Lecture 24

Comment: I made a mistake at the end of last time: the columns of the error syndrome table for Shor's code are *not* all distinct, but can nonetheless correct any single-qubit error, as we'll see today. So having distinct columns is a sufficient but non-necessary condition for being able to correct all the associated errors.

The column corresponding to Y_j is the product of the columns corresponding to X_j and Z_j. To see this, observe that

$$Y = iXZ. \tag{140}$$

So if a stabilizer P satisfies

$$PX_j = \alpha X_j P \quad \text{and} \quad PZ_j = \beta Z_j P, \tag{141}$$

then

$$PY_j = PiX_jZ_j = i\alpha X_j P Z_j = i\alpha X_j \beta Z_j P = \alpha\beta Y_j P. \tag{142}$$

Proposition 10. *Shor's nine-qubit code corrects any single-qubit X, Z, Y error.*

Proof. Consider errors on the first block of three qubits first. We have the following error syndrome table, where no sign means $+$.

	I	X_1	Z_1	Y_1	X_2	Z_2	Y_2	X_3	Z_3	Y_3
Z_1Z_2		-		-	-		-			
Z_2Z_3					-		-	-		-
Z_4Z_5										
Z_5Z_6										
Z_7Z_8										
Z_8Z_9										
$X_1X_2X_3X_4X_5X_6$			-	-		-	-		-	-
$X_4X_5X_6X_7X_8X_9$										

We observe that the syndrome (column) of X and Y errors are all distinct, and also distinct from that of Z errors. Therefore, X and Y errors can be corrected by directly inverting the error. The columns syndromes of the three Z errors are the same but consider correction by applying Z_1 if this common syndrome is measured. There are three cases

1. If error was actually Z_1, then this inverts the error since $Z_1Z_1 = I$
2. If the error was actually Z_2. Write $|\psi\rangle$ for the Shor-encoded 9-qubit state. Then this maps $Z_2|\psi\rangle$ to $Z_1Z_2|\psi\rangle$ but Z_1Z_2 is a stabilizer of $|\psi\rangle$ so $Z_1Z_2|\psi\rangle = |\psi\rangle$. So we've corrected the error.
3. If the error was actually Z_3. Then this maps $Z_3|\psi\rangle$ to $Z_1Z_3|\psi\rangle$ but Z_1Z_2 and Z_2Z_3 are stabilizers of $|\psi\rangle$ implies their product Z_1Z_3 is also a stabilizer. So $Z_1Z_3|\psi\rangle = |\psi\rangle$. So we've corrected the error.

Similarly, if we are promised that errors occur on any given block, we can correct it.

Finally, we observe that we can identify the block on which the error occurs:

1. If Z_1Z_2 or Z_2Z_3 rows have $-$ signs, it's first block.
2. If Z_4Z_5 or Z_5Z_6 rows have $-$ signs, it's second block.
3. If Z_7Z_8 or Z_8Z_9 rows have $-$ signs, it's third block.
4. If $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ rows take values $(-,+)$ it's first block; if $(-,-)$ it's second block; if $(+,-)$ it' third block.
5. Else no error has occured.

Therefore, we can correct any single-qubit X, Z, Y error. $\square$

Fact 13. If a quantum error correcting code can correct any single-qubit X, Z, Y error (as well as no error), then it can correct any single-qubit error.

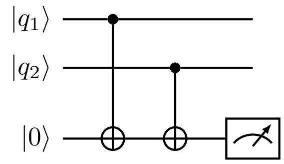
Proof sketch. Consequence of the fact that any single-qubit matrix (error) E can be written as a linear combination of I, X, Y, Z. That is, there exists $\alpha, \beta, \delta, \gamma \in \mathbb{C}$ such that

$$E = \alpha I + \beta X + \gamma Y + \delta Z \tag{143}$$

When syndrome is measured, error "collapses" to one of the four errors that is consistent with the measured syndrome, so can be corrected. (This can be proven formally using the full definition of measuring an observable on a state – not just on an eigenstate.) $\square$

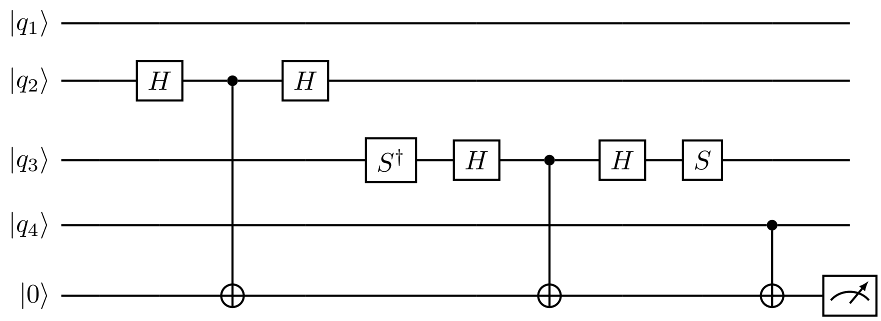
How to measure stabilizers Procedure and sketch of correctness by examples.

Say you wanted to measure $Z \otimes Z$ on 2 qubits:



Comment: Do example (i) $\alpha|00\rangle + \beta|11\rangle/\sqrt{2}$: when $Z_1 Z_2$ is measured on this should get $+1$. (ii) $\alpha|01\rangle + \beta|10\rangle/\sqrt{2}$: when $Z_1 Z_2$ is measured on this should get -1.

Say you wanted to measure $I \otimes X \otimes Y \otimes Z$ on 4 qubits. The conjugation on the ctrl "rotates" Z to either X or Y.



Five-qubit code smallest that can correct arbitrary single-qubit errors. (Easy to see for non-degenerate codes, separate argument needed for degenerate codes.)

Fun class exercise. Stabilizers of the five-qubit code:

$$\begin{array}{ccccccccccc} X & \otimes & Z & \otimes & Z & \otimes & X & \otimes & I & , \\ I & \otimes & X & \otimes & Z & \otimes & Z & \otimes & X & , \\ X & \otimes & I & \otimes & X & \otimes & Z & \otimes & Z & , \\ Z & \otimes & X & \otimes & I & \otimes & X & \otimes & Z & , \end{array} \tag{144}$$

Exercise: check it corrects arbitrary single-qubit errors.

Logical codeword of the five-qubit code

$$\begin{aligned} |0_L\rangle \coloneqq \frac{1}{4}(\, & |00000\rangle \\ & + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle + |00101\rangle \\ & - |11000\rangle - |01100\rangle - |00110\rangle - |00011\rangle - |10001\rangle \\ & - |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle - |11110\rangle \,) \end{aligned}$$

and

$$\begin{aligned} |1_L\rangle \coloneqq \frac{1}{4}(\, & |11111\rangle \\ & + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle + |11010\rangle \\ & - |00111\rangle - |10011\rangle - |11001\rangle - |11100\rangle - |01110\rangle \\ & - |10000\rangle - |01000\rangle - |00100\rangle - |00010\rangle - |00001\rangle \,). \end{aligned}$$