CPSC 536W: Homework 1

Due at the start of class on 22nd January 2024

Rules.

1. Please try to solve the problems yourself first. If you get stuck, you may consult any resources (books, internet, peers, office hours, etc.) for solutions. Provided you *acknowledge* these resources, no marks will be deducted.
2. Please write legibly, work that is illegible will be marked as incorrect. Latex is strongly recommended for legibility. (I also recommend using `https://www.overleaf.com/` if you're new to Latex.)
3. All answers should be justified.
4. The total number of points for non-bonus questions is $T = 28$. Credit policy for the bonus question: suppose you receive x points for the bonus question and y points in total for the non-bonus questions, then the total points you receive for this homework is $\min(x + y, T)$.

1 Correction to the definition of a projective measurement

Definition 1 (Projective measurement and its effect)**.** *Let Γ be an alphabet and $d \in \mathbb{N}$. A Γ-outcome projective measurement on $\mathbb{C}^d$ is a set of matrices $\mathcal{M} \coloneqq \{\Pi_i \mid i \in \Gamma\} \subseteq \mathbb{C}^{d\times d}$ labelled by elements in Γ such that $\forall i \in \Gamma, \Pi_i^\dagger = \Pi_i$, $\forall i, j \in \Gamma$, $\Pi_i\Pi_j = \delta_{i,j}\Pi_i$, and $\sum_{i\in\Gamma} \Pi_i = \mathbb{1}_d$.*

Given a (quantum) state $|\psi\rangle \in \mathbb{C}^d$, to measure $|\psi\rangle$ using $\mathcal{M}$ refers to a process that

1. *Outputs $i \in \Gamma$ with probability $\|\Pi_i |\psi\rangle\|^2$. This i is referred to as the measurement outcome.*
2. *Given the output is $i \in \Gamma$, $|\psi\rangle$ changes to*

$$|\psi'\rangle \coloneqq \frac{\Pi_i |\psi\rangle}{\|\Pi_i |\psi\rangle\|}. \tag{1}$$

Measurement in the computational basis on $\mathbb{C}^d$ refers to the $\{0, 1, \ldots, d-1\}$-outcome projective measurement defined by $\{|i\rangle\langle i| \mid i \in \{0, 1, \ldots, d-1\}\}$.

Remark 1. *Compared to the definition I gave in class, the projectors are labelled by elements in Γ, as opposed to $1, \ldots, |\Gamma|$. This is a somewhat minor correction. The major correction is the addition of the condition "$\forall i \in \Gamma, \Pi_i^\dagger = \Pi_i$" in the definition of a projective measurement. ($\dagger$ denotes the conjugate transpose, so $\Pi_i^\dagger = \Pi_i$ means Π_i is Hermitian.)*

2 Additional notation

For $s \in \{0, 1\}^n$, we write $|s\rangle$ for the n-qubit state

$$|s_1\rangle |s_2\rangle \ldots |s_n\rangle \in \mathbb{C}^{2^n}. \tag{2}$$

So, for example,

$$|01\rangle = |0\rangle |1\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

This notation will be used in Problems 4 and 5 (and also throughout the course).

3 Homework

1. **Randomized query complexity.**

   (a) (4 points.) For this question, recall $R(\cdot) = R_{1/3}(\cdot)$ by definition. Prove that $R(\mathrm{OR}_3) = 2$.
   [Hints: for ≤ 2: think intuitively about how you might compute OR_3 by exploiting randomness before formalizing the computation as an RDT; for ≥ 2: note the result we showed in class only gives ≥ 1 so we cannot use it here, instead directly show that any depth-1 RDT cannot compute OR_3 with bounded-error 1/3.]

   (b) (4 points.) Given $f\colon \{0,1\}^n \to \{0,1\}$, the sensitivity of f at $x \in \{0,1\}^n$, denoted $s_x(f)$, is defined to be the size of the set

   $$\{i \in [n] \mid f(x) \neq f(x^i)\}, \tag{4}$$

   where x^i denotes x with the ith bit flipped. (E.g., if $x = 001$, then $x^1 = 101$, $x^2 = 011$, $x^3 = 000$.) Then, the sensitivity of f is defined to be

   $$s(f) \coloneqq \max_{x \in \{0,1\}^n} s_x(f). \tag{5}$$

   Show that $s(\mathrm{OR}_n) = n$. Prove that for all $\epsilon \in (0, 1/2)$, and all $f\colon \{0,1\}^n \to \{0,1\}$, we have

   $$R_\epsilon(f) \geq (1 - 2\epsilon)s(f). \tag{6}$$

   (c) (2 points.) Let $f\colon \{0,1\}^{kl} \to \{0,1\}$ be defined by

   $$f(x_{11}, \ldots, x_{1l}, x_{21}, \ldots, x_{2l}, \ldots, x_{k1}, \ldots, x_{kl}) = (x_{11} \wedge \cdots \wedge x_{1l}) \vee (x_{21} \wedge \cdots \wedge x_{2l}) \vee \cdots \vee (x_{k1} \wedge \cdots \wedge x_{kl}). \tag{7}$$

