Qual Presentation

Daniel Khashabi

Outline

- My own line of research
- Papers:
  - Fast Dropout training, ICML, 2013
  - Distributional Semantics Beyond Words: Supervised Learning of Analogy and Paraphrase, TACL, 2013.

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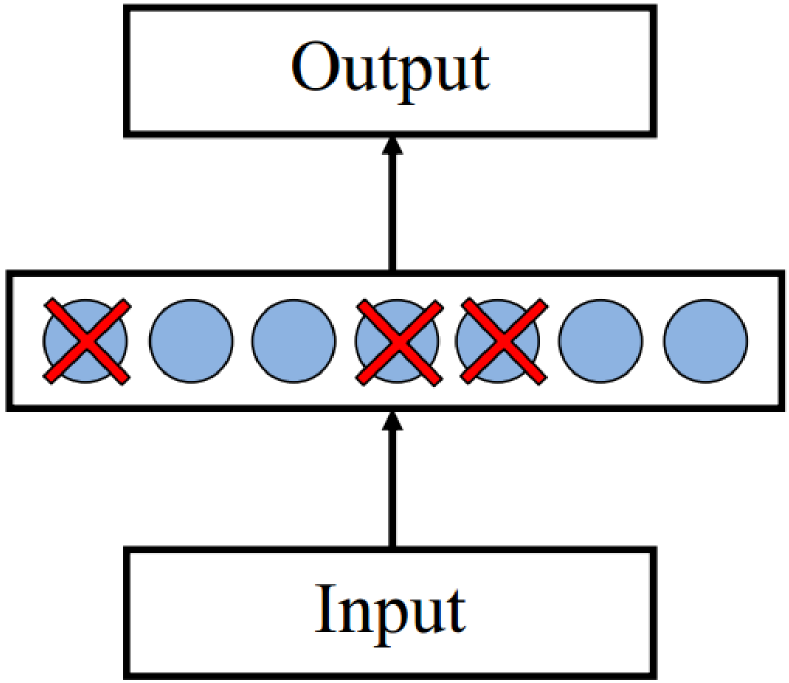
Dropout training

- Proposed by (Hinton et al, 2012)



- Each time decide whether to delete one hidden unit with some probability p

Dropout training

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  - Among 2^H models, with shared parameters
  - Only a few get trained
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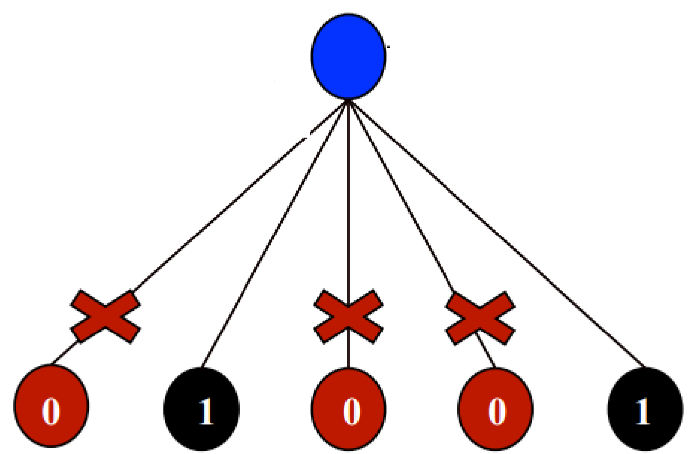
- What about the input space?
  - Do the same thing!



