Macroeconomics-2 - Assignment #2

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Question 1

Assume an individual who lives for infinite periods in a discrete time model. He has concave utility, earns an income of y each period. The subjective discount factor is equal to $\beta = 1/(1+\rho)$ and real interest rate is equal to r. Draw the consumption path of this individual for each of the following cases:
(1) $r_1 = r_2 = r_3 = \cdots = \rho$
(2) $r_1 = r_2 = r_3 = \cdots < \rho$
(3) $r_1 < \rho$ and $r_2 = r_3 = \cdots = \rho$
(4) $r_1 = r_5 = r_6 = \cdots = \rho$ and $r_2 = r_3 = r_4 < \rho$
(5) $r_1 = r_3 = r_5 = \cdots < \rho$ and $r_2 = r_4 = r_6 = \cdots = \rho$

Before solving every point of the task, lets derive the maximization task and its solution. Individual solves the following task, by choosing values of c_t:

$$\begin{cases} U(c) = \sum_{t=1}^{\infty} \dfrac{u(c_t)}{(1+\rho)^{t-1}} \to \max\limits_{\{c_t\}_{t=1}^{\infty}} \\ \sum_{t=1}^{\infty} \dfrac{c_t}{\prod_{i=1}^{t-1}(1+r_i)} = \sum_{t=1}^{\infty} \dfrac{y_t}{\prod_{i=1}^{t-1}(1+r_i)} \end{cases}$$

Lagrangian is the following:

$$\mathcal{L} = \sum_{t=1}^{\infty} \frac{u(c_t)}{(1+\rho)^{t-1}} + \lambda \left(\sum_{t=1}^{\infty} \frac{y_t}{\prod_{i=1}^{t-1}(1+r_i)} - \sum_{t=1}^{\infty} \frac{c_t}{\prod_{i=1}^{t-1}(1+r_i)} \right)$$

Taking derivatives yields:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial c_t} = 0: \quad \dfrac{u'(c_t)}{(1+\rho)^{t-1}} = \dfrac{\lambda}{\prod_{i=1}^{t-1}(1+r_i)} \qquad \forall t = 1 \ldots \\ \dfrac{\partial \mathcal{L}}{\partial c_{t+1}} = 0: \quad \dfrac{u'(c_{t+1})}{(1+\rho)^{t}} = \dfrac{\lambda}{\prod_{i=1}^{t}(1+r_i)} \qquad \forall t = 1 \ldots \end{cases}$$

Which is the **EE**:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1+r_t}{1+\rho} \qquad \forall t = 1\dots$$

1. $r_1 = r_2 = r_3 = \dots = \rho$

$$r_t = \rho \implies u'(c_t) = u'(c_{t+1}) \implies c_t = c_{t+1} \quad \forall t = 1\dots$$

2. $r_1 = r_2 = r_3 = \dots < \rho$

$$r_t < \rho \implies u'(c_t) < u'(c_{t+1})$$

Considering the concavity of utility function we get:

$$c_t > c_{t+1} \quad \forall t = 1\dots$$

3. $r_1 < \rho$ and $r_2 = r_3 = \dots = \rho$

$$r_1 < \rho \implies u'(c_1) < u'(c_2) \implies c_1 > c_2 = c_3 = \dots$$

4. $r_1 = r_5 = r_6 = \dots = \rho$ and $r_2 = r_3 = r_4 < \rho$

$$\begin{cases} r_2 = r_3 = r_4 < \rho \\ r_5 = r_6 = \dots \end{cases} \implies c_2 > c_3 > c_4 > c_5 = c_6 = \dots$$

and also applying that $r_1 = \rho$:

$$c_1 = c_2 > c_3 > c_4 > c_5 = c_6 = \dots$$

5. $r_1 = r_3 = r_5 = \dots < \rho$ and $r_2 = r_4 = r_6 = \dots = \rho$

$$\begin{cases} u'(c_{2t-1}) < u'(c_{2t}) \\ u'(c_{2t}) = u'(c_{2t+1}) \end{cases} \implies c_1 > c_2 = c_3 > c_4 = c_5 > \dots$$

Question 4

Consider two-period model of consumption. Let the utility function of an individual be

