Macroeconomics-2 - Assignment #1

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Problem 1

Question 1

Go to Wikipedia and read about the Hodrick-Prescott decomposition. Write down the main formula and briefly explain in your own words the idea behind it. What happens if the value of parameter λ is too large? What happens if it is too small?

(a) It is assumed that y_t is log of time-series for each period and y_t is made of trend component τ_t, cyclical component c_t and error component ε_t such that:

$$y_t = \tau_t + c_t + \varepsilon_t$$

Main formula for decomposition is:

$$\min_{\tau}\left(\sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda\sum_{t=2}^{T-1}[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2\right)$$

(b) In simple terms this decomposition is used to clean raw time-series data from cyclical component in data. We select such value of trend, that would minimise the distance from the y_t and would be as smooth as we desire by ranging the value for λ. The higher value for λ the more straightforward (less cyclical) would be our trend line. If λ is too high, minimisation problem would select τ_t being straight horizontal line. If λ equals to zero we minimise the following value:

$$\sum_{t=1}^{T}(y_t - \tau_t)^2$$

which leads us to solution where $y_t = \tau_t$ and therefore our trend component will perfectly repeat the dependable variable.

Question 2

Take levels and logs of each of the series and plots them (one graph for levels and one for logs). Compare shapes of their graphs, think what is better to use in practical analysis and why (any reasonable ideas would be graded). Perform questions 2-4 with both **logs** and **levels**.

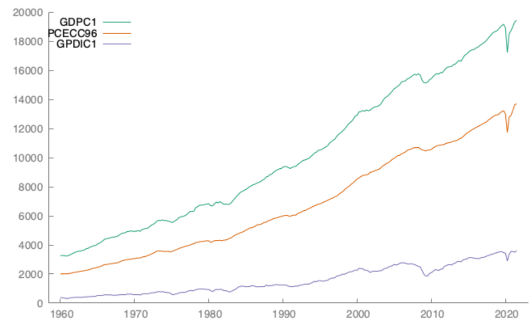


Figure 1: Variables.

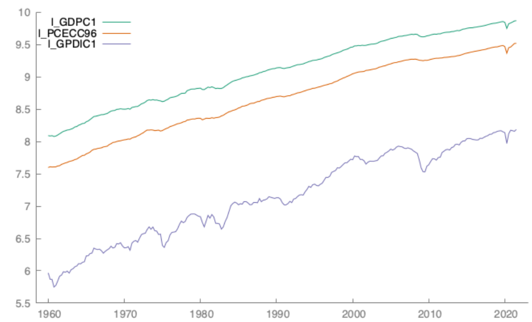


Figure 2: Logarithms.

As we can see, the shapes of RGDP and Real Personal Consumption Expenditures are highly correlated. It is also obvious that values for levels were rising extremely which would be inappropriate for some models because some of them using numerical optimization tools to find own parameters. Therefore logarithm transformation is more applicable in practical analysis.

Question 3

Use Hodrick-Prescott filter to smooth time series. Use $\lambda = 1600$ (default value). Extract residuals (original time series minus smoothed ones). Plot residuals for three time series on the same graph.

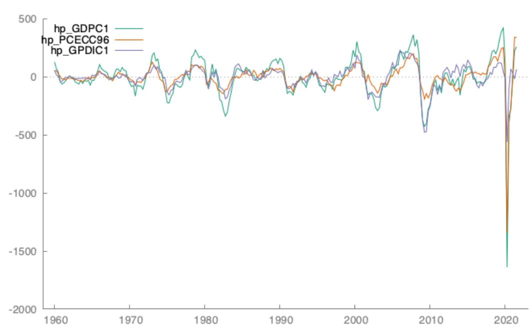


Figure 3: Residuals for levels.

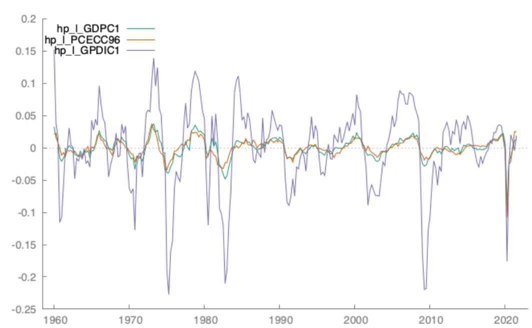


Figure 4: Residuals for logarithms.

Question 4

Plot the trends on the same graph. Describe the results.

