Cyclicity of $\mathbb{Z}_p^*$

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Definition 1 (Group)**.** A group $G = (S, \alpha)$ is defined by a set S and a function $\alpha\colon S \times S \to S$ with the following conditions. (For $g, h \in S$, we write $g \cdot h$ or simply gh as shorthand for $\alpha(g, h)$.)

1. Identity. There exists an $e \in S$ such that for all $g \in S$, $ge = eg = g$.
2. Inverse. For all $g \in S$, there exists $h \in S$ such that $gh = hg = e$.
3. Associativity. For all $g, h, k \in S$, $(gh)k = g(hk)$. (That is, $\alpha(\alpha(g, h), k) = \alpha(g, \alpha(h, k))$.)

The set S is known as the underlying set of G and the function α is known as the group operation of G. The *order* of G is the size of the set underlying G and is denoted $|G|$. We say G is a finite group if its size is finite.

If we additionally have $gh = hg$ for all $g, h \in S$, then we say G is an abelian group.

Remark 1. The definition implies: (i) the identity element e is unique, (ii) for all $g \in S$, there exists a unique $h \in S$ such that $gh = hg = e$ and we can denote it without ambiguity by g^{-1}. Good exercise to check.

Example 1. Our main working example is the group $\mathbb{Z}_p^*$, where p is prime. The underlying set is $\{1, \ldots, p-1\}$ and the group operation is *multiplication* modulo p. Consider $\mathbb{Z}_5^*$: the set is $\{1, 2, 3, 4\}$ and $3 \cdot 4 = 2$, $2 \cdot 3 = 1$, $3^{-1} = 2$, etc. Note that it's not obvious that the existence-of-inverse requirement of a group is satisfied, but it can be shown using Bézout's identity and the assumption that p is prime. The group is also abelian, since multiplication (modulo p) commutes.

Definition 2 (Subgroup)**.** Let $G = (S, \alpha)$ be a group. We say $T \subseteq S$ forms a subgroup of G if:

1. T contains the identity element of G.
2. T is closed under α, i.e., $g, h \in T \implies gh \in T$.
3. T contains inverses, i.e., $g \in T \implies g^{-1} \in T$.

This definition means that $(T, \alpha|_T)$ is a group, where $\alpha|_T\colon T \times T \to T$ is the natural restriction of α to T defined by $\alpha|_T(x, y) = \alpha(x, y)$ for all $x, y \in T$. We say $(T, \alpha|_T)$ is a subgroup of G. Often the function α is implicit in which case it is common to abuse language and identify the set S with the group G and the set T with the subgroup $(T, \alpha|_T)$. We write $H \leq G$ to mean H is a subgroup of G.

Definition 3 (Coset)**.** Let G be a group and H be a subgroup. A coset of H in G is a set of the form $gH \coloneqq \{gh \mid h \in H\}$.

Theorem 1 (Lagrange)**.** *Let G be a finite group and H be a subgroup. Then the order of H divides the order of G.*

Proof. The cosets of H partition G and each have size $|H|$. □

Definition 4. Let G be a finite group and $g \in G$. The order of g in G, denoted $o(g)$ or $\mathrm{ord}(g)$, is the minimum positive integer r such that $g^r = e$. The subgroup generated by g, denoted $\langle g \rangle$, is the subgroup formed by the subset $\{e, g^1, \ldots, g^{o(g)-1}\}$

Exercise: check $o(g)$ is well-defined and that $\langle g \rangle$ indeed forms a subgroup.

Corollary 1. *Let G be a finite group and $g \in G$, then $o(g)$ divides $|G|$, written $o(g) \mid |G|$.*

Proof. Follows from Lagrange's theorem because $\langle g \rangle$ is a subgroup of G of size $o(g)$. □

An immediate corollary of the above is:

Corollary 2. *Let G be a finite group and $g \in G$, then $g^{|G|} = e$. In particular, this implies Fermat's Little Theorem that for all $a \in \mathbb{Z}_p^*$, where p is prime, we have $a^{p-1} = 1$.*

Definition 5. Let n be a positive integer. We write $\mathbb{Z}_n \coloneqq \{0, 1, \ldots, n-1\}$. We write $\mathbb{Z}_n[X]$ for the set of polynomials with coefficients in $\mathbb{Z}_n$. Given $0 \neq P \in \mathbb{Z}_n[X]$, the degree of P is defined to be the exponent of the largest power of X that has a non-zero coefficient. We say $x \in \mathbb{Z}_n$ is a root of P if $P(x) = 0 \mod n$.

Example 2. The set $\mathbb{Z}_4[X]$ contains polynomials like $2X$, X^3, and $3X^{100}+X^{42}+1$, which are of degrees $1, 3, 100$, respectively. Note that 2 is a root of $2X$ and X^3; while $3X^{100}+X^{42}+1$ has no roots in $\mathbb{Z}_4$. Why?

