Ecuaciones Diferenciales Ordinarias

PhD. Alejandro Paredes

Ecuaciones Diferenciales Ordinarias (EDO)

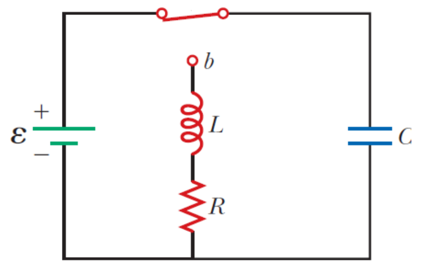
Ecuaciones que contienen una o más derivadas de una función que depende solamente de una variable. Estas ecuaciones sirven para modelar fenómenos en física, ingeniería, economía, etc.

Modelos de evolución población o PBI:

$$\frac{d}{dt}p(t) = rp(t)$$

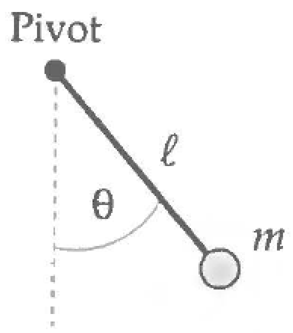
$$\frac{d}{dt}PBI(t) = rPBI(t)$$

Circuitos eléctricos:



$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

Péndulo simple:



$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}Sen(\theta)$$

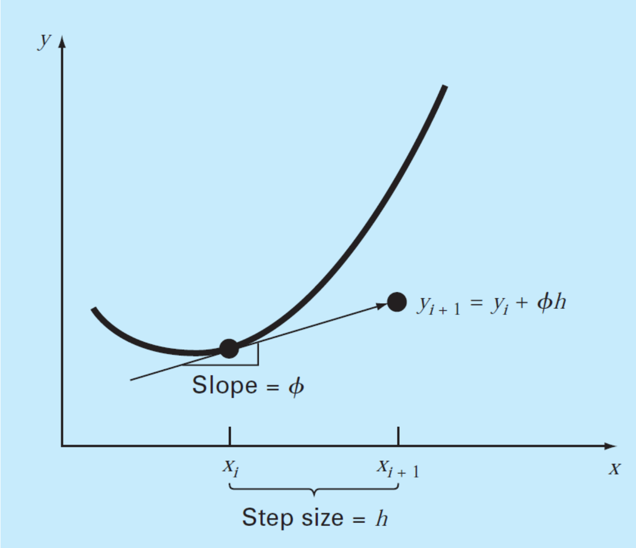
Clasificación de Métodos para EDO’s

Métodos de un solo paso

Runge-Kutta

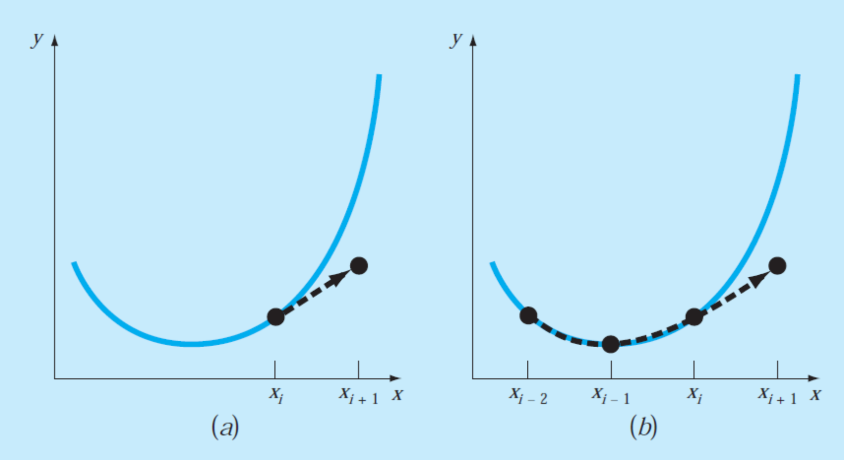
$$\frac{dy}{dx} = f(x, y) \quad y_{i+1} = y_i + \phi h$$

- Euler.
- Heun.
- Punto medio.
- RK2,RK4,RK5,RK6



Métodos multipasos

Se utiliza información precedente para estimar y_{t+1}.



- Heun modificado.
- Adam-Bashforth.
- Adams-Multon.

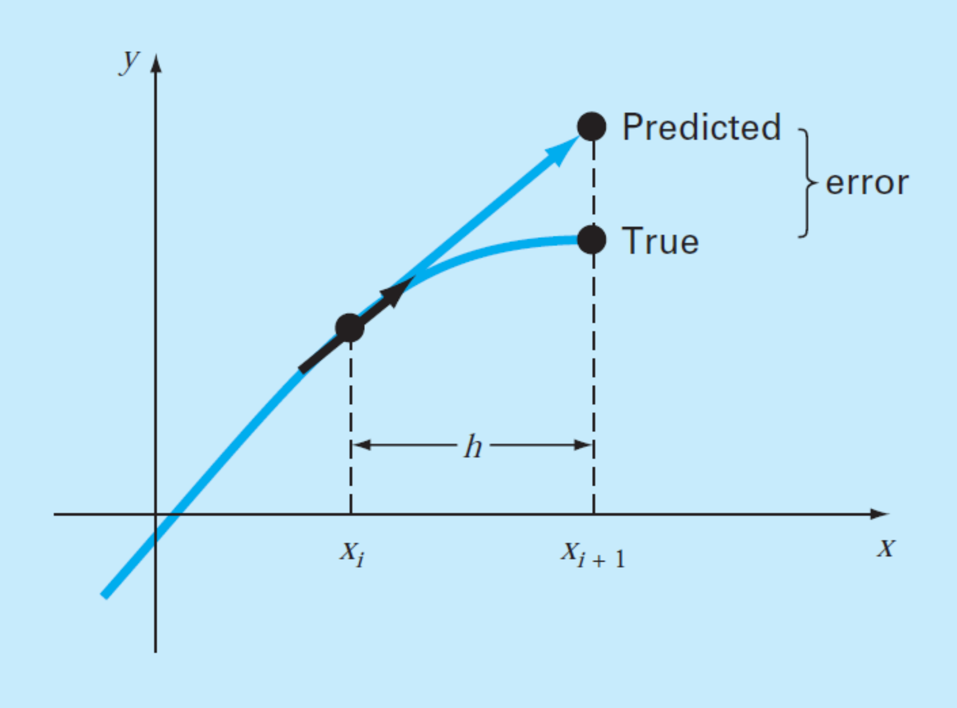
Método de Euler

$$\frac{dy}{dx} = f(x, y)$$

Tomamos a ϕ como la primera derivada

$$\phi = f(x_i, y_i)$$

$$y_{i+1} = y_i + f(x_i, y_i)h$$



$f(x_i, y_i)$ corresponde ala derivada en el punto inicial.
Nota: $f(x, y)$ depende de x e y y puede ser no lineal en general.