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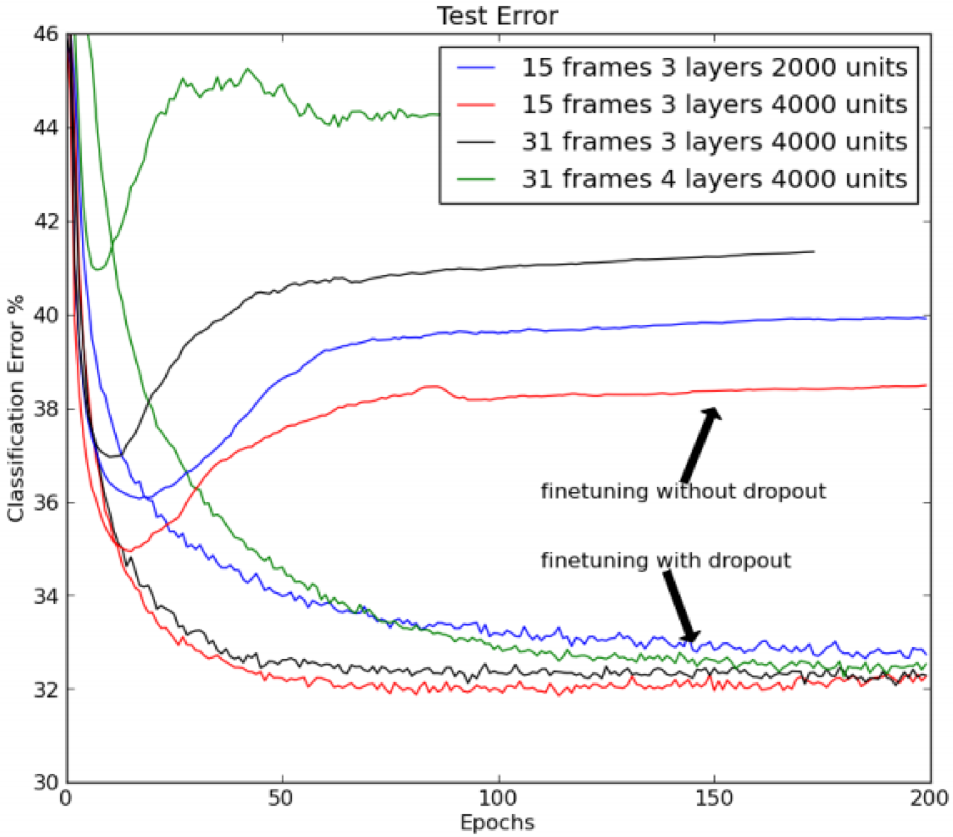

- Dropout of 50% of the hidden units and 20% of the input units (Hinton et al, 2012)

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  - Very easy to show for linear regression
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- Can we convert the sampling based update into a deterministic form?
  - Find expected form of updates

Linear Regression

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- Closed form solution:

$$w = \left(X^T X + \lambda I\right)^{-1} X^T y$$

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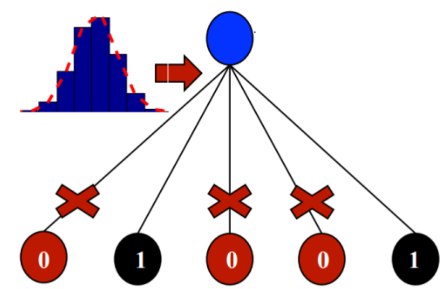

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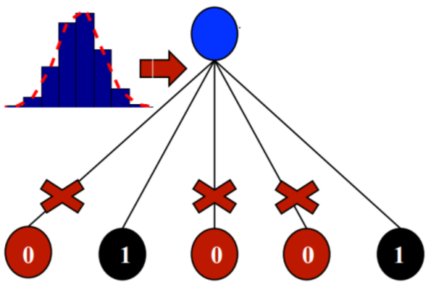

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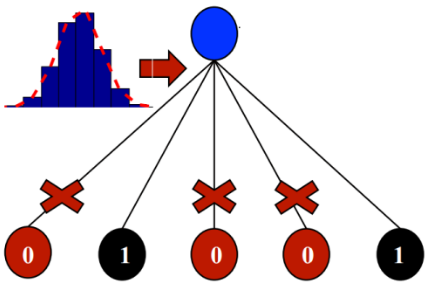

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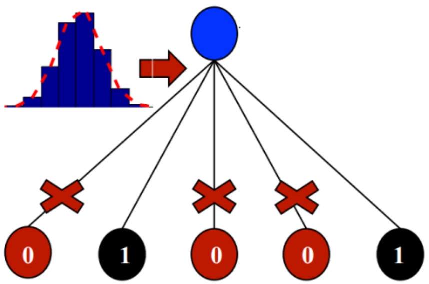

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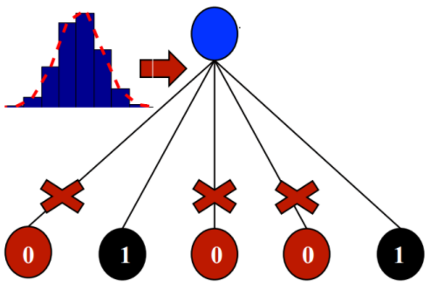

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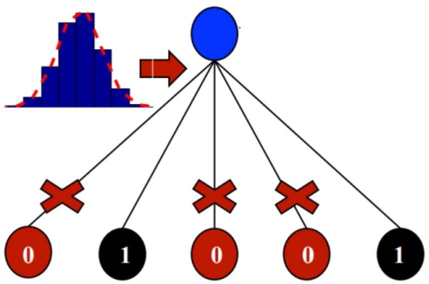

