

Cyclicity of \mathbb{Z}_p^*

Daochen Wang

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Definition 1 (Group). A *group* $G := (S, \alpha)$ is defined by a set S and a function $\alpha: S \times S \rightarrow S$ with the following conditions. (For $g, h \in S$, we write $g \cdot h$ or simply gh as shorthand for $\alpha(g, h)$.)

1. Existence of Identity. There exists an $e \in S$ such that for all $g \in S$, $ge = eg = g$.
2. Existence of Inverses. For all $g \in S$, there exists $h \in S$ such that $gh = hg = e$.
3. Associativity. For all $g, h, k \in S$, $(gh)k = g(hk)$. (That is, $\alpha(\alpha(g, h), k) = \alpha(g, \alpha(h, k))$.)

The set S is known as the underlying set of G and the function α is known as the group operation of G . The *order* of G is the size of the set underlying G and is denoted $|G|$. We say G is a *finite group* if its order is finite.

If we additionally have $gh = hg$ for all $g, h \in S$, then we say G is an abelian group.

Remark 1. The definition implies: (i) the identity element e is unique, (ii) for all $g \in S$, there exists a unique $h \in S$ such that $gh = hg = e$ and we can denote it without ambiguity by g^{-1} . **Good exercise to check.**

Example 1. Our main working example is the group \mathbb{Z}_p^* , where p is prime. The underlying set is $\{1, \dots, p-1\}$ and the group operation is *multiplication* modulo p . Consider \mathbb{Z}_5^* : the set is $\{1, 2, 3, 4\}$ and $3 \cdot 4 = 2$, $2 \cdot 3 = 1$, $3^{-1} = 2$, etc. Note that it's not obvious that the existence-of-inverse requirement of a group is satisfied, but it can be shown using Bézout's identity and the assumption that p is prime. The group is also abelian, since multiplication (modulo p) commutes.

Definition 2 (Subgroup). Let $G := (S, \alpha)$ be a group. We say $T \subseteq S$ forms a *subgroup* of G if:

1. T contains the identity element of G .
2. T is closed under α , i.e., $g, h \in T \implies gh \in T$.
3. T contains inverses, i.e., $g \in T \implies g^{-1} \in T$.

This definition means that $(T, \alpha|_T)$ is a group, where $\alpha|_T: T \times T \rightarrow T$ is the natural restriction of α to T defined by $\alpha|_T(x, y) = \alpha(x, y)$ for all $x, y \in T$. We say $(T, \alpha|_T)$ is a subgroup of G . Often the function α is implicit in which case it is common to abuse language and identify the set S with the group G and the set T with the subgroup $(T, \alpha|_T)$. We write $H \leq G$ to mean H is a subgroup of G .

Definition 3 (Coset). Let G be a group and H be a subgroup. A *coset* of H in G is a set of the form $gH := \{gh \mid h \in H\}$.

Theorem 1 (Lagrange). Let G be a finite group and H be a subgroup. Then the order of H divides the order of G .

Proof. The cosets of H partition G and each have size $|H|$. □

Definition 4. Let G be a finite group and $g \in G$. The *order* of g in G , denoted $o(g)$ or $\text{ord}(g)$, is the minimum positive integer r such that $g^r = e$. The *subgroup generated by g* , denoted $\langle g \rangle$, is the subgroup formed by the subset $\{e, g^1, \dots, g^{o(g)-1}\}$

Exercise: check $o(g)$ is well-defined and that $\langle g \rangle$ indeed forms a subgroup.

Corollary 1. Let G be a finite group and $g \in G$, then $o(g)$ divides $|G|$, written $o(g) \mid |G|$.

Proof. Follows from Lagrange's theorem because $\langle g \rangle$ is a subgroup of G of size $o(g)$. □

An immediate corollary of the above is:

Corollary 2. Let G be a finite group and $g \in G$, then $g^{|G|} = e$. In particular, this implies Fermat's Little Theorem that for all $a \in \mathbb{Z}_p^*$, where p is prime, we have $a^{p-1} = 1$.

Definition 5. Let n be a positive integer. We write $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$. We write $\mathbb{Z}_n[X]$ for the set of polynomials with coefficients in \mathbb{Z}_n . Given $0 \neq P \in \mathbb{Z}_n[X]$, the degree of P is defined to be the exponent of the largest power of X that has a non-zero coefficient. We say $x \in \mathbb{Z}_n$ is a root of P if $P(x) = 0 \pmod n$.

Example 2. The set $\mathbb{Z}_4[X]$ contains polynomials like $2X$, X^3 , and $3X^{100} + X^{42} + 1$, which are of degrees 1, 3, 100, respectively. Note that 2 is a root of $2X$ and X^3 in \mathbb{Z}_4 ; while $3X^{100} + X^{42} + 1$ has no roots in \mathbb{Z}_4 . **Why?**

Proposition 1. Let p be prime and $P \in \mathbb{Z}_p[X]$. If P has degree $d \geq 1$, then P has at most d distinct roots in \mathbb{Z}_p .

Proof. Proof by induction on d . For $d = 1$, the polynomial must be of the form $P = \alpha X + \beta$ for some $\alpha, \beta \in \mathbb{Z}_p$ with $\alpha \neq 0$. Since p is prime, this means α is invertible and the only root to $P(x) = 0$ is $-\alpha^{-1}\beta$. For $d > 1$, suppose x is a root of P , then use polynomial division to write $P = (X - x)Q + r$, where $Q \in \mathbb{Z}_p[X]$ has degree $d - 1$ and $r \in \mathbb{Z}_p$. Evaluating P at $X = x$ shows $r = 0$. Thus $P = (X - x)Q$. Suppose $y \in \mathbb{Z}_p$ is a root of P , then $(y - x)Q(y) = 0$, so $y = x$ or $Q(y) = 0$ as p is prime. (This uses the fact that if a prime divides a product of two integers, then it must divide at least one of them.) Therefore, by the inductive hypothesis, y can take one of at most $1 + (d - 1) = d$ possible values since Q has degree $d - 1$. This completes the proof. \square

Remark 2. Proposition 1 can be false if p is not prime:

1. The polynomial $2X \in \mathbb{Z}_4[X]$ has two distinct roots in \mathbb{Z}_4 , namely, 0 and 2.
2. The polynomial $X^2 - 1 \in \mathbb{Z}_8[X]$ has four distinct roots in \mathbb{Z}_8 , namely, 1, 3, 5, 7.