(1) $y_1 = 20, y_2 = 90$. Borrowing at $r_b = 150\%$, saving at 50%. In order to solve the consumer problem we should find the maximum of utility function for two cases and compare their values:

$$\begin{cases} \max \sqrt{c_1} + \sqrt{c_2} \\ c_1 + \dfrac{c_2}{1+1.5} = 20 + \dfrac{90}{1+1.5}, & \text{if } c_1 \geq 20 \\ c_1 + \dfrac{c_2}{1+0.5} = 20 + \dfrac{90}{1+0.5}, & \text{if } c_1 \leq 20 \end{cases}$$

(a) Case $c_1 \geq Y_1$

$$\mathcal{L} = \sqrt{c_1} + \sqrt{c_2} + \lambda_1 \left(20 + \frac{90}{2.5} - c_1 - \frac{c_2}{2.5} \right) + \lambda_2 (c_1 - 20)$$

FOCs:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial c_1} = 0 : & \dfrac{1}{2\sqrt{c_1}} - \lambda_1 + \lambda_2 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial c_2} = 0 : & \dfrac{1}{2\sqrt{c_2}} - \lambda_1/2.5 = 0 \\ \lambda_2[c_1 - 20] = 0 \end{cases} \quad \text{assuming } c_1 > 20 \implies \sqrt{\frac{c_2}{c_1}} = 2.5 \implies c_2 = 6.25c_1$$

Inserting to the budget constraint yields:

$$3.5c_1 = 56 \implies c_1 = 16 < Y_1 = 20$$

We got $c_1 = 16$, assuming that $c_1 > 20$, therefore what we got is not a maximum. The only case left is $c_1 = 20 \implies c_2 = 90$. Value of utility function at the maximum is $U(20, 90) = \sqrt{20} + \sqrt{90}$.

(b) Case $c_1 \leq Y_1$: FOCs:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial c_1} = 0 : & \dfrac{1}{2\sqrt{c_1}} - \lambda_1 - \lambda_2 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial c_2} = 0 : & \dfrac{1}{2\sqrt{c_2}} - \lambda_1/1.5 = 0 \end{cases} \quad \text{assuming } c_1 < 20 \implies \sqrt{\frac{c_2}{c_1}} = 1.5 \implies c_2 = 2.25c_1$$

Inserting to the budget constraint yields:

$$2.5c_1 = 80 \implies c_1 = 32 > Y_1 = 20$$

What we got is not a maximum. The only thing is left is $c_1 = 20$, $c_2 = 90$.

(2) $r_b = r_s = 0.5$.

$$\begin{cases} \max \sqrt{c_1} + \sqrt{c_2} \\ c_1 + \dfrac{c_2}{1+1.5} = 56 \end{cases}$$

$$\mathcal{L} = \sqrt{c_1} + \sqrt{c_2} + \lambda \left(56 - c_1 - \frac{c_2}{2.5}\right)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial c_1}: \quad \dfrac{1}{2\sqrt{c_1}} = \lambda \\ \dfrac{\partial \mathcal{L}}{\partial c_2}: \quad \dfrac{1}{2\sqrt{c_2}} = \dfrac{\lambda}{2.5} \end{cases} \implies c_2 = 6.25 c_1 \implies c_2 = 100, c_1 = 16, \quad s = Y_1 - c_1 = 4$$

(3) $r_b = r_s = 0.5, y_1 = 30, y_2 = 90$. Using results from the previous question $c_2 = 2.25c_1$

$$2.5c_1 = 66 \implies c_1 = 26.4 \quad c_2 = 165$$

MPC is the following:

$$c_1 = \frac{Y_1}{2.5} + 14.4 \implies c_1'(Y_1) = 0.4$$

(4)

Question 5

Two individuals who are optimizing their consumption path over an infinite horizon have different utility functions. While the utility of individual u is given by:

$$U(C) = \ln c_1 + \frac{\ln c_2}{1+\rho} + \frac{\ln c_3}{(1+\rho)^2} + \cdots = \sum_{t=1}^{\infty} \frac{\ln c_t}{(1+\rho)^{t-1}}$$

the utility of individual v is given by

$$V(C) = \sqrt[3]{c_1} + \frac{\sqrt[3]{c_2}}{1+\rho} + \frac{\sqrt[3]{c_3}}{(1+\rho)^2} + \cdots = \sum_{t=1}^{\infty} \frac{\sqrt[3]{c_t}}{(1+\rho)^{t-1}}$$

Individuals u and v have the same income path $\{y_t\}_{t=1}^{\infty}$. Moreover, they have the same time preferences and face the same interest rate.

