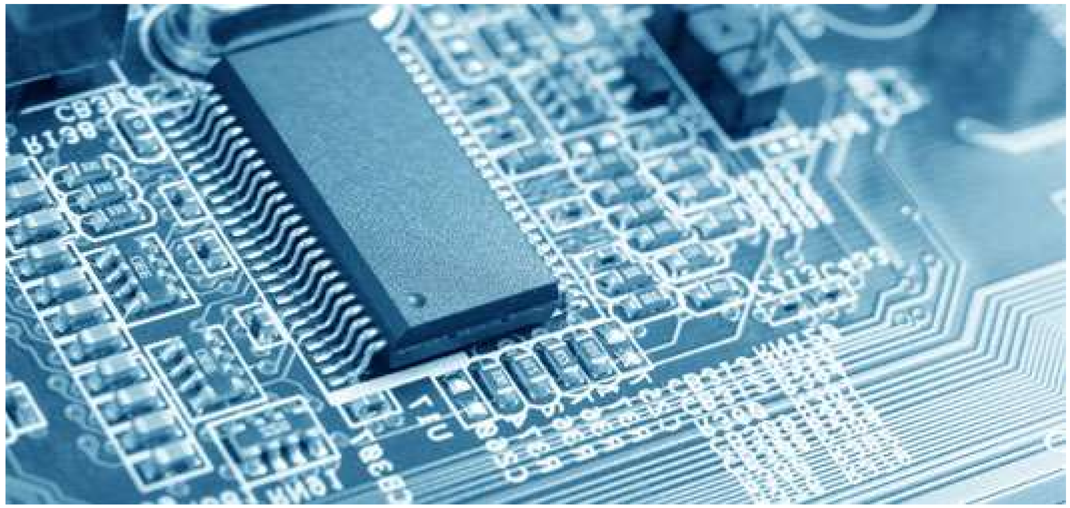
NAZARBAYEV UNIVERSITY
SCHOOL OF SCIENCE AND TECHNOLOGY



ROBT206 – Microcontrollers with Lab

Lectures 5-6 – Standard Forms

23-25 January, 2018

Topics

Today's Topics
Standard Forms

Course Logistics

Important Dates and Tasks
Reading Assignment: Mano Chapter 1& 2
Quiz #1 in Thursday 1 February
Homework #1 due to end of 4th of February (Sunday) Mano textbook: Problems: 1.4, 1.7, 1.9, 1.10, 1.16, 1.18, 1.23, 1.24, 2.1, 2.2, 2.3, 2.4, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.13, 2.14, 2.16, 2.21, 2.22

Overview – Canonical Forms

- What are Canonical Forms?
- Minterms and Maxterms
- Index Representation of Minterms and Maxterms
- Sum-of-Minterm (SOM) Representations
- Product-of-Maxterm (POM) Representations
- Representation of Complements of Functions
- Conversions between Representations

Canonical Forms

- It is useful to specify Boolean functions in a form that:
  - Allows comparison for equality.
  - Has a correspondence to the truth tables

- Canonical Forms in common usage:
  - **Sum of Minterms (SOM)**
  - **Product of Maxterms (POM)**

Minterms

- Minterms are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for *n* variables.
- Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:

  XY

  $X\overline{Y}$

  $\overline{X}Y$

  $\overline{X}\,\overline{Y}$

  Thus there are **four minterms** of two variables.

