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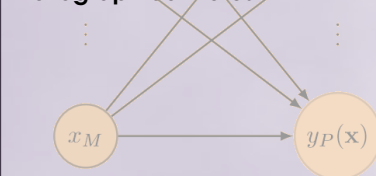
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1 — Expectation Propagation

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1.1 Introduction

Expectation Propagation(EP) is one of approaches for approximate inference which first formulated the way we see today at Minka (2001b) though the idea has roots in many previous works in various areas. It can be considered as a variant of message-passing where each of the individual messages are approximated while being transferred. To introduce EP, it is easier to first start with a couple of approximations (projections) for any arbitrary distribution. Here we start with Assumed Density Filtering(ADF).

1.2 Assumed Density Filtering

ADF is introduced independently in several areas at different times under different names like “Moment Matching”, “Weak Marginalization”, etc Lauritzen (1992); Koller et al. (1999); Maybeck (1979). The idea used is so much similar to the update equations in Kalman Filtering.

Assume that using the \mathbf{x} as observations we want to make inference about the latent variables \mathbf{y} . Now the goal is to find a an exact posterior $p(\mathbf{y}|\mathbf{x})$ and only keep the approximation to it $q(\mathbf{y})$, using a tractable form, say exponential form, and possibly use it for future calculation. This further approximation of the posterior is can be seen as *projection* of one distribution, over another family of distributions. There are many ways to project one distribution over another, but for here, let’s say we want to project any arbitrary distribution over exponential family using the KL-divergence², or,

$$\hat{q} = \text{proj}(p(\mathbf{y}|\mathbf{x}) \rightarrow q(\mathbf{y})) \triangleq \arg \min_q \text{KL}(p(\mathbf{y}|\mathbf{x}) || q(\mathbf{y}))$$

¹This is part of my notes; to find the complete list of notes visit <http://web.engr.illinois.edu/~khashab2/learn.html>. This work is licensed under a Creative Commons Attribution-NonCommercial 3.0 License. This document is updated on January 15, 2014.

²If you don’t know about the divergence measures, see this: <http://web.engr.illinois.edu/~khashab2/learn/info.pdf>

To do so, we choose the following exponential parametric form,

$$q_{\theta}(\mathbf{y}) = \frac{1}{Z(\theta)} \exp\left(\theta^{\top} \Phi(\mathbf{y})\right), \quad Z(\theta) = \int \exp\left(\theta^{\top} \Phi(\mathbf{y})\right) d\mathbf{y}$$

$\Phi(\mathbf{y})$ is natural statistic of \mathbf{y} . The most famous case is a Gaussian distribution with mean and covariance matrix. To reduce the difference between the posterior approximation, $q(\cdot)$ and the real posterior value, again we use the *KL-divergence*.

$$\begin{aligned} f(\theta) &= \text{KL}(p||q) = \mathbb{E}_p \log \frac{p}{q_{\theta}} = \mathbb{E}_p \log(p) + \mathbb{E}_p \log(Z(\theta)) - \mathbb{E}_p \left[\theta^{\top} \Phi(\mathbf{y}) \right] \\ \nabla_{\theta} f(\theta) &= 0 \Rightarrow \nabla_{\theta} f(\theta) = \nabla_{\theta} \log Z(\theta) - \mathbb{E}_p [\Phi(\mathbf{y})] = 0 \end{aligned} \quad (1.1)$$

We also have:

$$\nabla_{\theta} \log Z(\theta) = \frac{\nabla_{\theta} Z(\theta)}{Z(\theta)} = \frac{\nabla_{\theta} \int \exp(\theta^{\top} \Phi(\mathbf{y}))}{Z(\theta)} = \frac{\int \nabla_{\theta} \exp(\theta^{\top} \Phi(\mathbf{y}))}{Z(\theta)} = \mathbb{E}_q [\Phi(\mathbf{y})] \quad (1.2)$$

Combining the results from equations (1.2, 1.1) we have,

$$\nabla_{\theta} f(\theta) = 0 \Rightarrow \mathbb{E}_q [\Phi(\mathbf{y})] = \mathbb{E}_p [\Phi(\mathbf{y})] \quad (1.3)$$

Calculating the Hessian for $f(\theta)$ we can show that, the above solution is a *minimum* to $f(\theta)$.

