CPSC 436Q: Homework 1

Due on Gradescope by 23:59pm on September 27, 2024

Rules

1. Please try to solve the problems yourself first. If you get stuck, you may *consult* any resources (books, internet, peers, office hours, etc.) for solutions. Provided you *acknowledge* these resources, no marks will be deducted. However, you must write up your own solution *independently*, using your own words.
2. Please write legibly, work that is illegible will be marked as incorrect. Latex is strongly recommended for legibility. I also recommend using [Overleaf] if you're new to Latex. Please visit [link] to get the Latex code for this document. (To edit the code, you should copy it into a new project and edit it there.)
3. All answers should be justified.
4. The total number of points for this homework is 28.
5. If you spot any mistakes, please email me at `wdaochen@cs.ubc.ca`. Any corrections will be announced on Piazza.

Homework

1. **Prerequisites.**
   (a) (2 points) Show that for all $n \in \mathbb{N}$ and every Hermitian matrix $A \in \mathbb{C}^{n\times n}$, the equality $\|A^2\| = \|A\|^2$ holds. (Throughout this class, the notation $\|\cdot\|$ is reserved for the spectral norm, i.e., maximum singular value.)
   (b) (2 points) Is part (a) still true if we drop the word "Hermitian"? (If true, show it. If false, give a counterexample.)
   (c) Let $n \in \mathbb{N}$ be even. In class, we briefly discussed a randomized algorithm for determining whether an n-bit input $x = x_1 \dots x_n$ is of form (1) the all-zeros bitstring, or form (2) half of the bits are 0 and half are 1. It is promised that x is either of form (1) or (2). We will analyze that algorithm more carefully in this question. Recall the algorithm is described as follows for some $k \in \mathbb{N}$ with $k \le n$:
      i. Choose $i_1, \dots, i_k$ each independently and uniformly at random from $[n]$. (For $a \in \mathbb{N}$, $[a]$ denotes the set $\{1, \dots, a\}$.)
      ii. Examine bits $x_{i_1}, \dots, x_{i_k}$. If any examined bit is 1, output "form (2)", else output "form (1)".

   (1 point) Suppose the input x is actually of form (1), what is the success probability of the algorithm, i.e., the probability of the algorithm outputting "form (1)".
   (1 point) Suppose the input x is actually of form (2), what is the success probability of the algorithm, i.e., the probability of the algorithm outputting "form (2)". Write your answer as a function of k.

2. **Deterministic complexity of the NAND tree.** We sketched in class that the randomized complexity of the depth-h NAND tree on $n \coloneqq 2^h$ input bits is $O\big(((1+\sqrt{33})/4)^h\big) = O(n^{0.754})$.

   We now consider how well deterministic algorithms perform for this problem. Suppose the n input bits to the NAND tree are $x_1, \dots, x_n$ (located arbitrarily at the n leaves of the tree).

   (2 points) Suppose a deterministic algorithm examines bits $x_1, x_2 \dots, x_{n-1}$. Show that there exists an assignment of values to those bits (e.g., $x_1 = 0$, $x_2 = 1$, $\dots$, $x_{n-1} = 0$) such that the output value of the NAND tree, given this assignment, still *changes* depending on whether the unexamined bit x_n is 0 or 1.

   This argument shows that the deterministic complexity of the NAND tree on n input bits is at least n because a deterministic algorithm has to examine all n bits to know for sure what the output of the NAND tree is.[1]

[1] More precisely, it shows that the *deterministic query complexity* of the NAND tree is n. We will study query complexity later in the class.

3. **Eigenvalues and eigenvectors.** Let $\theta \in \mathbb{R}$ and $A \in \mathbb{C}^{2\times 2}$ be defined by

$$A := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{1}$$

   (a) (4 points.) Calculate the eigenvalues and eigenvectors of A. Therefore, write A in the form $A = UDU^\dagger$, where $U \in \mathbb{C}^{2\times 2}$ is unitary and $D \in \mathbb{C}^{2\times 2}$ is diagonal.