Maxterms

- Maxterms are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., $\overline{x}$) or complemented (e.g., x), there are 2^n maxterms for n variables.
- Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:

$X+Y$

$X+\overline{Y}$

$\overline{X}+Y$

$\overline{X}+\overline{Y}$

Standard Order

- Minterms and maxterms are designated with a subscript
- The subscript is a number, corresponding to a binary pattern
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the **same order** (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms: $(a + b + \bar{c})$, $(a + b + c)$
  - Terms: $(b + a + c)$, $a\,\bar{c}\,b$, and $(c + b + a)$ are NOT in standard order.
  - Minterms: $a\,\bar{b}\,c$, $a\,b\,c$, $a\,\bar{b}\,\bar{c}$
  - Terms: $(a + c)$, $\bar{b}c$, and $(\bar{a} + b)$ do not contain all variables

Index Example in Three Variables

- Example: (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The Index 0 (base 10) = 000 (base 2) for three variables).
- All three variables are complemented for minterm 0 ($\bar{X}, \bar{Y}, \bar{Z}$) and no variables are complemented for Maxterm 0 (X,Y,Z).
  - Minterm 0, called m_0 is $\bar{X}\bar{Y}\bar{Z}$
  - Maxterm 0, called M_0 is $(X + Y + Z)$.
  - Minterm 6 ?
  - Maxterm 6 ?

Minterm Truth Table

Minterms for Three Variables

X	Y	Z	Product Term	Symbol	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7
0	0	0	$\overline{X}\,\overline{Y}\,\overline{Z}$	m_0	1	0	0	0	0	0	0	0
0	0	1	$\overline{X}\,\overline{Y}Z$	m_1	0	1	0	0	0	0	0	0
0	1	0	$\overline{X}Y\overline{Z}$	m_2	0	0	1	0	0	0	0	0
0	1	1	$\overline{X}YZ$	m_3	0	0	0	1	0	0	0	0
1	0	0	$X\overline{Y}\,\overline{Z}$	m_4	0	0	0	0	1	0	0	0
1	0	1	$X\overline{Y}Z$	m_5	0	0	0	0	0	1	0	0
1	1	0	$XY\overline{Z}$	m_6	0	0	0	0	0	0	1	0
1	1	1	XYZ	m_7	0	0	0	0	0	0	0	1

Maxterm Truth Table

Maxterms for Three Variables

X	Y	Z	Sum Term	Symbol	M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7
0	0	0	$X+Y+Z$	M_0	0	1	1	1	1	1	1	1
0	0	1	$X+Y+\overline{Z}$	M_1	1	0	1	1	1	1	1	1
0	1	0	$X+\overline{Y}+Z$	M_2	1	1	0	1	1	1	1	1
0	1	1	$X+\overline{Y}+\overline{Z}$	M_3	1	1	1	0	1	1	1	1
1	0	0	$\overline{X}+Y+Z$	M_4	1	1	1	1	0	1	1	1
1	0	1	$\overline{X}+Y+\overline{Z}$	M_5	1	1	1	1	1	0	1	1
1	1	0	$\overline{X}+\overline{Y}+Z$	M_6	1	1	1	1	1	1	0	1
1	1	1	$\overline{X}+\overline{Y}+\overline{Z}$	M_7	1	1	1	1	1	1	1	0

Index Examples – Four Variables

Index i	Binary Pattern	Minterm m_i	Maxterm M_i
0	0000	$\bar{a}\bar{b}\bar{c}\bar{d}$	$a+b+c+d$
1	0001	$\bar{a}\bar{b}\bar{c}d$	?
3	0011	?	$a+b+\bar{c}+\bar{d}$
5	0101	$\bar{a}b\bar{c}d$	$a+\bar{b}+c+\bar{d}$
7	0111	?	$a+\bar{b}+\bar{c}+\bar{d}$
10	1010	$a\bar{b}c\bar{d}$	$\bar{a}+b+\bar{c}+d$
13	1101	$ab\bar{c}d$	?
15	1111	$abcd$	$\bar{a}+\bar{b}+\bar{c}+\bar{d}$

Minterm and Maxterm Relationship

- Review: DeMorgan's Theorem

  $\overline{\mathbf{x} \cdot \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$ and $\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} \cdot \overline{\mathbf{y}}$

- Two-variable example:

  $\mathbf{M}_2 = \overline{\mathbf{x}} + \mathbf{y}$ and $\mathbf{m}_2 = \mathbf{x}\,\overline{\mathbf{y}}$

  Thus M_2 is the complement of m_2 and vice-versa.
- Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables giving:

$$\mathbf{M}_i = \overline{\mathbf{m}}_i \text{ and } \mathbf{m}_i = \overline{\mathbf{M}}_i$$

Thus M_i is the complement of m_i.