$$[\nabla \nabla_{\theta} f(\theta)]_{ij} = \frac{\partial^2 \log Z(\theta)}{\partial \theta_j \partial \theta_i} = \frac{\partial}{\partial \theta_j} \frac{\int \Phi_i(\mathbf{y}) \exp(\theta^{\top} \Phi(\mathbf{y})) d\mathbf{y}}{Z(\theta)} \quad (1.4a)$$

$$= \mathbb{E}_q [\Phi_i(\mathbf{y}), \Phi_j(\mathbf{y})] - \mathbb{E}_q [\Phi_i(\mathbf{y})] \cdot \mathbb{E}_q [\Phi_j(\mathbf{y})] \geq 0 \quad (1.4b)$$

Using the above equation, we can conclude that, to get the best estimation for an arbitrary distribution (With *KL-divergence* as the difference between two distributions.) using an exponential distribution, it is enough to match their moments. Specifically if we assume having a Gaussian distribution for the approximating distributions, $q_{\theta}(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\mu, \Sigma)$ Results from equations (1.4, 1.3) are:

$$\begin{cases} \mu^* = \mathbb{E}_p [\mathbf{y}] \\ \Sigma^* = \mathbb{E}_p [\mathbf{y}\mathbf{y}^{\top}] - \mathbb{E}_p [\mathbf{y}] \mathbb{E}_p [\mathbf{y}]^{\top} \end{cases}$$

1.2.1 ADF for a factorized distribution

Now assume we can factorize the given distribution, $p(\mathbf{y}|\mathbf{x}) = p(\mathbf{x})p(\mathbf{x}, \mathbf{y}) = \prod_i t_i(\mathbf{y})$. The goal is to approximate each of the factors using the approximating distribution. It is relatively better to have less components, to keep further calculations minimum, while it is better to have a simple form for each of the factors, so to have easier approximation procedure for each of them. Thus there is a trade-off between the number of factors and simplicity of each factor. The goal is to get the least error by using the minimum number of approximating factors. Assume that the approximating distribution, $q_{\theta}(\mathbf{y})$ has a known exponential form, e.g. Gaussian. At each step, we will consider the factor t_i from our target distribution, and changed our approximation based on the added factor

t_i to get a better approximation. The distribution $\hat{p}(\mathbf{y})$ is an auxiliary distribution which we use during the algorithm.

$$\hat{p}(\mathbf{y}) = \frac{1}{\tilde{Z}(\theta)} t_i(\mathbf{y}) q_{\theta}^{old}(\mathbf{y}), \quad \tilde{Z}(\theta) = \int t_i(\mathbf{y}) q_{\theta}^{old}(\mathbf{y}) d\mathbf{y}. \quad (1.5)$$

$q_{\theta}^{old}(\mathbf{y})$ is the approximate-posterior distribution at the previous step. Based on the previous definition it is clear that, $\hat{p}(\mathbf{y})$ is the approximate posterior up to factor $t_i(\mathbf{y})$. Similar to what explained, by minimizing $\text{KL}(\hat{p}(\mathbf{y}) || q_{\theta}^{new}(\mathbf{y}))$ and assuming that $q_{\theta}^{new}(\mathbf{y})$ has an exponential form, we use the moments of the distributions to get new approximation, $q_{\theta}^{new}(\mathbf{y})$. To simplify the notation, we drop *old* from $q_{\theta}^{old}(\mathbf{y})$. Using the equation (1.5),

$$\begin{aligned} \nabla_{\theta} q_{\theta}(\mathbf{y}) &= \nabla_{\theta} \frac{1}{Z(\theta)} \exp(\theta^{\top} \Phi(\mathbf{y})) = \nabla_{\theta} \left[\frac{1}{Z(\theta)} \right] \exp(\theta^{\top} \Phi(\mathbf{y})) + \frac{1}{Z(\theta)} \cdot \nabla_{\theta} \exp(\theta^{\top} \Phi(\mathbf{y})) \\ &\Rightarrow \nabla_{\theta} q_{\theta}(\mathbf{y}) = -\frac{\nabla_{\theta} Z(\theta)}{Z(\theta)} q_{\theta}(\mathbf{y}) + \Phi(\mathbf{y}) q_{\theta}(\mathbf{y}) = -\mathbb{E}_q[\Phi(\mathbf{y})] + \Phi(\mathbf{y}) q_{\theta}(\mathbf{y}). \end{aligned}$$

