Algorithm for Electric Field for Cathodic Protection Simulation

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We assume that we have a soil, divided into a rectangle of $N \times M$ size (unit m^2). We take an example of 5×4.

A11	A12	A13	A14
A21	A22	A23	A24
A31	A32	A33	A34
A41	A42	A43	A44
A51	A52	A53	A54

Figure 1: 5×4 Soil Rectangle

Our goal would be to calculate the electric field for all the cells of figure 1.

Next, to summarize. Suppose that we put a source in the upper border of A13. Knowing that each cell of Figure 1, contains current sources and/or current sinks, what will be the electric field in each cell ? The problem here is not the computation of the electric field for each cell, but HOW to take into account the variables for each cell. I think that this could be done by calculating the electric potential first for each cell. The problem with this approach, is that it's true only if the sources are static, i.e $conservative\,electric\,field$.

Primary Idea : The simpliest case. (04/05/23)

Imagine that we take only one cell like the one in figure 2, The source is in the top right corner, while A and B are current source and current sink respectively. The charges of the source and the current source A are positive, while the charge of the current sink B is negative.

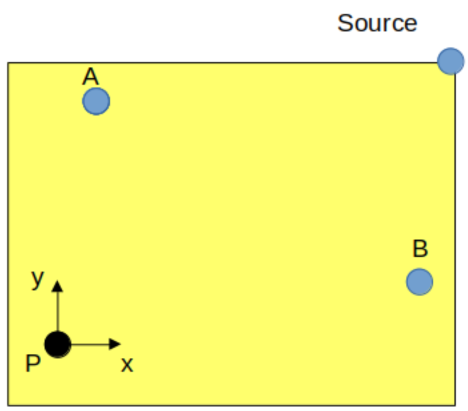

Figure 2: Representation of one cell

We want to estimate the Electric Field at the point P. For this we need first to define the point P as the origin of the our orthonormal and orthogonal basis $\left(O, \vec{i}, \vec{j}\right)$. Next, will with graphically define the electric fields acting on the point P.

Since the Electric fields are vectors, they can by definied in the cartesian coordinates as follow :

$$\vec{E}_S = E_{S_x}\vec{i} + E_{S_y}\vec{j}$$
$$\vec{E}_A = E_{A_x}\vec{i} + E_{A_y}\vec{j}$$
$$\vec{E}_B = E_{B_x}\vec{i} + E_{B_y}\vec{j}$$

The components E_{α_x} and E_{α_y} are the projections on the cartesian plan of the electric field of a source α. Thus we can write

$$\vec{E}_S \rightarrow \begin{cases} E_{S_x} = E_S \cos\alpha \\ E_{S_y} = E_S \sin\alpha \end{cases} \tag{1}$$

$$\vec{E}_A \rightarrow \begin{cases} E_{A_x} = E_A \sin\gamma \\ E_{A_y} = E_A \cos\gamma \end{cases} \tag{2}$$

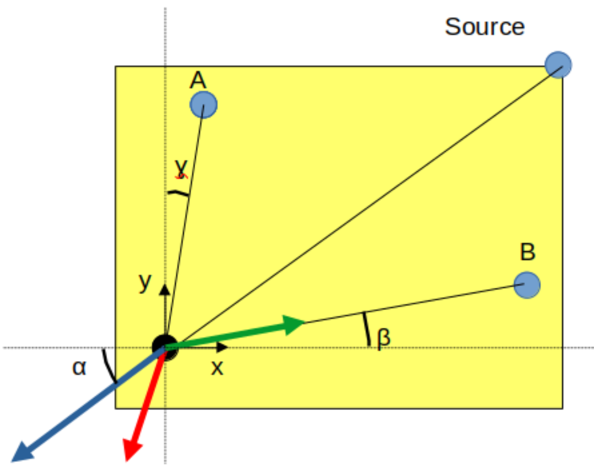

Figure 3: Graphical Representation of the Electric Fields. The blue is the electric field created by the source $\vec{E}_s$, the red is the electric field $\vec{E}_A$ created by the current source A and the green is the electric field $\vec{E}_B$ created by the current sink B.