Método de Euler

$$y_{i+1} = y_i + f(x_i, y_i)h$$

Expanción de Taylor

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!}h^2 + \cdots + \frac{y_i^{(n)}}{n!}h^n + R_n$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2!}h^2 + \cdots + \frac{f^{(n-1)}(x_i, y_i)}{n!}h^n + O(h^{n+1})$$

$$R_n = \frac{y^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

$$E_a = \frac{f'(x_i, y_i)}{2!}h^2$$

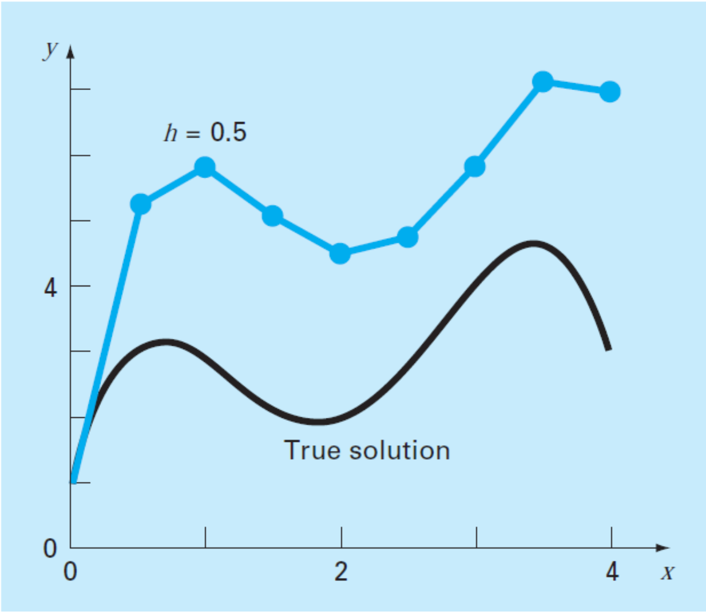
Método de Euler

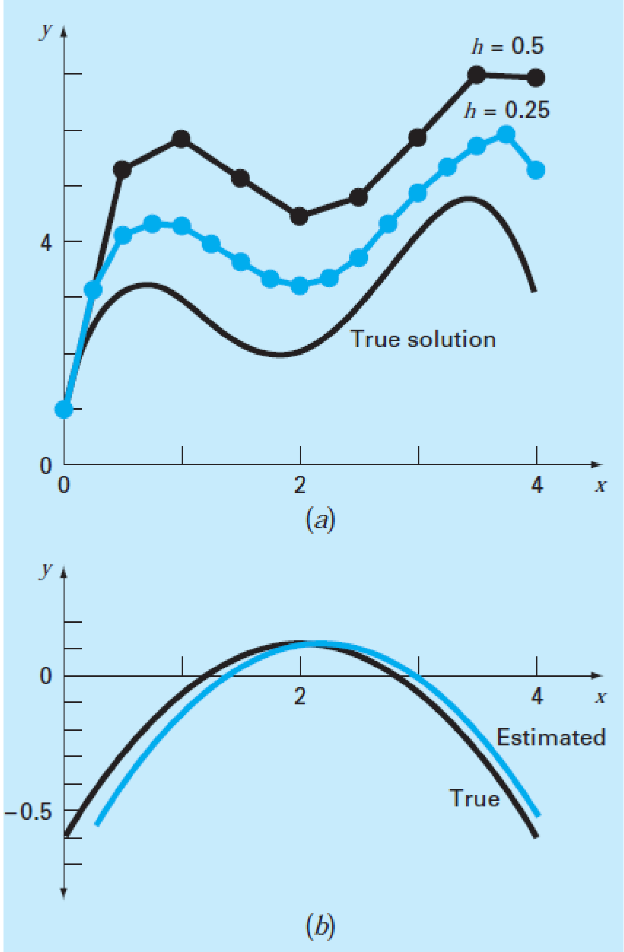
Ejemplo

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

x	y_{true}	y_{Euler}	Local
0.0	1.00000	1.00000	
0.5	3.21875	5.25000	−63.1
1.0	3.00000	5.87500	−28.0
1.5	2.21875	5.12500	−1.41
2.0	2.00000	4.50000	20.5
2.5	2.71875	4.75000	17.3
3.0	4.00000	5.87500	4.0
3.5	4.71875	7.12500	−11.3
4.0	3.00000	7.00000	−53.0





Método de Euler

```
'set integration range
xi = 0
xf = 4
'initialize variables
x = xi
y = 1
'set step size and determine
'number of calculation steps
dx = 0.5
nc = (xf − xi)/dx
'output initial condition
PRINT x, y
'loop to implement Euler's method
'and display results
DOFOR i = 1, nc
  dydx = −2x^3 + 12x^2 − 20x + 8.5
  y = y + dydx · dx
  x = x + dx
  PRINT x, y
END DO
```

```
Assign values for
y = initial value dependent variable
xi = initial value independent variable
xf = final value independent variable
dx = calculation step size
xout = output interval

x = xi
m = 0
xp_m = x
yp_m = y
DO
  xend = x + xout
  IF (xend > xf) THEN xend = xf
  h = dx
  CALL Integrator (x, y, h, xend)
  m = m + 1
  xp_m = x
  yp_m = y
  IF (x ≥ xf) EXIT
END DO
DISPLAY RESULTS
END
```

```
SUB Integrator (x, y, h, xend)
  DO
    IF (xend − x < h) THEN h = xend − x
    CALL Euler (x, y, h, ynew)
    y = ynew
    IF (x ≥ xend) EXIT
  END DO
END SUB


SUB Euler (x, y, h, ynew)
  CALL Derivs(x, y, dydx)
  ynew = y + dydx * h
  x = x + h
END SUB


SUB Derivs (x, y, dydx)
  dydx = . . .
END SUB
```

Método de Heund

La idea es mejorar la aproximación de la derivada

$$y'_i = f(x_i, y_i) \qquad y'_{i+1} = f\left(x_{i+1}, y^0_{i+1}\right)$$

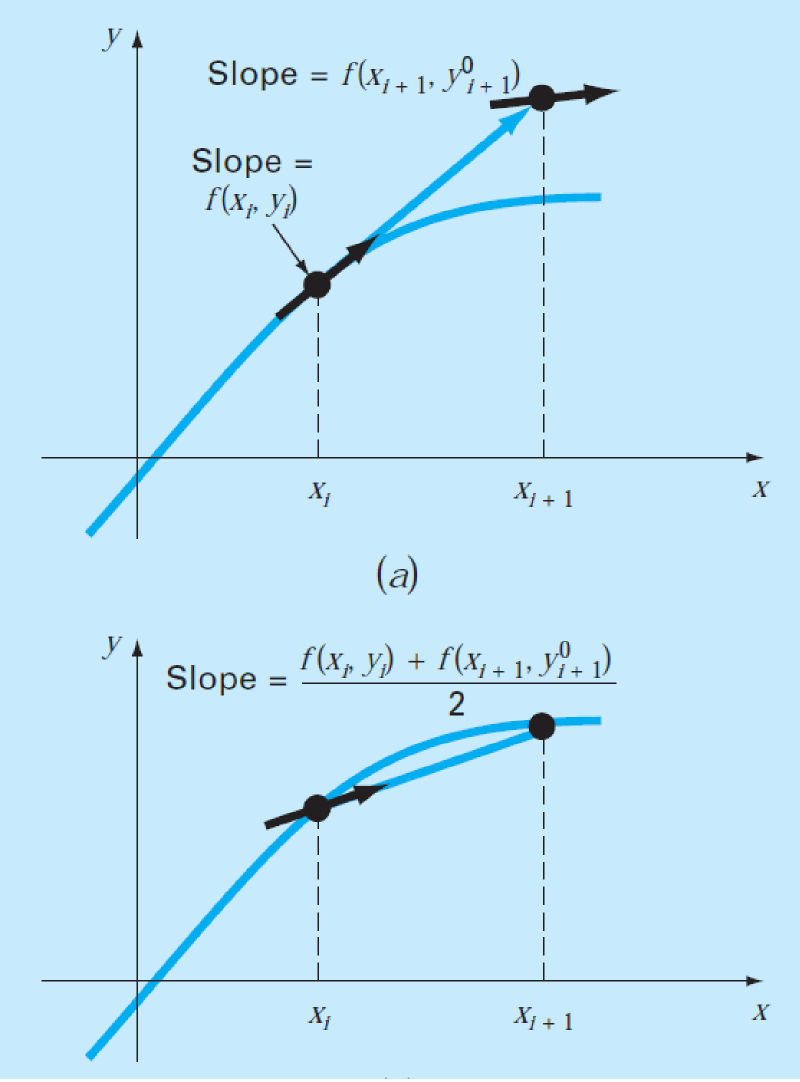
$$\bar{y}' = \frac{y'_i + y'_{i+1}}{2} = \frac{f(x_i, y_i) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}$$

