1 Labelled Transition System for Complete Head Linear Reduction

1.1 Notes

State is a tuple ⟨ λ-term with underlined node; context; list of pairs (argument, context) ⟩, where

- λ-term (a tree; by considering λ-term as a tree it becomes possible to cross arguments out of tree with a parent application node (denotes as $A_{\xcancel{B}}$)) is a usual lambda term with one underlined node;
- Context Γ is an ordered (first-in) set of elements of the following form ($variable \mapsto (term,\ Context)$);
- Δ is an ordered list of pairs (λ-term, Context) (one can also think about Δ as an ordered list of pointers to the corresponding subtree stored together with a corresponding context);
- Initial configuration is $\langle \lambda - term\ with\ underlined\ root,\ \emptyset,\ \$\rangle$;
- Final configuration is $\langle M\,[\underline{x}]$ (term in normal form), $\Gamma,\ \$\rangle$;
- Note, that Complete Head Linear Reduction (CHLR) can be seen as usual Head Linear Reduction (HLR) that is repeatedly applied to the all arguments;
- From the below LTS for CHLR.

1.2 Labelled transition system for CHLR

$$\langle A\,[e_1\underline{@}e_2]\,;\ \Gamma;\ \Delta\rangle \longrightarrow \big\langle A\,\big[\underline{e_1}@e_2\big]\,;\ \Gamma;\ (e_2,\Gamma)\bullet\Delta\big\rangle \qquad \text{[App]}$$

$$\langle A\,[\underline{\lambda x}.e]\,;\ \Gamma;\ \$\bullet\Delta\rangle \longrightarrow \langle A\,[\lambda x.\underline{e}]\,;\ \Gamma;\ \$\bullet\Delta\rangle \qquad \text{[Lam-Non-Elim]}$$

$$\langle A\,[\underline{\lambda x}.e]\,;\ \Gamma;\ (B,\Gamma')\bullet\Delta\rangle \longrightarrow \big\langle A_{\xcancel{B}}\big[\xcancel{\lambda x}.\underline{e}\big]\,;\ x\mapsto(B,\Gamma'),\,\Gamma;\ \Delta\big\rangle \qquad \text{[Lam-Elim]}$$

$$\langle A\,[\underline{x}]\,;\ x\mapsto(B,\Gamma'),\,\Gamma;\ \Delta\rangle \longrightarrow \langle A\,[\underline{B}]\,;\ \Gamma';\ \Delta\rangle \qquad \text{[BVar]}$$

$$\langle A\,[M\,[\underline{x}]\,@B]\,;\ \Gamma;\ (B,\Gamma')\bullet\$\bullet\Delta\rangle \longrightarrow \langle A\,[M[x]@\underline{B}]\,;\ \Gamma';\ \$\bullet\Delta\rangle,\ x\notin dom(\Gamma) \qquad \text{[FVar-0]}$$

$$\langle A\,[M\,[\underline{x}]\,@B]\,;\ \Gamma;\ (B,\Gamma')\bullet C\bullet\Delta\rangle \longrightarrow \langle A\,[M[x]@\underline{B}]\,;\ \Gamma';\ \$\bullet C\bullet\Delta\rangle,\ C\neq\$,\ x\notin dom(\Gamma) \qquad \text{[FVar-1]}$$

$$\langle A\,[M\,[\underline{x}]\,@B]\,;\ \Gamma;\ \$\bullet(B,\Gamma')\bullet\Delta\rangle \longrightarrow \langle A\,[M\,[x]\,@\underline{B}]\,;\ \Gamma';\ \$\bullet\Delta\rangle,\ \ x\notin dom(\Gamma) \qquad \text{[FVar-2]}$$

1.3 Correctness proof

Note that that labelled transition system for HLR can be obtained by removing [FVAR-*] rules from the LTS for CHLR. In other words first four rules describe HLR while [FVAR-*] rules describes a repeated application of HLR to the arguments.

LTS for HLR .
Remark: it is obvious that in [Lam-Non-Elim] rule for HLR we can safely replace $\$\bullet\Delta$ by empty list since there is no rule that pop \$ sign from Δ.

$$\langle A\,[e_1\underline{@}e_2]\,;\ \Gamma;\ \Delta\rangle \longrightarrow \big\langle A\,\big[\underline{e_1}@e_2\big]\,;\ \Gamma;\ (e_2,\Gamma)\bullet\Delta\big\rangle \qquad \text{[App]}$$

$$\langle A\,[\underline{\lambda x}.e]\,;\ \Gamma;\ []\rangle \longrightarrow \langle A\,[\lambda x.\underline{e}]\,;\ \Gamma;\ []\rangle \qquad \text{[Lam-Non-Elim]}$$

$$\langle A\,[\underline{\lambda x}.e]\,;\ \Gamma;\ (B,\Gamma')\bullet\Delta\rangle \longrightarrow \big\langle A_{\xcancel{B}}\big[\xcancel{\lambda x}.\underline{e}\big]\,;\ x\mapsto(B,\Gamma'),\,\Gamma;\ \Delta\big\rangle \qquad \text{[Lam-Elim]}$$

$$\langle A\,[\underline{x}]\,;\ x\mapsto(B,\Gamma'),\,\Gamma;\ \Delta\rangle \longrightarrow \langle A\,[\underline{B}]\,;\ \Gamma';\ \Delta\rangle \qquad \text{[BVar]}$$