$$\vec{E}_B \rightarrow \begin{cases} E_{B_x} = E_B \cos\beta \\ E_{B_y} = E_B \sin\beta \end{cases} \tag{3}$$

Therefore the total electric field resulting from $\vec{E}_s$, $\vec{E}_A$ and $\vec{E}_B$ is

$$E_T = E_{t_x}\vec{i} + E_{t_y}\vec{j}, \tag{4}$$

with

$$\begin{cases} E_{T_x} = E_{S_x} + E_{A_x} + E_{B_x} \\ E_{T_y} = E_{S_y} + E_{A_y} + E_{B_y} \end{cases} \tag{5}$$

What next ? The case presented above is as simple as possible. The next step would be to include current density in our calculations, using Gauss's law. The current density will allow us to introduce the soil resistivity, charges other than ponctual charges, the conductivity or any other variable in our cell.

The Gauss's law is defined by

$$\nabla.\vec{E}(\vec{r}) = \frac{\rho}{\epsilon_0}, \tag{6}$$

with ∇ in 2-dimension space is $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. In other way

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{e}_r}{r^2} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(s)\,(\vec{r}-\vec{s})}{|\vec{r}-\vec{s}|^3} d^2s \tag{7}$$

Using the face that the Dirac delta function $\delta(\vec{r})$ is defined by $\delta(\vec{r}) = \frac{1}{4\pi}\nabla.\left(\frac{\vec{r}}{|\vec{r}|^3}\right)$, then eq. 1 could be written as

$$\nabla.\vec{E}(\vec{r}) = \frac{\rho}{\epsilon_0}$$
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{\rho}{\epsilon_0}$$

The charge density ρ, will allow us to introduce the soil resistivity and current source and current sink.

For a known currents

First example : **Soil Resitivity and Source**

We can define the soil resistivity ρ by the Wenner method[1] [1] with $\rho = 2\pi a R$. Then we define the current as $J = \frac{I}{A}$,where I is the intensity and A the cross-section or surface of the soil. Combining the 2 equations leads to

$$\rho = \frac{E}{J} = \frac{E\,A}{I}, \tag{8}$$

I is the intensity and it could be given by $I = n\,|q|\,v_d$, where v_d is the drift velocity. The drift velocity and the n parameter (the density of free charges in the conductor) depends on the type of conductor. The charge density and the the current density are related by the relation $J = \rho v$. For more convinience we change the notation of the charge density from ρ to ψ. Then from the relation $J = \psi v$ and the relation of the soil resistivity we obtain an equation of the charge density of the soil with the needed variable :

$$\psi_{soil} = \frac{E}{\rho v} \tag{9}$$

Now it is important to understand what the electric field in eq. 2 represents. This electric field is the electric field created by the source, current sink B and current source A. Consider a cell without any other source than the external source, let's say a single phase generator G with an output of $120V$ and $50A$. For a soil cell of $1 \times 1m$, the surface is equal to 1, then the current is $J_s = \frac{I_S}{A} = 50A.m^{-2}$. We take as an example $\rho = 29\,\Omega.m$

Then

$$\begin{aligned} E_S &= \rho J_S \\ &= 29 \times 50 \\ &= 1450\,A.\Omega.m^{-1} \\ &= 1450\,V.m^{-1} \end{aligned}$$

[1] Other methods exist, the most interesting one is the electromagnetic measurements.