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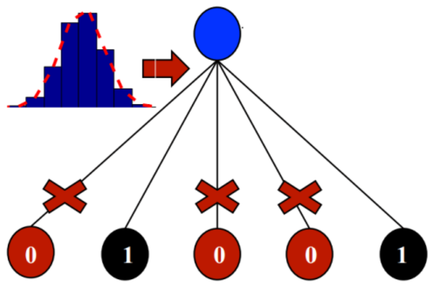

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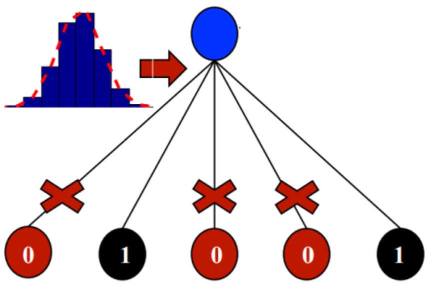

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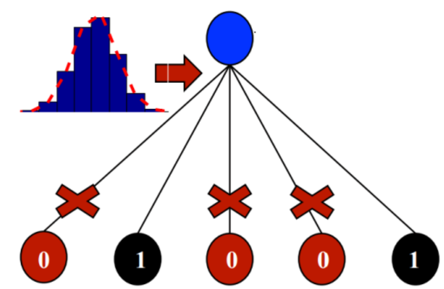

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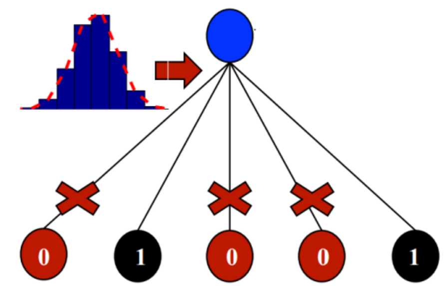

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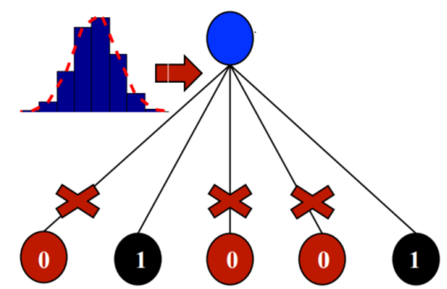

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$$= \sum_i \left(\mu_S^{(i)} - y^{(i)} \right)^2 + \lambda \sum_i c_i w_i^2, \quad c_i = \sum_j \left(x_i^{(j)} \right)^2$$

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$$\tilde{L}(w) = \mathbf{E} L(w) \simeq \sum_i \left(\mu_S^{(i)} - y^{(i)} \right)^2 + \sigma_S^{2\,(i)}$$

$$= \sum_i \left(\mu_S^{(i)} - y^{(i)} \right)^2 + \lambda \sum_i c_i w_i^2, \quad c_i = \sum_j \left(x_i^{(j)} \right)^2$$

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$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Some definitions useful equalities

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- We can find the following expectation in closed form:

$$\mathbf{E}_{S \sim N(\mu, \sigma^2)} \left[\sigma(S) \right]$$

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$$\mu_S = \mathbf{E}\left[w^T D_z x\right] = \sum_{i=1}^{m} w_i x_i \mathbf{E}[z_i] = \sum_{i=1}^{m} w_i x_i p_i$$

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$$\sigma_S^2 = \mathbf{Var}\left[w^T D_z x\right] = \sum_{i=1}^{m} (w_i x_i)^2 \,\mathbf{Var}\left[z_i\right] = \sum_{i=1}^{m} (w_i x_i)^2 \, p_i (1 - p_i)$$