Function Tables for Both

- Minterms of 2 variables

x y	m_0	m_1	m_2	m_3
0 0	1	0	0	0
0 1	0	1	0	0
1 0	0	0	1	0
1 1	0	0	0	1

Maxterms of 2 variables

x y	M_0	M_1	M_2	M_3
0 0	0	1	1	1
0 1	1	0	1	1
1 0	1	1	0	1
1 1	1	1	1	0

- Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i.

Sum of Minterms and Product of Maxterms

The minterm and maxterm representation of functions may look very different:

X	Y	**Minterm**	**Maxterm**	F
0	0	$\overline{X}\,\overline{Y}$	$X+Y$	0
0	1	$\overline{X}\,Y$	$X+\overline{Y}$	1
1	0	$X\,\overline{Y}$	$\overline{X}+Y$	1
1	1	$X\,Y$	$\overline{X}+\overline{Y}$	1

The sum of the minterms where the function is 1:

$$F = \overline{X}\,Y + X\,\overline{Y} + X\,Y$$

The product of the maxterms where the function is 0:

$$F = X + Y$$

Sum of Minterms Example

- Example: Find $F_1 = m_1 + m_4 + m_7$
- $F1 = \bar{x}\ \bar{y}\ z + x\ \bar{y}\ \bar{z} + x\ y\ z$

x y z	index	m1	+	m4	+	m7	= F1
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Sum of Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$

Product of Maxterms

- Example: Implement F1 in maxterms:

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z) \cdot (x + \bar{y} + z) \cdot (x + \bar{y} + \bar{z}) \cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F1$
0 0 0	0	$0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0$
0 0 1	1	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
0 1 0	2	$1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$
0 1 1	3	$1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 = 0$
1 0 0	4	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
1 0 1	5	$1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 = 0$
1 1 0	6	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 0 = 0$
1 1 1	7	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

Product of Maxterms Example

- $F(A, B, C, D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- $F(A, B, C, D) =$

Canonical Sum of Minterms

- Any Boolean function can be expressed as a Sum of Minterms.
  - For the function table, the minterms used are the terms corresponding to the 1's
  - For expressions, expand all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term ($\mathbf{v} + \overline{\mathbf{v}}$).
- Example: Implement $\mathbf{f} = \mathbf{x} + \overline{\mathbf{x}}\ \overline{\mathbf{y}}$ as a sum of minterms.

  First expand terms: $\mathbf{f} = \mathbf{x}(\mathbf{y} + \overline{\mathbf{y}}) + \overline{\mathbf{x}}\ \overline{\mathbf{y}}$

  Then distribute terms: $\mathbf{f} = \mathbf{xy} + \mathbf{x}\overline{\mathbf{y}} + \overline{\mathbf{x}}\ \overline{\mathbf{y}}$

  Express as sum of minterms: $\mathbf{f} = \mathbf{m_3} + \mathbf{m_2} + \mathbf{m_0}$

Another SOM Example

- Example: $\mathbf{F = A + \overline{B}\,C}$
- There are three variables, A, B, and C which we take to be the standard order.
- Expanding the terms with missing variables:
- Collect terms (removing all but one of duplicate terms):
- Express as SOM:

Shorthand Form for Sum of Minterms (SOM)

- From the previous example, we started with:

$$F = A + \overline{B}C$$

- We ended up with:

$$F = m_1 + m_4 + m_5 + m_6 + m_7$$

- This can be denoted in the formal shorthand:

$$F(A,B,C) = \Sigma m(1,4,5,6,7)$$

- Note that we explicitly show the standard variables in order and drop the "m" designators.

