Problem 1
(1.C.1) Consider the choice structure $(\mathscr{B}, C(\cdot))$ with $\mathscr{B} = (\{x,y\},\{x,y,z\})$ and $C(\{x,y\}) = \{x\}$. Show that if $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, then we must have $C(\{x,y,z\}) = \{x\}, = \{z\}$, or $= \{x,z\}$.

Solution. By the definition of WARP: if for any B, such that $x, y \in B$ and $B \subset \mathscr{B}$ it is true that $x \in C(B)$, then for all other B', such that $x, y \in B'$ the following should be satisfied:

$$x \in B'$$

Let's suppose, that $y \in C(\{x,y,z\})$, then:

$$\begin{cases} y \in \{x,y,z\} & \text{by condition} \\ y \in \{x,y\} & \text{by condition} \\ y \in C(\{x,y,z\}) & \text{by assumption} \end{cases} \xRightarrow{\text{WARP}} y \in C(\{x,y\}) \tag{1}$$

But conditions state, that $y \notin C(\{x,y\}) = \{x\}$, thus we got contradiction.

Problem 2
(1.D.3) Let $X = \{x,y,z\}$, and consider the choice structure $(\mathscr{B}, C(\cdot))$ with

$$\mathscr{B} = \{\{x,y\},\{y,z\},\{x,z\},\{x,y,z\}\}$$

and $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$ and $C(\{x,z\}) = \{z\}$. Show that $(\mathscr{B}, C(\cdot))$ must violate the weak axiom.

Solution.
If the WARP is not violated:

1. From the $C(\{x,y\}) = \{x\}$ it should follow, that $y \notin C(\{x,y,z\})$
2. From the $C(\{y,z\}) = \{y\}$ it should follow, that $z \notin C(\{x,y,z\})$
3. From the $C(\{x,z\}) = \{z\}$ it should follow, that $x \notin C(\{x,y,z\})$

As we can see from the points above, $C(\{x,y,z\}) = \emptyset$ which is contradictionary to the defintion of choice function, thus WARP is violated.

Problem 3
Let $X = \mathbb{R}$, and define a pair of binary relations, R_1 and R_2, on this set as follows: For any $x, y \in X$:

$$x \, R_1 \, y \iff x \geq y - 1$$
$$x \, R_2 \, y \iff x \geq y + 1$$

For $i = 1;2$, determine if R_i is complete, transitive or asymmetric. Find a simple expression for the symmetric part $R_i^{\sim}$, which is defined as $x \, R_i^{\sim} \, y \iff [x \, R_i \, y \text{ and } y \, R_i \, x]$. Explain your answer.

Solution.

Completeness.
The binary relation R_i is complete if the following holds:

$$\forall x, y \in X, \text{ we have that either } (x \, R_i \, y) \text{ or } (y \, R_i \, x)$$

In other words, if $X \setminus A = \emptyset$, where A – set of points, such that $(x \, R_i \, y)$ or $(y \, R_i \, x)$, then relation is complete.

1. $i = 1$
   $A = \{y \geq x-1\} \cup \{y \leq x+1\}$. A is obviously a union of two rays, with non-empty intersection at $y \in [x-1, x+1]$ thus it covers the whole $\mathbb{R} \implies R_1$ is complete.

2. $i = 2$
   $A = \{y \leq x-1\} \cup \{y \geq x+1\}$. A doesnt include points $x, y \in X$, such that $y \in (x-1, x+1) \implies R_2$ is not complete.

Assymetry.
Lets start with $i = 1$. If asymmetry holds, then the following should be true:

$$\begin{aligned} x\,R_1\,y &\implies \neg(y\,R_1\,x) \\ x \geq y-1 &\implies \neg(y \geq x-1) \\ y \leq x+1 &\implies \neg(y \geq x-1) \\ y \leq x+1 &\not\Longrightarrow y < x-1 \end{aligned}$$

We see that $y \leq x+1$ doesnt imply $y < x-1$ (for example points $x, y \in X$ such that $x = y$), therefore R_1 is not
By the analogy for $i = 2$:

$$\begin{aligned} x\,R_2\,y &\implies \neg(y\,R_2\,x) \\ x \geq y+1 &\implies \neg(y \geq x+1) \\ y \leq x-1 &\implies \neg(y \geq x+1) \\ y \leq x-1 &\implies \neg(y \geq x+1) \\ y \leq x-1 &\implies y < x+1 \end{aligned}$$

We see that $y \leq x-1$ implies $y < x+1$, therefore R_2 is assymetric.

Transitivity.
Lets take $x, y, z \in X$. From transitivity the following should apply:

1. $i = 1$

$$(x\,R_1\,y) \wedge (y\,R_1\,z) \implies x\,R_1\,z$$

$$(x\,R_1\,y) \wedge (y\,R_1\,z) \iff \begin{cases} y \leq x+1 \\ z \leq y+1 \end{cases} \implies z \leq x+2 \not\Longrightarrow x \geq z-1 \iff x\,R_1\,z$$

   Thus relation is not transitive.

2. $i = 2$

$$(x\,R_2\,y) \wedge (y\,R_2\,z) \implies x\,R_2\,z$$

$$(x\,R_2\,y) \wedge (y\,R_2\,z) \iff \begin{cases} y \leq x-1 \\ y \geq z+1 \end{cases} \implies z \leq x-2 \implies x \geq z+1 \iff x\,R_1\,z$$

   Thus relation is transitive.