Multiplying in $\frac{1}{\tilde{Z}(\theta)} \cdot t_i(\mathbf{y})$ and integrating with respect to \mathbf{y} we have,

$$\begin{aligned} \frac{t_i(\mathbf{y})}{\tilde{Z}(\theta)} \nabla_{\theta} q_{\theta}(\mathbf{y}) &= -\mathbb{E}_q[\Phi(\mathbf{y})] \cdot \frac{1}{\tilde{Z}(\theta)} t_i(\mathbf{y}) q_{\theta}(\mathbf{y}) + \Phi(\mathbf{y}) \cdot \frac{1}{\tilde{Z}(\theta)} t_i(\mathbf{y}) q_{\theta}(\mathbf{y}). \\ &\Rightarrow \frac{1}{\tilde{Z}(\theta)} \nabla_{\theta} \tilde{Z}(\theta) = -\mathbb{E}_q[\Phi(\mathbf{y})] + \mathbb{E}_{\hat{p}}[\Phi(\mathbf{y})]. \\ &\Rightarrow \mathbb{E}_{\hat{p}}[\Phi(\mathbf{y})] = \nabla_{\theta} \log(\tilde{Z}(\theta)) + \mathbb{E}_q[\Phi(\mathbf{y})]. \end{aligned}$$

For calculating $q_{\theta}^{new}(\mathbf{y})$ from $\text{KL}(q_{\theta}^{new}(\mathbf{y}) || \hat{p}(\mathbf{y}))$ we can use equations (1.4, 1.3) to match the moments for $q_{\theta}^{new}(\mathbf{y})$ and $\hat{p}(\mathbf{y})$:

$$\mathbb{E}_{q^{new}}[\Phi(\mathbf{y})] = \mathbb{E}_{\hat{p}}[\Phi(\mathbf{y})].$$

$$\Rightarrow \mathbb{E}_{q^{new}}[\Phi(\mathbf{y})] = \nabla_{\theta} \log(\tilde{Z}(\theta)) + \mathbb{E}_q[\Phi(\mathbf{y})].$$

If we assume an exponential distribution we can find $\nabla_{\theta} \log(\tilde{Z}(\theta))$ and $\mathbb{E}_q[\Phi(\mathbf{y})]$ in closed form. Assume a Gaussian distribution, $q_{\theta}(\mathbf{y}) = q(\mathbf{y}; \mu, \Sigma) = \mathcal{N}(\mathbf{y}; \mu, \Sigma)$ and $\tilde{Z}(\theta) = \tilde{Z}(\mu, \Sigma) = \int t(\mathbf{y}) \cdot q(\mathbf{y}; \mu, \Sigma) d\mathbf{y}$. We now have:

$$\begin{aligned} \nabla_{\mu} q(\mathbf{y}; \mu, \Sigma) &= \Sigma^{-1} (\mathbf{y} - \mu) q(\mathbf{y}; \mu, \Sigma) \\ &\Rightarrow \mathbf{y} \cdot q(\mathbf{y}; \mu, \Sigma) = \mu \cdot q(\mathbf{y}; \mu, \Sigma) + \Sigma \cdot \nabla_{\mu} q(\mathbf{y}; \mu, \Sigma) \end{aligned}$$

Multiplying both sides in $\frac{1}{\tilde{Z}} t_i(\mathbf{y})$ and integrating with respect to \mathbf{y} ,

$$\mu^* = \mu + \frac{1}{\tilde{Z}} \Sigma \cdot \nabla_{\mu} \int t(\mathbf{y}) q(\mathbf{y}) d\mathbf{y} \quad (1.6a)$$

$$= \mu + \frac{1}{\tilde{Z}} \Sigma \cdot \nabla_{\mu} \tilde{Z}(\mu, \Sigma) \quad (1.6b)$$

$$= \mu + \Sigma \cdot \nabla_{\mu} \log(\tilde{Z}(\mu, \Sigma)) \quad (1.6c)$$

$$= \mu + \Sigma \cdot \mathbf{g}, \quad \mathbf{g} \triangleq \nabla_{\mu} \log(\tilde{Z}(\mu, \Sigma)) \quad (1.6d)$$