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a Product of Maxterms (POM).
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law , "ORing" terms missing variable v with a term equal to 0 and then applying the distributive law $v \cdot \bar{v}$ again.
- Example: Convert to product of maxterms:

$$f(x, y, z) = x + \bar{x}\,\bar{y}$$

Apply the distributive law:

$$x + \bar{x}\,\bar{y} = (x + \bar{x})(x + \bar{y}) = 1 \cdot (x + \bar{y}) = x + \bar{y}$$

Add missing variable z:

$$x + \bar{y} + z \cdot \bar{z} = (x + \bar{y} + z)(x + \bar{y} + \bar{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- Convert to Product of Maxterms:

$$f(A,B,C) = A\overline{C} + BC + \overline{A}\,\overline{B}$$

- Use $x + y z = (x+y) \cdot (x+z)$ with

$$x = (A\overline{C} + BC),\ y = \overline{A} \quad \text{and} \quad z = \overline{B}$$

to get: $f = (A\overline{C} + BC + \overline{A})(A\overline{C} + BC + \overline{B})$

Then use $x + \overline{x}y = x + y$

to get: $f = (\overline{C} + BC + \overline{A})(A\overline{C} + C + \overline{B})$

Another POM Example

and a second time to get:

$$f = (\bar{C} + B + \bar{A})(A + C + \bar{B})$$

Rearrange to standard order,

$$f = (\bar{A} + B + \bar{C})(A + \bar{B} + C)$$

to give f = M_2* M_5

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x, y, z) = \Sigma_m(1,3,5,7)$

$$\bar{F}(x, y, z) = \Sigma_m(0,2,4,6)$$
$$\bar{F}(x, y, z) = \Pi_M(1,3,5,7)$$

Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.
- Example: Given F as before: $F(x, y, z) = \Sigma_m(1,3,5,7)$
- Form the Complement: $\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $F(x, y, z) = \Pi_M(0,2,4,6)$

Standard Forms

- Standard Sum-of-Products (SOP) form: equations are written as an OR of AND terms
- Standard Product-of-Sums (POS) form: equations are written as an AND of OR terms
- Examples:
  - SOP: $ABC + \overline{A}\,\overline{B}C + B$
  - POS: $(A+B)\cdot(A+\overline{B}+\overline{C})\cdot C$
- These "mixed" forms are neither SOP nor POS