$$y^0_{i+1} = y_i + f(x_i, y_i)h$$

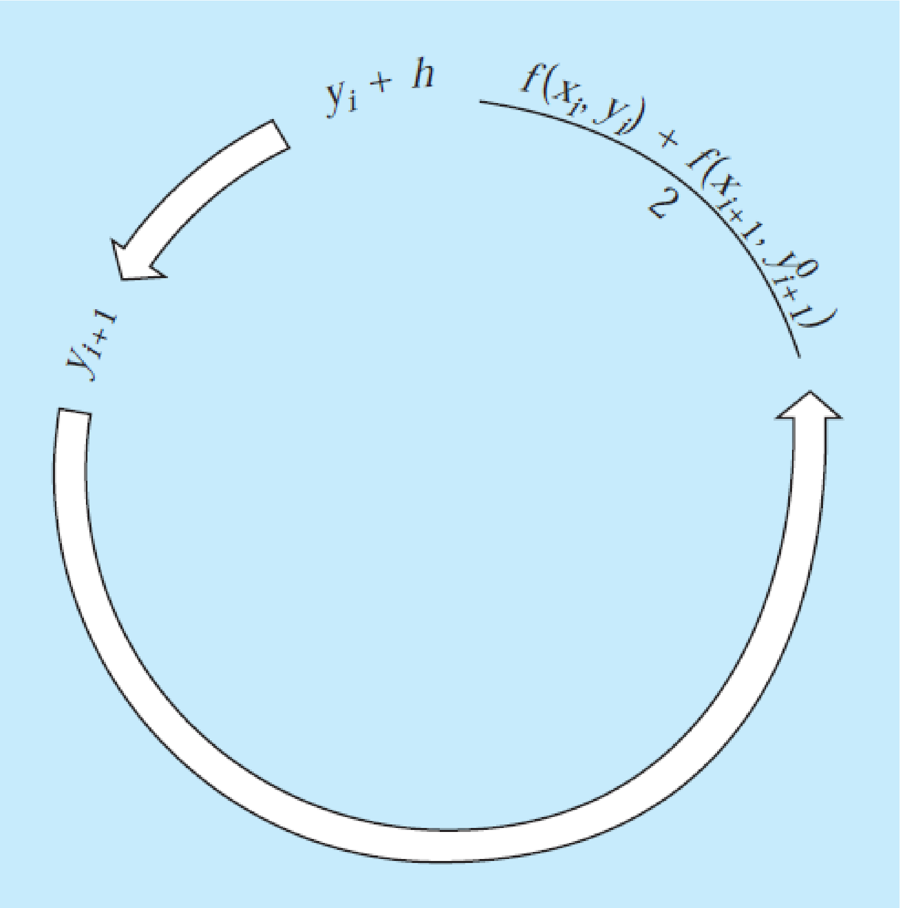
Predictor

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}h$$

Corrector



Método de Heun



$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f\left(x_{i+1}, y_{i+1}^0\right)}{2} h$$

Criterio de convergencia

$$|\varepsilon_a| = \left| \frac{y_{i+1}^j - y_{i+1}^{j-1}}{y_{i+1}^j} \right| 100\%$$

Método de Heun

Ejemplo: Obtener $y_1 = y(1)$
Integrar la ecuación con un paso $h = 1$.

$$y' = 4e^{0.8x} - 0.5y \qquad x = 0, y = 2$$

$$y = \frac{4}{1.3}(e^{0.8x} - e^{-0.5x}) + 2e^{-0.5x}$$

Valor real $\quad y(1) = 6.1946314$

Método

$$y^0_{i+1} = y_i + f(x_i, y_i)h$$

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}h$$

Calculamos $y_1 (i = 0)$, necesitamos

$$f(x_0, y_0) = \quad y'_0 = 4e^0 - 0.5(2) = 3$$

$$y^0_1 = 2 + 3(1) = 5$$

$$y'_1 = f\left(x_1, y^0_1\right) = 4e^{0.8(1)} - 0.5(5) = 6.402164$$

$$y_1 = 2 + 4.701082(1) = 6.701082$$

Método de Heun

La aproximación mejora $y_1 \rightarrow y_1^0$

Mejora aún más $y_1 \rightarrow y_1^0$

$$y_1 = 2 + \frac{[3 + 4e^{0.8(1)} - 0.5(6.701082)]}{2} 1 = 6.275811$$

$$y_1 = 2 + \frac{[3 + 4e^{0.8(1)} - 0.5(6.275811)]}{2} 1 = 6.382129$$

		Iterations of Heun's Method			
		1		15	
x	y_{true}	y_{Heun}	$\lvert\varepsilon_t\rvert$ (%)	y_{Heun}	$\lvert\varepsilon_t\rvert$ (%)
0	2.0000000	2.0000000	0.00	2.0000000	0.00
1	6.1946314	6.7010819	8.18	6.3608655	2.68
2	14.8439219	16.3197819	9.94	15.3022367	3.09
3	33.6771718	37.1992489	10.46	34.7432761	3.17
4	75.3389626	83.3377674	10.62	77.7350962	3.18

