图模型初步

Problem 1: Diagnoses

- The doctor has bad news and good news.

- The bad news is that you tested positive for a serious disease, and that **the test is 99% accurate** (i.e., the probability of testing positive given that you have the disease is 0.99, as is the probability of testing negative given that you don't have the disease).

- The good news is that this is a rare disease, striking only 1 in 10,000 people.

- What are the chances that you actually have the disease?

50 5

Problem 2: Monty Hall

On a game show, a contestant is told the rules as follows:

- There are three doors, labeled 1, 2, 3. A single big prize has been hidden behind one of them. The other two doors have goats. You get to select one door.
- Initially your chosen door will not be opened. Instead, the host will open one of the other two doors, and he will do so in such a way as not to reveal the prize.
- At this point, you will be given a fresh choice of door: you can either stick with your first choice, or you can switch to the other closed door. All the doors will then be opened and you will receive whatever is behind your final choice.

Imagine that the contestant chooses door 1 first; then the host opens door 3, revealing a goat behind the door. Should the contestant

(a) stick with door 1, 6
(b) switch to door 2, 50
(c) does it make no difference? 4

Bayes rule

Bayes rule enables us to reverse probabilities:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_A P(B|A)P(A)}$$

$$P(AB) = P(B|A)\,P(A)$$
$$= P(A|B)\,P(B)$$

$$P(B|A)P(A) = P(A|B)P(B)$$

$$\sum_B P(B|A) = 1 \qquad \sum_B P(B) = 1$$

Problem 1: Diagnoses

The test is 99% accurate: $P(T=1|D=1) = 0.99$ *and* $P(T=0|D=0) = 0.99$
*Where **T** denotes test and **D** denotes disease.*

The disease affects 1 in 10000: $P(D=1) = 0.0001$

$$P(D=1|T=1) = \frac{P(T=1|D=1)P(D=1)}{P(T=1|D=0)P(D=0)+P(T=1|D=1)P(D=1)}$$

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TED

$$\approx 0.0098$$

Problem 2: Monty Hall

*(i) $H=i$ denote the Hypothesis that the prize is behind door i. **A priori** all 3 doors are equally likely to have the prize:*

$$P(H=1) = P(H=2) = P(H=3) = 1/3$$

*(ii) Contestant **chooses door 1.***

Let's think. If the prize is truly behind door 1, the host is indifferent and will choose doors 2 or 3 with equal probability. If the prize is behind door 2 (or 3), host chooses 3 (or 2).

$$P(D=2|H=1) = \tfrac{1}{2},\quad P(D=3|H=1) = \tfrac{1}{2}$$
$$P(D=2|H=2) = 0,\quad P(D=3|H=2) = 1$$
$$P(D=2|H=3) = 1,\quad P(D=3|H=3) = 0$$

*(iii) The host **opens door 3 ($D=3$)**, revealing a goat behind the door. That is, the observation is $D=3$. **Now is the prize behind door 2 or 1?***

Problem 2: Monty Hall

$P(H=1) = P(H=2) = P(H=3) = 1/3$

$P(D=2|H=1) = ½$, $P(D=3|H=1) = ½$
$P(D=2|H=2) = 0$, $P(D=3|H=2) = 1$
$P(D=2|H=3) = 1$, $P(D=3|H=3) = 0$

We use Bayes rule to compute the probability of the hypothesis that the prize is behind door i (for $i=1,2,3$) given that the host has opened door 3 $(D=3)$. That is we compute $P(H=i|D=3)$.

$$P(H=1|D=3) = \frac{P(D=3|H=1)P(H=1)}{P(D=3)} = \frac{(1/2)(1/3)}{P(D=3)} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{3}} = \frac{1}{3}$$

$$P(H=2|D=3) = \frac{P(D=3|H=2)P(H=2)}{P(D=3)} = \frac{1\,(1/3)}{P(D=3)} = \frac{1/3}{1/6+1/3} = \frac{2}{3}$$

$$P(H=3|D=3) = \frac{P(D=3|H=3)P(H=3)}{P(D=3)} = \frac{0\,(1/3)}{P(D=3)} = 0$$

Speech recognition

$$P(\text{words} \mid \text{sound}) \;\; \alpha \;\; P(\text{sound} \mid \text{words})\, P(\text{words})$$

Final beliefs

Likelihood of data

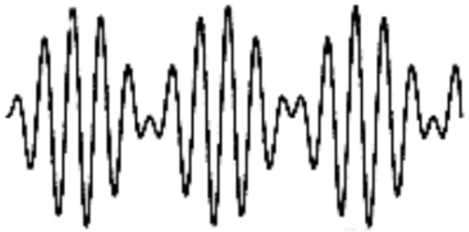
eg mixture of Gaussians

Prior language model

eg unigrams

Hidden Markov Model (HMM)

"Recognize speech"



"Wreck a nice beach"

Bayes and decision theory

Utilitarian view: We need models to make the right decisions under uncertainty. Inference and decision making are intertwined.

