Discrete Math
Rose

Name KEY

Well Ordering Principle ⇔ Mathematical Induction ⇔ Complete Induction

WOP → Mathematical Induction

Assume WOP
Every nonempty set of natural numbers has a least element.

Take arbitrary property P.

Assume $P(0) \wedge \forall K (P(K) \to P(K+1))$

Suppose $\sim \forall n P(n)$
Then $\exists n \sim P(n)$.
Let $S = \{x \mid x \in \mathbb{N} \wedge \sim P(x)\}$
Since $\exists n \sim P(n)$, $S \neq \emptyset$. Also $S \subset \mathbb{N}$
So apply the WOP to S.
S has a least element. Call it m.
$m \in S$, so $\sim P(m)$
$P(0)$ and $\sim P(m)$, so $m \neq 0$. So $m-1 \in \mathbb{N}$
But setting $K = m-1$ in $\forall K (P(K) \to P(K+1))$,
we have $P(m-1) \to P(m)$

Suppose $P(m-1)$
Then $P(m)$. But $\sim P(m)$
$\times$

So $\sim P(m-1)$. But then $m-1 \in S$.
$m-1 < m$, so this contradicts that m was the LEAST element of S. $\times$

$\forall n P(n)$

$(P(0) \wedge \forall K (P(K) \to P(K+1))) \to \forall n P(n)$

$\forall P[(P(0) \wedge \forall K (P(K) \to P(K+1))) \to \forall n P(n)]$

WOP ➔ Complete Induction

Assume WOP

Every nonempty set of natural numbers has a least element

Take arbitrary property P.

Assume $P(0) \wedge \forall K((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1))$

Suppose $\sim \forall n\, P(n)$

Then $\exists n \sim P(n)$

Let $S = \{x \mid x \in \mathbb{N} \text{ and } \sim P(x)\}$

Since $\exists n \sim P(n)$, $S \neq \emptyset$. Also $S \subset \mathbb{N}$.

So apply the WOP to S.

S has a least element. Call it m.

$m \in S$, so $\sim P(m)$

$P(0)$ and $\sim P(m)$, so $m \neq 0$. So $m-1 \in \mathbb{N}$

Since m is the least element of S, m is the least natural number w/o property P.

So we have $P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m-1)$

Setting $k = m-1$ in $\forall K((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K)) \rightarrow P(k+1))$

we have $(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m-1)) \rightarrow P(m)$

Thus $P(m)$

But $\sim P(m)$

$\divideontimes$

$\forall n\, P(n)$

$(P(0) \wedge \forall K((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)) \rightarrow \forall n\, P(n)$

$\forall P[(P(0) \wedge \forall K((P(0) \wedge P(1) \wedge \cdots \wedge P(K)) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$

Mathematical Induction ➔ Complete Induction

Assume MI

So $\forall Q[(Q(0) \wedge \forall K(Q(K) \rightarrow Q(K+1))) \rightarrow \forall n Q(n)]$

Take arbitrary property P

Suppose $P(0) \wedge \forall K((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K)) \rightarrow P(K+1))$

Let $Q(m) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m)$

Since $P(0)$, we have $Q(0)$

Try to prove $\forall K(Q(K) \rightarrow Q(K+1))$.

Suppose $Q(K)$ for abitrary k

$Q(K) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K)$

We know $(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K)) \rightarrow P(K+1)$

So $P(K+1)$.

$Q(K)$ and $P(K+1) \Rightarrow P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K) \wedge P(K+1)$

So $Q(K+1)$

So $\forall K(Q(K) \rightarrow Q(K+1))$

Thus we have $Q(0) \wedge \forall K(Q(K) \rightarrow Q(K+1))$

Applying MI, we have $\forall n Q(n)$

For any n, $Q(n)$ includes $P(n)$.

So we have $\forall n P(n)$

$(P(0) \wedge \forall K((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K)) \rightarrow P(K+1))) \rightarrow \forall n P(n)$

$\forall P[(P(0) \wedge \forall K((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(K)) \rightarrow P(K+1))) \rightarrow \forall n P(n)]$

CI

Complete Induction ➔ Mathematical Induction

Assume CI

Take arbitrary property P

Assume $P(0) \wedge \forall k(P(k) \rightarrow P(k+1))$

Take arbitrary m.

Suppose $P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m)$

Then we have $P(m)$

Setting $m = k$ above, we have $P(m) \rightarrow P(m+1)$

So $P(m+1)$ Follows

Thus we have shown $(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m)) \rightarrow P(m+1)$

Since m was arbitrary, $\forall m((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m)) \rightarrow P(m+1))$

Given $P(0)$ and $\forall m((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(m)) \rightarrow P(m+1))$,

we now apply CI.

$\forall n\, P(n)$ follows

So $(P(0) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$

$\forall P[(P(0) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)]$

MI

Complete Induction ➔ WOP

Assume CI

$\forall P[(P(0) \land \forall K((P(0) \land P(1) \land P(2) \land \cdots \land P(K)) \to P(K+1))) \to \forall n\, P(n)]$

Take arbitrary $S \subseteq \mathbb{N}$ with $S \neq \emptyset$

Suppose S doesn't have a least element.

Suppose $0 \in S$

Well, 0 is the least natural number.

So, since $0 \in S$, 0 is the least element of S.

But S has no least element.

$\times$

So $0 \notin S$.

Let $P(n)$ be the property that $n \notin S$. So $P(0)$.

Suppose $P(0) \land P(1) \land P(2) \land \cdots \land P(K)$ for arbitrary K

Then $0 \notin S \land 1 \notin S \land 2 \notin S \land \cdots \land K \notin S$

Suppose $K+1 \in S$.

Then $K+1$ would be the least element of S.

But S has no least element.

$\times$

So $K+1 \notin S$

So $P(K+1)$

Thus $\forall K((P(0) \land P(1) \land P(2) \land \cdots \land P(K)) \to P(K+1))$

So by CI, $\forall n\, P(n)$.

So for any n, $n \notin S$.

Thus $S = \emptyset$

$\times$

S has a least element

Every nonempty subset of $\mathbb{N}$ has a least element.

WOP

Mathematical Induction ➔ WOP

Assume MI

$\forall P[(P(0) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n)]$

Take arbitrary $S \subset \mathbb{N}$ with $S \neq \emptyset$

Suppose S doesn't have a least element

Suppose $0 \in S$

Well, 0 is the least natural number.

So, since $0 \in S$, 0 is the least element of S.

But S has no least element.

$*$

So $0 \notin S$

Let $P(n)$ be the property that $m \notin S$ for all m, $0 \leq m \leq n$.

Suppose $P(k)$ for arbitrary k

So $0 \notin S \wedge 1 \notin S \wedge 2 \notin S \wedge \cdots \wedge k \notin S$

Suppose $k+1 \in S$

Then $k+1$ would be the least element of S

But S has no least element.

$*$

So $k+1 \notin S$

Now we have $0 \notin S \wedge 1 \notin S \wedge 2 \notin S \wedge \cdots \wedge k \notin S \wedge k+1 \notin S$

So $P(k+1)$

Thus $\forall k(P(k) \rightarrow P(k+1))$. Also, trivially, $P(0)$.

So by MI, $\forall n P(n)$.

Now, for any n, we have $n \notin S$. So $S = \emptyset$

$*$

S has a least element

Every nonempty subset of $\mathbb{N}$ has a least element.

WOP