Método de Heun

En el caso que $y' = f(x) \Rightarrow$

$$y_{i+1} = y_i + \frac{f(x_i) + f(x_{i+1})}{2} h$$

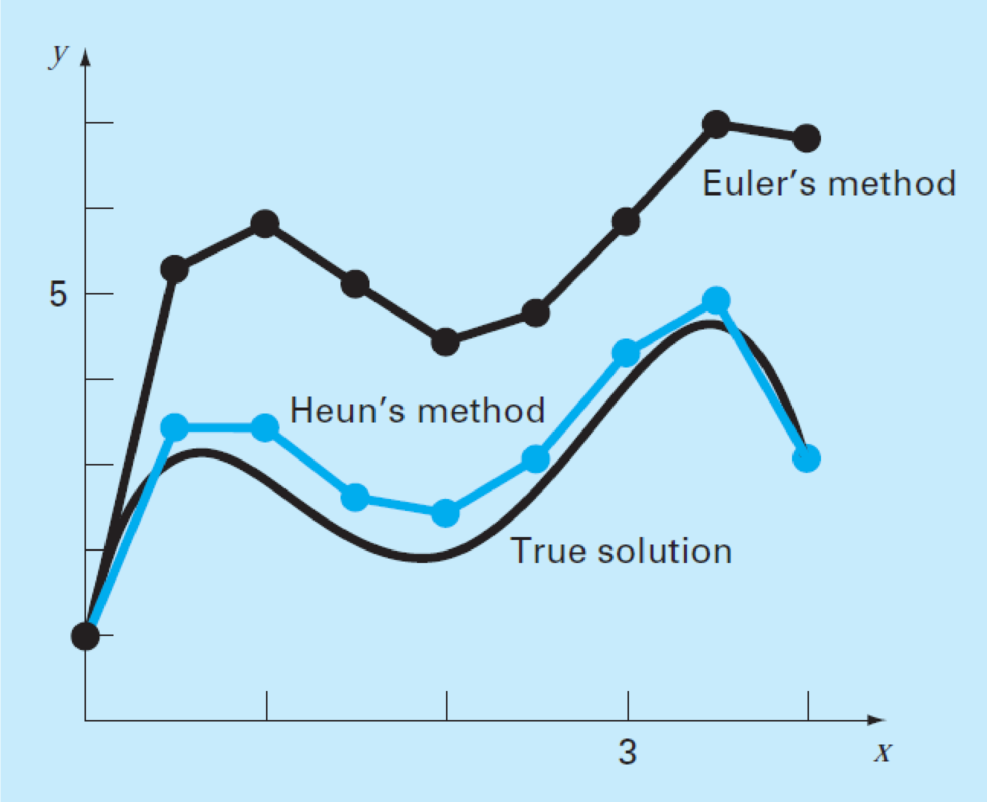
$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad \int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x)\, dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x)\, dx$$

$$\int_{x_i}^{x_{i+1}} f(x)\, dx \cong \frac{f(x_i) + f(x_{i+1})}{2} h$$

Regla del trapecio

$$y_{i+1} = y_i + \frac{f(x_i) + f(x_{i+1})}{2} h \qquad E_t = -\frac{f''(\xi)}{12} h^3$$



Método punto medio

La idea es utilizar el método de Euler evaluando la derivada en $y_{i+1/2}$.

$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2} \qquad y'_{i+1/2} = f(x_{i+1/2}, y_{i+1/2})$$

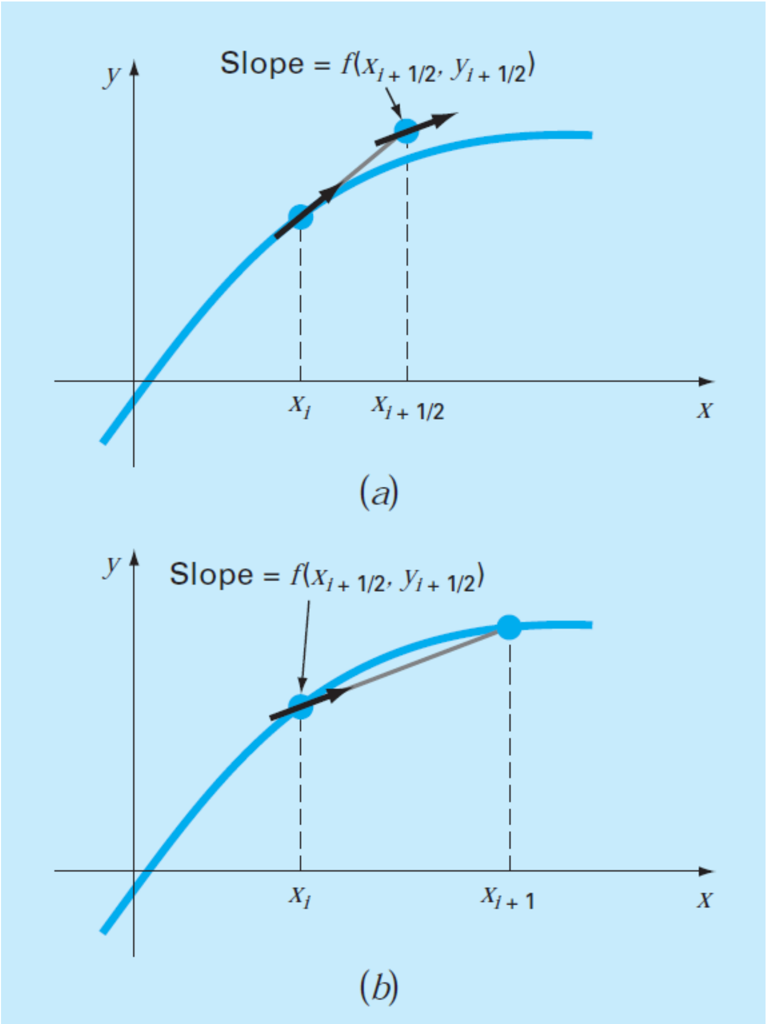
$$\boxed{y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})h}$$

$$\frac{dy}{dx} = f(x) \quad \Rightarrow \quad \int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x)\, dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x)\, dx \qquad \int_{x_i}^{x_{i+1}} f(x)\, dx \cong h f(x_{i+1/2})$$

$$\int_a^b f(x)\, dx \cong (b-a)\, f(x_1)$$

x_1 = punto medio



Pseudocódigos

(a) Simple Heun without Corrector

```
SUB Heun (x, y, h, ynew)
  CALL Derivs (x, y, dy1dx)
  ye = y + dy1dx · h
  CALL Derivs(x + h, ye, dy2dx)
  Slope = (dy1dx + dy2dx)/2
  ynew = y + Slope · h
  x = x + h
END SUB
```

(b) Midpoint Method

```
SUB Midpoint (x, y, h, ynew)
  CALL Derivs(x, y, dydx)
  ym = y + dydx · h/2
  CALL Derivs (x + h/2, ym, dymdx)
  ynew = y + dymdx · h
  x = x + h
END SUB
```

(c) Heun with Corrector

```
SUB HeunIter (x, y, h, ynew)
  es = 0.01
  maxit = 20
  CALL Derivs(x, y, dy1dx)
  ye = y + dy1dx · h
  iter = 0
  DO
    yeold = ye
    CALL Derivs(x + h, ye, dy2dx)
    slope = (dy1dx + dy2dx)/2
    ye = y + slope · h
    iter = iter + 1
```

$$ea = \left| \frac{ye - yeold}{ye} \right| 100\%$$

```
    IF (ea ≤ es OR iter > maxit) EXIT
  END DO
  ynew = ye
  x = x + h
END SUB
```

Métodos Runge-Kutta

Métodos que alcanzan la precisión de un desarrollo de Taylor sin calcular derivadas de orden superior

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

$$\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \\
k_3 &= f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\
&\ \ \vdots \\
k_n &= f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)
\end{aligned}$$

Nota:

- Los valores p y q son constantes.
- El valor k_n se obtiene por recurrencia.
- Eficiente para cálculos.
- El método RK para n=1 es el método de Euler.