Expansion function. To prove correctness of HLR we introduce an axiliary expansion function *exp*. This function being applied to some state of LTS for HLR results in some λ-term which we will call an expansion.

$$exp\ \langle M[\underline{A}];\ \Gamma,\ x \mapsto (B, \Gamma');\ \Delta\rangle = exp\ \langle M[\underline{A}[x/B[\Gamma']]];\ \Gamma;\ \Delta\rangle \tag{1}$$

$$exp\ \langle M_B[\underline{A}];\ \emptyset;\ (B, \Gamma') \bullet \Delta\rangle = exp\ \langle M_{B[\Gamma']}[\underline{A}];\ \emptyset;\ \Delta\rangle \tag{2}$$

$$exp\ \langle M;\ \emptyset;\ []\rangle = M \tag{3}$$

$$\text{where } B[\Gamma'] = exp\ \langle \underline{B};\ \Gamma';\ []\rangle \tag{4}$$

Note that (1) substitudes $B[\Gamma']$ instead of all occurence of variable x in the subtree of term M that has an underlined node as a root (i.e. subtree A). (2) substitudes a all variables with respect to the corresponding context in one of arguments (one call).

Now consider LTS for HLR. Note that:

- Only [Lam-Elim] rule can change an expansion;
- The number of transitions without applying [Lam-Elim] rule is limited since the definition of context, input term has a final size, and only [Lam-Elim] rule expands the context; Hereafter we will denote a sequence of transitions without applying [Lam-Elim] rule by $\overset{*}{\to}$;
- The expansion function exp cannot change a path from the root of term to the underlined node since:
  - (I) (1) can only change the subtree of the current term with underlined node as a root;
  - (II) (2) can only change arguments of the current term that are above the underlined node; moreover each application of (2) change exactly one argument (that corresponds to the first elemnt of the pair in Δ) leaving all other arguments and the subtree of the current term with underlined node as a root intact.

Theorem .

$$\begin{tikzcd}
\langle \dots \rangle \arrow[r, "{[Lam-Elim]}"] & \langle M_i;\ \Gamma_i;\ \Delta_i\rangle \arrow[r, "*"] \arrow[d, "exp"] & \langle A[\underline{\lambda x}.e];\ \Gamma;\ (B, \Gamma') \bullet \Delta\rangle \arrow[r, "{[Lam-Elim]}"] \arrow[dl, "exp"] & \langle A_{\cancel{B}}\cancel{[\lambda x.}\underline{e}];\ x \mapsto (B, \Gamma'), \Gamma;\ \Delta\rangle \arrow[d, "exp"] \\
\dots \arrow[r, "HR"] & M_i \arrow[rr, dotted, "HR"] & & M_{i+1}
\end{tikzcd}$$

Proof by induction on number of $[Lam - Elim]$ steps.

Base: trivial since first element being added to Γ is a head redex by definition.

Induction step. We know that after i^{th} application of rule $[Lam - Elim]$ we got some LTS state $\langle M_i;\ \Gamma_i;\ \Delta_i\rangle$ which expansion is some term M_i. As was mentioned above $\overset{*}{\to}$ has not change the expansion, the number of steps in $\overset{*}{\to}$ is finite. Thus, we have two possible cases: first, LTS got stuck and nothing to prove, and second, we can apply the $[Lam - Elim]$ rule. In the second case we know the form of the state (following the $[Lam-Elim]$ definition), i.e. $\langle A[\underline{\lambda x}.e];\ \Gamma;\ (B, \Gamma') \bullet \Delta\rangle$ (let us denote it (i)), and by assumption the result of applying $[Lam-Elim]$ rule is a state $\langle A_{\cancel{B}}\cancel{[\lambda x.}\underline{e}];\ x \mapsto (B, \Gamma'), \Gamma;\ \Delta\rangle$ (let us denote it (ii)). Now we have to show that applying function exp to that state results in the same term as a step of head reduction from term M_i.