Learned posterior

$$\begin{cases} P(x=\textbf{\textit{healthy}}|data) = 0.9 \\ P(x=\textbf{\textit{cancer}}|data) = 0.1 \end{cases}$$

Loss/Reward model *u(x,a)*

	$\mathbf{a} = \textit{no treatment}$	$\mathbf{a} = \textit{treatment}$
$\mathbf{x} = \textit{healthy}$	0	-30
$\mathbf{x} = \textit{cancer}$	-100	-20

We choose the action that maximizes the **expected utility**:

$$EU(a) = \sum_x u(x,a)\, P(x|data)$$

$EU(a=\textbf{\textit{treatment}}) = u(\text{treatment}, \text{healthy})\ 0.9 + u(\text{treatment}, \text{cancer})\ 0.1$
$= (-30)(0.9) + (-20)(0.1) = -29$

$EU(a=\textbf{\textit{no treatment}}) = (0)(0.9) + (-100)(0.1) = -10$ Don't treat

The curse of dimensionality

This curse tells us that to represent a joint distribution of **d** *binary variables, we need* 2^d *terms!*

$d = 3$

A	B	C	
0	0	0	$P(A=0, B=0, C=0)$
0	0	1	$P(A=0, B=0, C=1)$
0	1	0	$\vdots$
0	1	1	
1	0	0	
	$\vdots$		

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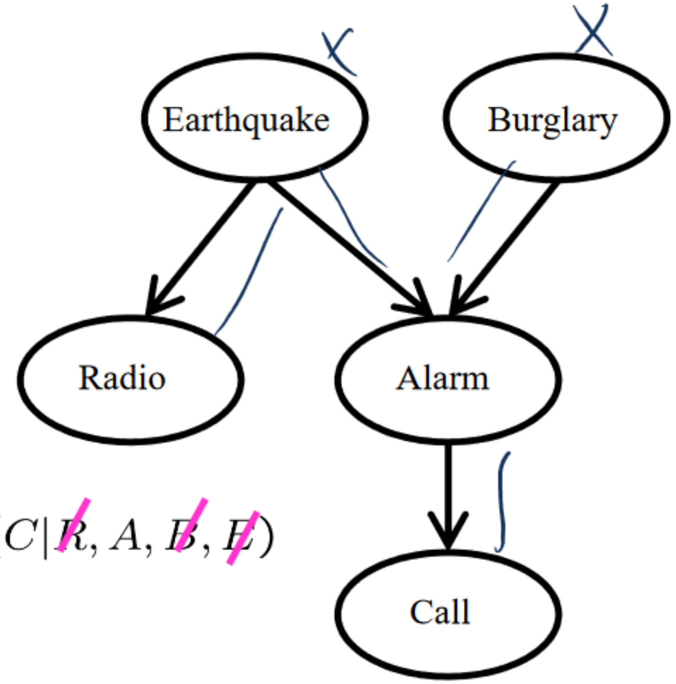
Directed probabilistic graphical models

1. Directed Acyclic Graph (DAG)
   Nodes – random variables
   Edges – direct influence ("causation")
2. X_i independent of $X_{ancestors} \mid X_{parents}$



drop

$$\begin{aligned} P(B,E,A,R,C) &= P(B)P(E|\cancel{B})P(A|B,E)P(R|\cancel{A},\cancel{B},E)P(C|\cancel{R},A,\cancel{B},\cancel{E}) \\ &= P(B)P(E)P(A|B,E)P(R|E)P(C|A) \end{aligned}$$