Método RK2

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$f(x_i + p_1 h, y_i + q_{11} k_1 h) = f(x_i, y_i) + p_1 h \frac{\partial f}{\partial x} + q_{11} k_1 h \frac{\partial f}{\partial y} + O(h^2)$$

$$y_{i+1} = y_i + a_1 h f(x_i, y_i) + a_2 h f(x_i, y_i) + a_2 p_1 h^2 \frac{\partial f}{\partial x} + a_2 q_{11} h^2 f(x_i, y_i) \frac{\partial f}{\partial y} + O(h^3)$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2!} h^2$$

$$f'(x_i, y_i) = \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} \frac{dy}{dx}$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \right) \frac{h^2}{2!}$$

$$y_{i+1} = y_i + [a_1 f(x_i, y_i) + a_2 f(x_i, y_i)]h + \left[a_2 p_1 \frac{\partial f}{\partial x} + a_2 q_{11} f(x_i, y_i) \frac{\partial f}{\partial y} \right] h^2 + O(h^3)$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

$$\Rightarrow$$

$$a_1 = 1 - a_2$$

$$p_1 = q_{11} = \frac{1}{2a_2}$$

Método RK2

Método de Heund (a_2=1/2)

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(x_i, y_i)$$

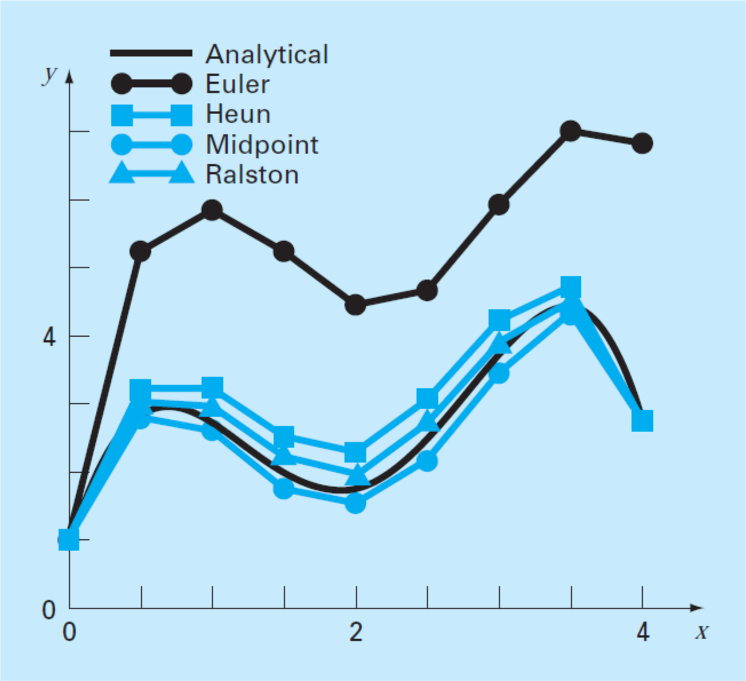
$$k_2 = f(x_i + h, y_i + k_1 h)$$

Método punto medio (a_2=1)

$$y_{i+1} = y_i + k_2 h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$



Método Ralston (a_2=2/3)

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

Método RK3

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

Se obtienen seis ecuaciones y ocho incognitas

Método RK4

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

Método RK5 (Butcher)

$$y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)h$$

$$k_1 = f(x_i, y_i)$$

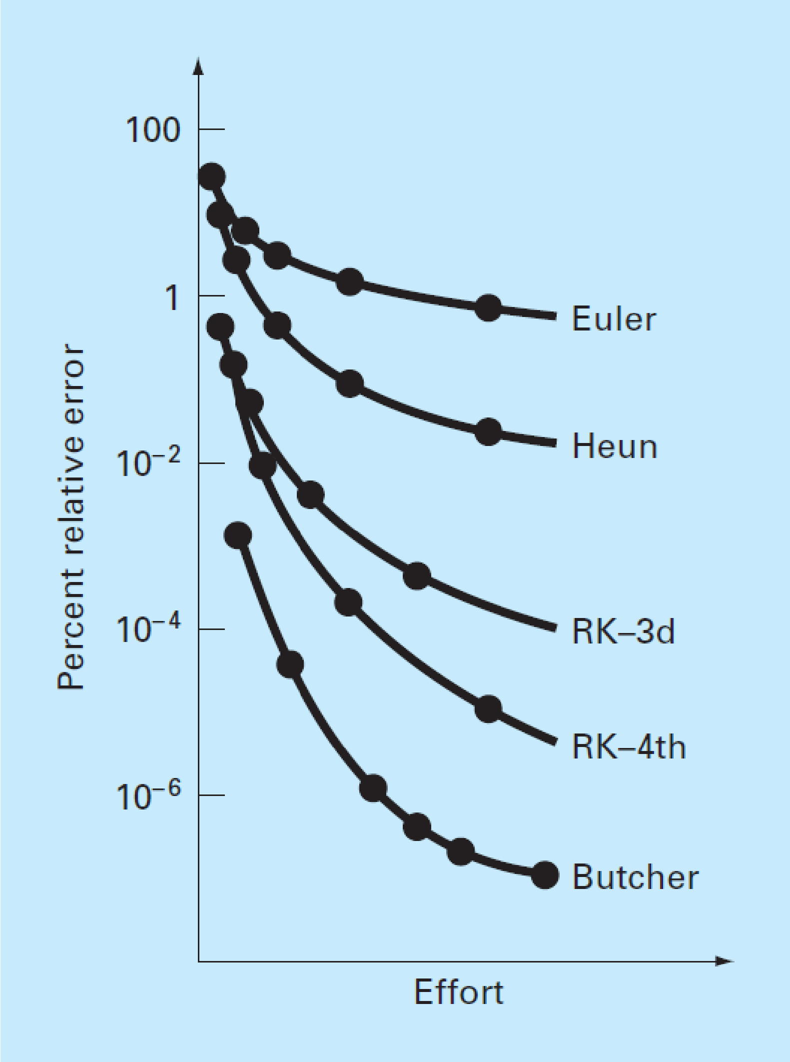
$$k_2 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h\right)$$

