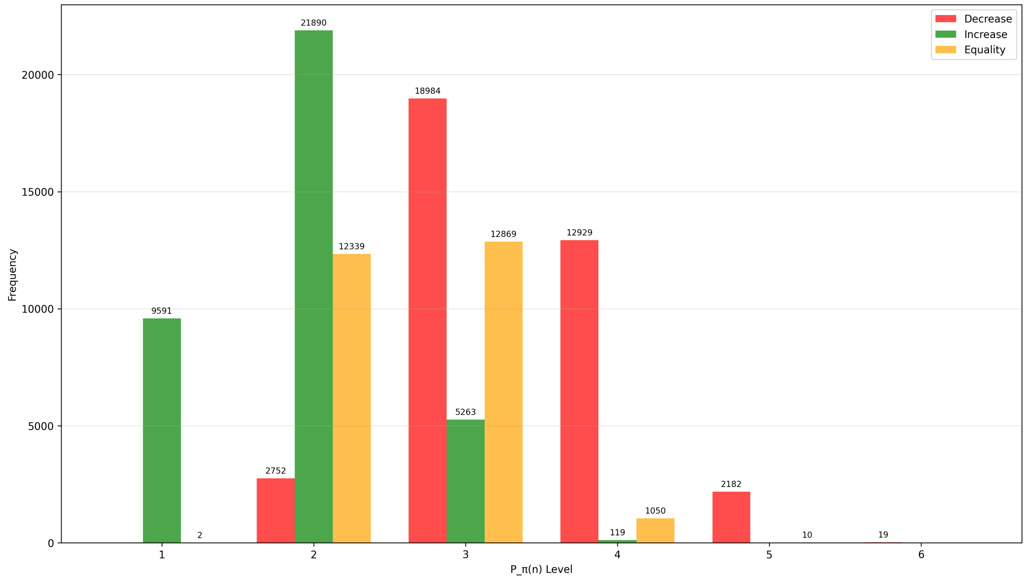


and consequently

$$1 = 2C^{(\pm)} + C^{(0)}.$$

Remark. The condition $|M - s_M| \ll M$ for $s_M \in \{d_{j_M^{(-)}}, e_{j_M^{(0)}}, i_{j_M^{(+)}}\}$ signifies that for a sufficiently large integer M, the largest terms $d_{j_M^{(-)}}$, $e_{j_M^{(0)}}$, and $i_{j_M^{(+)}}$ of the respective sequences that are less than or equal to M are indeed very close to M. This proximity ensures that the counts $j_M^{(-)}, j_M^{(0)}, j_M^{(+)}$ (i.e., the number of terms up to M) are representative of the density of these sequence terms across the interval $[1, M]$. Consequently, the ratios $C^{(-)} = j_M^{(-)}/d_{j_M^{(-)}}$, $C^{(0)} = j_M^{(0)}/e_{j_M^{(0)}}$, and $C^{(+)} = j_M^{(+)}/i_{j_M^{(+)}}$ (often approximated as $j_M^{(-)}/M$, $j_M^{(0)}/M$, $j_M^{(+)}/M$ due to $s_M \approx M$) become meaningful estimates of the true asymptotic densities. Table 3 provides empirical support for this condition, demonstrating that the relative differences $|M - s_M|/M$ are consistently small for large M.

Figure 5: Bar Chart of Distribution by Levels for values up to 10^5



2.1 Empirical Observations and Sequence Densities

The conjectured limits $C^{(-)}, C^{(0)}$, and $C^{(+)}$ can be interpreted as the asymptotic densities of integers m for which $P_\pi(m) > P_\pi(m+1)$, $P_\pi(m) = P_\pi(m+1)$, and $P_\pi(m) < P_\pi(m+1)$, respectively. Analysis of the data in Table 1 for values of M up to 9×10^5 reveals the following trends for the estimated densities:

- The density $C^{(-)}$ for the sequence of decrease (d_j) stabilizes around 0.3736.
- The density $C^{(0)}$ for the sequence of equality (e_j) stabilizes around 0.2525.
- The density $C^{(+)}$ for the sequence of increase (i_j) stabilizes around 0.3737.