(1) Show that both utility functions satisfy the standard assumptions assumed in our model.
(2) Find out the relationship between consumption in two subsequent periods for both individuals.
(3) Could it be that both individuals have the same consumption path? Explain
(4) Assume that the interest rate r is higher than ρ.
  (a) How does the consumption path look like?
  (b) Draw the consumption path of both individuals and compare them.
(5) Assume that individual u now knows that her income will increase by amount x starting from period 3.
  (a) How does the consumption path of individual u change?
  (b) Does you answer to the previous section depend on the whether $r >\!<\!= p$?
  (c) Draw all possible cases

(1) Standart assumptions about the utility function:

  (a) Non-decreasing by c_t and concavity with respect to c_t. Function $U(.)$ is concave iff $U''(.) < 0$:

$$\frac{\partial U(C)}{\partial c_t} = \frac{1}{c_t(1+\rho)^{t-1}} > 0 \qquad \frac{(\partial U(C))^2}{\partial^2 c_t} = -\frac{1}{c_t^2(1+\rho)^{t-1}} < 0 \quad \forall t = 1\ldots$$

$$\frac{\partial V(C)}{\partial c_t} = \frac{1}{3\sqrt[3]{c_t^2}(1+\rho)^{t-1}} > 0 \qquad \frac{(\partial U(C))^2}{\partial^2 c_t} = -\frac{2}{9c_t^{5/3}(1+\rho)^{t-1}} < 0 \quad \forall t = 1\ldots$$

(2) We need to solve the optimization problem:

(a) For agent u:

$$\begin{cases} \sum_{t=1}^{\infty} \frac{\ln c_t}{(1+\rho)^{t-1}} \to \max_{c_t} \\ \sum_{t=1}^{\infty} \frac{c_t}{\prod_{i=1}^{t-1}(1+r_i)} = \sum_{t=1}^{\infty} \frac{y_t}{\prod_{i=1}^{t-1}(1+r_i)} \end{cases}$$

Since the task is fully equivalent to the agent's task of question 1, we can use the **EE** from there:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1+r_t}{1+\rho} \quad \Longrightarrow \quad \frac{c_{t+1}}{c_t} = \frac{1+r_t}{1+\rho}$$

(b) For agent v:

$$\begin{cases} \sum_{t=1}^{\infty} \frac{\sqrt[3]{c_t}}{(1+\rho)^{t-1}} \to \max_{c_t} \\ \sum_{t=1}^{\infty} \frac{c_t}{\prod_{i=1}^{t-1}(1+r_i)} = \sum_{t=1}^{\infty} \frac{y_t}{\prod_{i=1}^{t-1}(1+r_i)} \end{cases}$$

and, again, using the **EE**:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1+r_t}{1+\rho} \quad \Longrightarrow \quad \frac{c_{t+1}}{c_t} = \sqrt[2]{\left(\frac{1+r_t}{1+\rho}\right)^3}$$

(3) Since both agents receive the sane value of income, difference in consumption is fully determined by **EE**. So, if **EE** is identical for all agents for every value of t, then consumers will have the same consumption path, or mathematically:

$$\forall t = 1, 2, \ldots \qquad \frac{1+r_t}{1+\rho} = \left(\frac{1+r_t}{1+\rho}\right)^{3/2}$$
$$1 = \left(\frac{1+r_t}{1+\rho}\right)^{1/2}$$
$$1+\rho = 1+r_t \iff \rho = r_t$$

So if both consumers face $r_t = \rho$, then they will have the same consumption path.

(4) Assuming that $r_t > \rho$

(a) Both eulers equations (thanks to concavity of functions $u(c_t)$ and $v(c_t)$) yield that next period consumption is higher than the present one, or:

$$r_t > \rho \implies c_{t+1} > c_t$$

But for the u and v growth of consumption gain is different: v's agent consumption is increasing faster:

$$\underbrace{c_{t+1} = c_t \frac{1+r_t}{1+\rho}}_{\text{Agent } u} \qquad \underbrace{c_{t+1} = c_t \left(\frac{1+r_t}{1+\rho}\right)^{3/2}}_{\text{Agent } v}$$

(5) (a) Since the y_t does not affect the Euler Equation, the c_{t+1} to c_t ratio will not change. But still the absolute value for c_t will be increased since more income is available.

(b) yes, if $r_t > \rho$, then consumption is increasing.