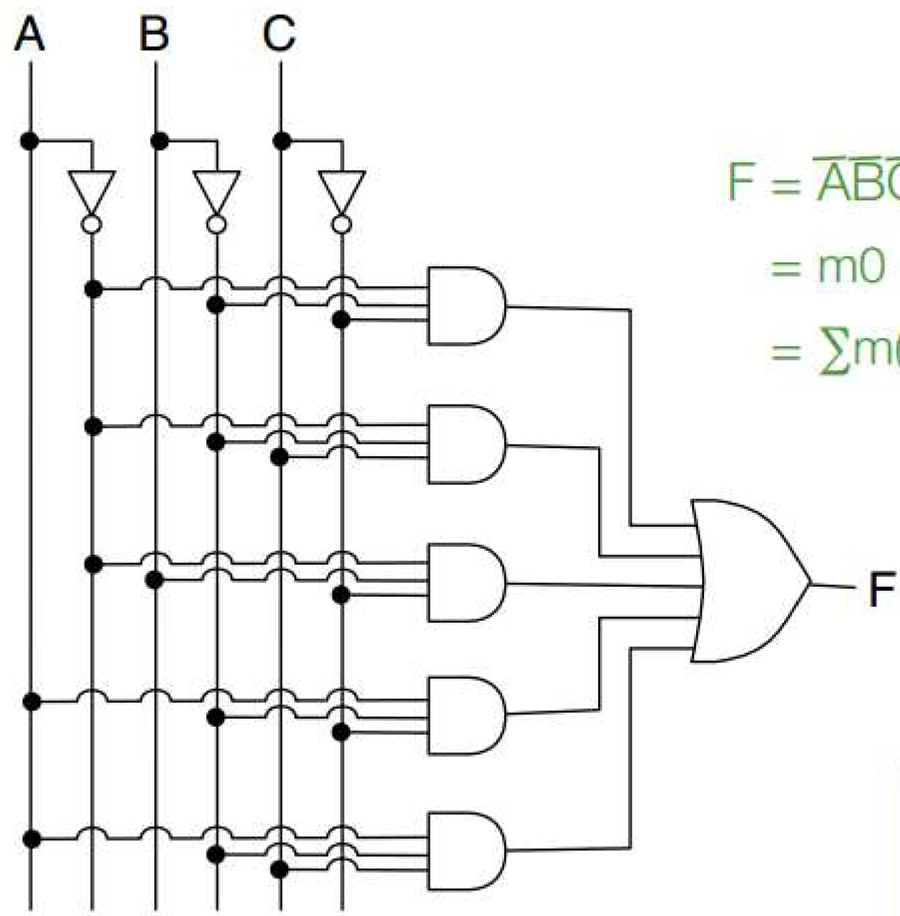
$$(AB+C)(A+C)$$

$$AB\overline{C}+AC(A+B)$$

Standard Sum-of-Products (SOP)

- A sum of minterms form for *n* variables can be written down directly from a truth table.
  - Implementation of this form is a two-level network of gates such that:
  - The first level consists of *n*-input AND gates, and
  - The second level is a single OR gate (with fewer than 2^n inputs).
- This form often can be simplified so that the corresponding circuit is simpler.

SOP as a Circuit



$$F = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$$
$$= m0 + m1 + m2 + m4 + m5$$
$$= \sum m(0,1,2,4,5)$$

Standard form is not minimal form!

Standard Sum-of-Products (SOP)

- A Simplification Example:

$$F(A,B,C) = \Sigma m(1,4,5,6,7)$$

- Writing the minterm expression:

$$F = \bar{A}\,\bar{B}\,C + A\,\bar{B}\,\bar{C} + A\,B\,\bar{C} + A\bar{B}C + ABC$$

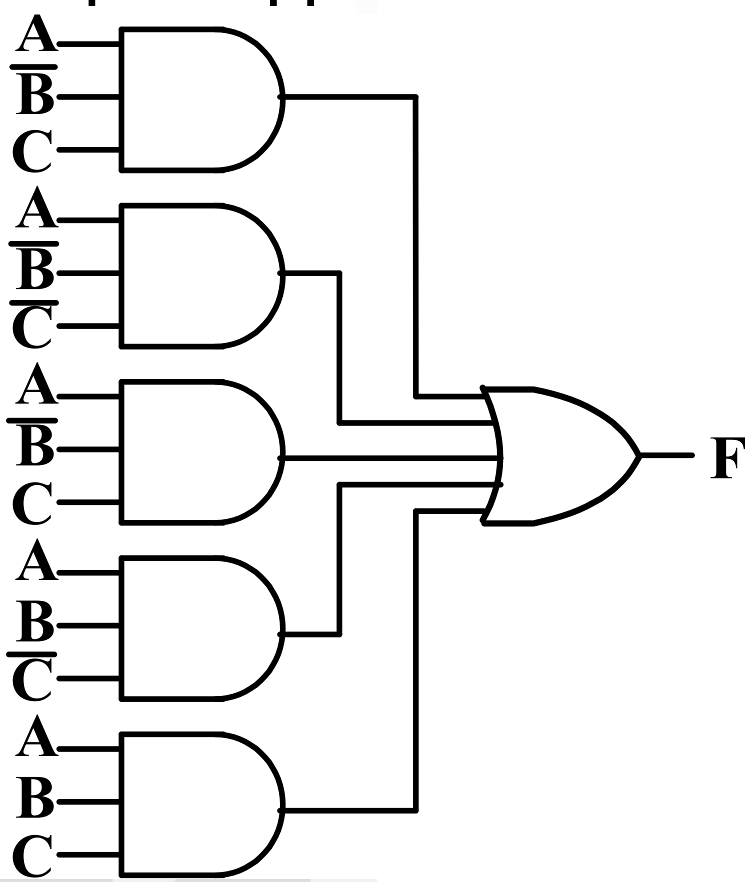
- Simplifying:

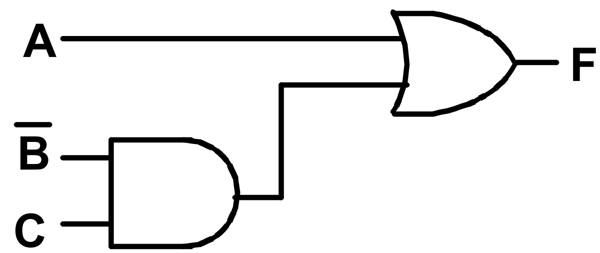
F =

- Simplified F contains 3 literals compared to 15 in minterm F

AND/OR Two-level Implementation of SOP Expression

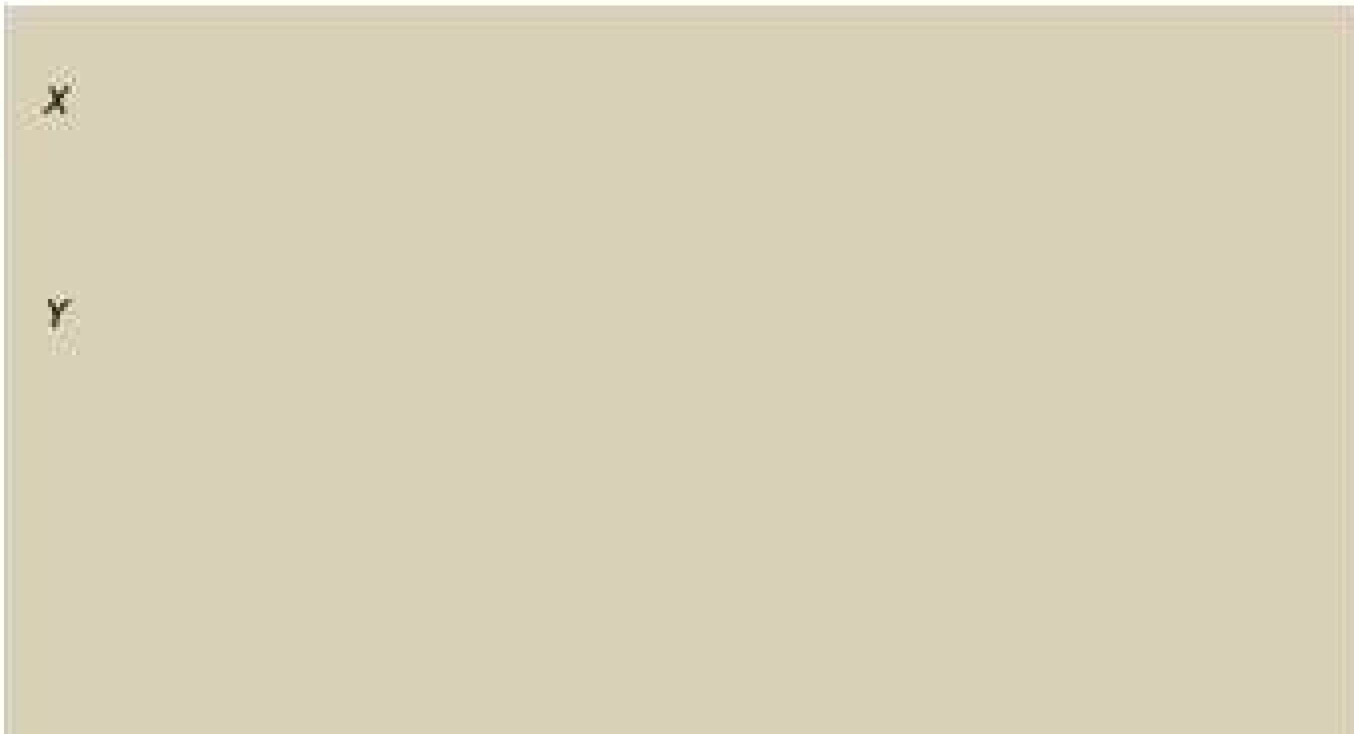
- The two implementations for F are shown below – it is quite apparent which is simpler!