   (b) (2 points.) For $k \in \mathbb{N}$, show that $A^k = UD^kU^\dagger$ and use the expression on the right-hand side to calculate A^k, simplifying your answer as much as possible.

4. **Kronecker product.** [Hint: it's easier to do the following problems if you use Dirac notation as much as possible.]

   (a) (2 points.) Let $A, B \in \mathbb{C}^{d\times d}$ and $u, v \in \mathbb{C}^d$. Show that

$$(A \otimes B)(u \otimes v) = Au \otimes Bv. \tag{2}$$

   You are allowed to use any property of the Kronecker product listed in `https://en.wikipedia.org/wiki/Kronecker_product`, *except* "the mixed-product property" — since that is stronger than what you're being asked to show.

   [Hint: first show eq. (2) for $A = |i_1\rangle\langle j_1|$, $B = |i_2\rangle\langle j_2|$, $u = |k\rangle$ and $v = |l\rangle$ where $i_1, j_1, i_2, j_2, k, l \in \{0, 1 \ldots, d-1\}$, then use other properties of the Kronecker product. (Recall that $|0\rangle, |1\rangle, \ldots, |d-1\rangle \in \mathbb{C}^d$ denote the computational basis vectors.)]

   (b) (2 points.) Define $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ by

$$|\psi\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle |i\rangle. \tag{3}$$

   Let $\mathbb{1}_d \in \mathbb{C}^{d\times d}$ denote the identity matrix. Show that for any $A \in \mathbb{C}^{d\times d}$, we have

$$A \otimes \mathbb{1}_d |\psi\rangle = \mathbb{1}_d \otimes A^\top |\psi\rangle, \tag{4}$$

   where $^\top$ denotes the transpose.

   (c) (2 points.) Let $|u_0\rangle, \ldots |u_{d-1}\rangle \in \mathbb{C}^d$ be an arbitrary orthonormal basis. Show that

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |u_i\rangle |u_i^*\rangle, \tag{5}$$

   where $|u_i^*\rangle$ denotes the (entry-wise) complex conjugate of $|u_i\rangle$.

5. **Quantum teleportation.**
   Let $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$ be defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha |0\rangle + \beta |1\rangle)(|00\rangle + |11\rangle), \tag{6}$$

   where $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha|^2 + |\beta|^2 = 1$.

   Now define the following 2-qubit states

$$|\psi_1\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{7}$$

$$|\psi_2\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{8}$$

$$|\psi_3\rangle := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{9}$$

$$|\psi_4\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{10}$$

   Then, for all $i \in [4]$, define $\Pi_i := |\psi_i\rangle\langle\psi_i| \otimes \mathbb{1}_2 \in \mathbb{C}^{8\times 8}$.

   [Hint: it's easier to do the following problems if you use Dirac notation as much as possible.]

   (a) (2 points) Show that $|\psi_i\rangle$ and $|\psi_j\rangle$ are orthogonal for all $i \neq j$.

(b) (2 points) Using part (a), or otherwise, show that $\mathcal{M} := \{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$ is a $[4]$-outcome projective measurement on $\mathbb{C}^8$.

(c) (4 points) For each $i \in [4]$, when we measure $|\psi\rangle$ using $\mathcal{M}$, what is the probability that the measurement outcome is i? Given the measurement outcome is i, compute the state $|\psi'\rangle$ that $|\psi\rangle$ changes to.

[If you did this question correctly, you should find that an i-dependent version of the state $\frac{1}{\sqrt{2}}(\alpha\,|0\rangle + \beta\,|1\rangle)$ will be transferred (or "teleported") from the first (left-most) qubit to the last (right-most) qubit in eq. (6) following measurement $\mathcal{M}$. We can imagine the first two qubits as belonging to Alice who lives on Earth and the last qubit as belonging to Bob who lives in the Andromeda Galaxy. Then $\mathcal{M}$ can be implemented by Alice locally due to its tensor product form. Unfortunately, the state teleported depends on i, so this does not actually allow Alice to communicate to Bob instantaneously.]