Dirichlet Functional

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Abstract

I present how it is equivalent to solve Poisson's equation and to minimize the Dirichlet Functional.

Introduction

Poisson's equation shows up often throughout both mathematics and physics. A few examples would be when finding the electric potential with a given boundary condition, finding the gravitational potential with a given boundary condition or when finding harmonic functions, and thus, analytic functions. Equating the problem of solving the partial differential equation (PDE) to solving a minimization problem allows one to gain a new perspective, and hopefully allows them to learn new information about the solutions. This method of equating solutions to PDEs to be minimizers of functionals is a standard technique in PDE analysis and Calculus of Variations. It brings two fairly distinct fields of mathematics together. A specific example of when this might be useful is in the case of when a ball of mass m drops due to gravity g (note that this is a positive number here). Due to Lagrange we can say that the action of our system is

$$S[x] = \int \frac{1}{2} m \dot{x}^2 - mgx$$

where x is the position of the ball. From this the principal of least action tells us that the path our ball takes $x(t)$ will be when S is minimized. The result of this paper shows that this is equivalent to solving the following differential equation

$$\ddot{x} = -g \implies ma = -mg \iff F = ma$$

which is exactly the differential equation that governs motion that Newton came up with in the first place.

Theorem

First let's start by remember what $\triangle f$ is for some *nice* f:

$$\triangle f = \operatorname{div}(\nabla f) = \sum_{i=1}^{3} f_{x_i x_i} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{1}$$

Now let $U \subset \mathbb{R}^3$ be a bounded open set, $u \in C^2(\overline{U})$ (meaning u is twice continuously differentiable, continuous on the boundary and defined on the closure of U), $g \in C(\partial U)$ and $f \in C(\overline{U})$ (meaning f, g are continuous and defined on $U, \partial U$ respectively). Next put $I : C^2(U) \to \mathbb{R}$ given by

$$I[u] = \int_U \left(\frac{1}{2} \|\nabla u\|^2 - fu \right) \tag{Dirichlet Functional}$$

and finally put

$$\mathcal{A} = \{u \in C^2(\overline{U}) \mid u = g \text{ on } \partial U\}$$

Then we have the following theorem:

Theorem. *$u \in \mathcal{A}$ is such that $I[u] \leq I[w]\ \forall\, w \in \mathcal{A}$ iff*

$$-\triangle u = f \text{ in } U \tag{2}$$
$$u = g \text{ on } \partial U \tag{3}$$

It's worth noting that (2) is Poisson's Equation.

Proof. First, let $u \in \mathcal{A}$ such that $I[u] \leq I[w] \; \forall \, w \in \mathcal{A}$. Now let $\phi \in C_c^\infty(\overline{U})$ (infinitely differentiable and zero on the boundary) and define $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(t) = I[u + t\phi]$$

This makes sense since $u + t\phi = u$ on the boundary (since by construction $\phi = 0$ on the boundary. Next notice that $\psi(t) = I[u + t\phi] \geq I[u] = \psi(0) \; \forall \, t \in \mathbb{R}$ so that $\psi(0)$ is a global minimum, thus we must have that $\psi'(0) = 0$. Now let's calculate $\psi'(t)$:

$$\psi'(t) = \frac{\mathrm{d}}{\mathrm{d}t} I[u] = \frac{\mathrm{d}}{\mathrm{d}t} \int_U \frac{1}{2} \|\nabla u + t \nabla \phi\|^2 - fu - tf\phi \tag{4}$$

Now we can commute the derivative with the integral (you can see this from Lebesgue's Dominated Convergence theorem) so that (4) becomes

$$\begin{aligned}
\int_U \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|\nabla u + t \nabla \phi\|^2 - fu - tf \right) \phi &= \int_U \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla u + t \nabla \phi, \nabla u + t \nabla \phi \rangle - f\phi \\
&= \int_U \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big(\langle \nabla u, \nabla u \rangle + 2t \langle \nabla u, \nabla \phi \rangle + t^2 \langle \nabla \phi, \nabla \phi \rangle \big) - f\phi \\
&= \int_U \langle \nabla u, \nabla \phi \rangle + t \langle \nabla \phi, \nabla \phi \rangle - f\phi
\end{aligned}$$

But now we have that $\psi'(0) = 0$ so

$$0 = \int_U \langle \nabla u, \nabla \phi \rangle - f\phi = \int_U (-\operatorname{div}(\nabla u) - f)\phi \tag{5}$$

where we used integration by parts for multivariable integrals. Now the fundamental theorem of calculus of varitations tells us that if we have $\int f\phi = 0$ for every $\phi \in C_c^\infty$ then $f = 0$ so that (5) becomes

$$-\operatorname{div}(\nabla u) - f = 0 \implies -\operatorname{div}(\nabla u) = f \tag{6}$$

And now we see that using (1) that (6) is the same as (2) and indeed (3) is satisfied since $u \in \mathcal{A}$.
Now consider $u \in C^2(\overline{U})$ such that u satisfies (2) and (3). Let's try to mimmick something like the reverse of the last proof, so let's start with $w \in \mathcal{A}$ and consider $\int_U (-\triangle u - f)(u - w)$, since this $u - w$ term is similar to the ϕ term used above (meaning it's 0 on the boundary). Now since u satisfies (2) we know that $0 = -\triangle u - f$ so that

$$\begin{aligned}
0 &= \int_U (-\triangle u - f)(u - w) \\
&= \int_U -\triangle u(u - w) - f(u - w) \\
&= \int_U \langle \nabla u, \nabla u \rangle - \langle \nabla u, \nabla w \rangle - fu + fw
\end{aligned}$$

Now let's move the terms with w to the over side to get a formula similar to the one $I[u] \leq I[w]$:

$$\int_U \|\nabla u\|^2 - fu = \int_U \langle \nabla u, \nabla w \rangle - fw \tag{7}$$

but now note that the cauchy schwartz inequality tells us

$$\int \langle f, g \rangle \leq \sqrt{\int \|f\|^2} + \sqrt{\int \|g\|^2}$$

and we have for real numbers a, b

$$(a-b)^2 \geq 0 \implies \frac{a^2}{2} + \frac{b^2}{2} \geq ab$$

so that (7) becomes

$$\int_U \|\nabla u\|^2 - fu \leq \int_U \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla w\|^2 - fw \implies \int_U \frac{1}{2}\|\nabla u\|^2 - fu \leq \int_U \frac{1}{2}\|\nabla w\|^2 - fw \tag{8}$$

and so comparing (8) to the Dirichlet Functional we can see that $I[u] \leq I[w]$, and then finally we note that $u \in \mathcal{A}$ so that the claim is shown. $\square$