Lecture 12

Turing model of computation.

Definition 14 (Classical and quantum algorithms in the Turing model (informal))**.** *Classical deterministic, classicla randomized, and quantum algorithms* for solving a problem in the Turing model are specified by uniform families of circuits composed of elementary gates.

1. *Classical deterministic*: classical circuits made of AND, OR, NOT, and FANOUT gates with input $x_1 \dots x_N \in \{0,1\}^N$.
2. *Classical randomized*: same as deterministic but with another possible gate: the COIN gate that outputs a bit 0 or 1 with probability 1/2 each.
3. *Quantum*: quantum circuits with input $|x_1 \dots x_N\rangle \in \mathbb{C}^{2^N}$ together with ancilla/workspace qubits[10] initialized in state $|0^k\rangle$ for some non-negative integer k made of X[11], CNOT, H, and T gates followed by computational basis measurement at the end.

The output of these algorithms can be a single bit or multiple bits depending on the problem. In the quantum case, if the problem has M-bit output, then can wlog take the output to be the last M bits (out of $k+N$ bits) of the measurement outcome.

Solving a problem. We say that deterministic algorithm solves the problem if the output of the circuit is always correct for every input. We say (randomized/quantum) algorithms solves the problem with error ϵ if the output of the circuit is correct with probability at least $1-\epsilon$ for every input: common to just say "solves the problem with bounded error" without qualification in which case ϵ is conventionally taken to be 1/3.

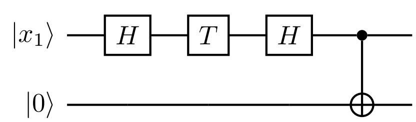
Uniform family means that there exists a classical Turing machine (this has a complicated formal definition but can think of as a program/computer code/circuit) that for every N generates the circuit that solves the problem on all inputs $x \in \{0,1\}^N$ of size N.

The *time complexity* of the (deterministic/randomized/quantum) algorithm is the *time of generation* (which is at least the number of gates).

Remark 5. The set of gates does not matter too much as long as they are universal.

1. In the deterministic case: any function of the input can be implemented by some circuit.
2. In the randomized case: any stochastic map can be approximated by some circuit.
3. In the quantum case: any unitary can be approximated by some circuit.

Example 3. *We analyzed the following example in class.*



Definition 15 (Complexity classes)**.** Given a decision problem $\mathcal{P}$ (decision means $\mathcal{P}$ has a single-bit so can be modelled as $\mathcal{P}\colon \{0,1\}^* \to \{0,1\}$), we say $\mathcal{P}$ is in $\{P, BPP, BQP\}$ if it can be solved by a {deterministic, randomized, quantum} algorithm with time complexity scaling polynomially in the input size N.

Transforming a classical circuit to a quantum one.

Proposition 6. *Suppose we have a classical circuit (implementing the function) $C\colon \{0,1\}^N \to \{0,1\}$. We can efficiently transform C to a quantum circuit Q implementing the unitary $Q\colon |x\rangle |b\rangle \mapsto |x\rangle |b \oplus C(x)\rangle$ for all $x \in \{0,1\}^N$ and $b \in \{0,1\}$ (note that this acts on $\mathbb{C}^{2^{N+1}}$).*

Proof. First assume quantum circuits also have Toffoli gates. (See HW2.)

AND can be simulated by Toffoli, OR can be simulated by Toffoli and X (think de Morgan's law), NOT is simulated by X, FANOUT is simulated by CNOT (with target set to $|0\rangle$).

Comment: Then draw U-copy-$U^\dagger$ and explain. □

Remark 6. This implies P is contained in BQP. In fact BPP is also contained in BQP: can simulate the COIN gate by the Hadamard gate (and use principle of deferred measurement).

[10] You might wonder why there were no ancilla/workspace bits in the deterministic/randomized case. The reason is that you could generate them yourself using FANOUT and NOT: $0 = x_1 \wedge \neg x_1$ but in the quantum case simulating FANOUT using the CNOT gate requires ancilla (see below).

[11] In class, we simulated the X gate using the CNOT gate with one ancilla qubit intialized to $|1\rangle$ but because I used the convention of having the ancilla being be in state $|0\rangle$ that construction is technically not allowed.