

# 1 Labelled Transition System for Complete Head Linear Reduction

## 1.1 Notes

State is a tuple  $\langle \lambda\text{-term with underlined node; context; list of pairs (argument, context) \rangle$ , where

- $\lambda$ -term (a tree; by considering  $\lambda$ -term as a tree it becomes possible to cross arguments out of tree with a parent application node (denotes as  $A_{\underline{x}}$ ) is a usual lambda term with one underlined node;
- Context  $\Gamma$  is an ordered (first-in) set of elements of the following form (*variable*  $\mapsto$  (*term*, *Context*));
- $\Delta$  is an ordered list of pairs ( $\lambda$ -term, Context) (one can also think about  $\Delta$  as an ordered list of pointers to the corresponding subtree stored together with a corresponding context);
- Initial configuration is  $\langle \lambda\text{-term with underlined root}, \emptyset, \$ \rangle$ ;
- Final configuration is  $\langle M[\underline{x}]$  (term in normal form),  $\Gamma, \$ \rangle$ ;
- Note, that Complete Head Linear Reduction (CHLR) can be seen as usual Head Linear Reduction (HLR) that is repeatedly applied to the all arguments;
- From the below LTS for CHLR.

## 1.2 Labelled transition system for CHLR

$$\langle A[e_1 @ e_2]; \Gamma; \Delta \rangle \longrightarrow \langle A[\underline{e_1} @ e_2]; \Gamma; (e_2, \Gamma) \bullet \Delta \rangle \quad [\text{App}]$$

$$\langle A[\underline{\lambda x}.e]; \Gamma; \$ \bullet \Delta \rangle \longrightarrow \langle A[\lambda x.\underline{e}]; \Gamma; \$ \bullet \Delta \rangle \quad [\text{Lam-Non-Elim}]$$

$$\langle A[\underline{\lambda x}.e]; \Gamma; (B, \Gamma') \bullet \Delta \rangle \longrightarrow \langle A_{\underline{x}}[\underline{\lambda x}.e]; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle \quad [\text{Lam-Elim}]$$

$$\langle A[\underline{x}]; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle \longrightarrow \langle A[\underline{B}]; \Gamma'; \Delta \rangle \quad [\text{BVar}]$$

$$\langle A[M[\underline{x}] @ B]; \Gamma; (B, \Gamma') \bullet \$ \bullet \Delta \rangle \longrightarrow \langle A[M[x] @ \underline{B}]; \Gamma'; \$ \bullet \Delta \rangle, x \notin \text{dom}(\Gamma) \quad [\text{FVar-0}]$$

$$\langle A[M[\underline{x}] @ B]; \Gamma; (B, \Gamma') \bullet C \bullet \Delta \rangle \longrightarrow \langle A[M[x] @ \underline{B}]; \Gamma'; \$ \bullet C \bullet \Delta \rangle, C \neq \$, x \notin \text{dom}(\Gamma) \quad [\text{FVar-1}]$$

$$\langle A[M[\underline{x}] @ B]; \Gamma; \$ \bullet (B, \Gamma') \bullet \Delta \rangle \longrightarrow \langle A[M[x] @ \underline{B}]; \Gamma'; \$ \bullet \Delta \rangle, x \notin \text{dom}(\Gamma) \quad [\text{FVar-2}]$$

## 1.3 Correctness proof

Note that that labelled transition system for HLR can be obtained by removing [FVAR-\*] rules from the LTS for CHLR. In other words, first four rules describe HLR while [FVAR-\*] rules describes a repeated application of HLR to the arguments.

LTS for HLR .

Remark: it is obvious that in [Lam-Non-Elim] rule for HLR we can safely replace  $\$ \bullet \Delta$  by empty list since there is no rule that pop  $\$$  sign from  $\Delta$ .

$$\langle A[e_1 @ e_2]; \Gamma; \Delta \rangle \longrightarrow \langle A[\underline{e_1} @ e_2]; \Gamma; (e_2, \Gamma) \bullet \Delta \rangle \quad [\text{App}]$$

$$\langle A[\underline{\lambda x}.e]; \Gamma; [] \rangle \longrightarrow \langle A[\lambda x.\underline{e}]; \Gamma; [] \rangle \quad [\text{Lam-Non-Elim}]$$

$$\langle A[\underline{\lambda x}.e]; \Gamma; (B, \Gamma') \bullet \Delta \rangle \longrightarrow \langle A_{\underline{x}}[\underline{\lambda x}.e]; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle \quad [\text{Lam-Elim}]$$

$$\langle A[\underline{x}]; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle \longrightarrow \langle A[\underline{B}]; \Gamma'; \Delta \rangle \quad [\text{BVar}]$$