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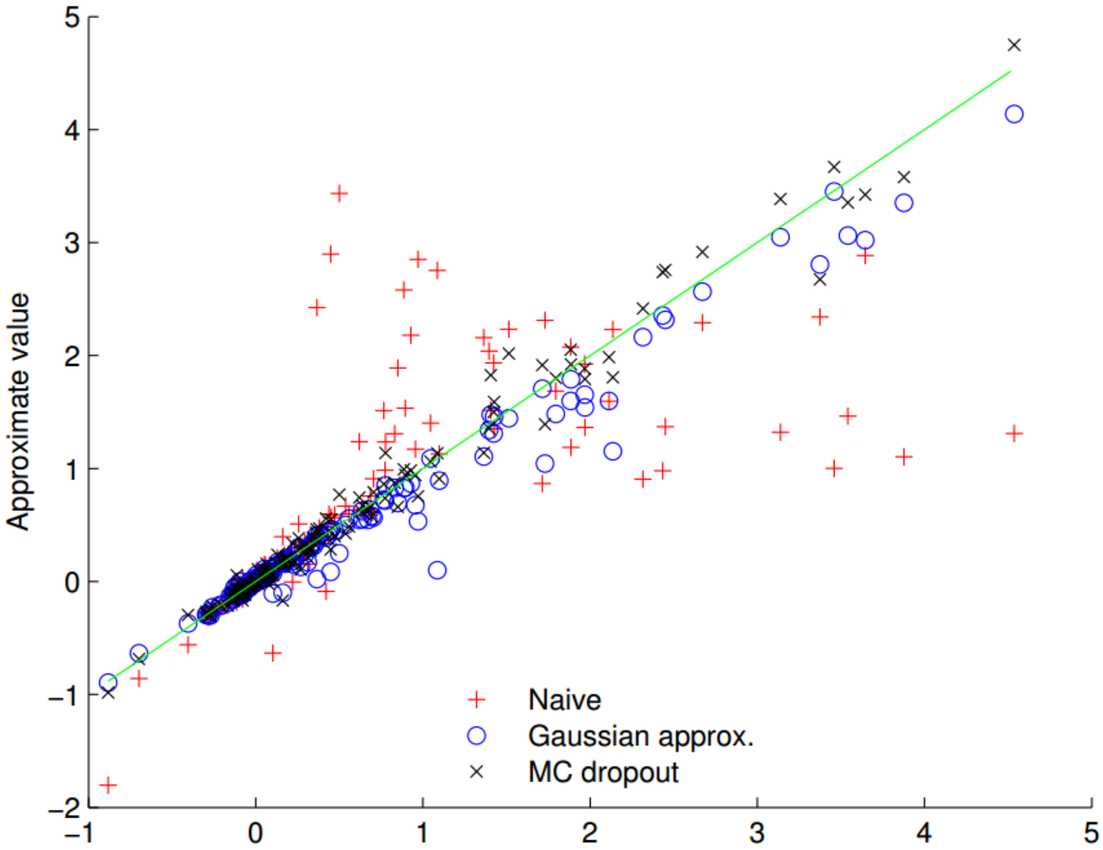
↓ Have closed forms but poor approximations

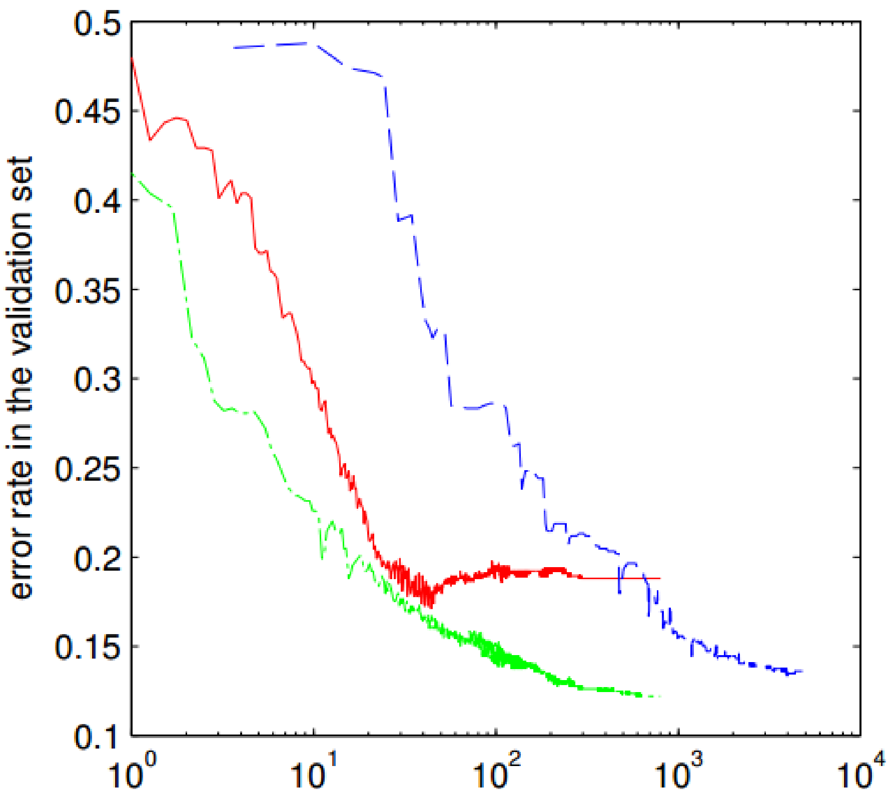
Experiment: evaluating the approximation

