Linear Control of Flexible Joint

Final Report for "Linear Control Laboratory"
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1 Modelling a flexible joint

Regarding the guidelines in lab sheet, we consider the fundamental equations of system as following:

$$\begin{cases} J_{eq}\ddot{\theta} + J_{arm}\left(\ddot{\alpha} + \ddot{\theta}\right) = T_{output} - B_{eq}\dot{\theta} \\ J_{arm}(\ddot{\theta} + \ddot{\alpha}) + K_{stiff}\alpha = 0 \end{cases} \tag{1}$$

The motor model is assumed as following:

$$T_{output} = \frac{\eta_m \eta_g K_t K_g (V_m - K_g K_m \dot{\theta})}{R_m} \tag{2}$$

In which constants are shown in Table.1.

Constant	Value
K_{stiff}	1.2485
B_{eq}	0.0040
J_{arm}	0.0035
J_{eq}	0.0026

Table 1. The constant values of system's dynamics

The motor model is assumed as shown in Table.2.

Constant	Value
R_m	2.6
K_g	70
$K_t = K_m$	0.00767
η_g	0.9
η_m	0.69

Table 2. The constant values of the motor model

2 State-space model

In this section we will derive the state-space model of the system.

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{AX}{+}\mathbf{BU} \\ \mathbf{Y} = \mathbf{CX} + \mathbf{DU} \end{cases}$$

$$\mathbf{X} = \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

By rearranging the fundamental equations, we can easily achieve to following state equations:

$$\begin{bmatrix}\dot{\alpha}\\ \ddot{\alpha}\\ \dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} 0 & 1\,0 & 0\\ -\frac{k_{stiff}(J_{eq}+J_{arm})}{J_{arm}J_{eq}} & 0\,0 & -\frac{\eta_m\eta_g K_t K_g^2 K_m}{R_m J_{eq}} - \frac{B_{eq}}{J_{eq}}\\ 0 & 0\,0 & 1\\ \frac{K_{stiff}}{J_{eq}} & 0\,0 & +\frac{\eta_m\eta_g K_t K_g}{R_m J_{eq}} + \frac{B_{eq}}{J_{eq}}\end{bmatrix} \begin{bmatrix}\alpha\\ \dot{\alpha}\\ \theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ \frac{\eta_m\eta_g K_t K_g}{R_m J_{eq}}\\ 0\\ -\frac{\eta_m\eta_g K_t K_g}{R_m J_{eq}}\end{bmatrix} V_m$$

Thus the state matrices become as what follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1\,0 & 0\\ -836.9 & 0\,1 & -28.0193\\ 0 & 0\,0 & 1\\ 480.19 & 0\,0 & 50.8\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}0\\ 49.3\\ 0\\ -49.3\end{bmatrix}$$

By defining the output to be $\mathbf{Y} = \begin{bmatrix}\alpha\\ \theta\end{bmatrix}$ we can find the output state-space equations as following:

$$\mathbf{C} = \begin{bmatrix}1\,0\,0\,0\\ 0\,0\,1\,0\end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix}0\end{bmatrix}$$

3 Transfer functions

In addition to state-space model, here we derive the transfer functions of the system parameters.

$$\left(J_{eq} + J_{arm}\right) s^2\theta + J_{arm}s^2\alpha + \left(\frac{\eta_m\eta_g K_t K_m K_g^2}{R_m} + B_{eq}\right) s\theta = \frac{\eta_m\eta_g K_t K_g}{R_m} V_m \tag{3}$$

$$J_{arm}s^2\theta + J_{arm}s^2\alpha + K_{stiff}\alpha = 0 \tag{4}$$

By combining equations (3) and (4), we can find:

$$\alpha = \frac{-J_{arm}s^2}{J_{arm}s^2 + K_{stiff}}\theta \tag{5}$$

$$\frac{\theta}{V_m} = \frac{(J_{arm}s^2 + K_{stiff})\frac{\eta_m\eta_g K_t K_g}{R_m}}{J_{arm}J_{eq}s^4 + J_{arm}\left(\frac{\eta_m\eta_g K_t K_m K_g^2}{R_m} + B_{eq}\right)s^3 + K_{stiff}(J_{eq}+J_{arm})s^2 + K_{stiff}\left(\frac{\eta_m\eta_g K_t K_m K_g^2}{R_m} + B_{eq}\right)s}$$

$$\frac{\alpha}{V_m} = \frac{-J_{arm}s^2\frac{\eta_m\eta_g K_t K_g}{R_m}}{J_{arm}J_{eq}s^4 + J_{arm}\left(\frac{\eta_m\eta_g K_t K_m K_g^2}{R_m} + B_{eq}\right)s^3 + K_{stiff}(J_{eq}+J_{arm})s^2 + K_{stiff}\left(\frac{\eta_m\eta_g K_t K_m K_g^2}{R_m} + B_{eq}\right)s}$$

