Rational degree is polynomially related to degree

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Degree

$f\colon \{0,1\}^n \to \{0,1\}$

Fact. For every $g: \{0,1\}^n \to \mathbb{R}$, there exists a unique multilinear real polynomial p that represents g, i.e., $p(x) = g(x)$ for all $x \in \{0,1\}^n$

Definition. The degree of f is defined to be the degree of the unique multilinear real polynomial that represents it. Denoted $\deg(f)$

Example. $n = 2, f(x_1, x_2) = x_1 \oplus x_2$. Represented by

$$p = -2x_1x_2 + x_1 + x_2$$

Rational degree

$f \colon \{0,1\}^n \to \{0,1\}$

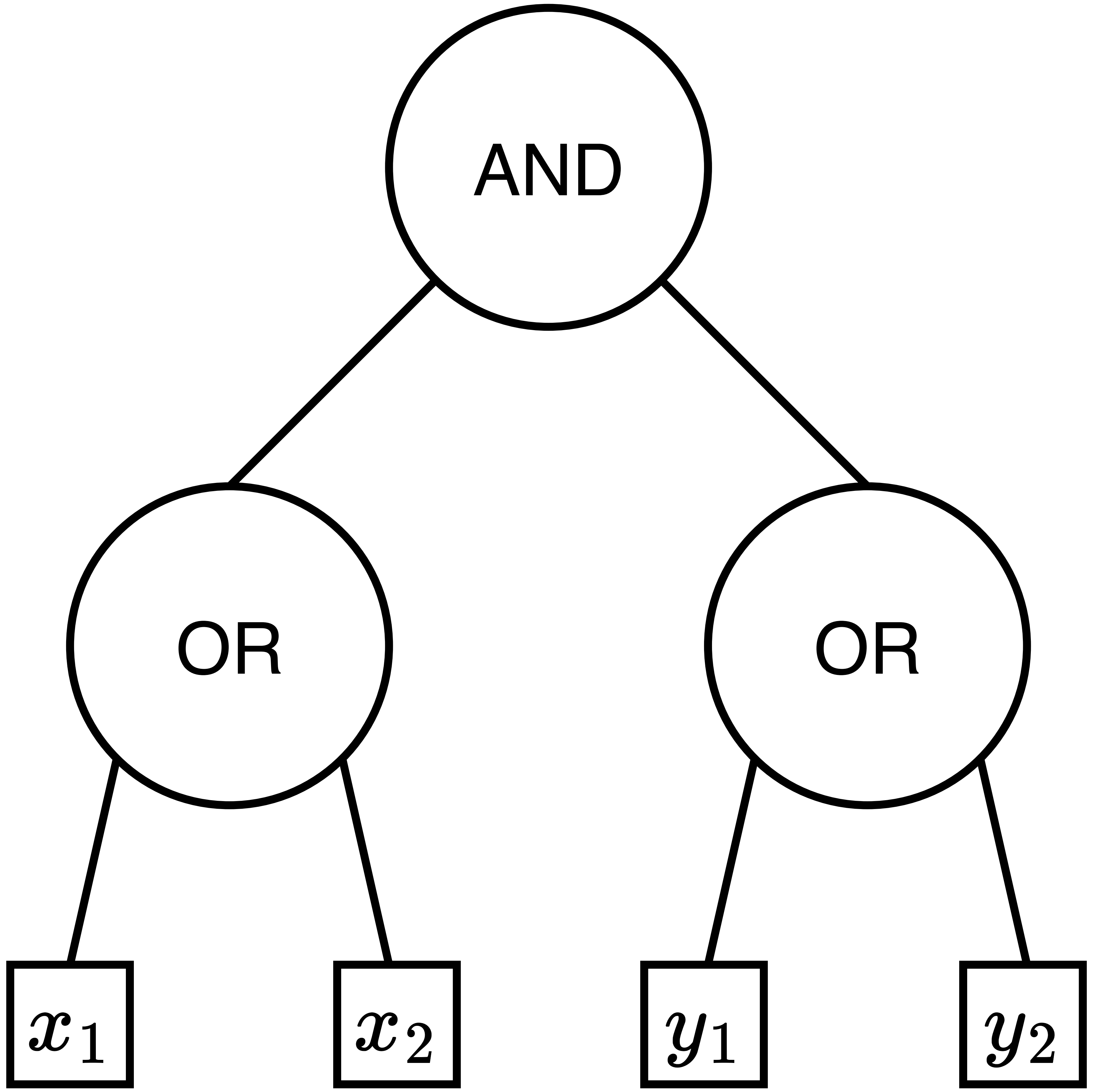
Definition. The rational degree of f is defined as the minimum value of $\max(\deg(p), \deg(q))$, where p, q are multilinear real polynomials such that p/q represents f, i.e., $p(x)/q(x) = f(x)$ for all $x \in \{0,1\}^n$. Denoted $\mathrm{rdeg}(f)$

