Lecture 11

Three comments on Nobel Prize in Phyics, 2025 to Clarke, Devoret, Martinis:

1. Part of origin story of superconducting qubits. They showed macroscopic objects (superconducting circuits) can exhibit a certain quantum phenomenon, namely quantum tunneling, where a quantum object can pass a barrier with less energy than any classical object.

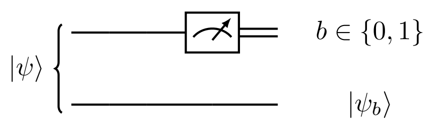
2. Martinis used to be head of hardware at Google's quantum computing lab (known as Google Quantum AI), and led the first "quantum supremacy" experiment – an experiment where a quantum computer does a *useless* task faster than any known classical algorithm [Nature paper]. Left to found quantum computing startup QoLab. Devoret is the new head of quantum hardware at Google.

3. The notion of quantum tunneling has a particularly striking manifestation in quantum algorithms: the glued-tree problem proposed by one of my PhD advisors Andrew Childs and collaborators: paper link. Briefly describe the problem. I plan to cover the analysis of this problem in my graduate course on quantum algorithms next semester [link to course] and I encourage you to register if interested. Alternatively, it's covered in Andrew's lecture notes and can serve as the basis of your project for this course.

Okay, back to regularly scheduled programming!
Last time we considered partial measurement of the first qubit of the state:

$$|\psi\rangle \coloneqq \sqrt{1/2}\,|00\rangle + \sqrt{1/4}\,|01\rangle + \sqrt{1/6}\,|10\rangle + \sqrt{1/12}\,|11\rangle\,. \tag{64}$$

Partial measurement of the first qubit in a circuit diagram.



Two possible measurement outcomes:

1. $b = 0$. Probability is $1/2 + 1/4 = 3/4$. Given this outcome, the state of second qubit collapses to

$$|\psi_0\rangle = \sqrt{2/3}\,|0\rangle + \sqrt{1/3}\,|1\rangle$$

. This state is referred to as the *post-measurement state* of the second qubit.

2. $b = 1$. Similar.

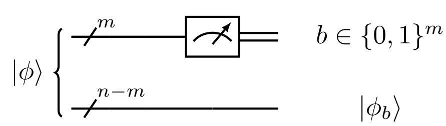
Proposition 4. *Let $|\phi\rangle$ be any n-qubit state. Consider the following two experiments:*

1. *Directly measure $|\phi\rangle$'s first m qubits, obtaining outcome $b \in \{0,1\}^m$ and post-measurement state $|\phi_b\rangle$.*

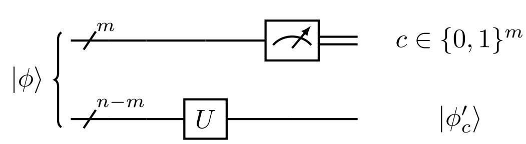
2. *First apply some $(n-m)$-qubit unitary U on the last $(n-m)$ qubits of $|\phi\rangle$ and then measure its first m qubits, obtaining outcome $c \in \{0,1\}^m$ and post-measurement state $|\phi'_c\rangle$.*

Then the distribution on b and c are the same and $|\phi'_c\rangle = U\,|\phi_c\rangle$.

First experiment



Second experiment



Proof. State of the first experiment just before measurement is $|\phi\rangle$. The probability of measuring $b \in \{0,1\}^m$ is

$$\|(\langle b| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle\,\|^2 \tag{65}$$

and given b, the post-measurement state is

$$|\phi_b\rangle := \frac{(\langle b| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle}{\|(\langle b| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle\,\|}. \tag{66}$$

The state of the second experiment just before the measurement is

$$(\mathbb{1}^{\otimes m} \otimes U)\,|\phi\rangle\,. \tag{67}$$

For $c \in \{0,1\}^m$, we have

$$(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})(\mathbb{1}^{\otimes m} \otimes U)\,|\phi\rangle = (\langle c| \otimes U)\,|\phi\rangle = (1 \otimes U)(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle = U(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle \tag{68}$$

But U is unitary. So probability of measuring $c \in \{0,1\}^m$ is

$$\|U(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle\,\| = \|(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle\,\|^2, \tag{69}$$

and given c, the post-measurment state is

$$|\phi'_c\rangle := \frac{U(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle}{\|(\langle c| \otimes \mathbb{1}^{\otimes(n-m)})\,|\phi\rangle\,\|} = U\,|\phi_c\rangle\,, \tag{70}$$

as required $\square$

Quantum algorithms and speedups.

Deutsch's problem. Historically, the *first* example of quantum speedup.

Given $f\colon \{0,1\} \to \{0,1\}$, ask balanced or constant.

1. Classically, assume the function f can only be accessed through queries. Unit cost per query. Classically, need two queries to f solve problem with certainty.
2. Quantumly, assume the ability to apply the *quantum oracle for* f, that is, the 2-qubit unitary defined by

$$O_f\,|x\rangle\,|b\rangle \to |x\rangle\,|b \oplus f(x)\rangle \tag{71}$$

where $x, b \in \{0,1\}$. Unit cost per application of O_f – also referred to as (quantum) query to f. Quantumly, can solve problem using a single query to f with certainty.

We'll motivate these assumptions later. Then do quantum analysis directly following Fig. 5.6 of Watrous notes. First note $H^2 = \mathbb{1}_2$, so

$$H\,|0\rangle = |+\rangle\,, \quad H\,|1\rangle = |-\rangle \implies H\,|+\rangle = |0\rangle\,, \quad H\,|-\rangle = |1\rangle\,. \tag{72}$$

Then[3]

$$\begin{aligned}
|0\rangle\,|1\rangle \mapsto& \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) \\
\mapsto& \frac{1}{2}(|0\rangle\,|f(0)\rangle - |0\rangle\,|f(0) \oplus 1\rangle + |1\rangle\,|f(1)\rangle - |1\rangle\,|1 \oplus f(1)\rangle) \\
=& \frac{1}{2}(|0\rangle\,(|f(0)\rangle - |f(0) \oplus 1\rangle) + |1\rangle\,(|f(1)\rangle - |1 \oplus f(1)\rangle)) \\
=& \frac{1}{2}((-1)^{f(0)}\,|0\rangle\,(|0\rangle - |1\rangle) + (-1)^{f(1)}\,|1\rangle\,(|0\rangle - |1\rangle)) \\
=& (-1)^{f(0)}\frac{1}{\sqrt{2}}(|0\rangle + (-1)^{f(1)-f(0)}\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
=& (-1)^{f(0)}\begin{cases} |+\rangle & \text{if } f(0) \neq f(1) \\ |-\rangle & \text{if } f(0) = f(1) \end{cases} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
\mapsto& (-1)^{f(0)}\begin{cases} |0\rangle & \text{if } f(0) \neq f(1) \\ |1\rangle & \text{if } f(0) = f(1) \end{cases} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\end{aligned}$$

Comment: I slightly rushed the analysis of the (partial) measurement at the end, and plan to say a bit more about it at the start of next lecture.

[3] In class, I wrote $(-1)^{f(1)-f(0)}$ as $(-1)^{f(0)-f(1)}$ at some point during the analysis. This was unnecessary, but it is still correct since $\forall a, b \in \{0,1\}$, we have $(-1)^{a-b} = (-1)^{b-a}$; and both sides equal $(-1)^{a\oplus b}$. (Can check directly or just observe that (-1) to the power of any integer only depends on the parity of that integer.)