Expansion function. To prove correctness of HLR we introduce an axiliary expansion function  $exp$ . This function being applied to some state of LTS for HLR results in some  $\lambda$ -term which we will call an expansion.

$$exp \langle M[\underline{A}]; \Gamma, x \mapsto (B, \Gamma'); \Delta \rangle = exp \langle M[\underline{A}[x/B[\Gamma']]]; \Gamma; \Delta \rangle \quad (1)$$

$$exp \langle M_B[\underline{A}]; \emptyset; (B, \Gamma') \bullet \Delta \rangle = exp \langle M_{B[\Gamma']}[\underline{A}]; \emptyset; \Delta \rangle \quad (2)$$

$$exp \langle M; \emptyset; [] \rangle = M \quad (3)$$

$$\text{where } B[\Gamma'] = exp \langle \underline{B}; \Gamma'; [] \rangle \quad (4)$$

Note that (1) substitutes term  $B[\Gamma']$  instead of all occurrence of variable  $x$  in the subtree of term  $M$  that has an underlined node as a root (i.e. subtree  $A$ ). Each recursive call of (2) substitutes all variables with respect to the corresponding context exactly in one argument term.

Now consider LTS for HLR. Note that:

- Only  $[Lam - Elim]$  rule can change an expansion;
- The number of transitions without applying  $[Lam - Elim]$  rule is limited since the definition of context, input term has a final size, and only  $[Lam - Elim]$  rule expands the context; Hereafter we will denote a sequence of transitions without applying  $[Lam - Elim]$  rule by  $\xrightarrow{*}$ ;
- The expansion function  $exp$  cannot change a path from the root of term to the underlined node since:
  - recursive call (1) can only change the subtree of the current term with underlined node as a root; (I)
  - recursive call (2) can only change arguments of the current term that are above the underlined node; Moreover, each application of (2) change exactly one argument (that corresponds to the first element of the pair in  $\Delta$ ) leaving all other arguments and the subtree of the current term with underlined node as a root intact. (II)

Theorem . HLR and HR coincide as follows:

$$\begin{array}{ccccc} \langle \dots \rangle & \xrightarrow{[Lam - Elim]} & \langle M_i; \Gamma_i; \Delta_i \rangle & \xrightarrow{*} & \langle A[\underline{\lambda x}.e]; \Gamma; (B, \Gamma') \bullet \Delta \rangle & \xrightarrow{[Lam - Elim]} & \langle \cancel{A_B[\underline{\lambda x}.e]}; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle \\ \downarrow exp & & \downarrow exp & \swarrow exp & & & \downarrow exp \\ \dots & \xrightarrow{HR} & M_i & \xrightarrow{HR} & M_{i+1} & & \end{array}$$

Proof by induction on number of  $[Lam - Elim]$  steps.

Base: trivial since first element being added to  $\Gamma$  is a head redex by definition.

Induction step. We know that after  $i^{th}$  application of rule  $[Lam - Elim]$  we got some LTS state  $\langle M_i; \Gamma_i; \Delta_i \rangle$  which expansion is some term  $M_i$ . As was mentioned above  $\xrightarrow{*}$  has not change the expansion and the number of steps in  $\xrightarrow{*}$  is finite. Thus, we have two possible cases:

first, LTS got stuck and nothing to prove,

and second, we can apply the  $[Lam - Elim]$  rule. In the second case we know the form of the state (following the  $[Lam - Elim]$  definition), i.e.  $\langle A[\underline{\lambda x}.e]; \Gamma; (B, \Gamma') \bullet \Delta \rangle$  (let us denote it (i)), and by assumption the result of applying  $[Lam - Elim]$  rule is a state  $\langle \cancel{A_B[\underline{\lambda x}.e]}; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle$  (let us denote it (ii)). Now we have to show that applying function  $exp$  to that state results in the same term as a step of head reduction from term  $M_i$ .