$$k_4 = f\left(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h\right)$$

$$k_5 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h\right)$$

$$k_6 = f\left(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h\right)$$

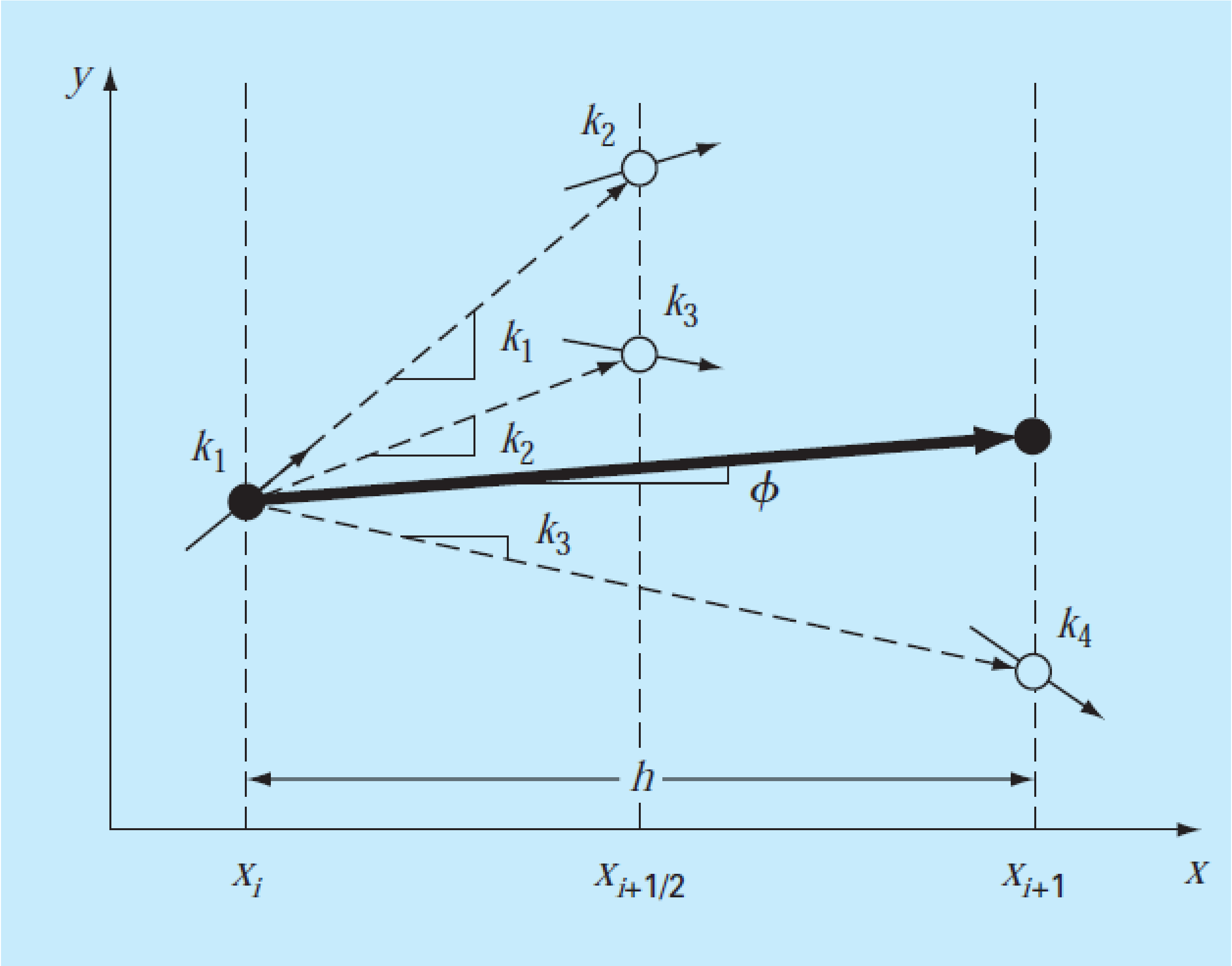


$$\text{Effort} = n_f \frac{b-a}{h}$$

n_f :numero de evaluaciones

Pseudocódigo RK4

```
SUB RK4 (x, y, h, ynew)
  CALL Derivs(x, y, k1)
  ym = y + k1 · h/2
  CALL Derivs(x + h/2, ym, k2)
  ym = y + k2 · h/2
  CALL Derivs(x + h/2, ym, k3)
  ye = y + k3 · h
  CALL Derivs(x + h, ye, k4)
  slope = (k1 + 2(k2 + k3) + k4)/6
  ynew = y + slope · h
  x = x + h
END SUB
```



Aplicación

Resolver la siguiente ecuación diferencial

$$\frac{dy}{dt} = yt^3 - 1.5y$$

- Analíticamente.
- Método de Euler.
- Método de Heund.
- Método de Ralston.
- RK4.

Sistemas de ecuaciones

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

Se necesitan n condiciones iniciales

Definimos $\boldsymbol{r} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$ y $\boldsymbol{F} = \begin{pmatrix} f_1(x, \boldsymbol{r}) \\ \vdots \\ f_n(x, \boldsymbol{r}) \end{pmatrix}$

RK4

$$\boldsymbol{r_{i+1}} = \boldsymbol{r_i} + \frac{1}{6}\left(\boldsymbol{K_1} + 2\boldsymbol{K_2} + 2\boldsymbol{K_3} + \boldsymbol{K_4}\right)h$$

$$\boldsymbol{K_1} = \boldsymbol{F}(x, \boldsymbol{r})$$

$$\boldsymbol{K_2} = \boldsymbol{F}\left(x + \frac{1}{2}h, \boldsymbol{r} + \frac{1}{2}\boldsymbol{K_1}\,h\right)$$

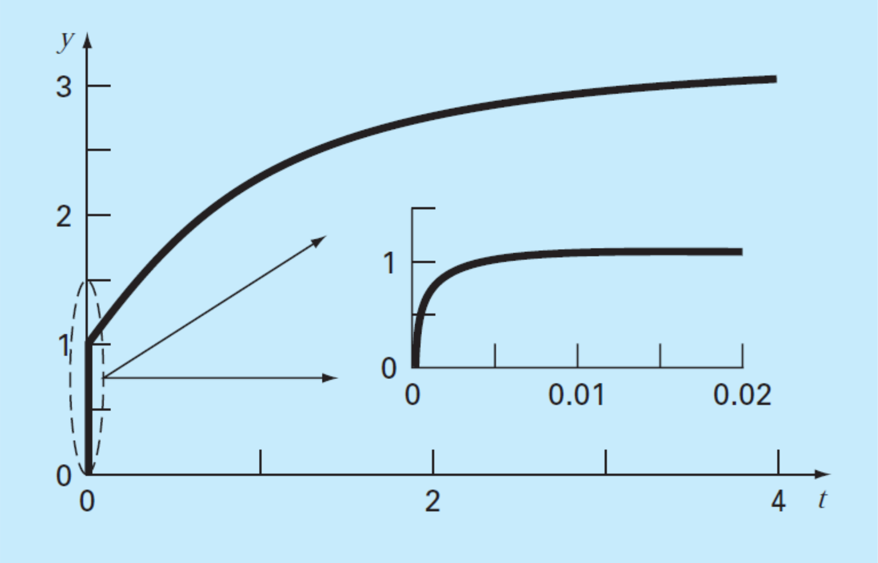
$$\boldsymbol{K_3} = \boldsymbol{F}\left(x + \frac{1}{2}h, \boldsymbol{r} + \frac{1}{2}\boldsymbol{K_2}\,h\right)$$

$$\boldsymbol{K_4} = \boldsymbol{F}\left(x + h, \boldsymbol{r} + \frac{1}{2}\boldsymbol{K_3}\,h\right)$$

Stiffness:

Son problemas donde la solución tiene periodos de evolución lentos y rapidos.

