The min-max construcions

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April 15, 2019

References: Colding-Minicozzi: A course in minimal surfaces (Width and mean curvature flow, first a few page, proof is 2 page, even more readable); Colding-De Lellis: The min-max construction of minimal surfaces; Xin Zhou: Introduction to the min-max theory of minimal surfaces.

1 Motivations

Let $f : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^2$ a compact domain be a function with saddle point. If $\min_{p\in\{p\}} \max_{t\in[0,1]} f(p(t))$ will pick up the critical point if $\{p\}$ is the set of paths that cross the middle parabola. (**PIC!**)

2 Existence of closed geodesics on 2-sphere

Set-up:

- Let M be a topological 2-sphere (also works for general closed Riemannian manifold).
- Λ, space of piecewise geodesics $\mathbb{S}^1 \to M$ with exactly L breaks. Each piece is of length $\leq 2\pi$. The topology is induced by $W^{1,2}$ norm.
- Let $G \subset \Lambda$ denote the set of immersed closed geodesics in M of length at most $2\pi L$.

Main idea (**PIC!**):

- Take minimizing sequence of closed piecewise geodesics.
- Apply Birkhoff path shortening process to tighten up.
- The tightened closed curve is close enough to the set of closed geodesics, and length is positive.
- Conclude that G contains non-constant geodesics.

Here we consider 2-sphere isometriclly embedded into some Euclidean space $\mathbb{R}^N$ and we can scale $\mathbb{R}^N$ so that we can cover every closed curves by finitely many normal neighborhood, so we are able to modify it to piecewise geodesics.

2.1 Birkhoff curve shortening process

Let $\gamma : \mathbb{S}^1(\text{or } [0, 2\pi]) \to M$ be a closed curve, parametrized to constant speed. Fix a partition of $\mathbb{S}^1$ by choosing $2L$ consecutive evenly spaced points $x_0, x_1, \dots, x_{2L}$ for some fixed large L.

Construct the shortening process in 4 steps:

(A1) Replace γ on each even interval, i.e., $[x_{2j}, x_{2j+2}]$ by a geodesic with the same endpoints to get a piecewise geodesic γ_e.

(B1) Reparametrize γ_e (fixing the image of x_0) to get the constant speed curve $\tilde{\gamma}_e$. This reparametrization moves the ponts x_j to new points $\tilde{x}_j$, i.e., $\gamma_e(x_j) = \tilde{\gamma}_e(\tilde{x}_j)$

(A2) Replace $\tilde{\gamma}_e$ on odd $\tilde{x}_j$ intervals to get $\tilde{\gamma}_o$.

(B2) Reparametrize $\tilde{\gamma}_o$ (fixing the image of x_0) to get the constant speed curve $\Psi(\gamma)$.

Remark 2.0.1. $\Psi(\gamma)$ is homotopic to γ (by straight-line homotopy). Key properties: There is an estimate of the upperbound of $\text{dist}^2(\gamma, \Psi(\gamma))$; If $\text{Length}(\Psi(\gamma))$ does not decrease too much, $\text{dist}(\gamma, G) < \epsilon$, i.e., γ is very close to G.

2.2 Existence of closed geodesics

Wrap-up: For curves with length close to the critical value, Ψ does not decrease the length too much, so the curve is close to G.

A few notions: $\Omega, \Omega_{\hat{\sigma}}, W(\hat{\sigma})$.

Let Ω be the set of continuous maps $\sigma : \mathbb{S}^1 \times [0, 1] \to M$ so that (**PIC!**)

1 For each t $\sigma(\cdot, t)$ is in $W^{1,2}$.

2 The map $t \to \sigma(\cdot, t)$ is continuous from $[0, 1]$ to $W^{1,2}$.

3 σ maps $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$ to points.

Given a map $\hat{\sigma} \in \Omega$, $\Omega_{\hat{\sigma}} := \{\sigma \in \Omega : \sigma \simeq \hat{\sigma}\} \subset \Omega$.

Definition 2.1. *The width $W = W(\hat{\sigma})$ associated to $\Omega_{\hat{\sigma}}$ is*

$$W = \inf_{\sigma \in \Omega_{\hat{\sigma}}} \max_{t \in [0,1]} E(\sigma(\cdot, t)),$$

where $E(\cdot, t) = \int_{\mathbb{S}^1} |\partial_x \sigma(x,t)|^2 \, dx$.

Remark 2.1.1. The width is nonnegative and positive if $\hat{\sigma}$ is not nulhomotopic. The width is continuous in the metric but the min-max curves that realizes it might not be. (**PIC!**)

Sweepouts: choose a sequence of maps $\hat{\sigma}^j \in \Omega_{\hat{\sigma}}$ with

$$\max_{t \in [0,1]} E(\hat{\sigma}^j(\cdot, t)) < W + \frac{1}{j}.$$

Do similar operation as Birkhoff path shortening to replace by piecewise geodesic (can, because uniform $C^{1/2}$). Then reparametrize to constant speed we get $\sigma^j(\cdot, t)$. Take $\gamma^j(\cdot, t) := \Psi(\sigma^j(\cdot, t))$.

Lemma 2.2. *Given $W \geq 0$ and $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in \Lambda$ and*

$$2\pi(W - \delta) < \mathit{Length}^2(\Psi(\gamma)) \leq \mathit{Length}^2(\gamma) < 2\pi(W + \delta),$$

then $\mathit{dist}(\Psi(\gamma), G) < \epsilon$.