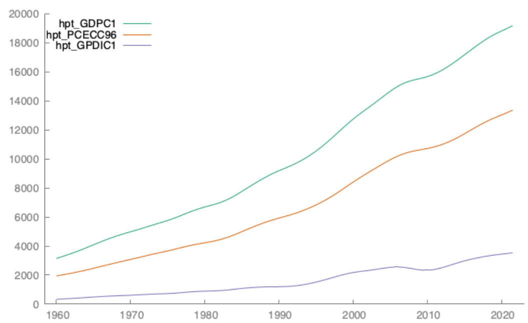


Figure 5: Trends for levels.

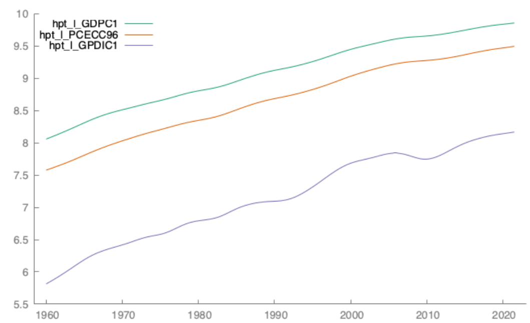


Figure 6: Trends for logarithms.

As we may see, the trend variables clearly illustrate trends for both levels and logarithms. Those trend lines are smooth and are merely fluctuating.

Question 5

Find the variance of cyclical components (residuals) of these three time series. Make a guess based on the data, which of the three time series had larger fluctuations.

Descriptional statistics for data 1960:1–2021:3

Var	Avg	Median	Var.	min.	max.
hp_GDPC1	−8.04e−12	−6.41	30,450.0	−1.64e+03	424.0
hp_PCECC96	−4.96e−12	−0.360	14,568.49	−1.34e+03	345.0
hp_GPDIC1	−1.13e−12	6.78	10,547.29	−559.0	228.0
hp_l_GDPC1	0.00	−0.000472	0.00024336	−0.0906	0.0373
hp_l_PCECC96	0.00	0.000143	0.000196	−0.108	0.0370
hp_l_GPDIC1	0.00	0.00789	0.004225	−0.227	0.153

From the table above we can see that the GDPC1 had larger fluctuations among level variables and GPDIC1 – among log variables.

Question 6

Find the correlation between cyclical components of GDP and consumption. Are the results you've received consistent with Keynesian consumption function ($C = a + bY$)?

We get the following correlation matrix:

Correlation matrix 1960:1–2021:3

hp_GDPC1	hp_PCECC96	
1.0000	0.9177	hp_GDPC1
	1.0000	hp_PCECC96

Which means that correlation between GDP residuals and consumption expenditures significant and *positive* and is around 0.9177 which coincides with Keynesian consumption function.

Question 7

Plot the histogram of each residual. Are they close to normal distribution? Compare distribution of levels and logs.

From the distribution pictures below we can see, that they look close to normal distribution, but actual statisticial test says they are not normally distributed. Statistics for normality presented in the upper left corner of each picture (zero hypothesis is such that there is normal distribution). We can also outline, that log residuals are more normal-looking than level residuals.

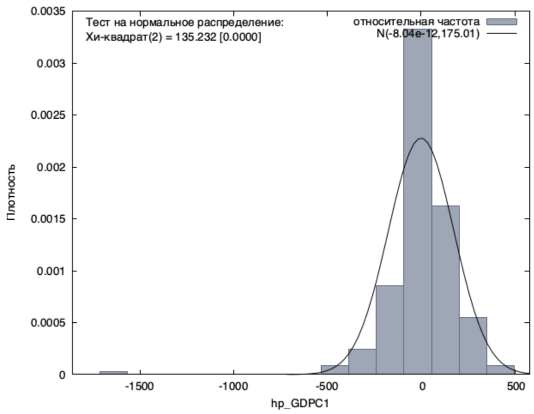


Figure 7: Level GDPC.

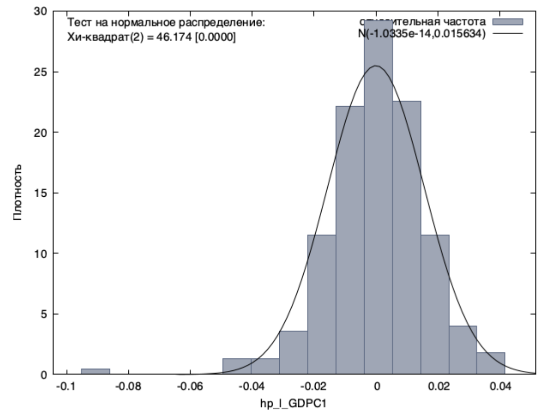


Figure 8: Log GDPC.

