$\frac{\ u_2}{\ })^2]]\}$

$-\mu)\}$

$$\Delta^2 = (x-\mu)^T \Sigma^{-1} (x-\mu)$$

$\Sigma = U\Lambda U^T$ with U: eigenvector matrix $(n\times n)$

Λ: eigenvalue matrix $(n\times n)$

Σ symmemic $\rightarrow$ U is orthorgonal $\rightarrow$ $U^{-1} = U^T$

$U^T U = I$.

$$\rightarrow \Sigma u_i = \lambda_i u_i \rightarrow \Sigma = U\Lambda U^T$$

$$= U \begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \lambda_2 \dots & 0 \\ \vdots & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \dots \\ u_n^T \end{bmatrix} = U \begin{bmatrix} \lambda_1 u_1^T \\ \lambda_2 u_2^T \\ \dots \\ \lambda_n u_n^T \end{bmatrix}$$

$$= [u_1 \ u_2 \dots u_n] \begin{bmatrix} \lambda_1 u_1^T \\ \lambda_2 u_2^T \\ \dots \\ \lambda_n u_n^T \end{bmatrix} = \sum \lambda_i u_i u_i^T$$

Since $Au = uA$ $\Sigma = U\Lambda U^T \rightarrow \Sigma U = U\Lambda$

$\rightarrow (\Sigma U)^{-1} = (U\Lambda)^{-1} \rightarrow U^T\Sigma^{-1} = \Lambda^{-1}U^T$

$\rightarrow \Sigma^{-1} = U\Lambda^{-1}U^T$

$= \sum \frac{1}{\lambda_i} u_i u_i^T$

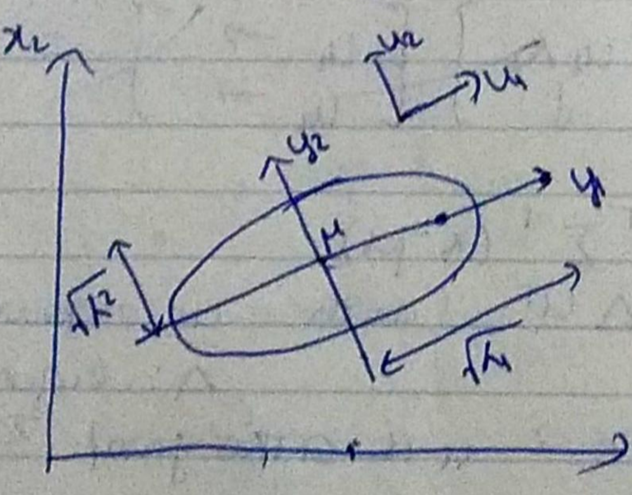
$$\Delta^2 = (x-\mu)^T \Sigma^{-1} (x-\mu)$$

$$= (x-\mu)^T \left(\sum \frac{1}{\lambda_i} u_i u_i^T \right)(x-\mu)$$

$$= (x-\mu)^T \sum \frac{1}{\lambda_i} u_i u^T (x-\mu)$$

$$= \sum \frac{1}{\lambda_i} (x-\mu)^T u_i \underbrace{u_i^T (x-\mu)}_{y_i}$$

$\Delta^2 = \sum \frac{y_i^2}{\lambda_i} \rightarrow$ constant number



$y_1 = u_1 \begin{bmatrix} x_1 - \mu \\ x_2 - \mu \end{bmatrix} \rightarrow y_1$ theo hệ tọa độ u

$\rightarrow \Delta^2 = \frac{y_1^2}{\lambda_1} + \frac{y_2^2}{\lambda_2}$

$\rightarrow p(y) = \prod_{j=1}^{k} \frac{1}{\sqrt{(2\pi\lambda_j)}} e\left\{\frac{-y_j^2}{2\lambda_j}\right\}$

where $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix}$; $y = u(x-$

~~$\int p(y)\,dy$~~ =

$$\int p(y)\,dy = \prod_{1}^{k} \int \frac{1}{\sqrt{2\pi\lambda_j}} \exp\left\{-\frac{y_j^2}{2\lambda_j}\right\} dy_j = 1$$

Conditional Gaussian distr.

