HOME ASSIGNMENT 3

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Question (1)**.** An unemployed agent is looking for a job. Regardless of whether the agent is employed or not, wage offers on the market follow a Markov process x_t that is governed by a Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.9 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \end{pmatrix}$$

and numeric values of the wages are given by the vector $W = (15\ 12\ 2)^T$. If the agent is unemployed and accepts the current offer, he earns that wage starting from the current period and until he departs.

The employer cannot terminate the worker; however, the worker can quit with a one-period advance notice. That is, in any period t in which he has a job, after learning x_t and getting paid for the current period, the worker may give his employer a notice, and in this case he enters the job market in period $t+1$ – draws an offer and decides whether to accept it or stay unemployed and try his luck in the period/periods. The agent's objective is to maximize $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t y_t\right]$, where y_t is her income – either ω if she is employed with the wage ω, or c if she is uemployed in period t. The discount factor $\beta = 4/5$ and the unemployment compensation $c = 3$.

(1) Write the problem recursively (the Bellman equation). Be careful and detailed in how you define your value function – ambiguity and omissions at this stage will be explicitly penalized.
(2) Find the optimal strategy of the worker and expected discounted present value of his income stream as functions of the state of the Markov process.

Solution. Let V_i, V_{ij}^e and V_i^u be components of the agent's value function defined as follows: each of these scalar variables is the maximum of agent's objective function at the beginning of the current period if he observes the curret state $i \in \{1, 2, 3\}$ and:

(1) V_i – the agent is about to decide whether to accept the offer (continue working) or reject the wage offer (leave the job) at wage W_i
(2) V^e_{ij}– the agent is employed at wage W_j
(3) V^u_i – agent has just rejected the offer W_j

Let $V = (V_1\ V_2\ V_3)^T$ and $V^e_j = \left(V^e_{1j}\ V^e_{2j}\ V^e_{3j}\right)^T$. Then V_i, V^e_{ij} and V^u_i should satisfy the following Bellman equation system:

$$\begin{cases} V_i = \max\{V^e_{ii}, V^u_i\} \\ V^e_{ij} = \max\{W_j + \beta P_i V^e_j,\ W_j + \beta P_i V\} \\ V^u_i = c + \beta P_i V \end{cases}$$

Firstly lets notice that, because $W_1 = 15$ is the highest salary possible, than at state $x_t = 1$ agent will choose employment:

$$V_1 = \max\{V^e_{11}, V^u_1\} \implies V_1 = \frac{15}{1 - 4/5} = 75 \tag{1}$$

By the same logic since $2 =: W_3 < c := 3$ we can state that $V_3 = V^u_3$ and as a result:

$$V_3 = \max\{V^e_{33}, V^u_3\} = V^u_3 = c + \beta\left(\sum_{i=1}^{3} P_{3i} V_i\right) = c + \beta\left(0.2V_1 + 0.8V_3\right)$$

Therefore we can derive the value for V_3 numerically:

$$V_3 = c + \beta\left(0.2V_1 + 0.8V_3\right) = \frac{1}{1 - 0.8\beta}\left(c + 0.2\beta V_1\right)$$

Substituting values for parameters and V_1:

$$V_3 = \frac{1}{0.36}\left(3 + 12\right) \sim 41.66 \tag{2}$$

The last thing to understand is the value for V_2:

$$\begin{cases} V_2 = \max\{V^e_{22}, V^u_2\} \\ V^e_{22} = \max\{W_2 + \beta P_2 V^e_2,\ W_2 + \beta P_2 V\} \\ V^u_2 = c + \beta P_2 V \end{cases}$$

Incorporating everything to the first equation yields:

$$V_2 = \max\{V^e_{22}, V^u_2\} = \max\{\max\{\underbrace{W_2 + \beta P_2 V^e_2}_{A},\ \underbrace{W_2 + \beta P_2 V}_{B}\}, \underbrace{c + \beta P_2 V}_{C}\}$$

Lets calculate every value separately:

(A)

$$W_2 + \beta P_2 V_2^e = 12 + \frac{4}{5}\left(\sum_{i=1}^{3} P_{2i} V_{i2}^e\right) = 12 + \frac{4}{5}\left(0.9V_{12}^e + 0.1V_{22}^e\right)$$

(B)

$$W_2 + \beta P_2 V = 12 + \frac{4}{5}\left(0.9\underbrace{V_1}_{75} + 0.1V_2\right) = 12 + \frac{4}{5}\left(67.5 + 0.1V_2\right)$$

(C)

$$\begin{aligned} c + \beta P_2 V &= 3 + \frac{4}{5}\left(\sum_{i=1}^{3} P_{2i} V_i\right) = 3 + \frac{4}{5}\left(0.9V_1 + 0.1V_2\right) \\ &= 3 + \frac{4}{5}\left(67.5 + 0.1V_2\right) \end{aligned}$$

Lets assume first that $B > A$, then:

$$V_2 = B = 12 + \frac{4}{5}\left(67.5 + 0.1V_2\right) \implies V_2 = 71.7 \tag{3}$$

If we assumed the opposite, namely $B < A$ then:

$$V_2 = 12 + \frac{4}{5}\left(0.9V_{12}^e + 0.1V_{22}^e\right)$$

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