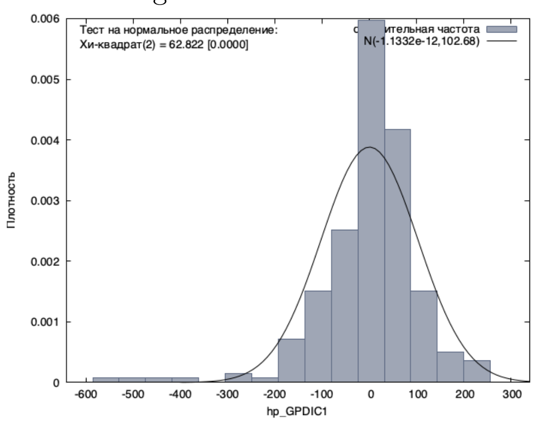


Figure 9: Level for GPDIC.

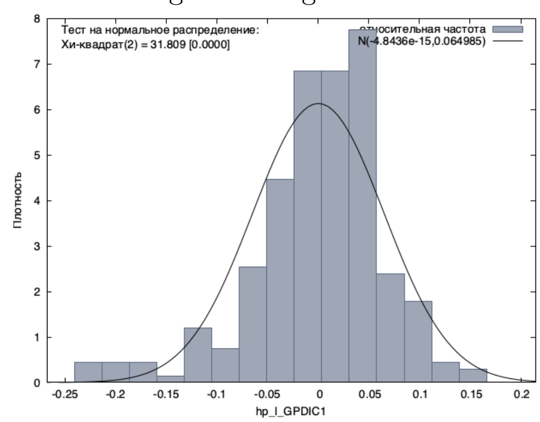


Figure 10: Log for GPDIC.

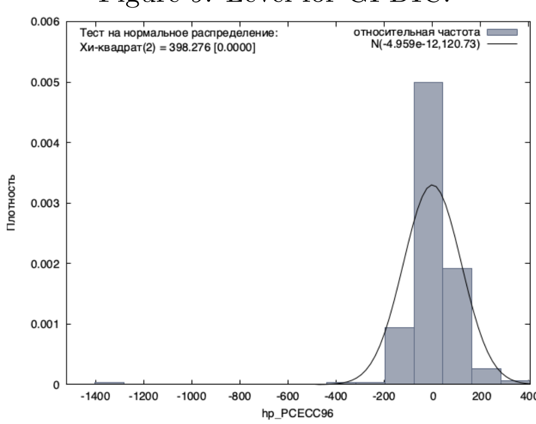


Figure 11: Level for PCECC.

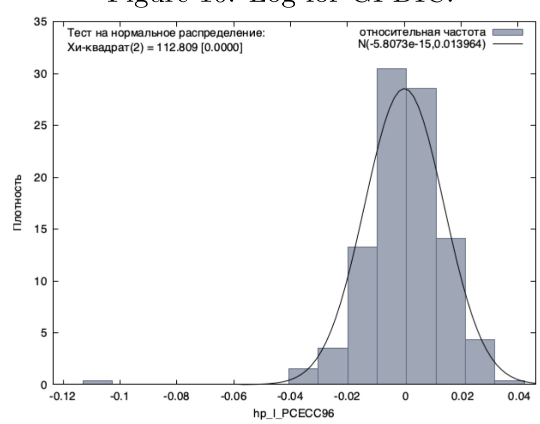


Figure 12: Log for PCECC.

Problem 2

Consider an continuation of Two-Period Model: a Three-Period Model. In this model the household receives income stream Q_1, Q_2, Q_3. One-period interest rate is r. Household can consume in each period. Denote the consumption stream as C_1, C_2, C_3.

Question 1

> Write down the intertemporal budget constraint. Discuss also the following possible cases: how the budget constraint will change if the household receives bequest b_0 in period 0 or leaves bequest b_3 in period 3?