   Show that $s(f) \geq \max(k, l)$. (I hope the definition of f is clear: informally, the input to f consists of k blocks of l bits each. Say the *value* of a block is the AND of all of its l bits, then the output of f is the OR of the values of all k blocks.)

2. **Eigenvalues and eigenvectors.** Let $\theta \in \mathbb{R}$ and $A \in \mathbb{C}^{2\times 2}$ be defined by

   $$A \coloneqq \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{8}$$

   (a) (4 points.) Calculate the eigenvalues and eigenvectors of A. Therefore, write A in the form $A = UDU^\dagger$, where $U \in \mathbb{C}^{2\times 2}$ is unitary and $D \in \mathbb{C}^{2\times 2}$ is diagonal.

   (b) (2 points.) For $k \in \mathbb{N}$, show that $A^k = UD^kU^\dagger$ and use the expression on the right-hand side to calculate A^k, simplifying your answer as much as possible.

3. **Kronecker product.**

   (a) (2 points.) Let $A, B \in \mathbb{C}^{d\times d}$ and $u, v \in \mathbb{C}^d$. Prove that

   $$(A \otimes B)(u \otimes v) = Au \otimes Bv. \tag{9}$$

   You are allowed to use any property of the Kronecker product listed in `https://en.wikipedia.org/wiki/Kronecker_product`, *except* "the mixed-product property" — since that is stronger than what you're being asked to prove.
   [Hint: first prove eq. (9) for $A = |i_1\rangle\langle j_1|$, $B = |i_2\rangle\langle j_2|$, $u = |k\rangle$ and $v = |l\rangle$ where $i_1, j_1, i_2, j_2, k, l \in \{0, 1 \ldots, d-1\}$, then use other properies of the Kronecker product. (Recall that $|0\rangle, |1\rangle, \ldots, |d-1\rangle \in \mathbb{C}^d$ denote the computational basis vectors.)]

   (b) (2 points.) Define $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ by

   $$|\psi\rangle \coloneqq \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle\,|i\rangle\,. \tag{10}$$

   Let $\mathbb{1}_d \in \mathbb{C}^{d\times d}$ denote the identity matrix. Show that for any $A \in \mathbb{C}^{d\times d}$, we have

   $$A \otimes \mathbb{1}_d\,|\psi\rangle = \mathbb{1}_d \otimes A^\top\,|\psi\rangle\,, \tag{11}$$

   where $^\top$ denotes the transpose.

   (c) (2 points.) Let $|u_0\rangle, \ldots |u_{d-1}\rangle \in \mathbb{C}^d$ be an arbitrary orthonormal basis. Show that

   $$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |u_i\rangle\,|u_i\rangle\,. \tag{12}$$

4. **Quantum teleportation.**

Let $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^8$ be defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha\,|0\rangle + \beta\,|1\rangle)(|00\rangle + |11\rangle), \tag{13}$$

where $\alpha, \beta \in \mathbb{C}^2$ are such that $|\alpha|^2 + |\beta|^2 = 1$.

Now define the following 2-qubit states

$$|\psi_1\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{14}$$

$$|\psi_2\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{15}$$

$$|\psi_3\rangle \coloneqq \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{16}$$

$$|\psi_4\rangle \coloneqq \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{17}$$

Then, for all $i \in [4]$, define $\Pi_i \coloneqq |\psi_i\rangle\langle\psi_i| \otimes \mathbb{1}_2 \in \mathbb{C}^{8\times 8}$.

(a) (2 points.) Show that $\mathcal{M} \coloneqq \{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$ is a [4]-outcome projective measurement on $\mathbb{C}^8$.

(b) (4 points.) For each $i \in [4]$, when we measure $|\psi\rangle$ using $\mathcal{M}$, what is the probability that the measurement outcome is i? Given the measurement outcome is i, explicitly compute the state $|\psi'\rangle$ that $|\psi\rangle$ changes to.

5. **CHSH game (bonus question).**

The CHSH game is defined to be the following game between 3 players: Referee, Alice, and Bob.

(a) Referee flips two fair coins to get two random bits $x, y \in \{0, 1\}$.

(b) Referee sends x to Alice and y to Bob.