By substituting values of the parameters, we have:

$$\frac{\theta}{V_m} = \frac{49.32s^2 + 1.759\times10^4}{s^4 + 28.02s^3 + 836.9s^2 + 9995s} \tag{6}$$

$$\frac{\alpha}{V_m} = \frac{-49.32s^2}{s^4 + 28.02s^3 + 836.9s^2 + 9995s} \tag{7}$$

4 Controller design and simulation

As it is demanded, the controller show afford the values in Table.3.

Parameter	Desired Value
Settling time	less than 0.7sec
Overshoot	less than 5%
Maximum deviation of α	9°

Table 3. Desired values for the controller

Simplifying the desired point, we can write:

$$T_s < 0.7sec \Rightarrow \frac{4}{\xi\omega_n} < 0.7 \Rightarrow \xi\omega_n > 5.72$$

$$e^{\frac{\xi\omega_n}{\sqrt{1-\xi^2}}} < 0.05 \Rightarrow \xi > 0.6901$$

By the of the MATLAB's SISO tool we design the following controllers:

Controller	$C(s) = 2.1\frac{\frac{s}{6}+1}{\frac{s}{5}+1}$	$C(s) = 2.1\frac{\frac{s}{7}+1}{\frac{s}{6}+1}$	$C(s) = 2.1\frac{\frac{s}{8}+1}{\frac{s}{7}+1}$	$C(s) = 2\frac{\frac{s}{10}+1}{\frac{s}{8}+1}$	$C(s) = 2\frac{\frac{s}{11}+1}{\frac{s}{9}+1}$	$C(s) = 2\frac{\frac{s}{12}+1}{\frac{s}{10}+1}$
T_s (sec)	0.49	0.49	0.49	0.49	0.49	0.49
$O.V.$ (%)	4.1	3.2	2.5	3.9	3.1	2.5
α_{max} (deg)	6.2	6.2	6.2	6.1	6.1	6.1

The controlling schematic of the system is shown in Fig.1.

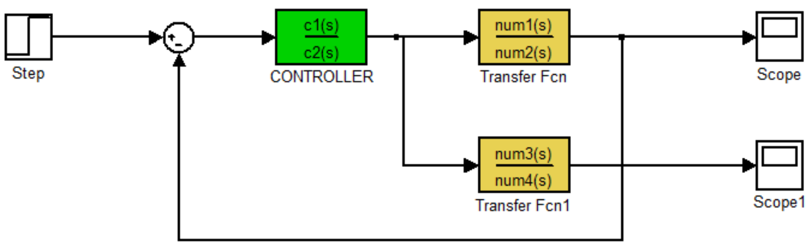

Fig. 1. The controlling system schematic.

A sample simulation of the system's response to unit pulse input is shown in Fig.2.

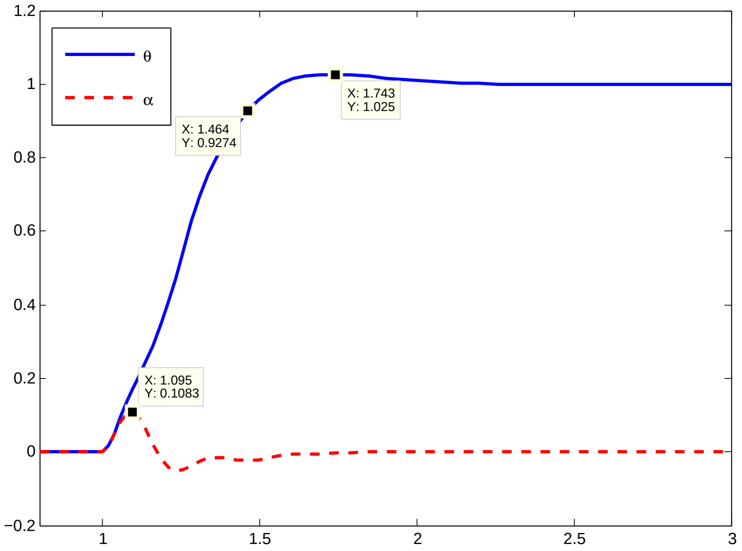

Fig. 2. Unit pulse response.