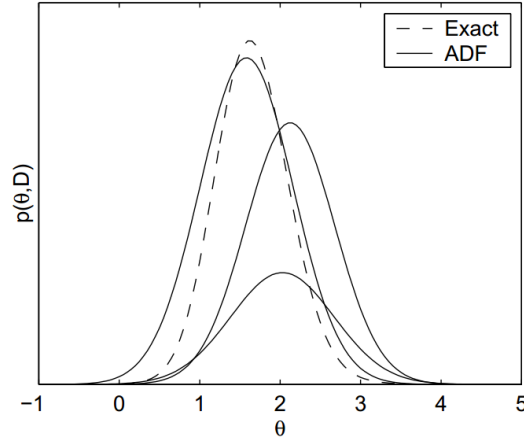


Figure 1.1: The effect of changing the ordering in factors of the target distribution, while approximating via ADF; Minka (2001c).

Similarly we for covariance (second moment) we have,

$$\begin{aligned} \Rightarrow \mathbf{y}\mathbf{y}^\top q(\mathbf{y}) &= 2\Sigma \cdot [\nabla_\Sigma q(\mathbf{y})] \\ \Rightarrow \langle \mathbf{y}\mathbf{y}^\top \rangle &= \Sigma + 2\Sigma \mathbf{G} \Sigma \langle \mathbf{y} \rangle_{\hat{p}(\mathbf{y})} \boldsymbol{\mu}^\top + \boldsymbol{\mu} \langle \mathbf{y} \rangle_{\hat{p}(\mathbf{y})}^\top - \boldsymbol{\mu} \boldsymbol{\mu}^\top, \quad \mathbf{G} = \nabla_\Sigma \log(\tilde{Z}(\boldsymbol{\mu}, \Sigma)) \\ \Rightarrow \Sigma^* &= \langle \mathbf{y}\mathbf{y}^\top \rangle_{\hat{p}(\mathbf{y})} - \langle \mathbf{y} \rangle_{\hat{p}(\mathbf{y})} \langle \mathbf{y} \rangle_{\hat{p}(\mathbf{y})}^\top = \Sigma - \Sigma (\mathbf{g}\mathbf{g}^\top - 2\mathbf{G}) \Sigma. \end{aligned}$$

Based the above formula, it should be clear that, the order in which we use the factors $\{t_i\}_i$ to create the approximation, will change the final answer, since to approximate the posterior we go through the factors linearly only once. While Expectation Propagation aims to solve this problem by finding more consistent approximation by passing the factors for several time, so that the approximation is not dependent on the order which we see the factors. One sample difference in approximation of posterior of a toy example is shown in Figure (1.1).

1.3 Expectation Propagation (EP)

The EP approximation first introduced in Minka (2001c). The method is so much similar to Automatic Density Filtering (ADF). In fact, EP is using update rules of in an intelligent iterative way until convergence. As mentioned the main problem in ADF is that, different order in approximating the factors will result in different approximations. While EP loops over the factors until convergence. Another difference is that, instead of applying the *KL-divergence* to $q_\theta(\mathbf{y})$ as in ADF, in EP we apply it to each factors $t_i(\mathbf{y})$ and then we update the approximation $p(\mathbf{y}|\mathbf{x})$. This way, by sweep over the factors for several time, the ordering of selecting the factors $\{t_i\}_i$ doesn't make any difference. The algorithm is shown in Algorithm (1).

We want to approximate $p(\mathbf{y}|\mathbf{x}) = \frac{1}{Z} \prod_i^n t_i(\mathbf{y})$ using $q_\theta(\mathbf{y}) = \frac{\prod_i^n \tilde{t}_i}{\int \prod_i^n \tilde{t}_i}$. We choose the approximating family, \tilde{t}_i to be exponential, to make it easier to work with comparing to t_i itself. Thus we want to approximate any factor t_i with \tilde{t}_i , such that the global difference between the exact distribution and the approximating family, $\text{KL}(p(\mathbf{y}|\mathbf{x}) || q(\mathbf{y})_\theta)$ is minimized. It can be shown that the global minimization could be achieved via a set of

local minimizations Minka et al. (2005). Note that in this formulation, \tilde{t}_i don't need to be normalized or a proper distribution. For example they can be Gaussians with negative variance (an improper distribution). But given the set of $\{\tilde{t}_i\}_{i=1}^n$ we can normalized their multiplication and get a proper distribution.