(c) Alice responds with bit $a \in \{0, 1\}$ and Bob responds with bit $b \in \{0, 1\}$.

(d) Alice and Bob win if and only if $a \oplus b = x \wedge y$.

($\oplus$ is the XOR, i.e., its truth table is $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, and $1 \oplus 1 = 0$. $\wedge$ is the AND, i.e., its truth table is $0 \wedge 0 = 0$, $0 \wedge 1 = 0$, $1 \wedge 0 = 0$, $1 \wedge 1 = 1$.)

(a) (2 points) For $f\colon \{0,1\} \to \{0,1\}$ and $g\colon \{0,1\} \to \{0,1\}$, let $w(f, g)$ denote the probability of Alice and Bob winning the game if they set $a = f(x)$ and $b = g(y)$ in step (c), where the probability is over the referee's coin flips in step (a). Evaluate the value of

$$\max_{f,g} w(f, g). \tag{18}$$

This value can be thought of as the maximum winning probability of any classical deterministic strategy if Alice and Bob are not allowed to communicate during the game.

(b) (4 points.) Let $|\psi\rangle$ denote the 2-qubit state

$$|\psi\rangle \coloneqq \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{19}$$

Define the following 1-qubit states:

$$|+\rangle \coloneqq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle \coloneqq \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{20}$$

$$|s_0\rangle \coloneqq \cos(\pi/8)\,|0\rangle + \sin(\pi/8)\,|1\rangle \quad |s_1\rangle \coloneqq -\sin(\pi/8)\,|0\rangle + \cos(\pi/8)\,|1\rangle\,, \tag{21}$$

$$|t_0\rangle \coloneqq \cos(\pi/8)\,|0\rangle - \sin(\pi/8)\,|1\rangle \quad |t_1\rangle \coloneqq \sin(\pi/8)\,|0\rangle + \cos(\pi/8)\,|1\rangle\,. \tag{22}$$

Define

$$\mathcal{A}_0 \coloneqq \{A_0 \coloneqq |0\rangle\langle 0| \otimes \mathbb{1}_2, \quad A_1 \coloneqq |1\rangle\langle 1| \otimes \mathbb{1}_2\}, \tag{23}$$

$$\mathcal{A}_1 \coloneqq \{A'_0 \coloneqq |+\rangle\langle +| \otimes \mathbb{1}_2, \quad A'_1 \coloneqq |-\rangle\langle -| \otimes \mathbb{1}_2\}, \tag{24}$$

$$\mathcal{B}_0 \coloneqq \{B_0 \coloneqq \mathbb{1}_2 \otimes |s_0\rangle\langle s_0|, \quad B_1 \coloneqq \mathbb{1}_2 \otimes |s_1\rangle\langle s_1|\}, \tag{25}$$

$$\mathcal{B}_1 \coloneqq \{B'_0 \coloneqq \mathbb{1}_2 \otimes |t_0\rangle\langle t_0|, \quad B'_1 \coloneqq \mathbb{1}_2 \otimes |t_1\rangle\langle t_1|\}. \tag{26}$$

You can assume that $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$ are $\{0,1\}$-outcome projective measurements (not hard to verify).

Consider the following quantum strategy for Alice and Bob. At step (c):

i. Alice sets a to be the measurement outcome resulting from measuring $|\psi\rangle$ using $\mathcal{A}_x$. Let $|\psi'\rangle$ be the state $|\psi\rangle$ changes to. (Note that $|\psi'\rangle$ depends on the measurement outcome).

ii. Bob sets b to be the measurement outcome resulting from measuring $|\psi'\rangle$ using $\mathcal{B}_y$.

(Here we are assuming Bob measures after Alice but it can be seen that the ordering does not in fact matter.)

Show that the probability of Alice and Bob winning the game under this quantum strategy is

$$\cos^2(\pi/8), \tag{27}$$

where the probability is over the referee's coin flips in step (a) as well as the randomness in the measurement process.

(Hint: systematically consider what happens when (x, y) is $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$.)

Remark 2. *If you did this question correctly, you should find that* $\cos^2(\pi/8) = 0.853...$ *is strictly greater than the value you computed in 5(a). This fact is known as a Bell-inequality violation. In 2022, the Nobel Prize in Physics was awarded to three experimental physicists for demonstrating such violations in the lab. (One of the laureates, Clauser, is responsible for the "C" in CHSH.)*

Alice and Bob's quantum strategy in part (b) does not require them to communicate because the measurements they perform are on "their own part of the state $|\psi\rangle$*" — this is mathematically captured by the* $\otimes \mathbb{1}_2$*s in the definitions of* $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1$.