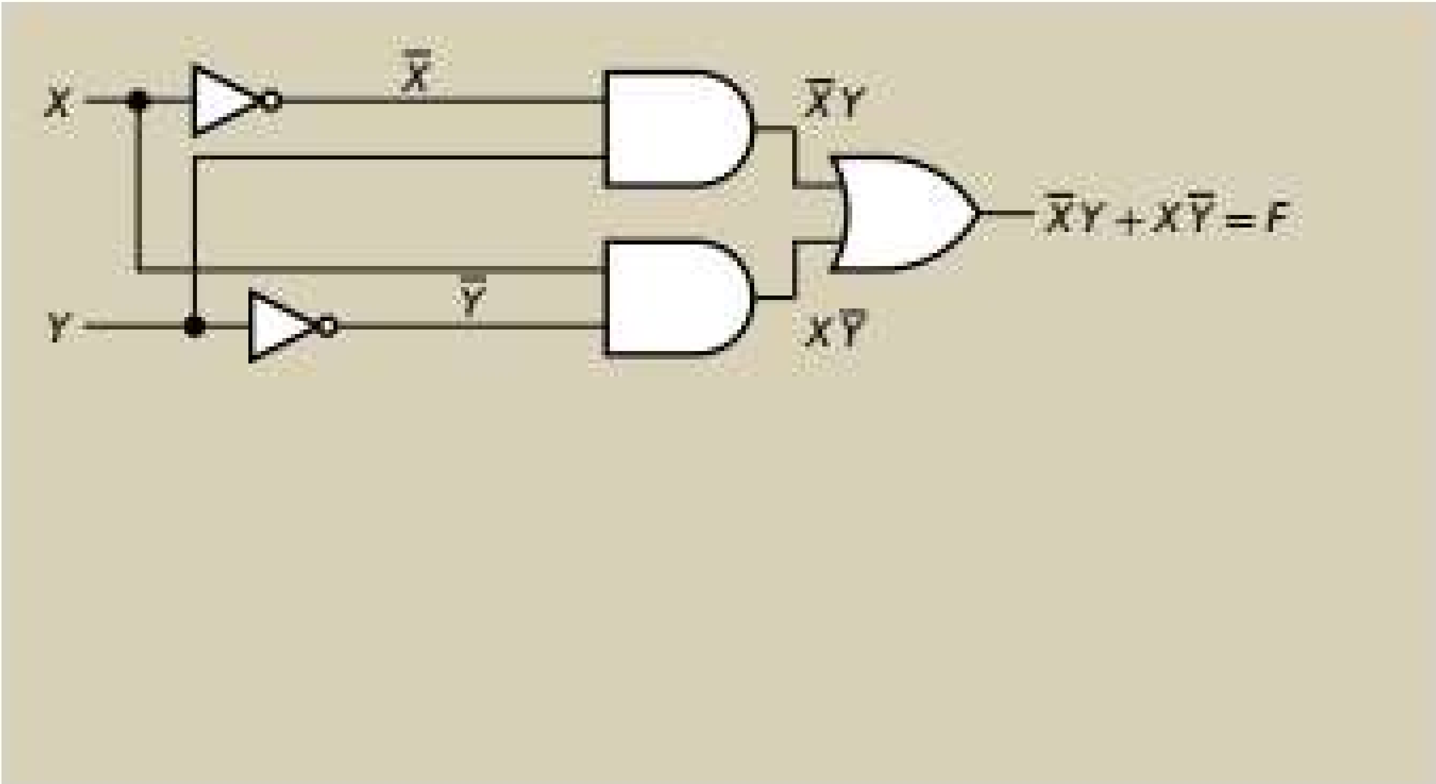
Expressions to Schematics

$$F = \overline{X}Y + X\overline{Y}$$



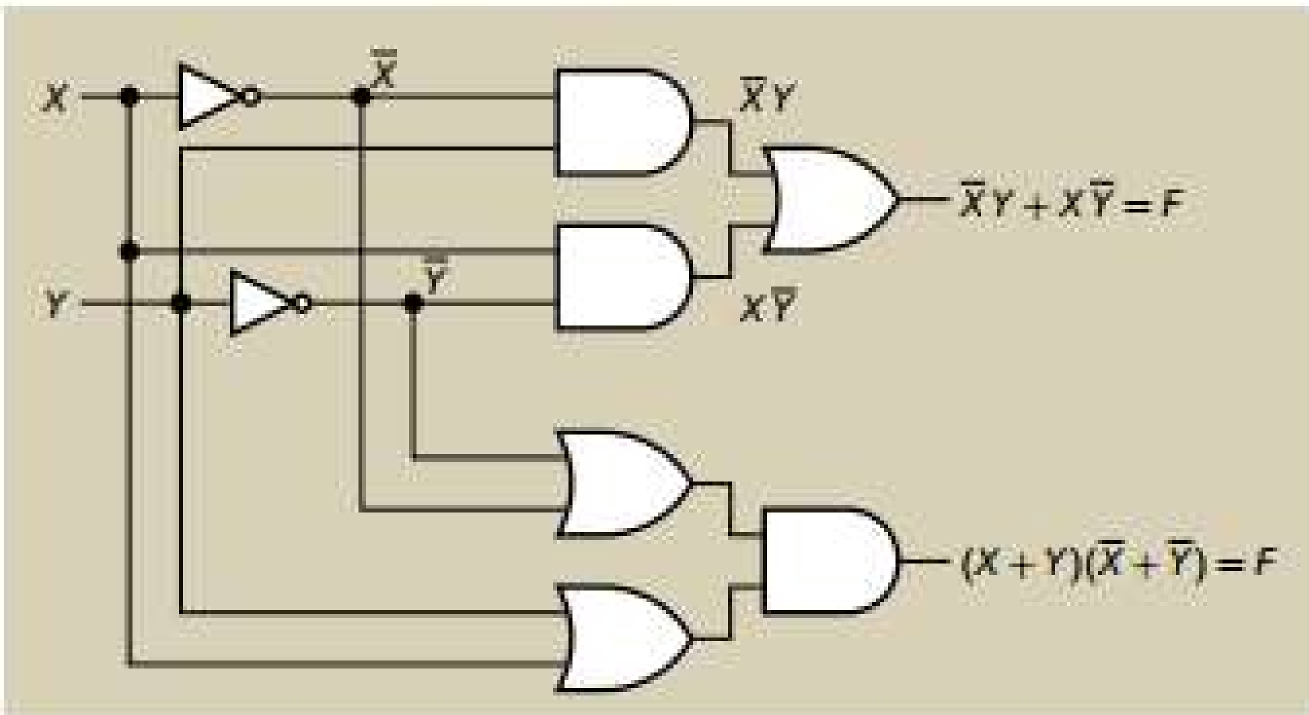
Expressions to Schematics

$$F = \overline{X}Y + X\overline{Y}$$



Expressions to Schematics

$$F = \overline{X}Y + X\overline{Y} = (X+Y)(\overline{X}+\overline{Y})$$



Any Questions?