Intertemporal budget constraint:

$$\begin{cases} C_1 + s_1 \leq Q_1 & \implies s_1 \leq Q_1 - C_1 \\ C_2 + s_2 \leq Q_2 + (1+r)s_1 & \implies s_1 \geq \frac{C_2 - Q_2}{1+r} + \frac{s_2}{1+r} \\ C_3 \leq Q_3 + (1+r)s_2 & \implies \frac{s_2}{1+r} \geq \frac{C_3 - Q_3}{(1+r)^2} \end{cases} \implies \frac{C_3 - Q_3}{(1+r)^2} \leq Q_1 - C_1 + \frac{Q_2 - C_2}{1+r}$$

Or more familiar:

$$C_1 + \frac{C_2}{1+r} + \frac{C_3}{(1+r)^2} \leq Q_1 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2}$$

If household receives bequest b_0, that means he will have more income, threrefore Intertemporal budget constraint will change:

$$\begin{cases} C_1 + s_1 \leq Q_1 + b_0 & \implies s_1 \leq Q_1 - C_1 + b_0 \\ C_2 + s_2 \leq Q_2 + (1+r)S_1 & \implies s_1 \geq \frac{C_2 - Q_2}{1+r} + \frac{s_2}{1+r} \\ C_3 \leq Q_3 + (1+r)s_2 & \implies \frac{s_2}{1+r} \geq \frac{C_3 - Q_3}{(1+r)^2} \end{cases} \implies \frac{C_3 - Q_3}{(1+r)^2} \leq Q_1 - C_1 + \frac{Q_2 - C_2}{1+r}$$

Or more familiar:

$$C_1 + \frac{C_2}{1+r} + \frac{C_3}{(1+r)^2} \leq Q_1 + b_0 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2}$$

If household leaves bequest b_3, that means he will have less income in the last period, threrefore Intertemporal budget constraint will change:

$$\begin{cases} C_1 + s_1 \leq Q_1 & \implies S_1 \leq Q_1 - C_1 \\ C_2 + s_2 \leq Q_2 + (1+r)s_1 & \implies s_1 \geq \frac{C_2 - Q_2}{1+r} + \frac{s_2}{1+r} \\ C_3 \leq Q_3 - b_3 + (1+r)s_2 & \implies \frac{s_2}{1+r} \geq \frac{C_3 - Q_3 + b_3}{(1+r)^2} \end{cases} \implies \frac{C_3 - Q_3 + b_3}{(1+r)^2} \leq Q_1 - C_1 + \frac{Q_2 - C_2}{1+r}$$

Or more familiar:

$$C_1 + \frac{C_2}{1+r} + \frac{C_3}{(1+r)^2} \leq Q_1 + \frac{Q_2}{1+r} + \frac{Q_3 - b_3}{(1+r)^2}$$

If we havent assumed for the presence of s_i, then the intertemporal constraints would be:

(1) No bequests:

$$\begin{cases} C_1 \leq Q_1 \\ C_2 \leq Q_2 \\ C_3 \leq Q_3 \end{cases}$$

(2) Received bequest b_0:

$$\begin{cases} C_1 \leq Q_1 + b_0 \\ C_2 \leq Q_2 \\ C_3 \leq Q_3 \end{cases}$$

(3) Left bequest b_3:

$$\begin{cases} C_1 \leq Q_1 \\ C_2 \leq Q_2 \\ C_3 + b_3 \leq Q_3 \end{cases}$$

Question 2

Assume that household has following utility function $$U(C_1, C_2, C_3) = \sqrt{C_1} + \frac{\sqrt{C_2}}{1+r} + \frac{\sqrt{C_3}}{(1+r)^2}$$ Formulate the problem of the household and find its FOCs.

(1) First case, no assumption about presence of savings: household solves the following optimisation problem. Notice that we turned the ineqaulities constraints into equations since constraints bind ($\partial U / \partial C_i > 0$)

$$U(C_1, C_2, C_3) = \sqrt{C_1} + \frac{\sqrt{C_2}}{1+r} + \frac{\sqrt{C_3}}{(1+r)^2} \to \max_{C_1, C_2, C_3}$$
$$s.t. \quad C_1 = Q_1 \quad C_2 = Q_2 \quad C_3 = Q_3$$

So lagrangian and the FOCs are the following:

$$\mathcal{L} = \sqrt{C_1} + \frac{\sqrt{C_2}}{1+r} + \frac{\sqrt{C_3}}{(1+r)^2} + \lambda_1(Q_1 - C_1) + \lambda_2(Q_2 - C_2) + \lambda_3(Q_3 - C_3)$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} = 0 \implies &\quad \frac{1}{2\sqrt{C_1}} - \lambda_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial C_2} = 0 \implies &\quad \frac{1}{2\sqrt{C_2}} - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial C_3} = 0 \implies &\quad \frac{1}{2\sqrt{C_3}} - \lambda_3 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \implies &\quad \lambda_i(Q_i - C_i) = 0, \quad \lambda_i \geq 0
\end{aligned}
$$

