Problem 1

1. Case 1. Quasilinear function.
2. Case 2. Homothetic function.

Solution

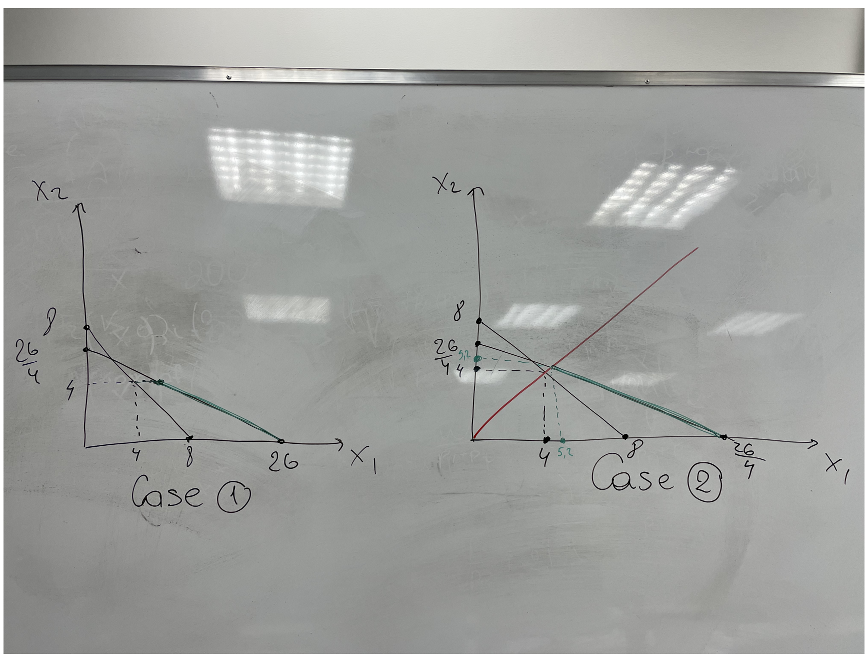
1. Case 1.

   Since utility function is quasilinear with respect to first good, it means that

   $$u(x_1, x_2) = x_1 + v(x_2)$$

   where $v(x_2)$ is a concave function. From UMP solution we get that $\partial v(x_2)/\partial x_2 = p_2/p_1$. Since concaves function derivative is decreasing with growth of x_2, than the higher p_2/p_1 the less x_2. We can see that in the period $t = 0$, x_2 was higher since price relation was equal to $p_2/p_1 = 1$ and x_2 in the period $t = 1$ would be lower, since price relation increased $p_2/p_1 = 4$. Ilustration is following

2. Case 2. Considering homothetic preferencies we must outline their specifics. First of all, it increases with respect to commodities growth proportianally. Secondly, the relation of marginal utilities is the same on a ray with angle 45 grad. Therefore we would expect it to have the same MRS on a new budget line. The picture of possible solutions on the image below:



Problem 2

Let $\succsim$ be a complete and transitive preference relation on a convex set $X \subset \mathbb{R}^L_+$. Suppose

$$y \succsim x \quad \Longrightarrow \quad \alpha y + (1-\alpha)x \succsim x \quad \forall \alpha \in (0,1)$$

Show that this implies convexity of $\succsim$, as defined at p. 44 in the textbook.

Solution

In the textbook, convexity of $\succsim$ by definition is the following: let $X \subset \mathbb{R}^L_+$ is a consumption set. Then $\forall x, y, z \in X$, if $y \succsim x$ and $z \succsim x$ follows:

$$\alpha y + (1-\alpha)x \succsim x \quad \forall \alpha \in (0,1)$$

To use book's notation we should provide $z \in X$, such that $z \succsim x$. Let $z = \alpha y + (1-\alpha)x$. It is obvious, that $z \in X$ and $z \succsim x$ (by the condition). Lets suppose, that by contradiction the following holds:

$$
\begin{aligned}
\alpha y + (1-\alpha)z &\prec x \\
\alpha y + (1-\alpha)(\alpha y + (1-\alpha)x) &\prec x \\
\alpha y + \alpha y + (1-\alpha)x - \alpha^2 y - \alpha(1-\alpha)x &\prec x \\
2\alpha y + x - \alpha x - \alpha^2 y - \alpha x + \alpha^2 x &\prec x \\
x\left(\alpha^2 - 2\alpha + 1\right) + y\left(2\alpha - \alpha^2\right) &\prec x \\
y\left(2\alpha - \alpha^2\right) &\prec x\left(2\alpha - \alpha^2\right) \implies y \prec x
\end{aligned}
$$

The last implication is certainly a contradiction, which means that for any $z = \alpha y + (1-\alpha)x$ convexity holds (under stated restrictions and completeness of preferences), which means that:

$$\alpha y + (1-\alpha)z \succsim x$$

Problem 3
Consider a differentiable demand function $x(p,w)$ associated with a (continously) defferentiable utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$. Find a simple expression for $\partial V(p,w)/\partial w$, where V is the inderect utility function.