*The DAG tells us that if we have **n** variables x_i, the joint distribution of these variables **factorizes** as follows:*

$$P(x_{1:n}) = P(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(x_i))$$

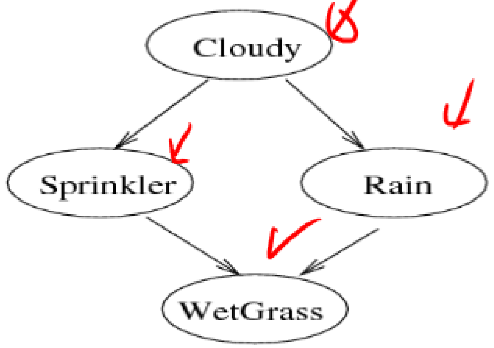
Joint vs Factorized joint distributions

c	s	r	w	prob
0	0	0	0	0.200
0	0	0	1	0.000
0	0	1	0	0.005
0	0	1	1	0.045
0	1	0	0	0.020
0	1	0	1	0.180
0	1	1	0	0.001
0	1	1	1	0.050
1	0	0	0	0.090
1	0	0	1	0.000
1	0	1	0	0.036
1	0	1	1	0.324
1	1	0	0	0.001
1	1	0	1	0.009
1	1	1	0	0.000
1	1	1	1	0.040

$$p(C,S,R,W) = p(C)p(S|C)p(R|C)p(W|S,R)$$

P(C=F)	P(C=T)
0.5	0.5

P(c)



C	P(S=F)	P(S=T)
F	0.5	0.5
T	0.9	0.1

P(S|C)

P(R|C)

C	P(R=F)	P(R=T)
F	0.8	0.2
T	0.2	0.8

S	R	P(W=F)	P(W=T)
F	F	1.0	0.0
T	F	0.1	0.9
F	T	0.1	0.9
T	T	0.01	0.99

P(W|R,S)

The sprinkler network

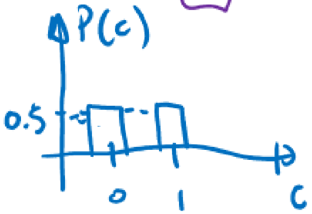
$P(C) = \begin{array}{cc} 0 & 1 \\ \boxed{0.5} & \boxed{0.5} \end{array}$ $\quad P(C=0)=0.5$

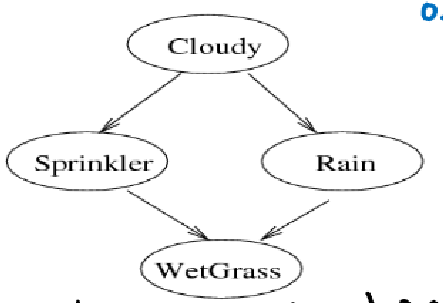
$P(S|C=1) = \begin{array}{cc} 0 & 1 \\ \boxed{0.9} & \boxed{0.1} \end{array}$

$\sum_S P(S|C=1) = 1$

$P(C)$

P(C=F)	P(C=T)
0.5	0.5



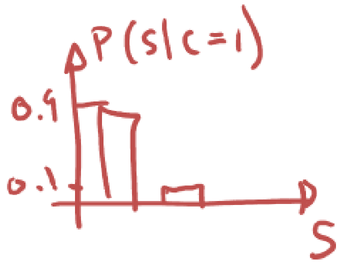


$P(S|C)$

C	P(S=F)	P(S=T)
F	0.5	0.5
T	0.9	0.1

C	P(R=F)	P(R=T)
F	0.8	0.2
T	0.2	0.8

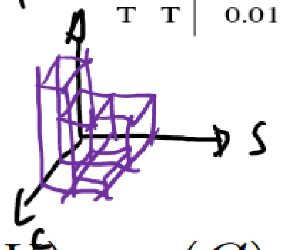
$P(C=0, S=0) = P(S=0|C=0)P(C=0)$



S	R	P(W=F)	P(W=T)
F	F	1.0	0.0
T	F	0.1	0.9
F	T	0.1	0.9
T	T	0.01	0.99

$P(W|S=0,R=1) = \begin{array}{cc} 0 & 1 \\ \boxed{0.1} & \boxed{0.9} \end{array}$

$P(C,S) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & .25 & .25 \\ 1 & 0.45 & 0.05 \end{array}$



$$p(C, S, R, W) = p(C)p(S|C)p(R|C)p(W|S, R)$$

Inference in DAGs

Let us use 0 to denote false and 1 to denote true.

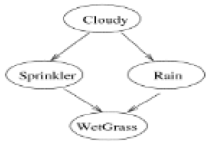
*What is the **marginal probability**, **P(S=1)**, that the sprinkler is on?*

$$P(S=1) = \sum_{C=0}^{1} \sum_{R=0}^{1} \sum_{W=0}^{1} P(C,R,W,S=1)$$

$$= \sum_{C} \sum_{R} \sum_{W} P(C)P(S=1|C)P(R|C)P(W|S=1,R)$$

$P(C)$

P(C=F)	P(C=T)
0.5	0.5



$P(R|C)$

C	P(R=F)	P(R=T)
F	0.8	0.2
T	0.2	0.8

$P(S|C)$

C	P(S=F)	P(S=T)
F	0.5	0.5
T	0.9	0.1

S	R	P(W=F)	P(W=T)
F	F	1.0	0.0
T	F	0.1	0.9
F	T	0.1	0.9
T	T	0.01	0.99

$P(W|R_S)$

Brute force (exponential) approach

What is the ***marginal probability***, ***P(S=1)***, *that the sprinkler is on?*

$$\sum_{C}\sum_{R}\sum_{W=0}^{1} P(W|S=1,R)\,P(S=1|C)\,P(R|C)\,\underline{P(C)}$$