(2) Second case, assuming presence of savings s_i Household solves the following optimisation problem:

$$
\begin{aligned}
&U(C_1, C_2, C_3) = \sqrt{C_1} + \frac{\sqrt{C_2}}{1+r} + \frac{\sqrt{C_3}}{(1+r)^2} \to \max_{C_1, C_2, C_3} \\
&s.t. \quad C_1 + \frac{C_2}{1+r} + \frac{C_3}{(1+r)^2} \leq Q_1 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2}
\end{aligned}
$$

So lagrangian and the FOCs are the following:

$$
\mathcal{L} = \sqrt{C_1} + \frac{\sqrt{C_2}}{1+r} + \frac{\sqrt{C_3}}{(1+r)^2} + \lambda \left(Q_1 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2} - C_1 - \frac{C_2}{1+r} - \frac{C_3}{(1+r)^2} \right)
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_1} = 0 \implies &\quad \frac{1}{2\sqrt{C_1}} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial C_2} = 0 \implies &\quad \frac{1}{2\sqrt{C_2}} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial C_3} = 0 \implies &\quad \frac{1}{2\sqrt{C_3}} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \implies &\quad \lambda \left(Q_1 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2} - C_1 - \frac{C_2}{1+r} - \frac{C_3}{(1+r)^2} \right) = 0, \quad \lambda \geq 0
\end{aligned}
$$

FOCs says that agent selects the same amount of consumption each period. This was predictable since his utility discount factor equals to amount of money he receives from savings. We should also notice that budget constraint binds, since utility function increases monotonically for each C_i. We therefore can calculate amount of good being consumed each period:

$$
C_i^* = \frac{(1+r)^2}{(2+r)(1+r)+1} \left(Q_1 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2} \right)
$$

Question 3

Assume that household makes savings in each period, denote them s_1, s_2, s_3. What is the optimal level of s_3?

Its obvious that agent lives only for 3 periods and his utility function doesnt include any bequests, so he has no incentives to save in the last period of his life $\implies s_3^* = 0$.

Question 4

Express consumption in period 3 in terms of Q_3 and s_2. Find s_2 from this equation.

$$C_3 = Q_3 + (1+r)s_2 \implies s_2 = \frac{C_3 - Q_3}{1+r} = \frac{C_3}{1+r} - \frac{Q_3}{1+r}$$

Question 5

Write down the second period budget constraint. Plug the expression for s_2 into this constraint and find s_1 from this equation.

$$C_2 + s_2 = Q_2 + (1+r)s_1 \implies C_2 + \frac{C_3}{1+r} - \frac{Q_3}{1+r} = Q_2 + (1+r)s_1$$
$$s_1 = \frac{C_2 - Q_2}{1+r} + \frac{C_3 - Q_3}{(1+r)^2}$$

Question 6

Repeat the same procedure as in previous question for the first period budget constraint. Does it look similar to what you found above?

$$C_1 + s_1 = Q_1 \implies s_1 = Q_1 - C_1$$
$$s_1 = \frac{C_2 - Q_2}{1+r} + \frac{C_3 - Q_3}{(1+r)^2} \implies Q_1 - C_1 = \frac{C_2 - Q_2}{1+r} + \frac{C_3 - Q_3}{(1+r)^2}$$

Yes, it is exactly the intertemporal constraint:

$$C_1 + \frac{C_2}{1+r} + \frac{C_3}{(1+r)^2} = Q_1 + \frac{Q_2}{1+r} + \frac{Q_3}{(1+r)^2}$$

Problem 3

Assume the following utility function:

$$U(C_1, C_2) = \log C_1 + \frac{\log C_2}{1+\rho}$$

Assume that share α of consumers receive income only in the first period (Y_1), while the rest $1-\alpha$ share of consumers – only in the second one (Y_2). Consumers can lend and borrow only from each other.