Let see what will happend if we apply exp function to state (i):

$$exp\ \langle A[\underline{\lambda x}.\ e];\ \Gamma;\ (B, \Gamma') \bullet \Delta\rangle \tag{5}$$

$$\overset{exp}{\to}{}^{*}\ exp\ \langle A'[\underline{\lambda x}.\ e[\Gamma]];\ \emptyset;\ (B, \Gamma') \bullet \Delta\rangle \qquad \text{by fully applying (1)} \tag{6}$$

$$\overset{exp}{\to}\ exp\ \langle A'_{B[\Gamma']}[\underline{\lambda x}.\ e[\Gamma]];\ \emptyset;\ \Delta\rangle \qquad \text{by one step of (2)} \tag{7}$$

$$\overset{exp}{\to}\ \dots \tag{8}$$

to state (ii):

$$exp\langle A_{\cancel{B}}\cancel{[\lambda x.}\ \underline{e}];\ x \mapsto (B, \Gamma'), \Gamma;\ \Delta\rangle \tag{9}$$

$$\overset{exp}{\to}{}^{*}\ exp\ \langle A_{\cancel{B}}\cancel{[\lambda x.}\ \underline{e}[\Gamma]];\ x \mapsto (B, \Gamma');\ \Delta\rangle \qquad \text{by elliminating } \Gamma \text{ following (1)} \tag{10}$$

$$\overset{exp}{\to}\ exp\ \langle A_{\cancel{B}}\cancel{[\lambda x.}\ \underline{e}[\Gamma][x/B[\Gamma']]];\ \emptyset;\ \Delta\rangle \qquad \text{by one step of (2)} \tag{11}$$

$$\overset{exp}{\to}\ \dots \tag{12}$$

It is easy to see that the head redex in (7) is $(\lambda x,\ B)$, that being applied (a step of head reduction) will results exactly in term $A_{\cancel{B}}\cancel{[\lambda x.}\ \underline{e}[\Gamma][x/B[\Gamma']]]$ that is equal to the first component of state (11). Following (II) and (I) if we will continue to apply exp function then elliminating the same Δ in both cases will lead no effect on argument B and underlined subtrees. Thus, $(\lambda x,\ B)$ is a head redex in M_i and the expansion of state (ii) is M_{i+1}.

Qed

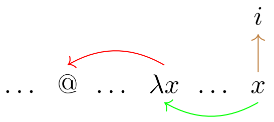
Consequences:

- if head linear reduction terminates then head reduction terminates;
- if head reduction terminates then head linear reduction terminates;
- Since expansion is not able to change a path from the root to the underlined node, the first component of the final state of LTS for head linear reduction contains a term which leftmost path is a part of the head normal form and the expansion of the final state is a node of the Boehm-tree of the input term;
  Thus, a repeated head linear reduction application to all arguments (complete head linear reduction) leeds to the normal form.

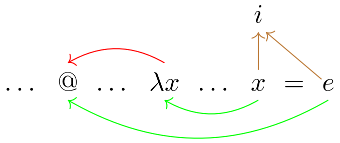
2 Labelled Transition System for UNP

This section contains LTS for UNP. This LTS has the same number of rules that the LTS for CHLR. Moreover, it is easy to see that rules with the same name corresponds to each other.

- $(BVar)$

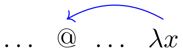


$\longrightarrow$ where e is is a right child of @

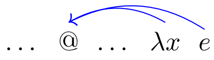


The added green pointer has to point to the binder if e is a bound variable.

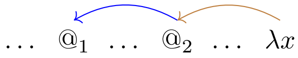
- $(Lam - Non - Elim)$



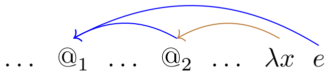
$\longrightarrow$ where e is a right child of @



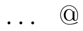
- $(Lam - Elim)$



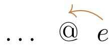
$\longrightarrow$ where e is a child of λx



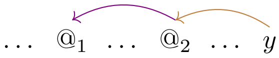
- (App)



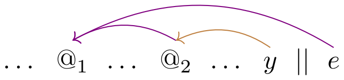
$\longrightarrow$ where e is a left child of @



- $(FVar - 0)$



$\longrightarrow$ where e is a right child of $@_2$



- $(FVar-1)$

$$\begin{tikzcd}[column sep=small]
\dots & @_1 & \dots & @_2 \arrow[ll, bend right] & \dots & y \arrow[ll, bend right]
\end{tikzcd}$$

$\longrightarrow$ where e is a right child of $@_2$

$$\begin{tikzcd}[column sep=small]
\dots & @_1 & \dots & @_2 \arrow[ll, bend right] & \dots & y \arrow[ll, bend right] & || & e \arrow[llllll, bend right]
\end{tikzcd}$$

- $(FVar-2)$

$$\begin{tikzcd}[column sep=small]
\dots & @_1 & \dots & @_2 \arrow[ll, bend right] & \dots & y \arrow[ll, bend right]
\end{tikzcd}$$

$\longrightarrow$ where e is a right child of $@_2$

$$\begin{tikzcd}[column sep=small]
\dots & @_1 & \dots & @_2 \arrow[ll, bend right] & \dots & y \arrow[ll, bend right] & || & e \arrow[llllll, bend right]
\end{tikzcd}$$