Proof uses Wirtinger inequality and properties of Ψ (to choose a small enough δ).

Theorem 2.3 (Colding-Minicozzi)**.** *Given $W \geq 0$ and $\epsilon > 0$, there exist $\delta > 0$ so that if $j > 1/\delta$ and for some t_0,*

$$2\pi E(\gamma^j(\cdot, t_0)) = \mathit{Length}^2(\gamma^j(\cdot, t_0)) > 2\pi(W - \delta),$$

then for this j we have $\mathit{dist}(\gamma^j(\cdot, t_0), G) < \epsilon$.

Proof. Let δ be given by the above lemma. Then using $j > 1/\delta$,

$$2\pi(W - \delta) < \mathrm{Length}^2(\gamma^j(\cdot, t_0)) \leq \mathrm{Length}^2(\sigma^j(\cdot, t_0)) < 2\pi(W + \delta).$$

Thus $\mathrm{dist}(\gamma^j(\cdot, t_0), G) < \epsilon$. ∎

In the case of M a 2-sphere, the width is positive and realized by a nontrivial closed geodesic.

3 Existence of closed embedded minimal surfaces in a closed 3-dimensional manifold

We want to construct closed embeded minimal surfaces in a closed 3-dimensional manifold via min-max arguments.

Definition 3.1 (minimal surface)**.** *Critical point of the area functional.*

Remark 3.1.1. Baby do Carmo: normal variation. (**PIC!**)

Set-up:

- Let M be a closed 3-manifold.
- A smooth family of surfaces in M $\{\Sigma_t\}_{t \in [0,1]}$. Λ, saturated set of generalized families, namely closed under all smooth isotopies.
- $m_0(\Lambda) = \inf_{\{\Sigma_t\} \in \Lambda} \left[\max_{t \in [0,1]} \mathcal{H}^2(\Sigma_t) \right]$, critical value. $\mathcal{H}^2$ the 2-dimensional Hausdorff (outer) measure.
- Minimizing sequence: $\{\Sigma_t\}^n$ such that $\lim_n \left[\max_{t \in [0,1]} \mathcal{H}^2(\Sigma_t^n) \right] = m_0(\Lambda)$.
- Min-max sequence: $\{\Sigma_{t_n}^n\}$ is a min-max sequence, if there is $\{t_n\}$ such that $\mathcal{H}^2(\Sigma_{t_n}^n) \to m_0(\Lambda)$.

Idea: We want to realize the critical value with the min-max sequence.

Theorem 3.2 (Simon-Smith)**.** *Let M be a closed 3-manifold with a Riemannian metric. For any saturated set of generalized families of surfaces Λ, there is a min-max sequence obtained from Λ and converging in the sense of varifolds to a smooth embedded minimal surface with area $m_0(\Lambda)$ (multiplicity allowed).*

Definition 3.3. *A 2-**varifold** V is a Radon measure on $G_2(U) := \{(x, s) : \; x \in U, \; s \text{ is a 2-dim plane of } T_x M\}$.*

Remark 3.3.1 (Moral). Basically area.

There is a notion of integral/integer rectifiable varifold, which says that the support of the 2-varifold is a countable union of closed subsets of C^1 surfaces, so it can be written as integrating over the surfaces. Probably from that information, we can recover the surface. Or from later proof, it seems if one concludes the support of the varifold is a smooth surface, then say, it is a surface.

Definition 3.4. *The **first variation** of a varifold is defined to be* $\delta V(X) := \frac{d}{dt}\big|_{t=0}\mathcal{H}(\varphi_{t\#}V)$ *where* φ_t *is the flow generated by the vector field* X *supported on* U*.*

Definition 3.5. V *is said to be **stationary** in* U *if* $\delta V(X) = 0$ *for all vector fields* X *supported in* U*.*

Remark 3.5.1. Still think of our very first example from baby do Carmo: X the normal vector field.

3.1 Idea of the proof

Step 1: Existence of good min-max sequences converging to a stationary varifold

Take $\{\{\Sigma_t\}^n\}$ minimizing sequence (can, because definition of $m_0(\Lambda)$ is an infimum).

- We can show that $\Sigma^n_{t_n} \to V_\infty$ a stationary varifold for some $\{t_n\}$.
- Since there may exists min-max sequence whose area converges to $m_0(\Lambda)$ but it actually not converge in the sense of varifold (bad slices far from V_∞) (**PIC!**).
- Tightening process, like Birkhoff path shortening, to get rid of bad slices. The min-max seqence did converge to V_∞.

Step 2: Regularity

- Defined a notion of almost minimizing sequence. Roughly speaking: any path $\{\Sigma_t\}$ such that $\Sigma_0 = \Sigma$ and Σ_1 has smaller area must necessarily pass through a surface with large area. (**PIC!**)
- Work with annulus: Allow singularities.
- Existence of good replacements (smooth surface) in annulus; glue replacements; remove the singularity in the center of annulus.
- We can find min-max sequence that each of which is almost miminizing. Use almost minimizing to construct replacements.

Summary: We find a min-max sequence that converge to a stationary varifold, and establish the regularity. Then from the smooth stationary varifold we can get the minimal surface.