Algorithm 1: Expectation Propagation

Initialize $\{\tilde{t}_i\}$

$$q_\theta(\mathbf{y}) = \frac{\prod_i \tilde{t}_i}{\int \prod_i \tilde{t}_i}$$

repeat

Message elimination: Choose a \tilde{t}_i to do approximation with.

 Remove the factor \tilde{t}_i from approximation, $q_\theta^{-i} = \frac{q_\theta}{\tilde{t}_i}$

Belief projection: Project the approximate posterior, with \tilde{t}_i replaced with t_i , on the approximating family,

$$q_\theta^{\text{new}}(\mathbf{y}) = \text{proj}(\hat{p}_i(\mathbf{y}) \rightarrow q_\theta(\mathbf{y})),$$

 where,

$$\hat{p}_i(\mathbf{y}) = \frac{1}{Z} q_\theta^{-i}(\mathbf{y}) t_i(\mathbf{y}), \quad Z = \int q_\theta^{-i}(\mathbf{y}) \times t_i(\mathbf{y}) d\mathbf{y}$$

Message update: Compute the new approximating factor,

$$\tilde{t}_i = Z \frac{q_\theta^{\text{new}}(\mathbf{y})}{q_\theta^{-i}(\mathbf{y})}$$

until all \tilde{t}_i converge;

In the case when the approximating family is from the exponential family, $q_\theta(\mathbf{y}) \propto \exp(\boldsymbol{\eta}^\top \boldsymbol{\phi}(\mathbf{y}))$, the projection in algorithm 1, $q_\theta^{\text{new}}(\mathbf{y}) = \text{proj}(\hat{p}_i(\mathbf{y}) \rightarrow q_\theta(\mathbf{y}))$ is equivalent to the following moment matching,

$$\mathbb{E}_{q_\theta(\mathbf{y})} [\boldsymbol{\phi}(\mathbf{y})] = \mathbb{E}_{\hat{p}_i(\mathbf{y})} [\boldsymbol{\phi}(\mathbf{y})] \quad (1.7)$$

Note that this moment matching is distributed over each factor.

To find the marginal likelihood, it is enough to find the following expression:

$$p(\mathbf{x}) \approx \int p(\mathbf{x}|\mathbf{y}) p(\mathbf{y}) d\mathbf{y} = \int \prod_i \tilde{t}_i(\mathbf{y}) d\mathbf{y}.$$

1.4 Problems with the standard EP

EP, though giving nice approximations in many applications is sensitive to outliers, and the cases when the approximating family is not close to the target distribution. This issue is addressed in Qi and Guo (2013) and a relaxed form of EP is introduced which has better convergence properties. Another problem with EP is that, there is no known convergence proof for it.

1.5 EP for inference in graphical models

In a graphical model we can interpret each \tilde{t}_i as an approximate message to the original message t_i . There is a nice discussion of using EP for general factor graphs in Sutton (2005). Minka et al. (2005) also provides a unifying view of different message passing algorithms on graphs. Here we give some examples on how to use EP for inference in graphical models.

We can represent the distribution in a factor-graph as following³,

$$p(\mathbf{x}) = \frac{1}{\mathcal{Z}} \prod_i f_i(\mathbf{x})$$

where \mathcal{Z} is the partition function. To compute \mathcal{Z} , since we need to sum over all of the variables \mathbf{x} , this could be quite computationally expensive. Given this product of factors, in which it is hard to marginalize over, we can create an approximate product of factors using EP. In other words, we create another graphical model, in which the marginals are as close as possible to the marginals in the original graph. But in this graph, the it is easier to find the marginals.

Note that it is always possible to approximate any factor with several variables with several approximating functions. For example if a factor $f(x_a, x_b, x_c)$ is a function of three variables, it could be approximated with three approximating factors (approximate it with a function of three variables is computationally intensive, and approximating with one single-variable function is insufficient). This is easy to visualize for pairwise factors, in a graphical model. Consider Figure 1.2. In the left-side there is a pairwise factor which is being approximated by two single-variable factors in the right.



