

Lecture 4

Analysis of the CHSH game in the case $a = 1$ and $b = 0$.

In this case:

1. Alice's measurement is $\{|+\rangle\langle+| \otimes \mathbb{1}_2, |-\rangle\langle-| \otimes \mathbb{1}_2\}$ (the first projector is labelled 0, the second is labelled 1.)
2. Bob's measurement is $\{\mathbb{1}_2 \otimes |s_0\rangle\langle s_0|, \mathbb{1}_2 \otimes |s_1\rangle\langle s_1|\}$ (the first projector is labelled 0, the second is labelled 1.) Recall that $|s_0\rangle := \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$ and $|s_1\rangle := -\sin(\pi/8)|0\rangle + \cos(\pi/8)|1\rangle$.

They perform their measurements on the EPR pair

$$|\text{EPR}\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (11)$$

Observe that

$$|\text{EPR}\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle). \quad (12)$$

because

$$\begin{aligned} \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \\ &= \frac{1}{2^{3/2}}(|00\rangle + |01\rangle + |01\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \end{aligned}$$

Let's assume Alice measures first (the analysis gives the same winning probability if Alice measures second)¹. There are two cases.

1. Alice measures 0. The probability of this happening (according to the measurement postulates) is

$$\| |+\rangle\langle+| \otimes \mathbb{1}_2 \cdot |\text{EPR}\rangle \|^2 = \frac{1}{2}. \quad (13)$$

The state then changes to

$$\frac{|+\rangle\langle+| \otimes \mathbb{1}_2 \cdot |\text{EPR}\rangle}{1/\sqrt{2}} = |++\rangle \quad (14)$$

using the observation in Eq. (12) and the fact that $\langle-|+\rangle = 0$.

Now for Alice and Bob to win, Bob needs to measure 0 (recall the case is $a = 1$ and $b = 0$ so Alice and Bob's outputs need to be the *same*). The probability of Bob measuring 0 given Alice measured 0 is

$$\begin{aligned} &\|(\mathbb{1}_2 \otimes |s_0\rangle\langle s_0|)|++\rangle\|^2 && \text{see HW 1, Q 4(a)} \\ &= \| |+\rangle \otimes |s_0\rangle \langle s_0|+\rangle \|^2 && \|\lambda u\| = |\lambda| \|u\| \text{ for scalar } \lambda \\ &= \| \langle s_0|+\rangle \|^2 \| |+\rangle \otimes |s_0\rangle \|^2 && \| |+\rangle \otimes |s_0\rangle \| = \| |+\rangle \| \cdot \| |s_0\rangle \| = 1 \cdot 1 = 1 \\ &= \| \langle s_0|+\rangle \|^2 && \text{definitions} \\ &= \left| \langle \cos(\pi/8)\langle 0| + \sin(\pi/8)\langle 1| \rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right|^2 && \langle 0|1\rangle = 0, \langle 0|0\rangle = \| |0\rangle \|^2 = 1, \langle 1|1\rangle = \| |1\rangle \|^2 = 1 \\ &= \left| \frac{1}{\sqrt{2}}(\cos(\pi/8) + \sin(\pi/8)) \right|^2 && \text{trigonometry} \\ &= \cos^2(\pi/8) \end{aligned}$$

So the winning probability in this case is $\cos^2(\pi/8)$.

2. Alice measures 1. (The analysis in this cases is really similar, but here are the details for completeness) The probability of this happening (according to the measurement postulates) is

$$\| |-\rangle\langle-| \otimes \mathbb{1}_2 \cdot |\text{EPR}\rangle \|^2 = \frac{1}{2}. \quad (15)$$

¹Mathematically, this is because Alice and Bob's measurement projectors commute as matrices.

The state then changes to

$$\frac{|-\rangle\langle -| \otimes \mathbb{1}_2 \cdot |\text{EPR}\rangle}{1/\sqrt{2}} = |--\rangle \quad (16)$$

using the observation in Eq. (12) and the fact that $\langle -|+\rangle = 0$.

Now for Alice and Bob to win, Bob needs to measure 1 (recall the case is $a = 1$ and $b = 0$ so Alice and Bob's outputs need to be the *same*). The probability of Bob measuring 1 given Alice measured 1 is

$$\begin{aligned} & \|(\mathbb{1}_2 \otimes |s_1\rangle\langle s_1|) |--\rangle\|^2 \\ &= \| |+\rangle \otimes |s_1\rangle \langle s_1| - \rangle \|^2 && \text{see HW 1, Q 4(a)} \\ &= |\langle s_1|-\rangle|^2 \| |-\rangle \otimes |s_1\rangle \|^2 && \|\lambda u\| = |\lambda| \|u\| \text{ for scalar } \lambda \\ &= |\langle s_1|-\rangle|^2 && \| |-\rangle \otimes |s_1\rangle \| = \| |-\rangle \| \cdot \| |s_1\rangle \| = 1 \cdot 1 = 1 \\ &= \left| (-\sin(\pi/8) \langle 0| + \cos(\pi/8) \langle 1|) \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right|^2 && \text{definitions} \\ &= \left| \frac{1}{\sqrt{2}} (-\sin(\pi/8) - \cos(\pi/8)) \right|^2 && \langle 0|1\rangle = 0, \langle 0|0\rangle = \| |0\rangle \|^2 = 1, \langle 1|1\rangle = \| |1\rangle \|^2 = 1 \\ &= \cos^2(\pi/8) && \text{trigonometry} \end{aligned}$$

So the winning probability in this case is $\cos^2(\pi/8)$.

So the overall winning probability in the case $a = 1$ and $b = 0$ is

$$\frac{1}{2} \cos^2(\pi/8) + \frac{1}{2} \cos^2(\pi/8) = \cos^2(\pi/8). \quad (17)$$