Let see what will happend if we apply  $exp$  function to state (i):

$$exp \langle A[\underline{\lambda x}.e]; \Gamma; (B, \Gamma') \bullet \Delta \rangle \quad (5)$$

$$\xrightarrow{exp}^* exp \langle A'[\underline{\lambda x}.e[\Gamma]]; \emptyset; (B, \Gamma') \bullet \Delta \rangle \quad \text{by fully applying (1)} \quad (6)$$

$$\xrightarrow{exp} exp \langle A'_{B[\Gamma']}[\underline{\lambda x}.e[\Gamma]]; \emptyset; \Delta \rangle \quad \text{by one step of (2)} \quad (7)$$

$$\xrightarrow{exp} \dots \quad (8)$$

to state (ii):

$$exp \langle \cancel{A_B[\underline{\lambda x}.e]}; x \mapsto (B, \Gamma'), \Gamma; \Delta \rangle \quad (9)$$

$$\xrightarrow{exp}^* exp \langle \cancel{A_B[\underline{\lambda x}.e[\Gamma]]}; x \mapsto (B, \Gamma'), \Delta \rangle \quad \text{by eliminating } \Gamma \text{ following (1)} \quad (10)$$

$$\xrightarrow{exp} exp \langle \cancel{A_B[\underline{\lambda x}.e[\Gamma][x/B[\Gamma']]]}; \emptyset; \Delta \rangle \quad \text{by one step of (2)} \quad (11)$$

$$\xrightarrow{exp} \dots \quad (12)$$

It is easy to see that term in state in (7),  $A'_{B[\Gamma']}[\underline{\lambda x}.e[\Gamma]]$ , has the head redex  $(\lambda x, B)$  (since context is already empty). Thus, a step of head reduction will results exactly in term  $\cancel{A_B[\underline{\lambda x}.e[\Gamma][x/B[\Gamma']]]}$  that is equal to the first component of state (11). Following (I) and (II), if we will continue to apply  $exp$  function then eliminating the same  $\Delta$  in both cases will lead no effect on argument  $B$  and underlined subtrees. Thus,  $(\lambda x, B)$  is a head redex in  $M_i$  and the expansion of

Consequences:

- if head linear reduction terminates then head reduction terminates;
- if head reduction terminates then head linear reduction terminates;
- Since expansion is not able to change a path from the root to the underlined node, the first component of the final state of LTS for head linear reduction contains a term which leftmost path is a part of the head normal form and the expansion of the final state is a node of the Boehm-tree of the input term;  
Thus, a repeated application of head linear reduction to all arguments (complete head linear reduction) leads to the normal form.

## 2 Labelled Transition System for UNP

This section contains LTS for UNP. This LTS has the same number of rules that the LTS for CHLR. Moreover, it is easy to see that rules with the same name corresponds to each other.

- (*BVar*)  

$$\dots @ \dots \lambda x \dots x \xrightarrow{\text{where } e \text{ is a right child of } @} \dots @ \dots \lambda x \dots x = e$$

The added green pointer has to point to the binder if  $e$  is a bound variable.

- (*Lam - Non - Elim*)  

$$\dots @ \dots \lambda x \xrightarrow{\text{where } e \text{ is a right child of } @} \dots @ \dots \lambda x e$$
- (*Lam - Elim*)  

$$\dots @_1 \dots @_2 \dots \lambda x \xrightarrow{\text{where } e \text{ is a child of } \lambda x} \dots @_1 \dots @_2 \dots \lambda x e$$
- (*App*)  

$$\dots @ \xrightarrow{\text{where } e \text{ is a left child of } @} \dots @ e$$
- (*FVar - 0*)  

$$\dots @_1 \dots @_2 \dots y \xrightarrow{\text{where } e \text{ is a right child of } @_2} \dots @_1 \dots @_2 \dots y \parallel e$$
- (*FVar - 1*)  

$$\dots @_1 \dots @_2 \dots y \xrightarrow{\text{where } e \text{ is a right child of } @_2} \dots @_1 \dots @_2 \dots y \parallel e$$
- (*FVar - 2*)  

$$\dots @_1 \dots @_2 \dots y \xrightarrow{\text{where } e \text{ is a right child of } @_2} \dots @_1 \dots @_2 \dots y \parallel e$$