For a point P, the electric field applied denoted $\vec{E}_S$ is (we took the figure 3 for an example).

$$\vec{E}_S \rightarrow \begin{cases} E_{S_x} = E_S \cos\alpha \\ E_{S_y} = E_S \sin\alpha \end{cases}$$

Suppose that $\alpha = \frac{7\pi}{6}$, then

$$\vec{E}_S \rightarrow \begin{cases} E_{S_x} = E_S \cos\alpha = E_S \times \left(-\frac{\sqrt{3}}{2}\right) \approx -1256\, V.m^{-1} \\ E_{S_y} = E_S \sin\alpha = E_S \times \left(-\frac{1}{2}\right) = -750\, V.m^{-1} \end{cases}$$

Then the components of the electric field are

$$\vec{E}_S = -1256\,\vec{i} - 750\,\vec{j}$$

This last solution is the Electric field in point P created by the source shown in figure 2. Where $1450 V.m^{-1}$ is the strength of the field and $1256 V.m^{-1}$ and $750 V.m^{-1}$ are the components on 2D plane in the cartesian coordinates. This components can give us the direction of the Electric field on the plane, the minus sign is for the projection of the on the $\left(O, \vec{i}, \vec{j}\right)$.

Second example : **Soil Resitivity, Source and Current Source**

For this example we add in our cell a current source A to the single phase generator G (as in the figure 2.). We assume that the the current source outputs are less than the generator G, let's say $80V$ and $15A$. At a point P the electric field created by both of sources are the sum of all the electric charges, thus

$$\vec{E}_p = \vec{E}_S + \vec{E}_A$$

Same as for $\vec{E}_S$, the electric field $\vec{E}_A$ is

$$\begin{aligned} E_A &= \rho J_A \\ &= 29 \times \frac{15}{1} \\ &= 435 V.m^{-1} \end{aligned}$$

From eq.2, with $\gamma = \frac{8\pi}{6}$

$$\vec{E}_A \rightarrow \begin{cases} E_{A_x} = E_A \sin\gamma = E_A \times \left(-\frac{\sqrt{3}}{2}\right) = -376.72 V.m^{-1} \\ E_{A_y} = E_A \cos\gamma = E_A \times \left(-\frac{1}{2}\right) = -217.5 V.m^{-1} \end{cases} \tag{10}$$

To check $\left\|\vec{E}_A\right\| = \sqrt{E_{A_x}^2 + E_{A_y}^2} = \sqrt{(376.72)^2 + (217.5)^2} \approx 435 V.m^{-1}$, with $\vec{E} = -376.72\,\vec{i} - 217.5\,\vec{j}$. The total electric field $\vec{E}_P$ following eq.5 is then given by

$$\begin{cases} E_{P_x} = E_{S_x} + E_{A_x} = -1632V.m^{-1} \\ E_{P_y} = E_{S_y} + E_{A_y} = -967.5V.m^{-1} \end{cases} \tag{11}$$

Then $E_P = \sqrt{(1632)^2 + (967.5)^2} \approx 1897.23V.m^{-1}$, and its directions is

$$\vec{E}_p = -1632\,\vec{i} - 967.5\,\vec{j}$$

Third example : **Soil Resitivity, Source, Current Source and Current Sink**

Now we add a current sink to our cell, let's call it B, and follow the example of figure 2. The current sink outputs are $76V$ and $19A$. Then $J_B = \frac{A}{I} = 19A$.

$$\begin{aligned} E_B &= \rho J_B \\ &= 29 \times 19 \\ &= 551V.m^{-1} \end{aligned}$$

from eq.3 with $\beta = \frac{\pi}{4}$

$$\vec{E}_B \rightarrow \begin{cases} E_{B_x} = E_B \cos\beta = E_B \times \left(\frac{\sqrt{2}}{2}\right) = 389.62V.m^{-1} \\ E_{B_y} = E_B \sin\beta = E_B \times \left(\frac{\sqrt{2}}{2}\right) = 389.62V.m^{-1} \end{cases} \tag{12}$$

and $\vec{E}_B = 389.62\,\vec{i} + 389.62$. Then the electric field at point P created by the 3 current charges are

$$\begin{cases} E_{P_x} = E_{S_x} + E_{A_x} + E_{B_x} = -1242.38V.m^{-1} \\ E_{P_y} = E_{S_y} + E_{A_y} + E_{B_y} = -577.8V.m^{-1} \end{cases}$$

with

$$\vec{E}_P = -1242.38\,\vec{i} - 577.8\,\vec{j}$$

and $\left\|\vec{E}_p\right\| \approx 1370.16V.m^{-1}$.