Solution

Since $x(p,w)$ is a demand function, then it is solution to UMP problem. Then, by the definition of inderect utility function:

$$V(p,w) = u(x(p,w))$$

then, assuming, that $x >> 0$, meaning that we have inner solution to UMP problem:

$$\frac{\partial V(p,w)}{\partial w} = \frac{\partial u(x(p,w))}{\partial w} = \frac{\partial u(x(p,w))}{\partial x(p,w)} \frac{\partial x(p,w)}{\partial w} \tag{1}$$

since the Walras law holds, which implies that $px(p,w) = w$ and therefore $\partial x(p,w)/\partial w = 1/p$ and considering inner solutions we have that at optimum $\nabla u(x(p,w)) = \lambda p$. Combining this, we can transform 1 to:

$$\lambda p \frac{1}{p} = \lambda$$

Therefore $\partial V(p,w)/\partial w = \lambda$

Problem 4
Compute the Walrasian demand and the indirect utility functions associated with the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ below:

1. $u(x_1, x_2) = \min\{\alpha x_1, x_2\}$, where $\alpha > 0$ is a parameter.
2. $u(x_1, x_2) = \left(\sqrt{x_1} + \sqrt{x_2}\right)^2$

Solution

1. Case $\min\{\alpha x_1, x_2\}$

   First of all, one can see, that the min function is maximized once $\alpha x_1 = x_2$. It follows from the fact, that if αx_1 greater (smaller) x_2, then consumer can substitute consumption of goods in order to achieve the higher utility. For example, suppose $px < w$ and $x_2 = x_1 + 2\varepsilon$. Then, utility function is $\min\{x_1, x_1 + 2\varepsilon\} = x_1$. Lets suppose, that now consumer can increase consumption of x_1 and decrease consumption of x_2 within his budget constraints on the value ε, so that

$$u(x_1 + \varepsilon, x_2 - \varepsilon) = x_1 + \varepsilon > x_1$$

From the logic above it becomes clear, that maximization points of utility function are such that:

$$\alpha x_1^* = x_2^*$$

It is also worth noting, that because $u(0,x_2) = u(x_1,0) = 0$, then solution would lie on a budget constraint and $x_1 > 0, x_2 > 0$. Demand functions:

$$x_1^*(p_1,p_2,w) = \frac{w}{p_1+\alpha p_2} \qquad x_2^*(p_1,p_2,w) = \frac{\alpha w}{p_1+\alpha p_2}$$

And indirect utility function is:

$$V(p_1,p_2,w) = \min\left\{\frac{\alpha w}{p_1+\alpha p_2}, \frac{\alpha w}{p_1+\alpha p_2}\right\}$$

2. Case $u(x_1,x_2) = \left(\sqrt{x_1}+\sqrt{x_2}\right)^2$:

   First, we notice, that we can make our life easier by applying monotonic transformation to the utility function: $u' = \sqrt{u} \implies$

$$u(x_1,x_2) = \sqrt{x_1}+\sqrt{x_2}$$

   Then UMP is the following:

$$UMP \implies \begin{cases} \max_{x_1,x_2\geq 0} \sqrt{x_1}+\sqrt{x_2} \\ p_1x_1+p_2x_2 \leq w \end{cases}$$

   We should also notice two facts about our target function. First of all, its restriction set is a compact set, which implies that task will have solutions. Secondly, function is a sum of two concave functions, therefore it is also concave (strictly), meaning that it will have inner solutions. It is also worth noting, that budget constraint binds, since utility function is increasing in both parameters. Then:

$$\mathscr{L} = \sqrt{x_1}+\sqrt{x_2}+\lambda(w-p_1x_1-p_2x_2) \to \max_{x_1,x_2>0}$$

$$\frac{\partial\mathscr{L}}{\partial x_1}: \quad \frac{1}{2\sqrt{x_1}} = \lambda p_1$$
$$\frac{\partial\mathscr{L}}{\partial x_2}: \quad \frac{1}{2\sqrt{x_2}} = \lambda p_2$$

   From F.O.C it follows that:

$$\frac{x_2^*}{x_1^*} = \left(\frac{p_1}{p_2}\right)^2$$

   Supplying to the budget constraint and obtaining demand functions:

$$p_1x_1+\frac{p_1^2}{p_2}x_1 = w \quad \implies \quad x_1^*(p_1,p_2,w) = \frac{w}{p_1(1+p_1/p_2)} = \frac{p_2}{p_1}\frac{w}{p_1+p_2} \quad x_2^*(p_1,p_2,w) = \frac{p_1}{p_2}\frac{w}{p_1+p_2}$$

   We can stop there because $u(x_1^*,x_2^*) > u(0,w/p_2)$ and also $u(x_1^*,x_2^*) > u(w/p_1,0)$ (we have only one inner solution). And the indirect utility function would be the following:

$$V(p_1,p_2,w) = \left(\sqrt{x_1^*}+\sqrt{x_2^*}\right)^2 = \left(\sqrt{\frac{p_2}{p_1}\frac{w}{p_1+p_2}}+\sqrt{\frac{p_1}{p_2}\frac{w}{p_1+p_2}}\right)^2$$