I. Conditional distr.

(X, Y) has a joint prob. function $f(X, Y)$

$\rightarrow$ conditional prob. density function of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$\Leftrightarrow f_{X|Y}(x|y)\,dx = \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$\approx P\{x \le X \le x + dx \mid y \le Y \le y + dy\}$$

$\hookrightarrow$ for small dx, dy, $f_{X|Y}(x|y)$ reprents the cond. prob. that X is between x and $x + dx$, given that Y " y " $y + dy$

Ex:

Joint density of X & Y:

$$f(x, y) = \begin{cases} \frac{12}{5} x(2 - x - y) ; & x, y \in (0,1) \\ 0 & \text{otw.} \end{cases}$$

$$f_{X|Y}\{x|y\} = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx}$$

$$= \frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx} = \frac{6x(2-x-y)}{4-3y}$$

II. Cond. Gaussian distr.

x is k-dim. gaussian distr $\sim N(x, \mu, \Sigma)$

Split x into 2 subsets:

$x_a \sim N(x_a, \mu_a, \Sigma_{aa})$

$x_b \sim N(x_b, \mu_b, \Sigma_{bb})$

$\rightarrow x = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$; $\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$; $\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$

Cal

$\mu_{a|b}$

$\Sigma_{a|}$

$\Sigma^T = \Sigma \rightarrow \Sigma_{aa}^T = \Sigma_{aa}$

$\Sigma_{bb}^T = \Sigma_{bb}$

$\Sigma_{ab}^T = \Sigma_{ba}$

Đặt $A = \Sigma^{-1}$

$A = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix}$

$p(x_a | x_b) = \dfrac{p(x_a, p x_b)}{p(x_b)} = \dfrac{p(x)}{p(x_b)}$

$p(x) = \dfrac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left\{ \dfrac{-1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) \right\}$

$p(x_b) = \dfrac{1}{\sqrt{(2\pi)^{k_b} |\Sigma_{bb}|}} \exp\left\{ \dfrac{-1}{2} (x_b - \mu_b)^T \Sigma_{bb}^{-1} (x_b - \mu_b) \right\}$

$\Sigma = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}$

To find $p(x_a | x_b) \rightarrow x_b$ is constant
tham số / const

$\Delta^2 = \dfrac{-1}{2} (x-\mu)^T \Sigma^{-1} (x-\mu) = \dfrac{-1}{2} (x-\mu)^T A (x-\mu)$

$= \dfrac{-1}{2} (x_a - \mu_a)^T A_{aa} (x_a - \mu_a) - \dfrac{1}{2} (x_a - \mu_a)^T A_{ab} (x_b - \mu_b)$

$- \dfrac{1}{2} (x_b - \mu_b)^T A_{ba} (x_a - \mu_a) - \dfrac{1}{2} (x_b - \mu_b)^T A_{bb} (x_b - \mu_b)$

$= \dfrac{-1}{2} x_a^T A_{aa} x_a + x_a^T A_{aa} \mu_a - \dfrac{1}{2} x_a^T A_{ab} (x_b - \mu_b)$

$- \dfrac{1}{2} (x_b - \mu_b)^T A_{ba} x_a + \text{const}$

$= \dfrac{-1}{2} x_a^T A_{aa} x_a + x_a^T A_{aa} \mu_a - x_a^T A_{ab} (x_b - \mu_b) + C$

$= \dfrac{-1}{2} x_a^T A_{aa} x_a + x_a^T (A_{aa} \mu_a - A_{ab} (x_b - \mu_b))$

Vì x_b đã đc đặt là tham số, ↑ là quadratic form của x_a. → $p(x_a|x_b)$ is normal distr

→ So sánh vs quadratic form của norm.

$$\Delta^2 = \frac{-1}{2} x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + c$$

→ $\Sigma_{a|b} = A_{aa}^{-1}$

$\cancel{\mu_{a|b}} = (A_{aa}\mu_a - A_{ab}(x_b - \mu_b)) = \Sigma^{-1}\mu_{a|b}$

→ $\mu_{a|b} = \Sigma_{a|b}(A_{aa}\mu_a - A_{ab}(x_b - \mu_b))$

$= \mu_a - A_{aa}^{-1} A_{ab}(x_b - \mu_b)$

$A_{aa} = (\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}$

$A_{ab} = -(\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}\Sigma_{ab}\Sigma_{bb}^{-1}$

→ $\mu_{a|b} = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$

$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$

→ $p(x_a|x_b) \sim \mathcal{N}(x_{a|b}, \mu_{a|b}, \Sigma_{a|b})$

với đk $p(x_a, x_b) \sim \mathcal{N}$

$p(x_a) \sim \mathcal{N}(x_a, \mu_a, \Sigma_{aa})$

(III. Marginal distr.

$$p(x_a) = \int p(x_a, x_b)\, d(x_b)$$

$$\Delta^2 = \frac{-1}{2}(x_a - \mu_a)^T A_{aa}(x_a - \mu_a) - \frac{1}{2}(x_a - \mu_a)^T A_{ab}(x$$

$$-\frac{1}{2}(x_b - \mu_b)^T A_{ba}(x_a - \mu_a) - \frac{1}{2}(x_b - \mu_b)^T A_{bb}($$