Proposition 1. *Let p be prime. Let d be a positive integer. A degree d polynomial P with coefficients in $\mathbb{Z}_p$ has at most d distinct roots in $\mathbb{Z}_p$.*

Proof. Proof by induction on d. For $d = 1$, the polynomial must be of the form $P = \alpha X + \beta$ for some $\alpha, \beta \in \mathbb{Z}_p$ with $\alpha \neq 0$. Since p is prime, this means α is invertible and the only root to $P(x) = 0$ is $-\alpha^{-1}\beta$. For $d > 1$, suppose x is a root of P, then use polynomial division to write $P = (X - x)Q + r$, where $Q \in \mathbb{Z}_p[X]$ has degree $d - 1$ and $r \in \mathbb{Z}_p$. Evaluating P at $X = x$ shows $r = 0$. Thus $P = (X - x)Q$. Suppose $y \in \mathbb{Z}_p$ is a root of P, then $(y - x)Q(y) = 0$, so $y = x$ or $Q(y) = 0$ as p is prime. (This uses the fact that if a prime divides a product of two integers, then it must divide at least one of them.) Therefore, by the inductive hypothesis, y can take one of at most $1 + (d - 1) = d$ possible values since Q has degree $d - 1$. This completes the proof. □

Remark 2. Proposition 1 can be false if p is not prime:

1. $2x$ has two distinct roots in $\mathbb{Z}_4$, namely, 0 and 2.
2. $x^2 - 1$ has four distinct roots in $\mathbb{Z}_8$, namely, $1, 3, 5, 7$.

Definition 6. For positive integers a, b, $\operatorname{lcm}(a, b)$ denotes the least common multiple of a and b.

Example 3. $\operatorname{lcm}(6, 21) = 42$. $\operatorname{lcm}(7, 5) = 35$. $\operatorname{lcm}(35, 7) = 35$.

Lemma 1. *Let G be a finite abelian group. Let $g, h \in G$. Suppose $o(g), o(h)$ are coprime, then $o(gh) = \operatorname{lcm}(o(g), o(h)) = o(g) \cdot o(h)$.*

Proof. Since $o(g)$ and $o(h)$ are coprime, it directly follows that $\operatorname{lcm}(o(g), o(h)) = o(g) \cdot o(h)$. Thus, it suffices to show $o(gh) = \operatorname{lcm}(o(g), o(h))$. Write $k := o(gh)$ and $\ell := \operatorname{lcm}(o(g), o(h))$.

For $k \leq \ell$: we have

$$\begin{aligned} (gh)^\ell &= g^\ell h^\ell && G \text{ abelian} \\ &= e \cdot e = e && \ell \text{ is a multiple of } o(g) \text{ and } o(h) \end{aligned}$$

so $k \leq \ell$ by the definition of k as the *order* of gh.

For $\ell \leq k$: as above, we have

$$(gh)^k = g^k h^k = e \tag{1}$$

and so

$$x := g^k = (h^{-1})^k \in \langle g \rangle \cap \langle h \rangle \tag{2}$$

Thus, Corollary 1 implies $o(x) \mid o(g)$ and $o(x) \mid o(h)$. But $o(g)$ and $o(h)$ are coprime so $o(x) = 1$, so $x = e$. Therefore, the definition of x means $g^k = e = h^k$. Therefore, $o(g) \mid k$ and $o(h) \mid k$ (to see this, list powers of g, h in a sequence) so k is a common multiple of $o(g)$ and $o(h)$ so $k \geq \ell$ by the definition of ℓ as the *least* common multiple. □

Remark 3. The coprimality assumption is crucial in Lemma 1. For example, consider the group $\mathbb{Z}_2$, i.e., $\{0, 1\}$ under addition modulo 2. Then $o(1 + 1) = o(0) = 1$ but $\operatorname{lcm}(o(1), o(1)) = 2$.

From Lemma 1, we deduce the next proposition. (Based on this StackExchange answer.)

Proposition 2. *Every finite abelian group G has an* lcm*-closed order set. That is, for all $x, y \in G$, there exists $z \in G$ such that*

$$o(z) = \operatorname{lcm}(o(x), o(y)). \tag{3}$$

Proof. Proof by induction on $o(x)o(y)$. If $o(x)o(y) = 1$, then we can choose $z = e$. Otherwise, we can wlog factorize

$$o(x) = AP, \quad o(y) = BP', \tag{4}$$

where $P = p^m > 1$ for some prime p coprime to A, B; and $P' \mid P$.

Then

$$o(x^P) = A \quad \text{and} \quad o(y^{P'}) = B \tag{5}$$

By induction there exists z with $o(z) = \operatorname{lcm}(A, B)$.

Now note that $o(x^A) = P$ and P is coprime to $o(z) = \operatorname{lcm}(A, B)$. Therefore,

$$\begin{aligned} o(x^A z) &= P \cdot \operatorname{lcm}(A, B) && \text{Lemma 1} \\ &= \operatorname{lcm}(AP, BP') && P' \mid P \\ &= \operatorname{lcm}(o(x), o(y)), \end{aligned}$$

as required. □

Definition 7 (Cyclic groups and generators)**.** Let G be a finite group, we say G is cyclic if there exists $g \in G$, such that $o(g) = |G|$. In which case, we call g a generator of G.

Theorem 2. *For p prime, $\mathbb{Z}_p^*$ is a cyclic group.*

Example 4. In $\mathbb{Z}_5^*$, we have $o(1) = 1$, $o(2) = 4$, $o(3) = 4$, $o(4) = 2$. So 2 and 3 are the only generators.

Proof. Let ℓ be the least common multiple of the orders of the elements of $\mathbb{Z}_p^*$. By Proposition 2, ℓ must be the order of some element in $\mathbb{Z}_p^*$. Thus it suffices to show $\ell = p - 1$.

By Corollary 1, $p - 1$ is a common multiple of the orders of the elements of $\mathbb{Z}_p^*$, so $\ell \leq p - 1$.

Moreover, the definition of ℓ implies that every element of $\mathbb{Z}_p^*$ is a root of $X^\ell - 1$. This is a degree ℓ polynomial, so $p - 1 \leq \ell$ by Proposition 1. Hence the theorem. $\square$