Problem 3.G.15
Consider the utility function

$$u = 2x_1^{1/2} + 4x_2^{1/2}$$

1. Find the demand functions for goods 1 and 2 as they they depend on prices and wealth.
2. Find the compensated demand function $h()$
3. find the expenditure function and verify that $h(p,u) = \nabla_p e(p,u)$
4. Find the indirect utility function, and verify Roy's identity

Solution

1. We can find Walras demand by solving the UMP task:

$$\mathcal{L} = 2\sqrt{x_1} + 4\sqrt{x_2} + \lambda(w - p_1x_1 - p_2x_2) \to \max_{x_1,x_2}$$

FOCs are:

$$\frac{\partial \mathcal{L}}{\partial x_1} : \frac{1}{\sqrt{x_1}} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} : \frac{2}{\sqrt{x_2}} - \lambda p_2 = 0$$

$$\frac{1}{2}\frac{\sqrt{x_2}}{\sqrt{x_1}} = \frac{p_1}{p_2} \implies x_u^* = 4x_1\left(\frac{p_1}{p_2}\right)^2 \quad x_1\left(p_1 + 4\frac{p_1^2}{p_2}\right) = w \quad x_{1u}^* = \frac{p_2w}{p_1p_2 + 4p_1^2} \quad x_{2u}^* = \frac{4p_2w}{p_1p_2 + 4p_1^2}\left(\frac{p_1}{p_2}\right)^2$$

2. We can find compensated demand from EMP:

$$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(u - 2\sqrt{x_1} - 4\sqrt{x_2}) \to \min_{x_1,x_2}$$

FOCs are:

$$\frac{\partial \mathcal{L}}{\partial x_1} : p_1 - \lambda\frac{1}{\sqrt{x_1}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} : p_2 - \lambda\frac{2}{\sqrt{x_2}} = 0$$

$$h_1^* = 4x_1\left(\frac{p_1}{p_2}\right)^2 = \left(\frac{up_2}{8p_1 + 2p_2}\right)^2 \quad h_2^* = \left(\frac{up_1}{4p_1 + p_2}\right)^2$$

3.

$$e(p,u) = p_1h(p,u) + p_2h(p,u) = p_1\left(\frac{up_2}{8p_1 + 2p_2}\right)^2 + p_2\left(\frac{up_1}{4p_1 + p_2}\right)^2 = \frac{p_1p_2u^2}{4(4p_1 + p_2)}$$

$$\frac{\partial e}{\partial p_1} = \frac{4p_2u^2(4p_1 + p_2) - 16p_1p_2u^2}{16(4p_1 + p_2)^2} = \frac{4p_2^2u^2}{16(4p_1 + p_2)^2} = h_1^*(.)$$

symmetrically for $h_2^*(.)$

4. Indirect utility function

$$v(p_1,p_2,w) = u(x_1(p,w), x_2(p,w)) = 2\sqrt{\frac{p_2w}{p_1p_2 + 4p_1^2}} + 4\sqrt{\frac{4p_2w}{p_1p_2 + 4p_1^2}\left(\frac{p_1}{p_2}\right)^2} = 2\left(\frac{w}{p_1} + 4\frac{w}{p_2}\right)^{0.5}$$

Problem 1
Given arbitrarily selected $p \gg 0$ and $\bar{u} > u(0)$ show that $\partial h_l(p, \bar{u})/\partial p_k$ is equal to a Slutsky substitution effect.

Solution Considering knowing, that:

$$h_l(p, u) = x_l(p, e(p, u))$$

and differentiating this equation with respect to p_k yields:

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, e(p, u))}{\partial p_k} = \frac{\partial x_l(p, e(p, u))}{\partial p_k} + \frac{\partial x_l(p, e(p, u))}{\partial e(p, u)} \frac{\partial e(p, u)}{\partial p_k}$$

also applying that

$$h(p, u) = \nabla_p e(p, u)$$

we get the following:

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, e(p, u))}{\partial p_k} = \frac{\partial x_l(p, e(p, u))}{\partial p_k} + \frac{\partial x_l(p, e(p, u))}{\partial e(p, u)} h_k(p, u)$$

and since $w = e(p, u)$ and $h_k(p, u) = x_k(p, e(p, u)) = x_k(p, w)$

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w)$$

Problem 2
Show that

$$\sum_{k=1}^{L} \frac{\partial h_l(p, u)}{\partial p_k} p_k = 0 \text{ for each commodity } l$$

Solution

Considering homogeneity of degree zero in prices:

$$h(ap, \bar{u}) = h(p, \bar{u})$$

$$\nabla_a h(ap, u) = \nabla_a h(p, u)$$
$$p \nabla_{ap} h(ap, u) = 0 \iff a = 1$$
$$p \nabla_p h(p, u) = 0$$

Problem 4
Show that this dataset violates SARP but satisfies WARP

Solution Considering the availability of goods at different prices, we get:

	x1	x2	x3
p1	8	7	9
p2	9	8	7
p3	7	9	8

This table means, that consumer spent 8 on x_1, while being able to buy x_2 and $p_1 x_2 < p_1 x_1 \implies x_1 \succ x_2$. With the same logic we get that:

$$x_1 \succ x_2 \quad x_2 \succ x_3 \quad x_3 \succ x_1$$

Surely transitivity is not fulfilled, but WARP is not violated $\implies$ WARP satisfies and SARP is violated.