Simmetric part.
So we define $R_i^{\sim}$ as $[(x\,R_i\,y) \wedge (y\,R_i\,x)]$

1. $i = 1$

$$\begin{aligned} R_1^{\sim} &\iff [(x\,R_1\,y) \wedge (y\,R_1\,x)] \\ &\iff \begin{cases} y \leq x+1 \\ x \leq y+1 \end{cases} \implies 0 \leq 2 \quad \forall x, y \in X \\ &\iff x, y \in X \end{aligned}$$

2. $i = 2$

$$
\begin{aligned}
R_2^{\sim} &\iff [(x\ R_2\ y) \wedge (y\ R_2\ x)] \\
&\iff \begin{cases} x \geq y+1 \\ y \geq x+1 \end{cases} \implies 0 \geq 2 \quad \forall x, y \in X \\
&\iff x, y \notin X
\end{aligned}
$$

Problem 4
Consider a binary relation P on a set X that is assymmetric and negatively transitive. Define a further binary relation $\succsim_p$ on X as follows:

$$x \succsim_p y \iff \neg(y\ P\ x)$$

Show that $\succsim_p$ is a complete and transitive binary relation.

Solution.
Completeness.

$$\forall x, y \in X, \text{ we have that either } (x\ P\ y) \text{ or } (y\ P\ x)$$

Lets notice that $x \succsim_p y \iff \neg(y\ P\ x)$ and $y \succsim_p x \iff \neg(x\ P\ y)$, applying assymetricity we get:

$$x \succsim_p y \iff \neg(y\ P\ x) \tag{2}$$
$$x \succsim_p y \iff x\ P\ y \tag{3}$$

So from point (3) we see, that $x \succsim_P y$ $[x\ P\ y]$ is actually the opposite for $x \precsim_P y$ $[\neg(x\ P\ y)]$ and therefore relations are complete.
Transitivity.

$$\forall x, y, z \in X, \text{ we have that if } (x \succsim_p y) \text{ and } (y \succsim_p z) \text{ then } (x \succsim_p z)$$

$$\begin{cases} \neg(y\ P\ x) \\ \neg(z\ P\ y) \end{cases} \xrightarrow{\text{neg. trans}} \neg(z\ P\ x) = \neg(z\ P\ x) \quad \forall x, y, z \in X$$

Strict part.

$$x \succ_p y \xrightarrow{\text{definition}} \forall x, y \in X, \text{ we have that } (x \succsim_p y) \text{ and not } (y \succsim_p x)$$

$$
\begin{aligned}
&\neg(y\ P\ x) \wedge x\ P\ y \\
&x\ P\ y \wedge x\ P\ y \iff x\ P\ y
\end{aligned}
$$

Therefore $x \succ_p y \xrightarrow{\text{definition}} P$.

Problem 5
Consider a choice structure $(\mathscr{B}, C)$ on a set X. Suppose that any subset of X that has two elements belongs to $\mathscr{B}$ (in addition to other sets, possibly). Let $\succsim^*$ denote the revealed preference relation derived from C, as we denoted in class. Show that if C satisfies WARP, then for any $x, y \in X$, we have $x \succsim^* y$ iff $x \in C\{x, y\}$

Solution. Firstly, lets show, that there is rational revealed preference relation $\succsim^*$. The preference is rational once it satisfies transitivity and completeness:

1. Completeness. Since $\mathscr{B}$ includes all pairs of elements in X, then $\forall x, y \in X$ either $C(\{x, y\})$ is either $\{x\}$ or $\{y\}$ or $\{x, y\}$. Thus, all elements from X can be sorted by preference.

2. Transitivity. Let $x \succsim^* y$ and $y \succsim^* z$ and lets consider $B = \{x, y, z\}$. If $\succsim^*$ is rational, then $x \in C(B)$, because by the transitivity $x \succsim^* z$. Because of the completeness, $C(B) \neq \emptyset$. Lets suppose $C(\cdot)$ is such that $y \in C(B)$, then, by the WARP and $x \succsim^* y$, the x must be in $C(B)$. Suppose instead $z \in C(B)$. Then, by the WARP and $\succsim^*$ we conclude that $y \in C(B)$. From the previous logic, we conclude that x is also in $C(B)$.

Now, lets consider rationalized choice structure $C^*(B, \succsim)$ as

$$C^*(B, \succsim) = \{x \in B : x \succsim y,\ \forall y \in B\} \text{ for all B in } \mathscr{B}$$

and lets prove, that $C(B) = C^*(B, \succsim)$ for all B in $\mathscr{B}$. First, let $x \in C(B)$, then $x \succsim^* y, \forall y \in B$. Therefore $x \in C^*(B, \succsim^*)$, which means, that $C(B) \subset C^*(B, \succsim^*)$. From the other hand, if $x \in C^*(B, \succsim^*)$, then $x \succsim^* y$ for all $y \in B$. So, for each $y \in B$ there must be some set $B_y \in \mathscr{B}$, which satisfies $x, y \in B_y$ and $x \in C(B_y)$. Due to WARP, $x \in C(B)$, therefore $C^*(B, \succsim) \subset C(B)$. From this point it can be concluded, that $C(B) = C^*(B, \succsim^*)$.

And as X consists of all pairs of elementary objects, thus there is unqiue rationalizing preference relation $\succsim^*$, therefore we can get from choice structure and WARP to rational preference relation, or in other words

$$x \succsim^* y \iff x \in C\{x, y\}$$