5. Find and orthonormal basis for the column space of the matrix $\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$.

Hallaremos una base para el espacio columna al verificar la forma escalonada de forma reducida por fila de A.

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/3 & 1/3 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 8/3 & 2/3 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 8/3 & 2/3 \\ 0 & 10/3 & -5/3 \\ 3 & -7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 8/3 & 2/3 \\ 0 & 10/3 & -5/3 \\ 0 & -2 & 7 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 1 & 1/4 \\ 0 & 10/3 & -5/3 \\ 0 & -2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \\ 0 & 10/3 & -5/3 \\ 0 & -2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & -5/2 \\ 0 & 0 & 15/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \\ 0 & 0 & 15/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \\ 0 & 0 & 15/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 15/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \quad A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad B_{C(A)} = \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} ; \begin{pmatrix} -5 \\ 1 \\ 5 \\ 7 \end{pmatrix} ; \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} \right\}$$

Usando el proceso de Gran-Schmidt

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} ; \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} ; \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} \right\} = \{ x_1 ; x_2 ; x_3 \}$$

tenemos un conjunto de vectores L.I $\{x_1, x_2, \dots\}$ y bases ortogonales

$V = \{ V_1, V_2, V_3 \}$

$$V_1 = X_1 \quad , \quad V_2 = X_2 - \frac{\langle X_2 , V_1 \rangle}{\|V_1\|^2} . V_1 \quad , \quad V_3 = X_3 - \frac{\langle X_3 , V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle X_3 , V_2 \rangle}{\|V_2\|^2} V_2$$

PASO 1:

$$\boxed{V_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}}$$

PASO 2:

$$V_2 = X_2 - \frac{\langle X_2 , V_1 \rangle}{\|V_1\|^2} . V_1 = \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} - \frac{(-40)}{(20)} . \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix}$$

$$\boxed{V_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix}}$$

PASO 3:

$$V_3 = X_3 - \frac{\langle X_3 , V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle X_3 , V_2 \rangle}{\|V_2\|^2} V_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{(30)}{(20)} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{(-10)}{20} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\boxed{V_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}}$$

$\rightarrow$ las bases ortogonales para la columna $= \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} , \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} , \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$

$$\boxed{\rightarrow \text{las bases ortonormales} \left\{ \begin{pmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{pmatrix} , \begin{pmatrix} 1/\sqrt{20} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \\ -1/\sqrt{20} \end{pmatrix} , \begin{pmatrix} -3/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ 3/\sqrt{20} \end{pmatrix} \right\}}$$