Question 6

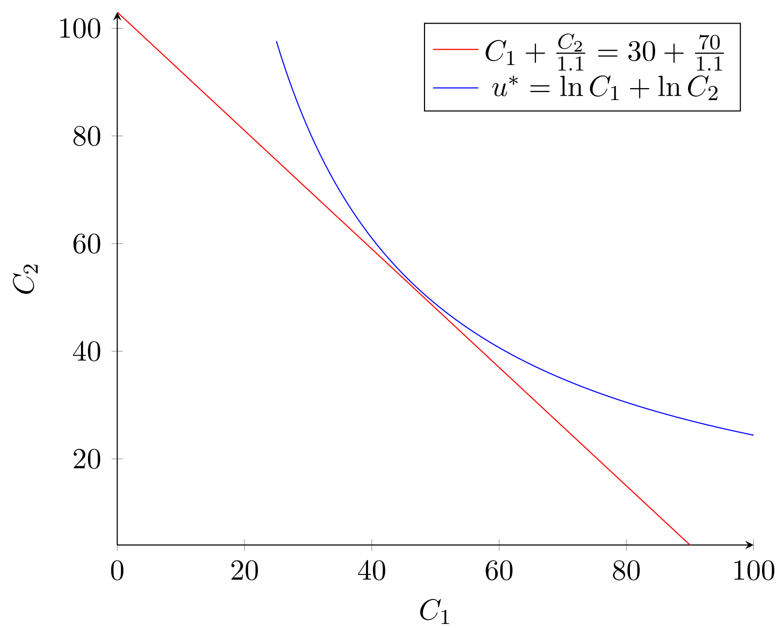
Find the supply and demand of loanable funds for the first period. Express it through income, discounting factor and real interest rate r.

Problem 4

Assume that a household has $Q_1 = 50$ and $Q_2 = 100$. It faces the following taxes: $T_1 = 20$, $T_2 = 30$. Disposable income equals $Q_i - T_i$. The household has utility function $U(C_1, C_2) = \ln C_1 + \ln C_2$. The market interest rate is $r = 0.1$.

Question 1

Show the budget constraint and indifference curve on the diagram. Is the consumer a net borrower or lender? Give a very brief explanation.



We can see that consumption C_1^* is higher than available disposable income $C_1^* > Q_1 - T_1 = 30$, thus agent borrows money and therefore he is net borrower.

Question 2

Evaluate the following statement: "as the subjective discount factor is one, individuals derive the same utility from any given unit of consumption in both periods. Therefore, $c_1 = c_2$ and it is not necessary to solve the maximization problem".

Well, the statement is latently assuming the fact that discount utility factor equals to $1+r$. Or in other words, solution to the more general task with discount factor β and borrow rate r and some utility function $u(.)$ is the **Euler equation**:

$$u'(c_1) = \beta(1+r)u'(c_2)$$

and if the following have been true $\beta = (1 + r)$ (which is not the case in our task), then the statement would have been true. So statement is false.

Question 3

Solve the maximization problem of the agent and find c_1, c_2. Are they equal? How is the result related to the answer to the previous question?

Agent's task:

$$\ln C_1 + \ln C_2 \to \max_{C_1, C_2}$$
$$s.t. \quad C_1 + \frac{C_2}{1.1} = 30 + \frac{70}{1.1}$$

Lagrangian and derivatives:

$$\mathcal{L} = \ln C_1 + \ln C_2 + \lambda \left(30 + \frac{70}{1.1} - C_1 - \frac{C_2}{1.1} \right)$$

$$\begin{cases} 1/C_1 = \lambda \\ 1/C_2 = \lambda/1.1 \end{cases} \implies C_2^* = 1.1 C_1^*$$

We can see clearly consumption is different in different periods.

Question 4

Now additionally assume that household can not borrow money. Repeat the previous question with this new assumption

Agent's task:

$$\ln C_1 + \ln C_2 \to \max_{C_1, C_2}$$
$$s.t. \quad C_1 + \frac{C_2}{1.1} = 30 + \frac{70}{1.1}$$
$$Y_1 - T_1 - C_1 \geq 0$$

Lagrangian and derivatives:

$$\mathcal{L} = \ln C_1 + \ln C_2 + \lambda_1 \left(30 + \frac{70}{1.1} - C_1 - \frac{C_2}{1.1} \right) + \lambda_2 (30 - C_1)$$

$$\begin{cases} 1/C_1 = \lambda_1 + \lambda_2 \\ 1/C_2 = \lambda_1/1.1 \end{cases} \implies 1/C_1 = 1.1/C_2 + \lambda_2 \implies C_2 = 1.1 C_1 + \lambda_2 C_1 C_2$$

If $Y_1 - T_1 - C_1 \geq 0$ binds, then we have that $C_1 = 30$, $C_2 = 70$ and $U = \ln 2100$. If it doesnt bind, then