Definition 6. For positive integers a, b , $\text{lcm}(a, b)$ denotes the least common multiple of a and b .

Example 3. $\text{lcm}(6, 21) = 42$. $\text{lcm}(7, 5) = 35$. $\text{lcm}(35, 7) = 35$.

Lemma 1. Let G be a finite abelian group. Let $g, h \in G$. Suppose $o(g), o(h)$ are coprime, then $o(gh) = \text{lcm}(o(g), o(h)) = o(g) \cdot o(h)$.

Proof. Since $o(g)$ and $o(h)$ are coprime, it directly follows that $\text{lcm}(o(g), o(h)) = o(g) \cdot o(h)$. Thus, it suffices to show $o(gh) = \text{lcm}(o(g), o(h))$. Write $k := o(gh)$ and $\ell := \text{lcm}(o(g), o(h))$.

For $k \leq \ell$: we have

$$\begin{aligned} (gh)^\ell &= g^\ell h^\ell && G \text{ abelian} \\ &= e \cdot e = e && \ell \text{ is a multiple of } o(g) \text{ and } o(h) \end{aligned}$$

so $k \leq \ell$ by the definition of k as the order of gh .

For $\ell \leq k$: as above, we have

$$(gh)^k = g^k h^k = e \tag{1}$$

and so

$$x := g^k = (h^{-1})^k \in \langle g \rangle \cap \langle h \rangle \tag{2}$$

Thus, Corollary 1 implies $o(x) \mid o(g)$ and $o(x) \mid o(h)$. But $o(g)$ and $o(h)$ are coprime so $o(x) = 1$, so $x = e$. Therefore, the definition of x means $g^k = e = h^k$. Therefore, $o(g) \mid k$ and $o(h) \mid k$ (to see this, list powers of g, h in a sequence) so k is a common multiple of $o(g)$ and $o(h)$ so $k \geq \ell$ by the definition of ℓ as the least common multiple. \square

Remark 3. The coprimality assumption is crucial in Lemma 1. For example, consider the group \mathbb{Z}_2 , i.e., $\{0, 1\}$ under addition modulo 2. Then $o(1 + 1) = o(0) = 1$ but $\text{lcm}(o(1), o(1)) = 2$.

From Lemma 1, we deduce the next proposition. (Based on this StackExchange answer.)

Proposition 2. Every finite abelian group G has an lcm-closed order set. That is, for all $x, y \in G$, there exists $z \in G$ such that

$$o(z) = \text{lcm}(o(x), o(y)). \tag{3}$$

Proof. Proof by induction on $o(x)o(y)$. If $o(x)o(y) = 1$, then we can choose $z = e$. Otherwise, $o(x)o(y) > 1$ and we can wlog factorize¹

$$o(x) = AP, \quad o(y) = BP', \tag{4}$$

where $P = p^m > 1$ for some prime p coprime to A, B ; and $P' \mid P$.

Then

$$o(x^P) = A \quad \text{and} \quad o(y^{P'}) = B \tag{5}$$

By induction there exists z with $o(z) = \text{lcm}(A, B)$.

Now note that $o(x^A) = P$ and P is coprime to $o(z) = \text{lcm}(A, B)$. Therefore,

$$\begin{aligned} o(x^A z) &= P \cdot \text{lcm}(A, B) && \text{Lemma 1} \\ &= \text{lcm}(AP, BP') && P' \mid P \\ &= \text{lcm}(o(x), o(y)), \end{aligned}$$

as required. \square

¹Intuition: this pulls out a fixed prime p from $o(x)$ and $o(y)$ as many times as possible. Can wlog assume $o(x)$ has at least one prime factor p since $o(x)o(y) > 1$, and also that p appears more times in the prime factorization of $o(x)$ than in that of $o(y)$ else can relabel $x \leftrightarrow y$.

Definition 7 (Cyclic groups and generators). Let G be a finite group, we say G is *cyclic* if there exists $g \in G$, such that $o(g) = |G|$. In this case, we call g a *generator* of G .

Theorem 2. For all p prime, \mathbb{Z}_p^* is a cyclic group.

Example 4. In \mathbb{Z}_5^* , we have $o(1) = 1$, $o(2) = 4$, $o(3) = 4$, $o(4) = 2$. So 2 and 3 are the only generators.

Proof. Let ℓ be the least common multiple of the orders of the elements of \mathbb{Z}_p^* . By Proposition 2, ℓ must be the order of some element in \mathbb{Z}_p^* . Thus it suffices to show $\ell = p - 1$.

By Corollary 1, $p - 1$ is a common multiple of the orders of the elements of \mathbb{Z}_p^* , so $\ell \leq p - 1$.

Moreover, the definition of ℓ implies that every element of \mathbb{Z}_p^* is a root of $P := X^\ell - 1 \in \mathbb{Z}_p[X]$ in \mathbb{Z}_p . Since P is of degree ℓ and \mathbb{Z}_p^* has $p - 1$ distinct elements, we must have $p - 1 \leq \ell$ by Proposition 1.

Hence $\ell = p - 1$ and the theorem follows. □