Problem 1
Consider a pair of numbers x and y with $x > y > 0$, and monetary lotteries defined. Show that the following pattern violates the independence axiom:

$$L_1 \succ L_1' \quad \text{and} \quad L_2 \prec L_2'$$

Solution

Preference choice $L_1 \succ L_1'$ implies that:

$$y > 0.1x + 0.8y \quad \Longrightarrow \quad 0.2y > 0.1x$$

whereas choice $L_2 \prec L_2'$ implies that:

$$0.2y < 0.1x$$

since we got impossible situation, independece axiom is violated.

Problem 2
Given a finite prize space X, let $\succsim$ be a preference relation on the set of all lotteries that satisfies the independence axiom. First prove statement (i) below. Then use (i) to prove statement (ii).

1. $L \succ L'$ implies $L \succ \gamma L + (1-\gamma)L' \succ L'$ for any $\gamma \in (0,1)$
2. $L \succ L'$ implies $\alpha L + (1-\alpha)L' \succ \beta L + (1-\beta)L'$ for any $\alpha > \beta \in (0,1)$

Solution

1. Lets assume, that $L \precsim \gamma L + (1-\gamma L')$ then by independence axiom:

   $$\gamma L + (1-\gamma)L \precsim \gamma L + (1-\gamma)L' \Longrightarrow L \precsim L', \quad \text{but } L \succ L' \quad \Longrightarrow \text{ contradiction}$$

   Now lets assume, that $\gamma L + (1-\gamma L') \precsim L'$ then by independence axiom:

   $$\gamma L + (1-\gamma)L' \precsim \gamma L' + (1-\gamma)L' \Longrightarrow L \precsim L', \quad \text{but } L \succ L' \quad \Longrightarrow \text{ contradiction}$$

   From two contradictions above we can conclude the following:

   $$L \succ \gamma L + (1-\gamma)L'$$
   $$\gamma L + (1-\gamma)L' \succ L'$$

   Thus $L \succ L'$ implies $L \succ \gamma L + (1-\gamma)L' \succ L'$ for any $\gamma \in (0,1)$

2. Let $\beta > \alpha$. Lets rewrite the following:

   $$\alpha L + (1-\alpha)L' = \gamma L + (1-\gamma)[\beta L + (1-\beta)L'] \quad \Longrightarrow \quad \gamma = \frac{\alpha - \beta}{1-\beta}$$
   $$\gamma L + (1-\gamma)(\beta L + (1-\beta)L') \succ \beta L + (1-\beta)L'$$

   Assuming that $\alpha < \beta$, by the argument proven previously, we must have then

   $$\beta L + (1-\beta)L' \succ \alpha L + (1-\alpha)L'$$