Task 1. Prove the following results about conditional means, forecasts and forecast errors:

(1) Let W be a randdom variable with mean μ_W and variance σ_W^2 and let c be a constant. Show that $\mathbb{E}\left[(W-c)^2\right] = \sigma^2 + (\mu_W - c)^2$

(2) Consider the problem of forecasting Y_t, using data on $Y_{t-1}, Y_{t-2}, \ldots$. Let f_{t-1} denote some forecast of Y_t, where the subscript $t-1$ on f_{t-1} indicates that the forecast is a function of data through date $t-1$. Let $\mathbb{E}[(Y_t - f_{t-1})^2 \mid Y_{t-1}, Y_{t-2}, \ldots]$ be the conditional mean squared error of the forecast f_{t-1}, conditional on values of Y observed through date $t-1$. Show that the conditional mean squared forecast error is minimized when $f_{t-1} = Y_{t|t-1}$, where $Y_{t|t-1} = \mathbb{E}(Y_t| \, Y_{t-1}, Y_{t-2}, \ldots)$

(3) Let u_t denote the error in the equation

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_p Y_{t-p} + u_t$$

where $\mathbb{E}(u_t | Y_{t-1} \ldots) = 0$. Show that $Cov(u_t, u_{t-j}) = 0 \quad \forall j \neq 0$

Solution: Lets <u>redefine</u> $\mathbb{E}(f(Y_t)|Y_{t-1}, Y_{t-2} \ldots)$ as $\mathbb{E}_{t-1}[f(Y_t)]$ (not to type a lot)

(1)

$$\mathbb{E}\left[(W-c)^2\right] = \mathbb{E}\left[W^2 - 2cW + c^2\right] = \mathbb{E}\left[W^2\right] - 2c\mathbb{E}\left[W\right] + c^2 =$$
$$= \mathbb{E}\left[W^2\right] - 2c\mu_W + c^2 = \sigma_W^2 + \mu_W^2 - 2c\mu_W + c^2 = \sigma_W^2 + (\mu_W - c)^2$$

(2) Keeping in mind that $\mathbb{E}\left(f_{t-1}|Y_{t-1}\ldots\right) = f_{t-1}$ we may take use of derivations from the previous point, namely:

$$\mathbb{E}_{t-1}[(Y_t - f_{t-1})^2] = \sigma_{Y_t}^2 + \underbrace{(\mathbb{E}_{t-1}[Y_t] - f_{t-1})}_{\text{We can influence}}$$

So in order to conditional mean to be the smallest we select $f_{t-1} = \mathbb{E}_{t-1}[Y_t]$ thus:

$$\mathbb{E}_{t-1}[(Y_t - \mathbb{E}_{t-1}\left[Y_t\right])^2] = \sigma_{Y_t}^2$$

(3) Lets notice that by LIME: $\mathbb{E}\left[u_t\right] = \mathbb{E}\left[\mathbb{E}_{t-1}\left[u_t\right]\right] = \mathbb{E}\left[\mathbb{E}_{t-j-1}\left[u_{t-j}\right]\right] = 0$ thus

$$\text{Cov}(u_t, u_{t-j}) = \mathbb{E}\left[\left(u_t - \mathbb{E}\left[u_t\right]\right)\left(u_{t-j} - \mathbb{E}\left[u_{t-j}\right]\right)\right] = \mathbb{E}\left[u_t u_{t-j}\right] = \mathbb{E}\left[u_{t-j}\mathbb{E}_{t-1}\left[u_t\right]\right] = 0$$

Task 2. Suppose that Y_t follows the stationary AR(1) model

$$Y_t = 2.5 + 0.7Y_{t-1} + u_t$$

where u_t is i.i.d. with $\mathbb{E}(u_t) = 0$ and $\text{Var}(u_t) = 9$.