Figure 1.2: Decomposition of the pairs edges in factor graph

In Figure 1.2 (left) an HMM is represented in factor-graph notation. Each of the factor could be decomposed into two disjoint factors. In practice, when creating \hat{p}_i (in the Algorithm 1) we decompose only one of the factors and use its approximation in our updates.

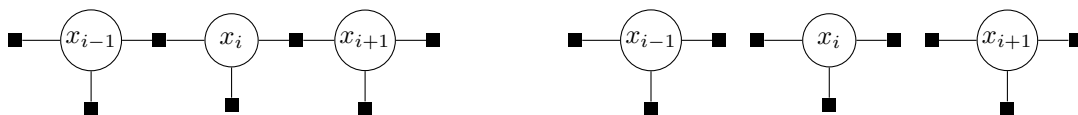


Figure 1.3: Decomposition of the factors in an HMM

³If you have problem with factor graphs, see this: web.engr.illinois.edu/~khashab2/learn/graph.pdf



Figure 1.4: Decomposing each pairwise edge at each step of EP.

1.6 EP energy function

There is a discussion of primal/dual energy minimization for EP in Minka (2001a). The primal energy is give in the following lemma.

■ **Lemma 1.1 — EP primal.** The primal energy function is the following,

$$\min_{\hat{p}_i} \max_q \left[\sum_i \int_{\mathbf{y}} \hat{p}_i(\mathbf{y}) \log \frac{\hat{p}_i(\mathbf{y})}{t_i(\mathbf{y})p(\mathbf{y})} d\mathbf{y} - (n-1) \int_{\mathbf{y}} q_{\theta}(\mathbf{y}) \log \frac{q_{\theta}(\mathbf{y})}{p(\mathbf{y})} d\mathbf{y} \right] \quad (1.8)$$

with the constraints that

$$\mathbb{E}_{q_{\theta}(\mathbf{y})} [\phi(\mathbf{y})] = \mathbb{E}_{\hat{p}_i(\mathbf{y})} [\phi(\mathbf{y})], \forall i \quad (\text{The local moment matching, in Equation 1.7}) \quad (1.9)$$

Basically by assigning energy function to a graphical model, we assign a global function to it, that is being optimized in a distributed fashion. This approach has been previously used to model the behaviour of other distributed inference algorithms like Belief Propagation. For more details on energy minimization schemas see Yedidia et al. (2005).

Justification of the above energy function is not hard. Basically we are minimizing a linear combination of several objectives, using an arbitrary divergence measure $D(\cdot||\cdot)$:

$$\begin{cases} \min_{\{\hat{p}_i\}} \min_q \sum_i \alpha_i D(\hat{p}_i||q) + \beta D(q||p) \\ \mathbb{E}_{q_{\theta}(\mathbf{y})} [\phi(\mathbf{y})] = \mathbb{E}_{\hat{p}_i(\mathbf{y})} [\phi(\mathbf{y})], \forall i \end{cases}$$

Choosing $D(\cdot||\cdot)$ to be the *KL*-divergence, $\alpha = 1$ and $\beta = (n-1)$ will give the desired result. Note that we chose these values intentionally in a way that would result in our distributed updates. Note that this is not a proof, this is a justification⁴. The proof of this is by finding the model evidence. To see the details, see Equation (66) at Minka et al. (2005), when $\alpha = 1$.

Using the primal form, one can obtain the dual form, and by setting the gradient of the dual form to zero, one can find the fixed-point updates introduced in the previous sections. In the rest, we show how to obtain this function.

⁴In fact, a very bad justification!

■ **Lemma 1.2 — Variational lower bound on KL divergence.** If $p(\cdot)$ and $q(\cdot)$ are proper distributions defined on \mathcal{X} , we can show that,

$$\text{KL}(p||q) = \int_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} dx = \max_v \left[\int_{x \in \mathcal{X}} p(x) v(x) dx - \log \int_{x \in \mathcal{X}} q(x) e^{v(x)} \right]$$

Proof. Easy enough to take functional derivatives with respect to $v(\cdot)$ and observe that $v(x) = \log \frac{p(x)}{q(x)} + c$ is indeed a global maximizer for the terms inside the $\max[\cdot]$ function. ■

Now we derive the dual EP energy.

