1 Labelled Transition System for repeated Head Linear Reduction

1.1 Notes

State is a tuple ⟨ λ-term with underlined node, context, list of arguments ⟩, where

- λ-term (a tree; by considering λ-term as a tree it becomes possible to cross arguments out of tree with a parent application node (denotes as $A_{\cancel{B}}$)); underlined node is a usual lambda term with one underlined position;
- Context Γ is an unordered list of pairs ($variable : term$);
- List of arguments Δ is an ordered list of λ-terms (one can also think about Δ as an ordered list of pointers to the corresponding subtree);
- Term that is received by throwing out all elements that are crossed out is a residual term in the normal form;
- Initial configuration is $\langle \lambda - term\ with\ underlined\ root,\ \emptyset,\ \$\rangle$;
- Final configuration is $\langle M\left[\underline{x}\right],\ \Gamma,\ \$\rangle$.

1.2 Rules

$$\langle A\left[e_1\underline{@}e_2\right];\ \Gamma;\ \Delta\rangle \longrightarrow \left\langle A\left[\underline{e_1}@e_2\right];\ \Gamma;\ e_2\bullet\Delta\right\rangle \qquad \text{[App]}$$

$$\langle A\left[\underline{\lambda x}.e\right];\ \Gamma;\ \$\bullet\Delta\rangle \longrightarrow \langle A\left[\lambda x.\underline{e}\right];\ \Gamma;\ \$\bullet\Delta\rangle \qquad \text{[Lam-Non-Elim]}$$

$$\langle A\left[\underline{\lambda x}.e\right];\ \Gamma;\ B\bullet\Delta\rangle \longrightarrow \left\langle A_{\cancel{B}}\left[\cancel{\lambda x}.\underline{e}\right];\ x:B,\Gamma;\ \Delta\right\rangle,\ B\neq\$ \qquad \text{[Lam-Elim]}$$

$$\langle A\left[\underline{x}\right];\ x:B,\Gamma;\ \Delta\rangle \longrightarrow \langle A\left[\underline{B}\right];\ x:B,\Gamma;\ \Delta\rangle \qquad \text{[BVar]}$$

$$\langle A\left[M\left[\underline{x}\right]@B\right];\ \Gamma;\ B\bullet\$\bullet\Delta\rangle \longrightarrow \langle A\left[M[x]@\underline{B}\right];\ \Gamma;\ \$\bullet\Delta\rangle,\ B\neq\$,\ x\notin dom\,\Gamma \qquad \text{[FVar-0]}$$

$$\langle A\left[M\left[\underline{x}\right]@B\right];\ \Gamma;\ B\bullet C\bullet\Delta\rangle \longrightarrow \langle A\left[M[x]@\underline{B}\right];\ \Gamma;\ \$\bullet C\bullet\Delta\rangle,\ C\neq\$,\ B\neq\$,\ x\notin dom\,\Gamma \qquad \text{[FVar-1]}$$

$$\langle A\left[M\left[\underline{x}\right]@B\right];\ \Gamma;\ \$\bullet B\bullet\Delta\rangle \longrightarrow \langle A\left[M\left[x\right]@\underline{B}\right];\ \Gamma;\ \Delta\rangle,\ B\neq\$,\ x\notin dom\,\Gamma \qquad \text{[FVar-2]}$$

1.3 Correctness proof

1.3.1 Residual term $M\left[\underline{x}\right]$ has no redexes

Proof We will prove by induction (with step size equal to 2) on number of steps that there is no way for redexes to appear in a path from the root to the current underlined node:

- Base Straightforward:
  1. $n == 0$, empty term has no redexes;
  2. $n == 1$, path from root to itself contains no redexes.
- Induction Hypothesis There is no redexes for all paths with a length less than n ($2 \leq n$).
- Induction Step ()
  Redex is an application node with λ-expression as a left child. We will show that there is no way to construct such subtree without crossing application node out by case analisys on a current (the underlined one) node form:
  - For $[Lam - *]$, $[BVar]$, $[FVar - *]$ rules this is obvious: they do not add λ as a left or down child of some node.
  - The only interesting case is $[App]$ rule. I.e. the rule that was applied on previous step is an $[App]$-rule. Here we have two possibilities:
    * $e_1 \neq \lambda\ x$; in this case no redexes appear.
    * $e_1 == \lambda\ x$, for some x. In this case a redex appears but by $[App]$-rule we have: $\langle A\left[e_1\underline{@}e_2\right];\ \Gamma;\ \Delta\rangle \longrightarrow \left\langle A\left[\underline{e_1}@e_2\right];\ \Gamma;\ e_2\bullet\Delta\right\rangle$.
      Note, that there exists a rule that can be applied next — $[Lam - Elim]$, and no other rule can be applied. This rule will cross application node with its right subterm out.

  Thus, no new redexes can appear.

1.3.2 Only $\langle M\,[\underline{x}]\,,\ \Gamma,\ \$\rangle$ can be a final state

Proof

- It is easy to see that the state $\langle M\,[\underline{x}]\,,\ \Gamma,\ \$\rangle$ is a final state: there is no rules that can be applied.
- Let's prove that there is no other final states (with respect to the form of the state). Consider another state which either :
  - If current term is not equal to $M\,[\underline{x}]$ then either $[App]$ or $[Lam - *]$ can be applied. (together they have no restrictions on Γ and Δ)
  - If $\Delta \neq \$$. Note that Δ can not be empty because there is no rule that can move \$ that is on the bottom of it. Thus, $\Delta = A \ldots \$$ where either $A == \$$ or $A \neq \$$. In this case one of $[FVar - *]$ rules can be applied.