These numerical results suggest that integers m marking a decrease $(P_\pi(m) > P_\pi(m+1))$ and those marking an increase $(P_\pi(m) < P_\pi(m+1))$ in the P_π function occur with nearly identical and the highest frequencies. Integers m where $P_\pi(m) = P_\pi(m+1)$ are less frequent. The sum $2C^{(\pm)} + C^{(0)}$ (approximating $2 \times 0.3736 + 0.2525 \approx 0.7472 + 0.2525 = 0.9997$ for the largest M values) is consistently close to 1, as detailed in Table 2. This empirically validates the conjecture that these three types of local behavior partition the set of positive integers, and leads to the two following conclusions:

- The condition $|M - s_M| \ll M$ (Table 3) holds robustly, indicating that the chosen M values do not terminate prematurely within long stretches devoid of sequence terms. This supports the reliability of density calculations.
- The convergence of $C^{(-)}, C^{(0)}, C^{(+)}$ appears relatively smooth with increasing M.

Further investigation could explore the distribution of lengths of consecutive runs of d_j, e_j, or i_j type integers, or analyze the behavior of these densities over different scales and special values of M.

Table 1: Statistical Convergence Analysis of the Conjecture 2.1 for Various Values of M up to 10^6

M	$j_M^{(-)}$	$d_{j_M^{(-)}}$	$j_M^{(0)}$	$e_{j_M^{(0)}}$	$j_M^{(+)}$	$i_{j_M^{(+)}}$	$C^{(-)}$	$C^{(0)}$	$C^{(+)}$
38838	14226	38838	10379	38837	14233	38835	0.366291	0.267245	0.366499
49904	18295	49900	13326	49904	18283	49903	0.366633	0.267033	0.366371
63946	23496	63945	16964	63946	23486	63944	0.367441	0.265286	0.367290
64480	23685	64480	17119	64478	23676	64479	0.367323	0.265501	0.367189
75000	27588	75000	19830	74997	27582	74999	0.367840	0.264411	0.367765
1e+05	36867	100000	26270	99996	36863	99999	0.368670	0.262711	0.368634
1e+05	45690	123805	32436	123806	45681	123807	0.369048	0.261991	0.368969
1e+05	46103	124944	32731	124941	46110	124943	0.368989	0.261972	0.369048
2e+05	55394	150000	39201	149998	55405	149999	0.369293	0.261343	0.369369
2e+05	80409	217065	56321	217066	80337	217067	0.370437	0.259465	0.370102
2e+05	92675	250000	64738	249998	92587	249999	0.370700	0.258954	0.370349
4e+05	130047	350000	89896	349995	130057	349999	0.371563	0.256849	0.371592
4e+05	149291	401536	102901	401532	149345	401537	0.371800	0.256271	0.371933
5e+05	199947	536700	136788	536694	199965	536699	0.372549	0.254871	0.372583
6e+05	239123	641396	163090	641398	239185	641387	0.372816	0.254273	0.372918
7e+05	262535	703872	178696	703870	262642	703873	0.372987	0.253876	0.373138
8e+05	279870	750000	190149	749997	279981	749999	0.373160	0.253533	0.373308
9e+05	338604	906176	228881	906175	338692	906177	0.373663	0.252579	0.373759

3 Conclusion and Future Work

The function $P_\pi(n)$ and its associated local behavior sequences (d_j), (e_j), and (i_j) provide a unique framework for analyzing multiplicative partitions constrained by the number of their parts. The empirical data presented strongly supports the Convergence Conjecture, suggesting that these three types of local transitions partition the set of positive integers with stable asymptotic densities, where $C^{(-)} \approx C^{(+)} \approx 0.373$ and $C^{(0)} \approx 0.252$ based on current estimates.