■ **Lemma 1.3 — The dual EP energy.** The dual energy function for EP is,

$$\min_v \max_{\lambda} \left[(n-1) \log \int_{\mathbf{y}} p(\mathbf{y}) \exp \left(\mathbf{v}^\top \phi(\mathbf{y}) \right) d\mathbf{y} - \sum_{i=1}^n \log \int_{\mathbf{y}} \hat{t}_i(\mathbf{y}) p(\mathbf{y}) \exp \left(\lambda_i^\top \phi(\mathbf{y}) \right) d\mathbf{y} \right], \quad (1.10)$$

with the constraint that,

$$(n-1)\mathbf{v} = \sum_i \lambda_i.$$

Proof. The dual can be found by applying Lemma 1.2 for several times. Consider the first part of equation 1.8, which can be lower-bounded using Lemma 1.2.

$$\int_{\mathbf{y}} \hat{p}_i(\mathbf{y}) \log \frac{\hat{p}_i(\mathbf{y})}{t_i(\mathbf{y}) p(\mathbf{y})} d\mathbf{y} = \max_{\lambda} \left[\int_{\mathbf{y}} \hat{p}_i(\mathbf{y}) \lambda_i(\mathbf{y}) d\mathbf{y} - \log \int_{\mathbf{y}} t_i(\mathbf{y}) p(\mathbf{y}) \exp \lambda_i(\mathbf{y}) d\mathbf{y} \right]$$

Define $\lambda_i(\mathbf{y}) = \lambda_i^\top \phi(\mathbf{y})$, also using the constraint in equation 1.9,

$$\int_{\mathbf{y}} p_i(\mathbf{y}) \log \frac{\hat{p}_i(\mathbf{y})}{t_i(\mathbf{y}) p(\mathbf{y})} d\mathbf{y} = \max_{\lambda} \left[\sum_i \lambda_i \int_{\mathbf{y}} \hat{q}_\theta(\mathbf{y}) \phi_i(\mathbf{y}) d\mathbf{y} - \log \int_{\mathbf{y}} t_i(\mathbf{y}) p(\mathbf{y}) \exp \left(\lambda_i^\top \phi(\mathbf{y}) \right) d\mathbf{y} \right],$$

where $\phi_i(\mathbf{y})$ is the i -th element in the $\phi(\mathbf{y})$. Now consider the second part of equation 1.8 which can also be lower-bounded using Lemma 1.2,

$$\begin{aligned} \int_{\mathbf{y}} q_\theta(\mathbf{y}) \log \frac{q_\theta(\mathbf{y})}{p(\mathbf{y})} d\mathbf{y} &= \max_v \left[\int_{\mathbf{y}} q_\theta(\mathbf{y}) v(\mathbf{y}) d\mathbf{y} - \log \int_{\mathbf{y}} p(\mathbf{y}) \exp v(\mathbf{y}) d\mathbf{y} \right] \\ \Rightarrow - \int_{\mathbf{y}} q_\theta(\mathbf{y}) \log \frac{q_\theta(\mathbf{y})}{p(\mathbf{y})} d\mathbf{y} &= \min_v \left[- \int_{\mathbf{y}} q_\theta(\mathbf{y}) v(\mathbf{y}) d\mathbf{y} + \log \int_{\mathbf{y}} p(\mathbf{y}) \exp v(\mathbf{y}) d\mathbf{y} \right]. \end{aligned}$$

Similarly define $v_i(\mathbf{y}) = v_i^\top \phi(\mathbf{y})$,

$$- \int_{\mathbf{y}} q_\theta(\mathbf{y}) \log \frac{q_\theta(\mathbf{y})}{p(\mathbf{y})} d\mathbf{y} = \min_v \left[- \sum_i v_i \int_{\mathbf{y}} q_\theta(\mathbf{y}) \phi_i(\mathbf{y}) d\mathbf{y} + \log \int_{\mathbf{y}} p(\mathbf{y}) \exp \left(\mathbf{v}^\top \phi(\mathbf{y}) \right) d\mathbf{y} \right].$$