Observation. $\mathrm{rdeg}(f) \leq \deg(f)$

Example. $n = 2, f(x_1, x_2) = x_1 \oplus x_2$. Can be represented by

$$\frac{p}{q} = \frac{x_1 - x_2}{1 - 2x_2} \quad \text{or} \quad \frac{p}{q} = \frac{x_2 - x_1}{1 - 2x_1}$$

Rational degree can be quadratically smaller than degree



For n square, $f = \mathrm{AND}_{\sqrt{n}} \circ \mathrm{OR}_{\sqrt{n}}$

$$\deg(f) = n \quad \text{and} \quad \mathrm{rdeg}(f) \leq \sqrt{n}$$

Example: $n = 4$

$$\frac{p}{q} = \frac{(x_1 + x_2)(y_1 + y_2)}{(x_1 + x_2)(y_1 + y_2) + [(1 - x_1)(1 - x_2) + (1 - y_1)(1 - y_2)]}$$

[IJKKKSWW '23]

Rational degree can be much smaller than degree if error allowed

$$f = x_1 \vee x_2 \vee \ldots \vee x_n$$

$$\frac{p}{q} = \frac{x_1 + x_2 + \ldots + x_n}{\epsilon + x_1 + x_2 + \ldots + x_n}, \ \epsilon > 0$$

Interpretations of rdeg and deg

rdeg: equals exact postselected quantum query complexity $\mathrm{PostQ_E}$

[Mahadev and de Wolf ’15]

[Aaronson ’05]

deg: polynomially related to almost all other complexity measures

e.g., block sensitivity bs and decision tree complexity D

[Nisan and Szegedy ’94]

A question of Fortnow, Nisan, and Szegedy (1994)

For all $f\colon \{0,1\}^n \to \{0,1\}$, $\deg(f) \leq \mathrm{poly}(\mathrm{rdeg}(f))$?

(Quantumness + postselection isn't much more powerful than deterministic?)

Prior results:

1. Randomness + postselection isn't much more powerful than deterministic: $\mathrm{D}(f) \leq \mathrm{C}(f)^2 = \mathrm{PostR_E}(f)^2$ [Nisan '89, Cade '20]

2. Yes for f symmetric or monotone [IJKKKSWW '23]

We resolve the question affirmatively

Theorem

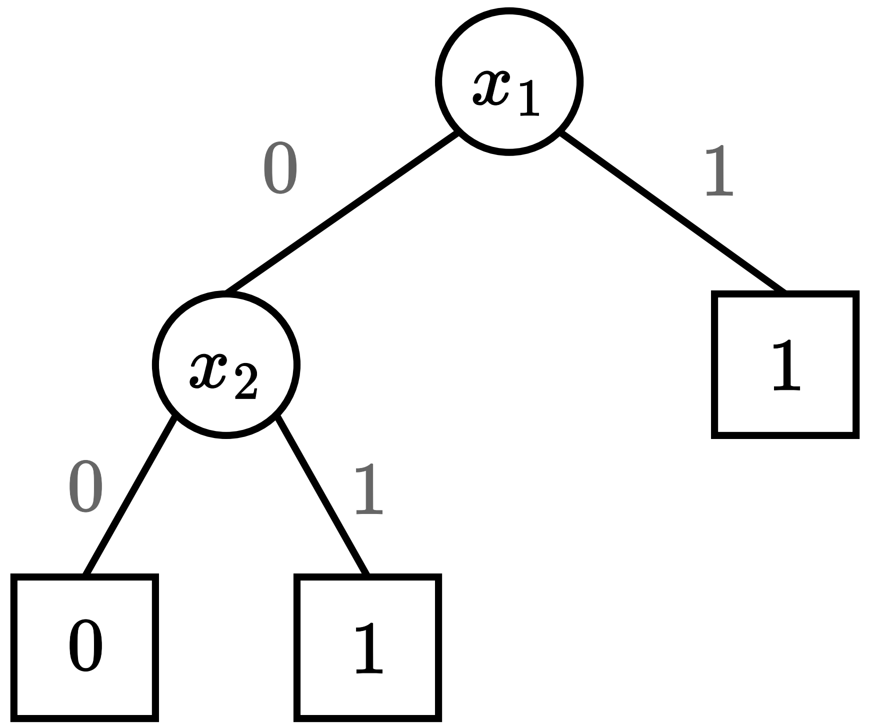
For all $f\colon \{0,1\}^n \to \{0,1\}$, $\deg(f) \le 2\,\mathrm{rdeg}(f)^4$

