Discrete	Math
Rose	

KEX

Well Ordering Principle ←→ Mathematical Induction ←→ Complete Induction

WOP -> Mathematical Induction

Assure NOP Every nonempty set of natural numbers has a least element. Take arbitrary property P. Assume P(0) 1 +K (P(K) -> P(K+1)) Suppose ~ In P(n) then In ~ P(n). Let S= {x | x EN 1 ~ P(x) } Since In ~P(n), S ≠ Ø, Also SCM So apply the WOP to S. 5 has a least element. Call it m. mes, so rp(m) P(0) and P(m), so  $m \neq 0$ . So  $m-1 \in \mathbb{N}$ But setting K = m-1 in  $\forall K (P(K) \rightarrow P(K+1))$ . we have P(m-1) -> P(m) Suppose P(m-1) Then P(m), But ~P(m) SorP(m-1). But then m-1 es. m-1 < m, so this contradicts that m was the LEAST element of S. \* FnP(n)

(PLO) N KK (P(K) -> P(K+1))) -> GuP(n) YP[ (P(O) N YK (P(K) → P(K+1))) → Yn P(n)

WOP → Complete Induction Assume WOP Every nonempty set of natural numbers has a least element Take arbitrary property P. Assume PCO) A YK ((PCO) A PCDA PCDA PCK)) -> P(K+1)) Suppose ~ Yn P(n) Then In NP(n) LetS = {x | x ∈ N and ~P(x)} Since In ~P(n), S + Ø. Also S CN. So apply the WOP to S. Shasa least element. Call it m. m ∈ S, so ~ P(m) P(0) and NP(m), so m = 0. So m-1 eN Since misthe least element of S, misthe least nortural number w/o property P. So we have P(0) 1 P(1) 1 P(2) 1... 1 P(m-1) Setting K=m-1 in YK ((Plon P(1)A P(Z)A... AP(K)) -> P(K+1) we have (P(0)1P(1)1P(2)1...1P(m-1)) -> P(m) Thus P(m) But NP(m) Yn P(n)

P(O) A +K ((P(O) APCI) AP(Z) A... AP(K)) → P(K+1)) → HAP(A)

HP[(P(O) A +K ((P(O) APCI) A... A P(K)) → P(K+1))) → HAP(A)

Assure MI So YQ[(Q(O) 1 YK(Q(K) →Q(K+1))) >> YnQ(n)/ Take arbitrary property P Suppose P(O) 1 YK ((P(O) NP(D) A P(Z) 1" NP(K)) ->P(K+1)) Let Q(m) = P(O) AP(1) AP(Z) A... AP(m) Since Pro), we have Q(0) Try to prove YK (Q(K) -> Q(K+1)). Suppose Q(K) for a bitrary K Q(K) = P(O) 1 P(1) 1 P(2) 1 ... 1 P(K) We know (P(O) AP(I) AP(Z) A. A P(K)) -> P(K+1) So P(K+1) Q(K) and P(K+1) >> P(O) AP(1) AP(2) A. AP(K) AP(K+1) So Q(K+1) So YK (Q(K) -> Q(K+1)) Thus we have Q(O) 1 YK (Q(K) - Q(K+1)) Applying MI, we have VnQ(n) For any n, Q(n) includes P(n). So we have  $\forall n P(n)$ 

(P(O) N HK ( (P(O) AP(I) AP(Z) N -- AP(K)) -> P(K+1))) -> Hn P(n) YP[(P(O) A YK((P(O) A P(I) 1 P(Z) 1 "1 P(K)) -> P(K+1))) -> HAP(A)]

Complete Induction > Mathematical Induction Assume (I Take arbitrary property P Assume P(O) 1 YK (P(K) -> P(K+1)) Talk arbitrary m. Suppose P(O) A P(1) A P(2) A... A P(m) Then we have P(m) Setting m= Kabove, we have P(m) -> P(m+1) So P(m+1) Follows Thus we have shown (PlOMP(1)MP(2)A"MP(m)) - Plant) Since in was arbitrary, In ((POMP(1) AP(2) 1... AP(m)) -> P(m)) Given P(O) and Vm ((P(O) / P(I) / P(D) / P(m)) -> P(m+1)), We now apply CI. Vn P(n) follows SO (P(O) N YK (P(K) > P(K+1))) -> Yn P(n) YP (P(O) NYK (P(K) → P(K+1))) → YnP(n)

Complete Induction > WOP Assume CI HP[(P(O) N HK((P(O) N P(I) N P(Z) N ··· N P(K)) -> P(K+1))) -> Vn P(n)] Take arbitrary SCN with S + 8 Suppose S doesn't have aleast element. Suppose OES Well, O is the least natural number. So, since O∈S, O is the least element of S. But Shas no least element. SO OES. Let P(n) be the property that n €5, So P(o). Suppose PODAP(1) AP(2) A. AP(K) For arbitrary K Then OFS 1 14512 &S1 ... 1 K&S Suppose K+1 ES Then K+1 would be the least element of S. But Shas no least element. So KHIES So P(K+1) Thus YK ((P(O) AP(1) AP(2) A MA P(K)) -> P(K+1)) Soby CI, YnP(n). So for any n, n €S. Thus S = Ø Shas a least element

Every nonempty subset of N has a least elevent.

Mathematical Induction → WOP Assure MI YPL(P(O) 1 YK (P(K) > P(K+1))) -> YnP(n) Take arbitrary SCN with S + O Suppose 5 doesn't have a least element Suppose OES Well, O is the least natural number. So, since OES. O is the least element of S. Buts has no least element. So 0 \$5 Let P(n) be the property that m & S for all m, 0=m=n. Suppose P(K) for arbitrary K SO O \$ \$ 1 | \$ \$ 1 2 \$ 5 1 ... 1 K \$ 5 Suppose KHIES Then K+1 would be the least elevent of S But 5 has no least elevent. SOK+1 45 Now we have 0\$51 1\$51 2\$51 ... 1 K\$51 K+1 & S SO P(K+1) Thus YK(P(K) -> P(K+1)). Also, trivially, P(O). So by MI, the P(n). Now, for any n, we have n \$5. So 5=0 S has a least element

Every nonempty subset of M has a least element. WOP