This study opens up several promising directions for future research:

3.1 Properties of $P_\pi(n)$ and Derived Sequences

- **Analytical Understanding of $P_\pi(n)$:** Developing analytical bounds (beyond trivial ones), determining the average or maximal order of $P_\pi(n)$, or finding an asymptotic formula for $P_\pi(n)$ would be significant theoretical advancements.
- **Characterizing Sequence Members:** A deeper dive into the number-theoretic properties (e.g., prime factorization structure, divisibility, values of other arithmetic functions like $\Omega(n)$ or $\sigma_0(n)$) of integers within each sequence (d_j), (e_j), (i_j) could illuminate underlying reasons for their classification.

Table 2: Statistical Conjecture 2.1 Relation Analysis: $2C^{(\pm)} + C^{(0)} = 1$

M	$C^{(-)}$	$C^{(0)}$	$C^{(+)}$	$2C^{(-)}$	$2C^{(-)} + C^{(0)}$	$2C^{(+)} + C^{(0)}$	$\lvert 1 - (2C^{(-)} + C^{(0)}) \rvert$
38838	0.366291	0.267245	0.366499	0.732581	0.999827	1.000244	0.000173
49904	0.366633	0.267033	0.366371	0.733267	1.000299	0.999774	0.000299
63946	0.367441	0.265286	0.367290	0.734882	1.000168	0.999867	0.000168
64480	0.367323	0.265501	0.367189	0.734646	1.000148	0.999880	0.000148
75000	0.367840	0.264411	0.367765	0.735680	1.000091	0.999940	0.000091
1e+05	0.368670	0.262711	0.368634	0.737340	1.000051	0.999978	0.000051
1e+05	0.369048	0.261991	0.368969	0.738096	1.000087	0.999929	0.000087
1e+05	0.368989	0.261972	0.369048	0.737979	0.999950	1.000068	0.000050
2e+05	0.369293	0.261343	0.369369	0.738587	0.999930	1.000082	0.000070
2e+05	0.370437	0.259465	0.370102	0.740875	1.000340	0.999670	0.000340
2e+05	0.370700	0.258954	0.370349	0.741400	1.000354	0.999653	0.000354
4e+05	0.371563	0.256849	0.371592	0.743126	0.999975	1.000034	0.000025
4e+05	0.371800	0.256271	0.371933	0.743600	0.999871	1.000138	0.000129
5e+05	0.372549	0.254871	0.372583	0.745098	0.999969	1.000038	0.000031
6e+05	0.372816	0.254273	0.372918	0.745633	0.999906	1.000109	0.000094
7e+05	0.372987	0.253876	0.373138	0.745974	0.999850	1.000153	0.000150
8e+05	0.373160	0.253533	0.373308	0.746320	0.999853	1.000150	0.000147
9e+05	0.373663	0.252579	0.373759	0.747325	0.999904	1.000098	0.000096

- **Investigating Runs and Patterns:** Studying the distribution of lengths of consecutive integers belonging to the same sequence type (e.g., a long run of m where $P_\pi(m) = P_\pi(m+1)$) could reveal insights into the local stability and correlational structure of $P_\pi(n)$.

3.2 Level Distributions and Conditional Frequencies

- **Behavior within $P_\pi(n) = k$ Strata:** For a fixed integer $k \geq 1$, let $S_k = \{m : P_\pi(m) = k\}$. Analyzing the conditional frequencies of $m \in S_k$ also belonging to (d_j), (e_j), or (i_j) could be very revealing. For example, how does the proportion of $m \in S_k$ that are points of increase (i.e., $m \in (i_j)$) change as k varies? This could shed light on how the function typically transitions away from a certain level k.

- **Specific Case $P_\pi(n) = 1$:** Integers m with $P_\pi(m) = 1$ (which include all primes, and other numbers like $1, 4, 6, 9, \dots$) cannot be in (d_j) because $P_\pi(m+1)$ must be at least 1. Thus, these m are either in (e_j) (if $P_\pi(m+1) = 1$) or (i_j) (if $P_\pi(m+1) > 1$). The relative frequencies of these outcomes for the subset of integers with $P_\pi(m) = 1$ would be a focused area of study.