$ab+ac$

$a(b+c)$

$$\overset{000}{=} \overset{0.1}{P(W=0|S=1,R=0)} \times \overset{0.5}{P(S=1|C=0)} \times \overset{0.8}{P(R=0|C=0)} \times \overset{0.5}{P(C=0)}$$

$$\overset{001}{+}\, P(W=1|S=1,R=0)\,P(S=1|C=0)\,P(R=0|C=0)\,P(C=0)$$

$$\overset{010}{+}\, P(W=0|S=1,R=1)\,P(S=1|C=0)\,P(R=1|C=0)\,P(C=0)$$

$$\overset{011}{+} \cdots$$

$$\vdots$$

$$\overset{111}{+}$$

P(C=F)	P(C=T)
0.5	0.5

C	P(S=F)	P(S=T)
F	0.5	0.5
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F	F	1.0	0.0
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Brute force (exponential) approach

What is the ***marginal probability****,* ***P(S=1)****, that the sprinkler is on?*

```
PROD
FOR R=0:1:1
    FOR C=0:1:1
        FOR W=0:1:1
            PROD = PROD + P(C)P(R|C)P(S=1|C)P(W|S=1,R)
        END
    END
END
```

Smart approach: variable elimination, *aka* dynamic programming, *aka* distributive law

*What is the **marginal probability**, **P(S=1)**, that the sprinkler is on?*

$$P(S=1) = \sum_C \sum_R \sum_W P(W|S=1,R)\,P(R|C)\,P(C)\,P(S=1|C)$$

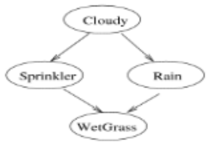
$$= \sum_C \sum_R P(R|C)P(C)P(S=1|C)\boxed{\cancelto{1}{\sum_W P(W|S=1,R)}} \sim \phi = 1$$

$$= \sum_C P(C)P(S=1|C)\boxed{\cancelto{1}{\sum_R P(R|C)}} \sim \psi = 1$$

$$= P(S=1|C=0)P(C=0) + P(S=1|C=1)P(C=1)$$

$$= 0.3$$

P(C=F)	P(C=T)
0.5	0.5



C	P(S=F)	P(S=T)
F	0.5	0.5
T	0.9	0.1

C	P(R=F)	P(R=T)
F	0.8	0.2
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S	R	P(W=F)	P(W=T)
F	F	1.0	0.0
T	F	0.1	0.9
F	T	0.1	0.9
T	T	0.01	0.99

Smart approach: variable elimination, *aka* dynamic programming, *aka* distributive law

*What is the **marginal probability**, **P(S=1)**, that the sprinkler is on?*

$\psi = 0$

$\phi = 0$

$\Theta = 0$

FOR $W=0:1:1$

$\quad \phi_R = \phi_R + P(W|S=1,R)$

END

FOR $R=0:1:1$

$\quad \psi_C = \psi_C + P(R|C)\phi_R$

END

FOR $C=0:1:1$

$\quad \Theta = \Theta + P(S=1|C)P(C)\psi_C \rightarrow \Theta = 0.3$

END

Inference in DAGs

What is the ***posterior probability****,* ***P(S=1|W=1)****, that the sprinkler is on given that the grass is wet?*

$$P(S=1 \mid W=1) = \frac{P(S=1, W=1)}{P(W=1)}$$

$$P(W=1) = \sum_{S} \sum_{C} \sum_{R} P(S, W=1, C, R)$$

$$P(\underline{S=1, W=1}) = \sum_{C} \sum_{R} P(\underline{S=1, W=1, C, R})$$

Inference in DAGs

What is the ***posterior probability***, ***P(S=1|W=1,R=1)***, *that the sprinkler is on given that the grass is wet and it is raining?*

$$P(S=1|W=1,R=1) = P(S=1 \mid (W=1,R=1))$$

$$= \frac{P(S=1,W=1,R=1)}{P(W=1,R=1)}$$

Naïve Bayes as a special case…