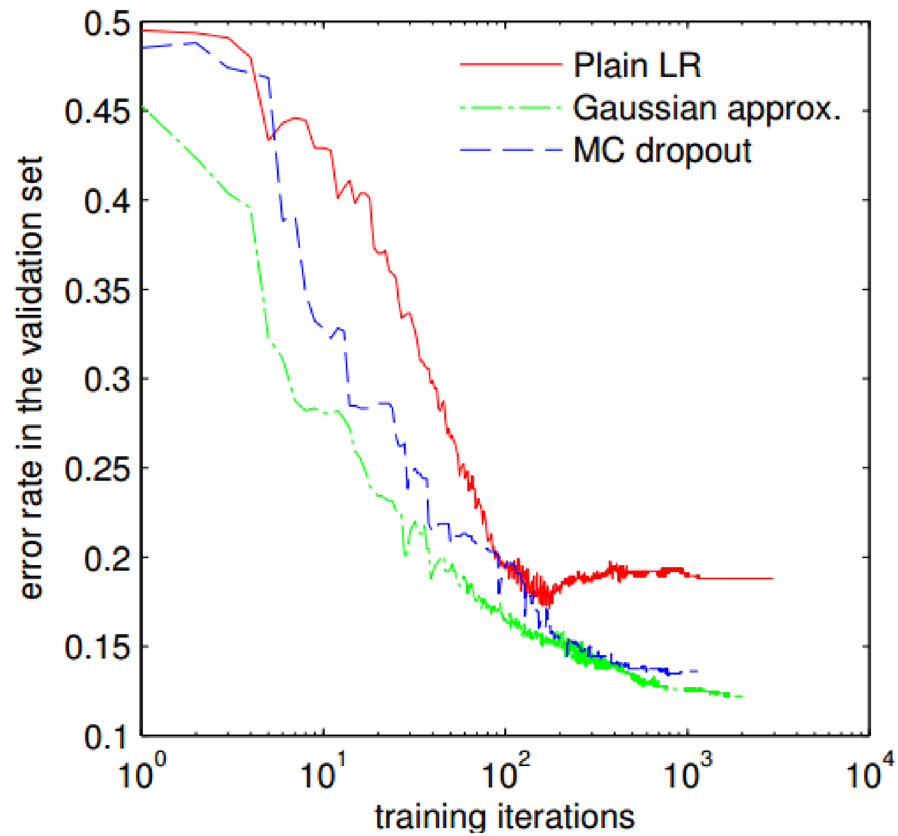
- The quality of approximation for $\Delta w_{\log}$



Experiment: Document Classification

- 20-newsgroup subtask *alt.atheism vs. religion.misc*





Experiment: Document Classification(2)

Methods\ Datasets	MR-2k	IMDB	RTs	Subj	AthR	CR	MPQA	Average
Real (MC) dropout	89.8	91.2	79.2	93.3	86.7	82.0	86.0	86.88
training time	*6400*	*6800*	*2300*	*2000*	*130*	*580*	*420*	*2700*
Gaussian dropout	89.7	91.2	79.0	93.4	87.4	82.1	86.1	86.99
training time	*240*	*1070*	*360*	*320*	*6*	*90*	*180*	*320*
Fast (closed-form) dropout	89.5	91.1	79.1	93.6	86.5	81.9	86.3	86.87
training time	*120*	*420*	*130*	*130*	*3*	*28*	*35*	*120*
plain LR	88.2	89.5	77.2	91.3	83.6	80.4	84.6	84.97
training time	*140*	*310*	*81*	*68*	*3*	*17*	*22*	*92*
Previous results								
TreeCRF(Nakagawa et al., 2010)	-	-	77.3	-	-	81.4	86.1	-
Vect. Sent.(Maas et al., 2011)	88.9	88.9	-	88.1	-	-	-	-
RNN(Socher et al., 2011)	-	-	77.7	-	-	-	86.4	-
NBSVM(Wang & Manning, 2012)	89.4	91.2	79.4	93.2	87.9	81.8	86.3	87.03
$\|\{i : x_i > 0\}\|$	788	232	22	25	346	21	4	

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$\downarrow$

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$$\mu_{S_i} = \mu_S + \mathbf{E}\left[-w_i x_i z_i + w_i x_i\right]$$

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- Has closed form!