$$\frac{dy}{dt} = -1000y + 3000 - 2000e^{-t}$$



$$y = 3 - 0.998e^{-1000t} - 2.002e^{-t}$$

Solución parte homogénea

$$\frac{dy}{dt} = -ay \qquad y = y_0 e^{-at}$$

$$y_{i+1} = y_i + \frac{dy_i}{dt}h$$

Estabilidad $\quad |1 - ah| < 1$

$$y_{i+1} = y_i - ay_i h$$

$$y_{i+1} = y_i(1 - ah)$$

Si $h > 2/a$, entonces

$y_i \to \infty$ cuando $i \to \infty$

Condición de estabilidad

$$h < 0.002$$

Euler implícito

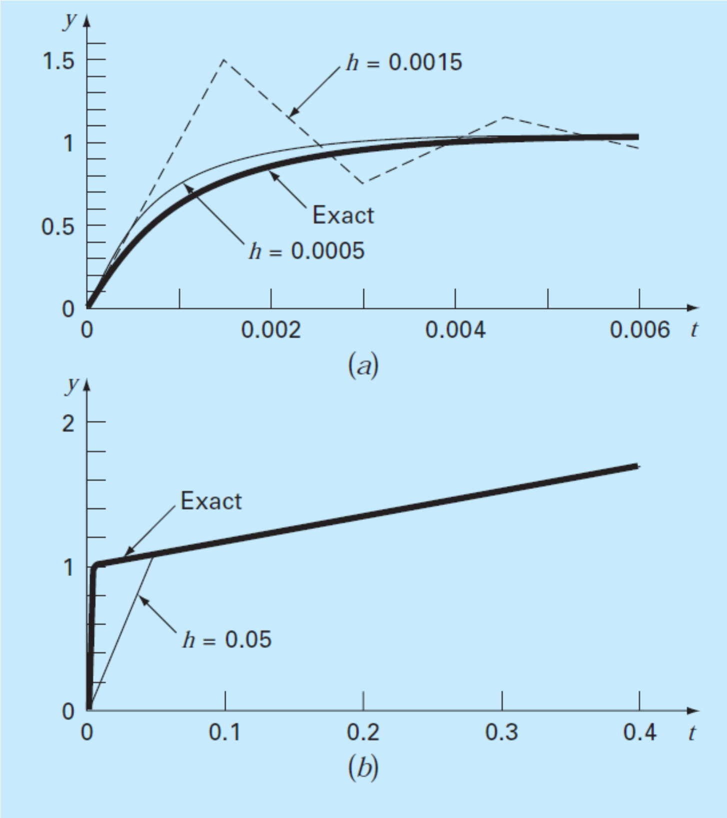
$$y_{i+1} = y_i + \frac{dy_{i+1}}{dt} h \qquad y_{i+1} = y_i + (-1000 y_{i+1} + 3000 - 2000 e^{-t_{i+1}}) h$$

$$y_{i+1} = y_i - a y_{i+1} h \qquad y_{i+1} = \frac{y_i + 3000h - 2000he^{-t_{i+1}}}{1 + 1000h}$$

$$y_{i+1} = \frac{y_i}{1 + ah}$$

Incondiconalmente stable

$$\frac{1}{|1 + ah|} < 1$$



Euler implícito (sistema ODE)

$$\frac{dy_1}{dt} = -5y_1 + 3y_2$$

$$\frac{dy_2}{dt} = 100y_1 - 301y_2$$

$$y_1 = 52.96e^{-3.9899t} - 0.67e^{-302.0101t}$$

$$y_2 = 17.83e^{-3.9899t} + 65.99e^{-302.0101t}$$

$$y_{1,i+1} = y_{1,i} + (-5y_{1,i+1} + 3y_{2,i+1})h$$

$$y_{2,i+1} = y_{2,i} + (100y_{1,i+1} - 301y_{2,i+1})h$$

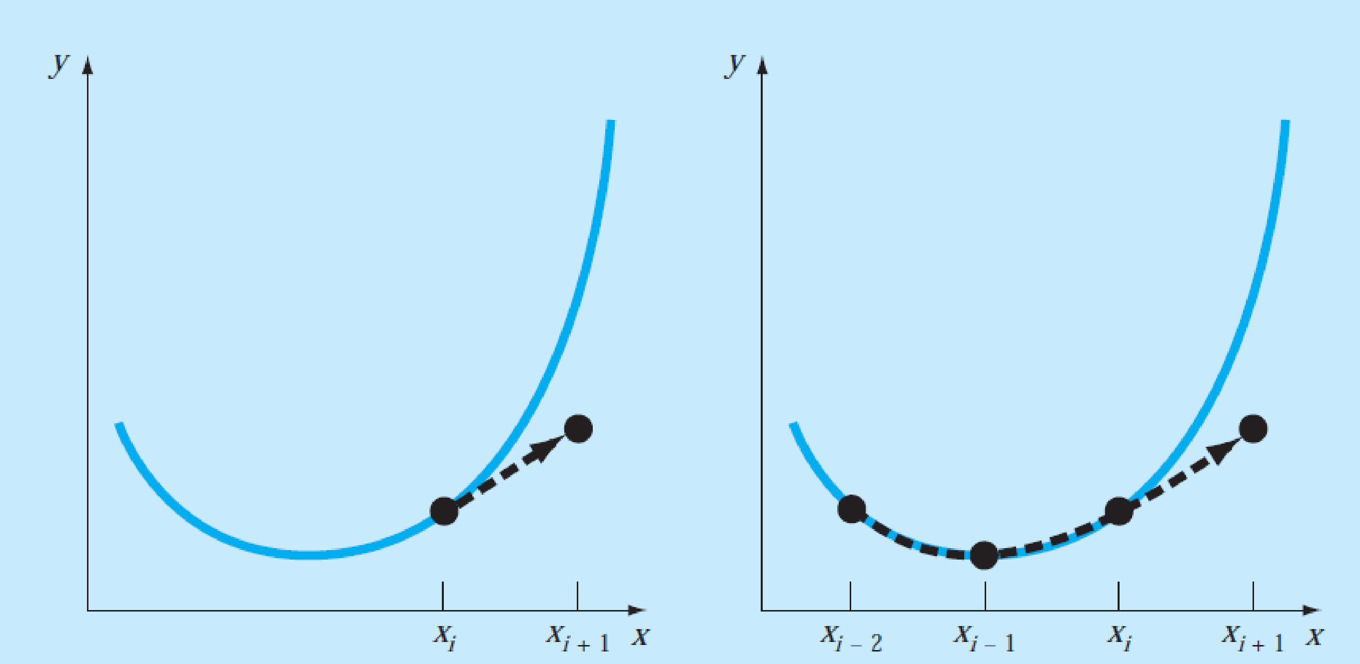
$$(1+5h)y_{1,i+1} - 3hy_{2,i+1} = y_{1,i}$$

$$-100hy_{1,i+1} + (1+301h)y_{2,i+1} = y_{2,i}$$

$\Leftarrow$ Sistema Lineal a resolver

Métodos multipasos

Le objetivo es utilizar información precedente sobre la curvatura de la solución.



(a)
(b)

Método de Heun modificado

Método de Heun $$y^0_{i+1} = y_i + f(x_i, y_i)h \qquad O(h^2)$$

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}h \qquad O(h^3)$$

Modificación $$y^0_{i+1} = y_{i-1} + f(x_i, y_i)2h \qquad O(h^3)$$

$$y^0_{i+1} = y^m_{i-1} + f\left(x_i, y^m_i\right)2h$$

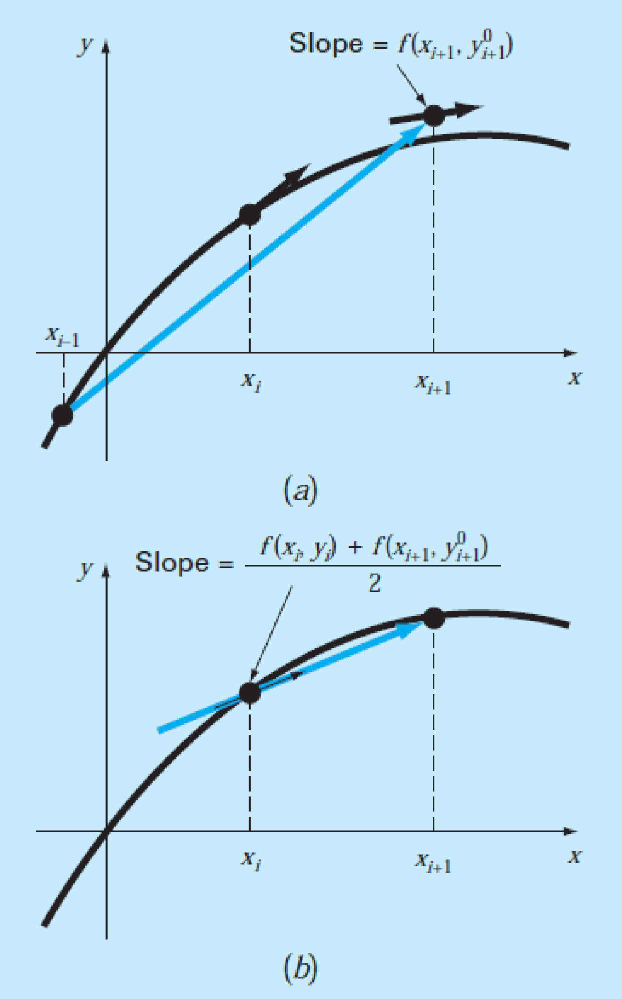
$$y^j_{i+1} = y^m_i + \frac{f\left(x_i, y^m_i\right) + f\left(x_{i+1}, y^{j-1}_{i+1}\right)}{2}h$$

$$(\text{for } j = 1, 2, \ldots, m)$$

Corrector con proceso iterativo

$$|\varepsilon_a| = \left|\frac{y^j_{i+1} - y^{j-1}_{i+1}}{y^j_{i+1}}\right| 100\%$$

