HOME ASSIGNMENT 2

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Question (1)**.** *John and Peter are bidding for an indivisible object. The rules of the auction are as follows. The individual who places the highest bid wins the object. If John wins he pays the amount of money Peter bade. If Peter wins he pays the amount of money he bade. Both players place their bids simultaneously. For each individual $i \in \{J, P\}$ let $v_i \geq 0$ denote their valuation for the object and let $b_i \geq 0$ denote the bid they placed. Their preferences are captured by the following payoff functions for John and Peter respectively:*

$$u_J(b_J, b_P) = \begin{cases} v_J - b_P & \text{if } b_J > b_P \\ \dfrac{1}{2}(v_J - b_P) & \text{if } b_J = b_P \\ 0 & \text{if } b_J < b_P \end{cases}$$

$$u_P(b_J, b_P) = \begin{cases} v_P - b_P & \text{if } b_P > b_J \\ \dfrac{1}{2}(v_P - b_P) & \text{if } b_P = b_J \\ 0 & \text{if } b_P < b_J \end{cases}$$

The information above (including valuations) is common knowledge. Does either player have a weakly dominant strategy to bid his valuation? Can you find values for the parameters $(v_J, v_P) \in \mathbb{R}^2_+$ such that the pair of strategies $(b_J, b_P) = (v_J, v_P)$ constitutes a Nash Equilibrium? For each answer you give provide the relevant demonstration

Solution. The solution to this taks bases on the common knowledge assumption, which in simple words describes how agents select their actions. In simple terms, John will select his bid considering the fact that Peter knows that John knows that ... knows others' actions are to maximise the utility. It is clear that optimal strategy of all bidders depends on the parameters v_P and v_J (and thus whether this auction has NE). In order to construct the solution, lets find out how players are going to choose their strategies keeping in mind possible actions of opponents with different values of v_P and v_J.

(1) **Case** $v_P > v_J$.
Logic of infinite "i know that he knows that i know ..." suggests

us the following reasoning. Whereas Peter values object more then John, he will keep increasing the bid until John gives up. This is exactly the bid value of v_J. But Peter doesnt want to bid the same amount as John, so he will bid a little bit more to win the auction and achieve more utility. Therefore predicted result of the game (with some remarks described futher) is the following in this case:

$$b^* = (b_J^*, b_P^*) = ([0, v_J], v_J + \varepsilon)$$

Since we cant assume that the auction has the fixed amount for bid size we cant say for sure that b^* is an equilibrium in this case just because Peter has incentives to decrease the value of ε as far as possible. For example he can do better if he selects $\varepsilon^* = \varepsilon/2$. So without assumption of the minimal bid size there is no NE in this case. *And there is weekly dominant strategy for John, but no for Peter.*

(2) **Case** $v_P = v_J = v$.
If both players valuate the object at the same value $v = v_P = v_J$, then both players have incentives to increase their bids until the level of v. Once they reach this level, they will not increase possible payoff somehow by deviating unilaterally. Therefore predicted result of the game (which is also NE) in this case is the following:

$$b^* = (b_J^*, b_P^*) = (v_J, v_P)$$

As I have already mentioned, the result is weak NE since nobody can strictly increase their payoff by deviating unilaterally (they can onlly lower their bids and get the same value of utility, which equals to zero). *In this case there is weekly dominant strategy for both John and Peter.*

(3) **Case** $v_P < v_J$.
In this case Peter will stop increasing his bid after reaching the value of v_P, whereas John's utility doesnt decrease with respect to his bid size, which implies that he can choose bid as huge as he wishes. Therefore predicted results of the game are:

$$b^* = (b_J^*, b_P^*) = ((v_P, +\infty), [0, v_P])$$

In this case there is weekly dominant strategy for both John and Peter.

Therefore only John has weakly dominant strategy in every case. And as we have previously seen, only if $v_P = v_J$, then there is weak NE such that:

$$(b_J^*, b_P^*) = (v_J, v_P)$$

. □

Question (2)**.** *Prove the following statement In a finite simultaneous move game representable in normal form,*

$$\Gamma = (I,\ \times_{i\in I} S_i,\ (u_i)_{i\in I})$$

if some pure strategy $s_i \in S_i$ strictly dominated by some $\sigma_i \in \Delta(S_i)$, then for any Nash Equilibrium of the game $\sigma^* \in \times_{i\in I}\Delta(S_i)$ we have $\sigma_i^*(s_i) = 0$ (that is, the equilibrium strategy of player i assigns zero probability to pure strategy s_i).

Hint: Proceed by contradiction. As a first step, show that $\sigma_i^*(s_i) \neq 1$. If you show as much, you get half the credit for this exercise. Of course, any proof that is correct and diverges from my suggestion will get full credit.

Solution. Let us suppose that $\sigma^*(s_i) = 1$ and s_i is strictly dominated pure strategy. Then we can conclude that by the definition of weak NE: $\forall \sigma_i \in \Delta(S_i)$ the following true:

$$u(s_i, \sigma_{-i}^*) \geq u(\sigma_i, \sigma_{-i}^*)$$

But since we assumed that s_i is dominated strategy, that implicates that $\exists \sigma'$ such that

$$u(\sigma'_{-i}, \sigma_{-i}^*) > u(s_i, \sigma_{-i}^*)$$

and therefore we got contradiction, by assuming that $\sigma^*(s_i) = 1$ □

Question (3)**.** *There are $n < +\infty$ firms in an industry. Each can try to convince Congress to give the industry a subsidy. Let h_i denote the hours of effort put by some industry $i \in \{1, \dots, n\}$. Let $c(h_i) = w_i h_i^2$, with $w_i > 0$, be the cost of this effort to firm i. For each profile of efforts $(h_1, \dots, h_n) \in \mathbb{R}_+^n$, the value of the subsidy that gets approved equals*

$$\alpha \sum_{i\in\{1\dots n\}} h_i + \beta \left(\prod_{i\in\{1\dots n\}} h_i \right)$$

where $\alpha, \beta \geq 0$. The approved subsidy is shared equally among the firms in the industry. Suppose that all firms choose the effort level simultaneously. Show that each firm has a strictly dominant strategy if and only if $\beta = 0$. What is firm i's strictly dominant strategy when this is so?

Solution. Firms decide how much time is devoted to approving subsidy is by profit maximization task, where $w_i h_i^2$ is the value needed to pay for efforts put. Then the profit is the following:

$$\pi(h_i) = \alpha \sum_i h_i + \beta \left(\prod_i h_i \right) - w_i h_i^2$$

Then derivatives look like:

$$\frac{\partial \mathcal{L}}{\partial h_i} : \quad \alpha + \beta \prod_{j \neq i}^{n} h_j - 2 w_i h_i = 0$$

From where it follows:

$$h_i^* = \frac{1}{2w_i} \left(\alpha + \beta \prod_{j \neq i}^{n} h_j \right)$$

We can see that h_i^* depends on the decisions of the competitors, thus in order to have unique solution we need to be independent from them, which is possible whereas $\beta = 0$. If this is the case, then the resulting optimal value for h_i is:

$$h_i^* = \frac{\alpha}{2w_i}$$

□