[KWY '26]

Proof

Decision tree complexity and nondeterministic degree

$f\colon \{0,1\}^n \to \{0,1\}$



$\mathrm{D}(f)$: minimum depth of a decision tree computing f

Fact. $\deg(f) \le D(f)$. **Proof by example.** $x_1 + (1 - x_1)\, x_2$

$\mathrm{ndeg}(f)$: minimum degree of a nondeterministic polynomial p for f, i.e., p multilinear real and $p(x) = 0 \iff f(x) = 0$ for all $x \in \{0,1\}^n$

Fact. $\mathrm{ndeg}(f), \mathrm{ndeg}(\neg f) \le \mathrm{rdeg}(f)$. **Proof.** $f = p/q \Longrightarrow \mathrm{ndeg}(f) \le \deg(p), \mathrm{ndeg}(\neg f) \le \deg(p - q)$

Theorem. For all $f\colon \{0,1\}^n \to \{0,1\}$, $\mathrm{D}(f) \le 2\,\mathrm{ndeg}(f)^2\,\mathrm{ndeg}(\neg f)^2$

A decision tree

$f\colon \{0,1\}^n \to \{0,1\}$; let p be a nondeterministic polynomial for f of degree $\mathrm{ndeg}(f)$

Repeat

1. query variables in p such that its degree decreases by at least one

2. set variables in p to their queried values, call resulting polynomial p

3. if p is a constant c, output 0 if $c = 0$, output 1 if $c \neq 0$

Observation

If can query no more than N variables in step 1, then $\mathrm{D}(f) \leq N \cdot \mathrm{ndeg}(f)$

Keep querying disjoint maxonomials of p

Definition. A maxonomial of p is a monomial of p with degree $\deg(p)$. Two maxonomials are said to be disjoint when they do not share any variable

Example: $p = 2X_1X_2X_3 + 3X_3X_4X_5 - X_5X_6X_7 - X_1X_2 - X_3$

Two possibilities: $\{X_1X_2X_3, X_5X_6X_7\}$ or $\{X_3X_4X_5\}$

Observations. (i) degree of p must decrease by at least 1, (ii) number of queries is at most $\deg(p) \times$ (number of disjoint maxonomials)

[Nisan and Smolensky '98]

Bounding number of disjoint maxonomials

Definition. The block sensitivity of $f\colon \{0,1\}^n \to \{0,1\}$ at $x \in \{0,1\}^n$ is the maximum k such that there exist disjoint $B_1, \ldots, B_k \subseteq \{1,\ldots,n\}$ such that $f(x) \neq f(x^{B_i})$ for all $i \in \{1,\ldots,k\}$. Denoted $\mathrm{bs}_x(f)$

[Nisan '89]

Example: $n = 3$ and $f(x) = x_1 \vee x_2 \vee x_3$

1. $\mathrm{bs}_{000}(f) = 3$: $B_1 = \{1\}, B_2 = \{2\}, B_3 = \{3\}$
2. $\mathrm{bs}_{111}(f) = 1$: $B_1 = \{1,2,3\}$

Bounding number of disjoint maxonomials

Recall $f\colon \{0,1\}^n \to \{0,1\}$; p a nondeterministic polynomial for f

Lemma. Suppose f is nonconstant. Then, the number of disjoint maxonomials of p is at most $\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f)$

Proof by example. Consider the $f\colon \{0,1\}^7 \to \{0,1\}$ nondeterministically represented by $p = 2X_1X_2X_3 + 3X_3X_4X_5 - X_5X_6X_7 - X_1X_2 - X_3$

Fix arbitrary $x \in f^{-1}(0)$. Consider maxonomial $2X_1X_2X_3$: set remaining variables in p according to x. Resulting polynomial nonzero. Uniqueness of multilinear representation $\Longrightarrow \exists a_1a_2a_3 \in \{0,1\}^3\colon p(a_1a_2a_3x_4x_5x_6x_7) \neq 0$

Bounding minimum block sensitivity

Lemma. Let $f\colon \{0,1\}^n \to \{0,1\}$. Suppose f is nonconstant, then

$$\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \leq 2\,\mathrm{ndeg}(\neg f)^2 .$$

False if $\min$ replaced by $\max$

Counterexample: n even; $f(x) = 1 \iff |x| = n/2$; $\neg f(x) = 0 \iff |x| = n/2$

A nondeterministic polynomial for $\neg f$: $p = x_1 + x_2 + \ldots + x_n - n/2$

$$\max_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \geq \mathrm{bs}_{0^{n/2-1}1^{n/2+1}}(f) \geq n/2 + 1$$

$$\boxed{\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \leq \mathrm{bs}_{0^n}(f) \leq 2}$$