3.3 Analysis of Convergence and the $|M - s_M|$ Gap

- **The Nature of the Gap $|M - s_M|$:** Table 3 indicates that for $s_M \in \{d_{j_M^{(-)}}, e_{j_M^{(0)}}, i_{j_M^{(+)}}\}$, the absolute difference $|M - s_M|$ is small. The maximum such difference observed in the provided table is 11. A key question is whether this difference $|M - s_M|$ remains bounded as $M \to \infty$, or if it grows, how slowly (e.g., $\log M$, $\log \log M$). This is crucial for the rigor of using M as the denominator in density calculations.

Table 3: Statistical Analysis of Differences $|M - s_M|$ for Convergence in Conjecture 2.1

M	$\lvert M - d_{j_M^{(-)}}\rvert$	$\lvert M - e_{j_M^{(0)}}\rvert$	$\lvert M - i_{j_M^{(+)}}\rvert$	$\frac{\lvert M-d_{j_M^{(-)}}\rvert}{M}$	$\frac{\lvert M-e_{j_M^{(0)}}\rvert}{M}$	$\frac{\lvert M-i_{j_M^{(+)}}\rvert}{M}$
38838	0	1	3	0.000000	0.000026	0.000077
49904	4	0	1	0.000080	0.000000	0.000020
63946	1	0	2	0.000016	0.000000	0.000031
64480	0	2	1	0.000000	0.000031	0.000016
75000	0	3	1	0.000000	0.000040	0.000013
1e+05	0	4	1	0.000000	0.000040	0.000010
1e+05	2	1	0	0.000016	0.000008	0.000000
1e+05	0	3	1	0.000000	0.000024	0.000008
2e+05	0	2	1	0.000000	0.000013	0.000007
2e+05	2	1	0	0.000009	0.000005	0.000000
2e+05	0	2	1	0.000000	0.000008	0.000004
4e+05	0	5	1	0.000000	0.000014	0.000003
4e+05	1	5	0	0.000002	0.000012	0.000000
5e+05	0	6	1	0.000000	0.000011	0.000002
6e+05	2	0	11	0.000003	0.000000	0.000017
7e+05	1	3	0	0.000001	0.000004	0.000000
8e+05	0	3	1	0.000000	0.000004	0.000001
9e+05	1	2	0	0.000001	0.000002	0.000000

- **Rate of Convergence of Densities:** Quantifying the speed at which the ratios $j_M^{(-)}/M$, $j_M^{(0)}/M$, and $j_M^{(+)}/M$ approach their respective limits $C^{(-)}$, $C^{(0)}$, and $C^{(+)}$ could provide further insights, possibly involving error terms related to M.

3.4 Further Explorations and Generalizations

- **Modifying Partition Constraints:** One could explore variations of the $P_\pi(n)$ function by altering the fundamental constraint $d_i \geq k$. For instance, what if $d_i \geq c \cdot k$ for some constant $c \neq 1$, or $d_i \geq k - c$, or $d_i \geq P_\pi(k)$ for some other slowly growing function f? How would such changes impact the existence and values of the corresponding densities?
- **Probabilistic Modeling:** Attempting to model the sequence of differences $P_\pi(m+1) - P_\pi(m)$ using a probabilistic framework (e.g., a type of random walk or a Markov chain on the values of $P_\pi(n)$) might offer a path to theoretically derive or explain the observed densities.
- **Relationship to Highly Composite Numbers:** Since $P_\pi(n)$ relates to factorizations, exploring its behavior for specific classes of numbers, such as highly composite numbers, primorials, or perfect powers, might reveal interesting patterns or extremal behaviors.

Pursuing these research avenues could substantially enhance our comprehension of this particular type of multiplicative partition and the rich statistical patterns that emerge from its local behavior.

References

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