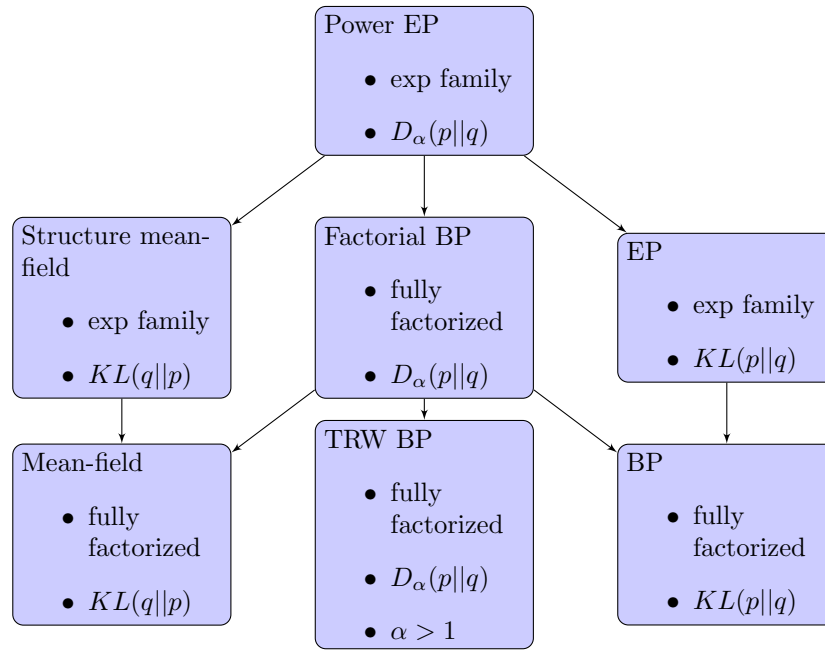


Figure 1.5: Hierarchies of different inference methods (From: <http://research.microsoft.com/en-us/um/people/minka/papers/message-passing/minka-message-passing-slides.pdf>.)

Combining these two results we get,

$$\begin{aligned} \text{Primal} = \max_{\hat{p}_i, q, \lambda} \min_v \{ & \sum_j \sum_i \lambda_i \int_{\mathbf{y}} q_{\theta}(\mathbf{y}) \phi_i(\mathbf{y}) d\mathbf{y} - \log \int_{\mathbf{y}} t_i(\mathbf{y}) p(\mathbf{y}) \exp \left(\lambda_i^{\top} \phi(\mathbf{y}) \right) d\mathbf{y} \\ & - (n-1) \sum_i v_i \int_{\mathbf{y}} q_{\theta}(\mathbf{y}) \phi_i(\mathbf{y}) d\mathbf{y} + (n-1) \log \int_{\mathbf{y}} p(\mathbf{y}) \exp \left(v^{\top} \phi(\mathbf{y}) \right) d\mathbf{y} \}. \end{aligned}$$

By defining $\sum_i \lambda_i = (n-1)v$, and dropping $q_{\theta}(\mathbf{y})$ as a result, we will end up with the dual form in Equation 1.10. ■

Stable fixed points of EP correspond to the *local* minima of its energy function. By taking derivatives of the dual EP energy function, we will end-up with the updates in Algorithm 1. We show this fact in the following.

1.6.1 EP updates from dual energy

[TBW]

1.7 BP vs. EP

[More later]

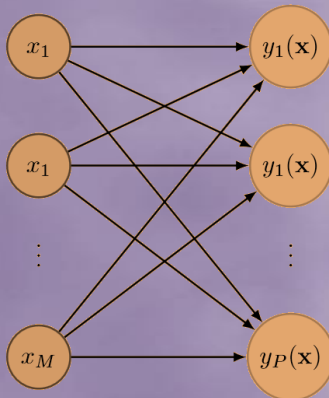
1.8 Further improvements on EP

Another future work could be in making the inference more efficient. In Seeger and Nickisch (2010) a faster way with convergence guarantees is proposed. Also at Pacheco

and Sudderth (2012); Opper et al. (2008) a faster scheme is presented. Minka (2004) is using the α -divergence for mapping the messages. It also showed that, it includes Fractional Belief Propagation Wiegerinck et al. (2003), EP and variational Bayes as special cases. In Kuss and Rasmussen (2005) EP has successfully used for Gaussian Process classification.

1.9 Bibliographical notes

Thanks to Doan Thanh Nam for reporting mistakes.



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