Microeconomics-2 - Assignment #1

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Question 2

Consider the single-output case, two-input case. Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ take the form $f(z_1, z_2) = -A + z_1^a z_2^b$, where $A, a, b \geq 0$. Define the set of feasible input-output vectors as follows: $Y = \{(z_1, z_2, q) \in R^2_+ \times R : q \leq f(z_1, z_2)\}$. Determine whether Y satisfies No Free lunch, Possibility of Inaction, Free Disposal, Non-increasing Returns to Scale and whether it is a convex cone. If your response depends on the parameters $A, a, b \geq 0$ explain how so.

I would like to use notations of "netputs" in there, meaning that y is a vector of inputs and outputs, where inputs are negative and outputs are positive.

(1) ***No free lunch***: to satisfy no free lunch, zero inputs should not produce any output or equivalently we should prove the following implication:

$$\begin{cases} q \leq f(z_1, z_2) \\ 0 \leq f(z_1, z_2) \\ z_1 = z_2 = 0 \end{cases} \implies f(z_1, z_2) = 0$$

This transforms to

$$\begin{cases} q \leq -A \\ 0 \leq -A \end{cases}$$

For no free lunch to perform the following should imply:

$$f(z_1, z_2) = 0$$

And this is of cource implies:

$$q \leq 0 \iff f(z_1, z_2) = 0$$

(2) ***Possibility of inaction***:

$$0 \in Y \iff \overline{0} \in Y \iff 0 \leq f(0, 0) \implies 0 \leq -A$$

Since $A \geq 0$, inaction is possible only when $A = 0$.

(3) ***Non-increasing returns to Scale***: We should proof the following implication:

$$q \leq f(z_1, z_2) \implies \begin{cases} kq \leq f(kz_1, kz_2) \\ k \in [0,1] \end{cases}$$

Lets look at $f(kz_1, kz_2)$:

$$f(kz_1, kz_2) = -A + k^{a+b} z_1^a z_2^b$$

Summing everything up:

$$\begin{cases} kq \leq -A + k^{a+b} z_1^a z_2^b \\ kq \leq -kA + k z_1^a z_2^b \\ k \in [0,1] \end{cases} \implies 0 \leq \underbrace{(1-k)A}_{\geq 0} + (k^{a+b} - k) \underbrace{z_1^a z_2^b}_{\geq 0} \implies 0 \leq \frac{(1-k)A}{k - k^{a+b}} \leq z_1^a z_2^b$$

(4) ***Free disposal***: We should prove the following implication:

$$\begin{cases} q \leq f(z_1, z_2) \\ z_1' \geq z_1 \\ z_2' \geq z_2 \\ q' \leq q \end{cases} \implies q' \leq f(z_1', z_2')$$

Lets start from:

$$q' \leq q \leq f(z_1, z_2) \quad \implies \quad q' \leq -A + z_1^a z_2^b$$

and

$$q' \leq f(z_1', z_2') \iff -A + z_1^a z_2^b \leq -A + (z_1')^a (z_2')^b$$

or when

$$a \underbrace{\left(\ln \frac{z_1}{z_1'} \right)}_{<0} + b \underbrace{\left(\ln \frac{z_2}{z_2'} \right)}_{<0} \leq 0$$

which is done for every value of parameter. ***Convex cone***: The following should be true:

$$\begin{cases} q \leq f(z_1, z_2) \\ q' \leq f(z_1', z_2') \\ k, h \geq 0 \end{cases} \implies kq + hq' \leq kf(z_1, z_2) + hf(z_1', z_2')$$

Using our production function:

$$\begin{cases} q \leq -A + z_1^a z_2^b \\ q' \leq -A + (z_1')^a (z_2')^b \\ k, h \geq 0 \end{cases} \implies kq + hq' \leq k(-A + z_1^a z_2^b) + h(-A + (z_1')^a (z_2')^b)$$

which is true for all parameters

Question 3

For each of the following production functions
(1) $f(z_1, z_2) = z_1 + z_2$
(2) $f(z_1, z_2) = \min\{z_1, z_2\}$
(3) $f(z_1, z_2) = (z_1^\rho + z_2^\rho)^{1/\rho}$, with $\rho \leq 1$
determine whether there exists a non-empty interval $P \in (0, +\infty)^3$ such that such that for each $p \in P$, either $(0,0,0) \in y(p)$ or $\pi(p) = +\infty$. Moreover, for each of these cases determine whether $y(p)$ is a singleton for each $p \in P^3_{++}$. [Hint: you do not need to solve the PMP]

Firstly, lets notice, that prices could be such that whether $p \cdot y > 0$ or $p \cdot y \leq 0$:

$$p \cdot y = p_3 f(z_1, z_2) - p_1 z_1 - p_2 z_2$$

(1) $f(z_1, z_2) = z_1 + z_2$ has nondecreasing returns to scale, which is easy to show:

$$f(az_1, az_2) = a(z_1 + z_2) = af(z_1, z_2)$$

Therefore, if the $p \cdot y > 0$, then we can increase usage of inputs in order to increase output infinetly (for example by increasing inputs a times, where $a > 1$). If the $p \cdot y \leq 0$, then the maximum of profit is indeed 0. So there is actually nonempty interval $P \subseteq (0, \infty)^3$ such that whether $\pi(p)$ is infinite or zero. For example we can take

$$p_1 = p_1^* > 0 \qquad p_2 = p_2^* > 0 \qquad p_3 = p_1^* + p_2^* + 1$$

More generally, in order for profit to be infinite it must be prices such that

$$p_3 > p_1 \qquad \text{or} \qquad p_3 > p_2$$

If prices such that

$$p_3 \leq p_1 \leq p_2 \qquad \text{or} \qquad p_3 \leq p_2 \leq p_1$$

then indeed each additional unit of output will bring less money than the cost of inputs and $\pi(p) = 0$. It is also obvious that for each p there will be set of $y(p)$ thus supply is not a singletone.

(2) Second case is still nondecreasing returns to scale:

$$f(az_1, az_2) = \min\{az_1, az_2\} = a \min\{z_1, z_2\} = af(z_1, z_2)$$

therefore logic with $p \cdot y > 0$ and $p \cdot y \leq 0$ applies: exist some prices at which profit is either infinity or zero.

(3) Its production function is convex and has nonincreasing returns to scale:

$$f(az_1, az_2) = (a^\rho(z_1^\rho + z_2^\rho))^{1/\rho} = (z_1^\rho + z_2^\rho)^{1/\rho} = f(z_1, z_2)$$

Thus there exists internal solution and there is no price at which profit is either 0 or infinite. It is also clear, that supply is singleton.