Criterio de convergencia



Ejemplo método de Heun modificado

Resolver para $h = 1.0$ asumiendo que

y(x=-1)= 0.3929953, y(x=0)=2.0

$$y' = 4e^{0.8x} - 0.5y \qquad x = 0 \text{ to } x = 4$$

$$y_{i+1}^0 = y_{i-1}^m + f(x_i, y_i^m)\,2h$$

$$y_{i+1}^j = y_i^m + \frac{f(x_i, y_i^m) + f(x_{i+1}, y_{i+1}^{j-1})}{2}h$$

$$(\text{for } j = 1, 2, \ldots, m)$$

$$y_1^0 = -0.3929953 + \left[4e^{0.8(0)} - 0.5(2)\right]2 = 5.607005$$

$$y_1^1 = 2 + \frac{4e^{0.8(0)} - 0.5(2) + 4e^{0.8(1)} - 0.5(5.607005)}{2}1 = 6.549331$$

$$y_1^2 = 2 + \frac{3 + 4e^{0.8(1)} - 0.5(6.549331)}{2}1 = 6.313749$$

$$|\varepsilon_a| = \left|\frac{6.313749 - 6.549331}{6.313749}\right| 100\% = 3.7\%$$

$$y_2^0 = 2 + \left[4e^{0.8(1)} - 0.5(6.360865)\right]2 = 13.44346 \qquad \varepsilon_t = 9.43\%$$

Error en las formulas predictor-corrector

Deseamos resolver $$\frac{dy}{dx} = f(x, y)$$

$$\int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

Método del trapecio $h = x_{i+1} - x_i$

$$\int_{x_i}^{x_{i+1}} f(x, y)\, dx = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} h$$

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} h$$

Error del corrector

$$E_c = -\frac{1}{12} h^3 y^{(3)}(\xi_c) = -\frac{1}{12} h^3 f''(\xi_c)$$

Error en las formulas predictor-corrector

Deseamos resolver $$\frac{dy}{dx} = f(x, y)$$

$$\int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

Método del punto medio

$$\int_{x_{i-1}}^{x_{i+1}} f(x, y)\, dx = 2h\, f(x_i, y_i)$$

$$y_{i+1} = y_{i-1} + 2h\, f(x_i, y_i)$$

Error del predictor

$$E_p = \frac{1}{3} h^3 y^{(3)}(\xi_p) = \frac{1}{3} h^3\, f''(\xi_p)$$

Formula de Adam-Bashforth

$$y_{i+1} = y_i + f_i h + \frac{f_i'}{2} h^2 + \frac{f_i''}{6} h^3 + \cdots$$

$$y_{i+1} = y_i + h\left(f_i + \frac{h}{2} f_i' + \frac{h^2}{3!} f_i'' + \cdots \right)$$

$$f_i' = \frac{f_i - f_{i-1}}{h} + \frac{f_i''}{2} h + O(h^2)$$

$$y_{i+1} = y_i + h\left\{ f_i + \frac{h}{2}\left[\frac{f_i - f_{i-1}}{h} + \frac{f_i''}{2} h + O(h^2) \right] + \frac{h^2}{6} f_i'' + \cdots \right\}$$

$$y_{i+1} = y_i + h\left(\frac{3}{2} f_i - \frac{1}{2} f_{i-1} \right) + \frac{5}{12} h^3 f_i'' + O(h^4)$$

Formula de segundo orden

Formulas de Adam-Bashforth

$$y_{i+1} = y_i + h \sum_{k=0}^{n-1} \beta_k f_{i-k} + O(h^{n+1})$$

Order	β_0	β_1	β_2	β_3	β_4	β_5	Local Truncation Error
1	1						$\frac{1}{2}h^2 f'(\xi)$
2	3/2	−1/2					$\frac{5}{12}h^3 f''(\xi)$
3	23/12	−16/12	5/12				$\frac{9}{24}h^4 f^{(3)}(\xi)$
4	55/24	−59/24	37/24	−9/24			$\frac{251}{720}h^5 f^{(4)}(\xi)$
5	1901/720	−2774/720	2616/720	−1274/720	251/720		$\frac{475}{1440}h^6 f^{(5)}(\xi)$
6	4277/720	−7923/720	9982/720	−7298/720	2877/720	−475/720	$\frac{19,087}{60,480}h^7 f^{(6)}(\xi)$

Formula de Adams-Multon

Aproximación de Taylor alrededor de y_{i+1}

$$y_i = y_{i+1} - f_{i+1}h + \frac{f'_{i+1}}{2}h^2 - \frac{f''_{i+1}}{3!}h^3 + \cdots$$

$$y_{i+1} = y_i + h\left(f_{i+1} - \frac{h}{2} f'_{i+1} + \frac{h^2}{6} f''_{i+1} + \cdots \right)$$

$$f'_{i+1} = \frac{f_{i+1} - f_i}{h} + \frac{f''_{i+1}}{2}h + O(h^2)$$

$$y_{i+1} = y_i + h\left(\frac{1}{2} f_{i+1} + \frac{1}{2} f_i\right) - \frac{1}{12}h^3 f''_{i+1} - O(h^4)$$

$$y_{i+1} = y_i + h \sum_{k=0}^{n-1} \beta_k f_{i+1-k} + O(h^{n+1})$$

Formula de Adams-Multon

Order	β_0	β_1	β_2	β_3	β_4	β_5	Local Truncation Error
2	1/2	1/2					$-\frac{1}{12}h^3f''(\xi)$
3	5/12	8/12	−1/12				$-\frac{1}{24}h^4f^{(3)}(\xi)$
4	9/24	19/24	−5/24	1/24			$-\frac{19}{720}h^5f^{(4)}(\xi)$
5	251/720	646/720	−264/720	106/720	−19/720		$-\frac{27}{1440}h^6f^{(5)}(\xi)$
6	475/1440	1427/1440	−798/1440	482/1440	−173/1440	27/1440	$-\frac{863}{60,480}h^7f^{(6)}(\xi)$

Ejercicio

- Resolver con el método Euler implícito y explícito para
- $x_1(0)=x_2(0)=1$
- t [0,0.2]
- h=0.05

$$\frac{dx_1}{dt} = 999x_1 + 1999x_2$$

$$\frac{dx_2}{dt} = -1000x_1 - 2000x_2$$

Ejercicio

- Resolver con el método Heun modificado.
- $y(1.5)=5.222138$
- $y(2.0)=4.143883$
- $t\ [2,3]$
- $h=0.5$
- $\varepsilon_S=0.01\%$

$$\frac{dy}{dt} = -0.5y + e^{-t}$$