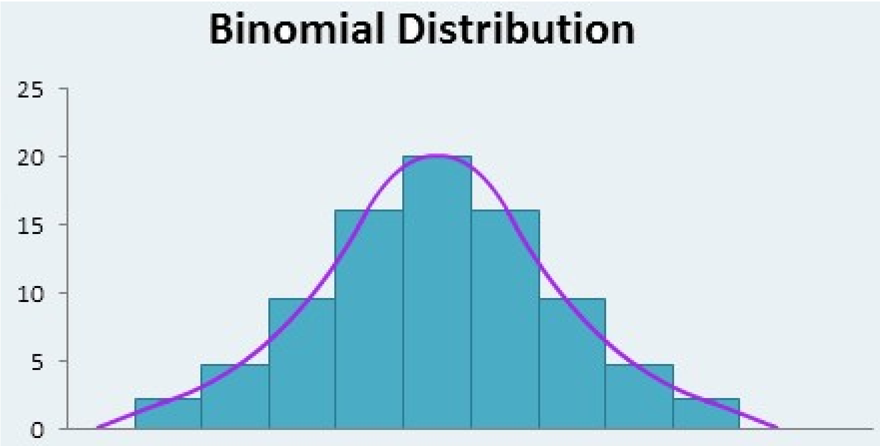
Binomial to Normal



PDF of the Normal Distribution

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

Theorem 1 [2]

According to de Moivre's Laplace limit theorem, the binomial distribution function can be approximated by

$$\binom{n}{k} p^x q^{n-x} \approx \frac{1}{\sqrt{2\pi npq}} \exp\left[\frac{-(x-np^2)}{2npq}\right]$$

where

- p, q are probabilities and $p + q = 1$ and $p, q > 0$
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Theorem 2 [1]

$$P(\text{between } a \text{ and } b \text{ successes}) = \phi\Big(\frac{b - np}{\sqrt{np(1-p)}}\Big) - \phi\Big(\frac{a - np}{\sqrt{np(1-p)}}\Big)$$

where

- ϕ is the CDF of the Normal Distribution
- np is the mean
- $\sqrt{np(1-p)}$ is the standard deviation
- $\dfrac{np}{\sqrt{np(1-p)}}$ is the standardization formula

Theorem 3 [1]

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{t^2}{2}\right] dt$$

Also, alternatively,

$$\Phi(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

where

- $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$

Derivation of Error Function [3]

Given

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

Let

$$I = \int_0^\infty e^{-x^2}\, dx$$

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Then,

$$I^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdr\, d\theta$$

Now, as the inner integral doesn't depend on θ, let $r^2 = s$ (and so, $rdr = \frac{ds}{2}$) to get

$$\begin{aligned} I^2 &= \frac{\pi}{2} \int_0^\infty e^{-s} \frac{ds}{2} \\ &= \frac{\pi}{4} \left[-e^{-s}\right]_0^\infty \\ &= \frac{\pi}{4} (-e^{-\infty}) - (-e^{-0}) \\ &= \frac{\pi}{4} \end{aligned}$$

Therefore, we have that

$$I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

And

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx$$

$$\frac{2}{\sqrt{\pi}} I$$

$$\frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1$$

Changing Variables in Double Integral

Given

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

The double integral change of variables formula is

$$\int_R f(x,y)\,dx\,dy = \int_D f(r\cos(\theta), r\sin(\theta)) \cdot r\,dr\,d\theta$$

Let $f(x,y) = 1$. Then,

$$dy\,dx = \int_R 1\,dx\,dy = \int_D 1 \cdot r\,dr\,d\theta$$

Now, we need to find the Jacobian determinant $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|$. The transformation matrix is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

Evaluating each partial derivative:

$$\frac{\partial x}{\partial r} = \cos(\theta)$$

$$\frac{\partial x}{\partial \theta} = -r\sin(\theta)$$

$$\frac{\partial y}{\partial r} = \sin(\theta)$$

$$\frac{\partial y}{\partial \theta} = r\cos(\theta)$$

The Jacobian determinant is

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |(\cos(\theta))(r\cos(\theta)) - (-r\sin(\theta))(\sin(\theta))| = r$$

Substituting this into the integral:

$$dy\,dx = \int_D 1 \cdot r\,dr\,d\theta$$

Thus

$$dy\,dx = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = r\,dr\,d\theta$$

Explanation for $dydx = r\ dr\ d\theta$

In polar coordinates, we have,

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Given the two functions x, y we apply partial differentiation with respect to r and θ

$$\frac{\partial x}{\partial r} = \cos(\theta)$$

$$\frac{\partial x}{\partial \theta} = -r\sin(\theta)$$

$$\frac{\partial y}{\partial r} = \sin(\theta)$$

$$\frac{\partial y}{\partial \theta} = r\cos(\theta)$$

Then, changing variables in a double integral,

$$dy\,dx = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

Applying the determinant of the Jacobian matrix,

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |(\cos(\theta))(r\cos(\theta)) - (-r\sin(\theta))(\sin(\theta))|$$

Simplifying

$$|(\cos(\theta))(r\cos(\theta)) - (-r\sin(\theta))(\sin(\theta))|$$

Expand the terms

$$\left| r \times (\cos^2(\theta) + \sin^2(\theta)) \right|$$

Use the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$

$$|r \cdot 1| = |r|$$

Therefore

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |r|$$

$$dy\,dx = r\,dr\,d\theta$$

Explaining $\frac{\partial x}{\partial r} = \cos(\theta)$

Given

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

The partial derivative of x is

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos(\theta))$$

Applying the product rule of differentiation (uv')

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}x = \frac{\partial}{\partial r}(r\cos(\theta)) = \frac{\partial r}{\partial r}\cos(\theta) + \frac{\partial\cos(\theta)}{\partial r}r$$

where

- $\frac{d}{dx}(uv) = \frac{dv}{dx}u + \frac{du}{dx}v$ is the product differentiation rule
- $u = r$
- $v = \cos(\theta)$

Then, we have

- $\frac{\partial r}{\partial r} = 1$ and

- $\frac{\partial \cos(\theta)}{\partial r} = 0$ (since $\cos(\theta)$ is not a function of r)

Simplifying

$$\frac{\partial x}{\partial r} = \cos(\theta)$$

Reference(s)

1. https://www.math.utah.edu/~davar/math5010/summer2010/L7.pdf
2. https://www.scirp.org/journal/paperinformation.aspx?paperid=100627
3. https://math.stackexchange.com/questions/364112/how-to-prove-that-integration-of-exp-x2-is-error-function