Random Graphs

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4-Cliques:

1. Expected number of 4-cliques in $G(n,p)$ as a function of n and p? If X denotes the number of the 4-cliques:

$$g(p) = \mathbb{E}[X] = \binom{n}{4} p^6$$

2. If X denotes the number of the 4-cliques. Give a function $f(n)$ such that $\mathbb{E}[X] = \Theta(1)$, for $p = f(n)$. If $g(p) = \Theta(1) \Rightarrow p = f(n) = \Theta\left(\binom{n}{4}^{-\frac{1}{6}}\right) = \Theta\left(n^{-\frac{2}{3}}\right)$. Therefore we observe that

   - $p \gg n^{-\frac{2}{3}}$ then $\mathbb{E}X \to +\infty$.
   - $p \ll n^{-\frac{2}{3}}$ then $\mathbb{E}X \to 0$.

   Next we will show that there is a sharp transition at this value of p. In other words, $p = n^{-\frac{2}{3}}$ is a threshold for counting the number of cliques.

3. Using the first-moment, show that for $p = o(f(n))$, $\mathbb{P}(X > 0) = o(1)$.

$$\mathbb{P}(X > 0) = \mathbb{P}(X \geq 1) \leq \mathbb{E}[X]$$

   Since $p \ll n^{-\frac{2}{3}}$ then $\mathbb{E}X \to 0$, which gives the desired result.

4. Using the second-moment, show that for $p = \omega(f(n))$, $\mathbb{P}(X = 0) = o(1)$. If $X = 0$, then definitely $|X - \mathbb{E}X| \geq \mathbb{E}X$. Therefore:

$$\mathbb{P}(X = 0) = \mathbb{P}(|X - \mathbb{E}X| \geq \mathbb{E}X)$$

   Using Chebychev's inequality,

$$\mathbb{P}(X = 0) = \mathbb{P}(|X - \mathbb{E}X| \geq \mathbb{E}X) \leq \frac{\mathrm{Var}(X)}{(\mathbb{E}X)^2}$$

   Now we bound $\mathrm{Var}(X)$ in order to get a bound on $\frac{\mathrm{Var}(X)}{(\mathbb{E}X)^2}$. Define the decomposition $X = \sum_C X_C$, where C is clique, and X_C is an indicator variable with value one when clique C

exists in the graph.

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}\left[X - \mathbb{E}X\right]^2 \\
&= \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 \\
&= \sum_C \mathbb{E}X_C^2 - \sum_C (\mathbb{E}X_C)^2 + \sum_{C \neq D} \mathbb{E}X_C X_D - \sum_{C \neq D} \mathbb{E}X_C \mathbb{E}X_D \\
&= \sum_C \operatorname{Var}(X_C) + \sum_{C \neq D} \operatorname{Cov}(X_C, X_D)
\end{aligned}$$

First, $\sum_C \operatorname{Var}(X_C) = \sum_C \mathbb{E}X_C^2 - \sum_C (\mathbb{E}X_C)^2 \leq \sum_C \mathbb{E}X_C^2 = \mathbb{E}X = O(n^4p^6)$. For the covariances, the values are all zeros if there is no overlap between C and D. For each value of C and D, $\operatorname{Cov}(X_C, X_D) \leq \mathbb{E}X_C X_D$. If there is overlap there are two possible cases:

- $|C \cap D| = 1$, then still no edge overlap and covariance is zero.
- $|C \cap D| = 2$, then there is one edge overlap.

  $$\operatorname{Cov}(X_C, X_D) \leq \mathbb{E}X_C X_D = \mathbb{P}\left(C \text{ and } D \text{ both cliques with one edge overlap}\right)$$

  Then coveriance $\sum_{C \neq D, |C \cap D| = 2} \operatorname{Cov}(X_C, X_D)$ is upperbounded by $O(n^6p^{11})$.
- $|C \cap D| = 3$, then there is two edges overlap.

  $$\operatorname{Cov}(X_C, X_D) \leq \mathbb{E}X_C X_D = \mathbb{P}\left(C \text{ and } D \text{ both cliques with two edge overlap}\right)$$

  The coveriance $\sum_{C \neq D, |C \cap D| = 3} \operatorname{Cov}(X_C, X_D)$ is upperbounded by $O(n^5p^9)$.

Then

$$\operatorname{Var}(X) = O(n^4p^6) + O(n^6p^{11}) + O(n^5p^9) \in O(n^6p^6)$$

If $p \ll n^{-2/3}$, then

$$\operatorname{Var}(X) \ll O(n^4/n^4) + O(n^6/n^{22/3}) + O(n^5/n^6) \in O(1)$$

Then

$$\frac{\operatorname{Var}(X)}{(\mathbb{E}X)^2} \leq O\left(\frac{1}{n^8p^{12}}\right)$$

Therefore for $p \ll n^{-2/3}$, $\frac{\operatorname{Var}(X)}{(\mathbb{E}X)^2}$ goes to zero.