Bounding minimum block sensitivity

Lemma. Let $f\colon \{0,1\}^n \to \{0,1\}$. Suppose f is nonconstant, then

$$\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \le 2\,\mathrm{ndeg}(\neg f)^2 .$$

Proof. Write $b = \min_{x \in f^{-1}(0)} \mathrm{bs}_x(f)$. Take nondeterministic polynomial p for $\neg f$, so $p(x) \neq 0 \iff f(x) = 0$. Let $h = \max_{x \in f^{-1}(0)} |p(x)|$ and $z \in f^{-1}(0)$ be such that $|p(z)| = h$. By definition, $\mathrm{bs}_z(f) \ge b$

Minsky-Papert symmetrization "around z" yields real univariate polynomial P:
$\deg(P) \le \deg(p)$, $|P(w)| \le h$ for all $w \in \{0,1,\ldots,b\}$, $|P(0)| = h$, $P(1) = 0$

Markov's inequality $\implies b \le 3\deg(P)^2 \le 3\deg(p)^2$

Can improve to $b \le 2\deg(p)^2$
[Shi '02, AKKT '20, Kothari '26]

Putting things together

$f\colon \{0,1\}^n \to \{0,1\}$; let p be a nondeterministic polynomial for f of degree $\mathrm{ndeg}(f)$

Repeat

1. query variables in p such that its degree decreases by at least one

2. set variables in p to their queried values, call resulting polynomial p

3. if p is a constant c, output 0 if $c = 0$, output 1 if $c \neq 0$

Observation

If can query no more than N variables in step 1, then $D(f) \leq N \cdot \mathrm{ndeg}(f)$

$N = \deg(p) \cdot (2\,\mathrm{ndeg}(\neg f)^2)$ works so $\mathrm{D}(f) \leq 2\,\mathrm{ndeg}(f)^2 \cdot \mathrm{ndeg}(\neg f)^2$

Future directions

Generalize beyond the hypercube

Let $X \subseteq \mathbb{R}^n$. For $f: X \to \{0,1\}$, define $\deg(f), \mathrm{rdeg}(f)$ analogously

Question. For given X, does $\deg(f) \le \mathrm{poly}(\mathrm{rdeg}(f))$ hold?

Our result shows answer is yes for $X = \{0,1\}^n$

Our paper also shows answer is no for some subset $X \subseteq \{0,1\}^n$

Questions of this flavor are studied in math: effective Nullstellensatz [Brownawell '87, Kollár '88, Alon '99, Jelonek '05]

Gotsman-Linial conjecture

[GL '90]

$f\colon \{0,1\}^n \to \{0,1\}$

Definition. The threshold degree of f is defined as the minimum degree of a multilinear real polynomial p such that $p(x) \neq 0$ and $f(x) = \mathrm{sign}(p(x))$ for all $x \in \{0,1\}^n$. Denoted $\deg_{\pm}(f)$

Definition. The sensitivity of f at $x \in \{0,1\}^n$ is the size of the set $\{i \in \{1,\ldots,n\} : f(x) \neq f(x^i)\}$. Denoted $s_x(f)$

Gotsman-Linial conjecture. For all $f\colon \{0,1\}^n \to \{0,1\}$,

$$\mathrm{E}_x[s_x(f)] \leq O\left(\sqrt{n} \cdot \deg_{\pm}(f)\right)$$

State-of-the-art
$\mathrm{E}_x[s_x(f)] \leq \sqrt{n} \cdot 2^{O(d^2 \log d)}$
$d = \deg_{\pm}(f)$ [Kane '12]

Gotsman-Linial conjecture

[GL '90]

$f\colon \{0,1\}^n \to \{0,1\}$

Observation. Have $\min_{x \in \{0,1\}^n} \mathrm{bs}_x(f) \leq 2\deg_{\pm}(f)^2$ [Kothari '26]

This yields:

1. $\min_{x \in \{0,1\}^n} s_x(f) \leq 2\deg_{\pm}(f)^2$
2. $D(f) \leq O(\mathrm{rdeg}(f)^2 \cdot \deg_{\pm}(f)^2)$

Recall conjecture

$\mathrm{E}_x[s_x(f)] \leq O\left(\sqrt{n} \cdot \deg_{\pm}(f)\right)$

Open questions

1. For given X, does $\deg(f) \leq \mathrm{poly}(\mathrm{rdeg}(f))$ hold?

2. Gotsman-Linial conjecture?

3. Is $\mathrm{D}(f) \leq \mathrm{poly}(\max(\mathrm{ndeg}_\epsilon(f), \mathrm{ndeg}_\epsilon(\neg f)))$?

4. Is $\mathrm{D}(f) \leq O(\mathrm{rdeg}(f)^4)$ tight?

Thank you for your attention!