(1) Compute the mean and variance of Y_t

(2) Compute the first two autocovariances of Y_t

(3) Compute the first two autocorrelations of Y_t

(4) Suppose that $Y_T = 102.3$ Compute $Y_{T+1|T} = \mathbb{E}_T(Y_{T+1})$

Solution:

(1) Stationarity implies that $\mathbb{E}Y_t = \mu_Y$, and $\text{Var}(Y_t) = \sigma_Y^2$ for all t, thus:

$$\mathbb{E}Y_t = 2.5 + 0.7\mathbb{E}Y_{t-1} \implies \mu_Y = 2.5 + 0.7\mu_Y \implies \mu_Y = \frac{2.5}{0.3} \sim 8.3$$
$$\text{Var}\left(Y_t\right) = 0.7^2\text{Var}(Y_{t-1}) + \text{Var}(u_t) \implies \sigma_Y^2 = \frac{\text{Var}(u_t)}{1 - 0.7^2} = \frac{9}{0.51} \sim 17.65$$

(2) By definition $\gamma(2) = \text{Cov}(Y_t, Y_{t-2})$:

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-2}) &= \text{Cov}(2.5 + 0.7(2.5 + 0.7Y_{t-2} + u_{t-1}) + u_t,\ Y_{t-2}) \\
&= 0.7^2\text{Cov}(Y_{t-2}, Y_{t-2}) + 0.7\underbrace{\text{Cov}(u_{t-1}, Y_{t-2})}_{=0} + \underbrace{\text{Cov}(u_t, Y_{t-2})}_{=0} \\
&= 0.7^2\gamma(0)
\end{aligned}$$

From previous calculations it is obvious that $\gamma(1) = 0.7\gamma(0)$

(3) Autocorrelation by definition is:

$$\rho_1 = \frac{\text{Cov}(Y_t,\ Y_{t-1})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-1})}} = \frac{\text{Cov}(Y_t,\ Y_{t-1})}{\text{Var}(Y_t)} = \frac{0.7\gamma(0)}{\gamma(0)} = 0.7$$

$$\rho_2 = \frac{\text{Cov}(Y_t,\ Y_{t-2})}{\sqrt{\text{Var}(Y_t)\text{Var}(Y_{t-2})}} = \frac{\text{Cov}(Y_t,\ Y_{t-2})}{\text{Var}(Y_t)} = \frac{0.7^2\gamma(0)}{\gamma(0)} = 0.7^2$$

(4)

$$\mathbb{E}_T\left[Y_{T+1}\right] = 2.5 + 0.7Y_T = 2.5 + 0.7 \times 102.3 = 74.11$$

Task 3. Consider the stationary AR(1) model $Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t$, where ut is i.i.d. with mean 0 and variance σ_u^2. The model is estimated using data from time periods $t = 1$ through $t = T$, yielding the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$. You are interested in forecasting the value of Y at time $T+1$ that is, Y_{T+1}. Denote the forecast by $\hat{Y}_{T+1|T} = \hat{\beta}_0 + \hat{\beta}_1 Y_T$.

(1) Show that the forecast error is $Y_{T+1} - \hat{Y}_{T+1|T} = u_{T+1} - \left[(\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1)Y_T\right]$
(2) Show that u_{T+1} is independent of Y_T
(3) Show that u_{T+1} is independent of $\hat{\beta}_0$, $\hat{\beta}_1$
(4) Show that $\text{Var}(Y_{T+1|T} - \hat{Y}_{T+1|T}) = \sigma_u^2 + \text{Var}\left[(\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1)Y_T\right]$

Solution.

(1)

$$Y_{T+1} - \hat{Y}_{T+1|T} = \beta_0 + \beta_1 Y_T + u_{T+1} - \hat{\beta}_0 - \hat{\beta}_1 Y_T = u_{T+1} - \left[(\hat{\beta}_0 - \beta_0) + (\hat{\beta}_1 - \beta_1)Y_T\right]$$

(2) We can try to proof it by calculating covariance $\text{Cov}(u_{T+1},\ Y_T)$:

$$\text{Cov}(u_{T+1},\ Y_T) = \mathbb{E}\left[u_{T+1}Y_T\right] = \mathbb{E}\left[Y_T\mathbb{E}_T\left[u_{T+1}\right]\right] = 0$$

(3)

(4)

$$\text{Var}(Y_{T+1|T} - \hat{Y}_{T+1|T}) = \text{Var}\left(\beta_0 + \beta_1 Y_T - \hat{\beta}_0 - \hat{\beta}_1 Y_T\right) = \text{Var}\left((\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1)Y_T\right)$$