Lecture 4

Analysis of the CHSH game in the case $a = 1$ and $b = 0$.

In this case:

1. Alice's measurement is $\{|+\rangle\langle+| \otimes \mathbb{1}_2, |-\rangle\langle-| \otimes \mathbb{1}_2\}$ (the first projector is labelled 0, the second is labelled 1.)
2. Bob's measurement is $\{\mathbb{1}_2 \otimes |s_0\rangle\langle s_0|, \mathbb{1}_2 \otimes |s_1\rangle\langle s_1|\}$ (the first projector is labelled 0, the second is labelled 1.) Recall that $|s_0\rangle \coloneqq \cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle$ and $|s_1\rangle \coloneqq -\sin(\pi/8) + \cos(\pi/8)\,|1\rangle$.

They perform their measurements on the EPR pair

$$|\mathrm{EPR}\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{11}$$

Observe that

$$|\mathrm{EPR}\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle). \tag{12}$$

because

$$\begin{aligned}\frac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle) =& \frac{1}{\sqrt{2}}\Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\cdot\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\cdot\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\Big)\\ =& \frac{1}{2^{3/2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle)\\ =& \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).\end{aligned}$$

Let's assume Alice measures first (the analysis gives the same winning probability if Alice measures second)[1].

There are two cases.

1. Alice measures 0. The probability of this happening (according to the measurement postulates) is

   $$\| |+\rangle\langle+| \otimes \mathbb{1}_2 \cdot |\mathrm{EPR}\rangle \|^2 = \frac{1}{2}. \tag{13}$$

   The state then changes to

   $$\frac{|+\rangle\langle+| \otimes \mathbb{1}_2 \cdot |\mathrm{EPR}\rangle}{1/\sqrt{2}} = |{+}{+}\rangle \tag{14}$$

   using the observation in Eq. (12) and the fact that $\langle -|+\rangle = 0$.

   Now for Alice and Bob to win, Bob needs to measure 0 (recall the case is $a = 1$ and $b = 0$ so Alice and Bob's outputs need to be the *same*). The probability of Bob measuring 0 given Alice measured 0 is

   $$\begin{aligned}&\|(\mathbb{1}_2 \otimes |s_0\rangle\langle s_0|)\,|{+}{+}\rangle\|^2 &&\\ =&\|\,|+\rangle \otimes |s_0\rangle\,\langle s_0|+\rangle\,\|^2 && \text{see HW 1, Q 4(a)}\\ =&|\,\langle s_0|+\rangle\,|^2 \|\,|+\rangle \otimes |s_0\rangle\,\|^2 && \|\lambda u\| = |\lambda|\|u\| \text{ for scalar } \lambda\\ =&|\,\langle s_0|+\rangle\,|^2 && \|\,|+\rangle \otimes |s_0\rangle\,\| = \|\,|+\rangle\,\|\cdot\|\,|s_0\rangle\,\| = 1\cdot 1 = 1\\ =&\left|(\cos(\pi/8)\,\langle 0| + \sin(\pi/8)\,\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right|^2 && \text{definitions}\\ =&\left|\frac{1}{\sqrt{2}}(\cos(\pi/8) + \sin(\pi/8))\right|^2 && \langle 0|1\rangle = 0,\ \langle 0|0\rangle = \|\,|0\rangle\,\|^2 = 1,\ \langle 1|1\rangle = \|\,|1\rangle\,\|^2 = 1\\ =&\cos^2(\pi/8) && \text{trignometry}\end{aligned}$$

   So the winning probability in this case is $\cos^2(\pi/8)$.

2. Alice measures 1. (The analysis in this case is really similar, but here are the details for completeness.) The probability of this happening (according to the measurement postulates) is

   $$\| |-\rangle\langle-| \otimes \mathbb{1}_2 \cdot |\mathrm{EPR}\rangle \|^2 = \frac{1}{2}. \tag{15}$$

[1] Mathematically, this is because Alice and Bob's measurement projectors commute as matrices.

The state then changes to

$$\frac{|-\rangle\langle -| \otimes \mathbb{1}_2 \cdot |\mathrm{EPR}\rangle}{1/\sqrt{2}} = |--\rangle \tag{16}$$

using the observation in Eq. (12) and the fact that $\langle -|+\rangle = 0$.

Now for Alice and Bob to win, Bob needs to measure 1 (recall the case is $a = 1$ and $b = 0$ so Alice and Bob's outputs need to be the *same*). The probability of Bob measuring 1 given Alice measured 1 is

$$\begin{aligned}
&\|(\mathbb{1}_2 \otimes |s_1\rangle\langle s_1|)\, |--\rangle\, \|^2 && \\
=&\|\, |+\rangle \otimes |s_1\rangle \langle s_1|-\rangle\, \|^2 && \text{see HW 1, Q 4(a)} \\
=&|\, \langle s_1|-\rangle\, |^2 \|\, |-\rangle \otimes |s_1\rangle\, \|^2 && \|\lambda u\| = |\lambda| \|u\| \text{ for scalar } \lambda \\
=&|\, \langle s_1|-\rangle\, |^2 && \|\, |-\rangle \otimes |s_1\rangle\, \| = \|\, |-\rangle\, \| \cdot \|\, |s_1\rangle\, \| = 1 \cdot 1 = 1 \\
=&\left|(-\sin(\pi/8)\, \langle 0| + \cos(\pi/8)\, \langle 1|)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right|^2 && \text{definitions} \\
=&\left|\frac{1}{\sqrt{2}}(-\sin(\pi/8) - \cos(\pi/8))\right|^2 && \langle 0|1\rangle = 0,\ \langle 0|0\rangle = \|\, |0\rangle\, \|^2 = 1,\ \langle 1|1\rangle = \|\, |1\rangle\, \|^2 = 1 \\
=&\cos^2(\pi/8) && \text{trignometry}
\end{aligned}$$

So the winning probability in this case is $\cos^2(\pi/8)$.

So the overall winning probability in the case $a = 1$ and $b = 0$ is

$$\frac{1}{2}\cos^2(\pi/8) + \frac{1}{2}\cos^2(\pi/8) = \cos^2(\pi/8). \tag{17}$$