Conclusion

The electric field $\vec{E}_p$ shows how the total electric field for a known current will be calculated in a cell for a choosen point P. The eqs. 1, 2 and 3 are general relations from which the resulting electric field of a total cell could be calculated. Take as an example the figure 4. The cell 1 is the cell in which the electric field have been calculated for the point P, the task here is to estimate the electric field at the point P' with another current source A' and another current sink B' contained in the new cell, i.e cell 2. To write a more general model, we introduce a new parameter, an old pipe in the soil with a high corrosion. This pipe will play the role of a linear conductor on a surface conductor. The interest behind

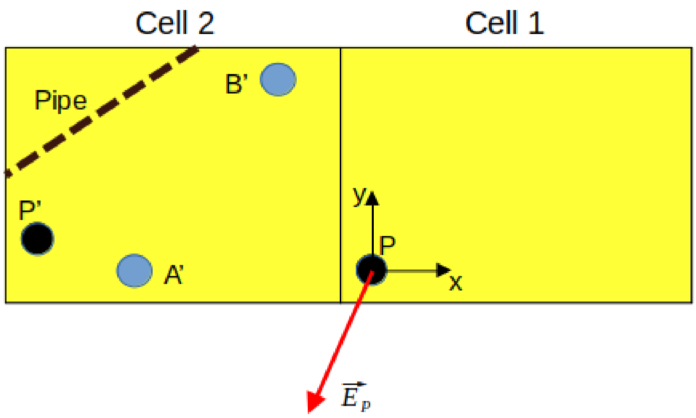

Figure 4: In the right the cell in figure 3, side by side with another cell

the linear conductor, is that he could later directly introduce other parameters, such as a metal frame on the soil, a water source, etc.

Before moving to the 2 cells problem. We generalize the model for 1 cell.

Generalization of the model for 1 cell

We start by calling back the solutions :

$$\vec{E}_T = E_{t_x}\vec{i} + E_{t_y}\vec{j},$$

with

$$\begin{cases} E_{T_x} = E_{S_x} + E_{A_x} + E_{B_x} \\ E_{T_y} = E_{S_y} + E_{A_y} + E_{B_y} \end{cases}$$

For a known current the equations will be written as

$$\vec{E}_T = \vec{E}_S + \vec{E}_A + \vec{E}_B \tag{13}$$

$$\left\|\vec{E}_S\right\| = \rho J_S = \rho\frac{I_S}{A}$$
$$\left\|\vec{E}_A\right\| = \rho J_A = \rho\frac{I_A}{A}$$
$$\left\|\vec{E}_B\right\| = \rho J_B = \rho\frac{I_B}{A}$$

Since we're dealing with continuous current, we can use the E-field/Current density relation $J = \frac{E}{\rho}$. The current density resulting from cell 1 could be written as

$$J_{T_1} = \frac{E_{T_1}}{\rho},$$

where E_{T_1} is the Electrical Field in eq. 13.

Single cell with a pipe

We add a pipe to the single cell with length L and conductivity σ, recall that $\sigma = \frac{1}{\rho}$.

$$V = EL, \quad J = \sigma E$$

then from the last 2 equations

$$E = \frac{V}{L} \rightarrow J = \sigma \frac{V}{L} = \frac{I}{A}$$

$$\frac{I}{A} = \sigma \frac{V}{L} \rightarrow V = \frac{IL}{\sigma A} = IR,$$

with $R = \frac{L}{\sigma A}$. A is the area of the cell and J the current density created by the source, current source and current sink in the cell. The electric potential V is

$$V = k\frac{q}{r}$$

Combining the 2 equations

$$V = IR = k\frac{q}{r}$$
$$I = k\frac{q}{rR}$$
$$I = k\frac{\sigma A q}{rL}$$

New Modelisation

$$V = \frac{IL}{\sigma A} = IR$$

References

[1] P. R. Roberge, Handbook of Corrosion Engineering, McGraw-Hill edition.