APSTA-GE 2351

Topic 0: Calculus Review

Calculus is an essential tool in probability and statistics. To be specific, there are many calculus applications in cumulative density functions (CDF), and in finding the expected value (or mean), variance (or standard deviation), and median of a distribution. Let's review some important concepts on differentiation and integral.

Differentiation (Part I)

1. General Ideas:

   - The derivative of a constant is 0.

   $$\frac{d}{du}c = 0$$

   - The derivative of a sum is the sum of the derivatives.

   $$\frac{d}{du}(f(u) + g(u)) = f'(u) + g'(u)$$

   - The derivative of u to a constant power:

   $$\frac{d}{du}u^n = n * u^{n-1}$$

2. Special Cases:

   - The derivative of e:

   $$\frac{d}{du}e^u = e^u$$

   - The derivative of log:

   $$\frac{d}{du}log(u) = \frac{1}{u}$$

Examples:

1. $\dfrac{d}{dt}(3t+7) =$ ________________ $\dfrac{d}{dt}3t + \dfrac{d}{dt}7 = 3$

2. $\dfrac{d}{dy}5y^3 =$ ________________ $3 * 5y^{3-1} = 15y^2$

3. $\dfrac{d}{dz}e^{4z} =$ ________________ $e^{4z} * 4 = 4e^{4z}$

4. $\dfrac{d}{dw}2log(w) =$ ________________ $\dfrac{2}{w}$

Differentiation (Part II)

1. Useful Rules:

   - Product rule:

   $$\frac{d}{du}(f(u) * g(u)) = f'(u)g(u) + g'(u)f(u)$$

   - Quotient rule:

   $$\frac{d}{du}\left(\frac{f(u)}{g(u)}\right) = \frac{f'(u)g(u) - g'(u)f(u)}{(g(u))^2}$$

   - Chain rule: Do the derivative of the outside and then do the derivative of the inside.

   $$\frac{d}{du}(f(g(u))) = f'(g(u)) * g'(u) * du$$

2. Derivatives of Trigonometric Functions

$$\begin{array}{ll} \frac{d}{du}sin(u) = cos(u)du & \frac{d}{du}cos(u) = -sin(u)du \\ \frac{d}{du}tan(u) = sec^2(u)du & \frac{d}{du}cot(u) = -csc^2(u)du \\ \frac{d}{du}sec(u) = sec(u)tan(u)du & \frac{d}{du}csc(u) = -csc(u)cot(u)du \end{array}$$

Examples:

1. $\dfrac{d}{dx}(5xlog(x)) =$ _______________

   Let $f(x) = 5x$ and $g(x) = log(x)$.

   $$f'(u)g(u) + g'(u)f(u) = 5 * log(x) + 5x * \frac{1}{x} = 5 + 5log(x)$$

2. $\dfrac{d}{du}\left(\dfrac{7u}{e^{3u}}\right) ==$ _______________

   Let $f(u) = 7u$ and $g(u) = e^{3u}$.

   $$\frac{f'(u)g(u) - g'(u)f(u)}{(g(u))^2} = \frac{7 * e^{3u} - 3e^{3u} * 7u}{(e^{3u})^2} = \frac{7e^{3u} - 21ue^{3u}}{e^{6u}}$$

3. $\dfrac{d}{dv}log_e(e^v) =$ _______________ $\dfrac{d}{dv}v = dv = 1$

4. $\dfrac{d}{dv}log_e(v^2e^v) =$ _______________ $\dfrac{d}{dv}log_e(v^2) + \dfrac{d}{dv}log_e(e^v) = \dfrac{2v}{v^2} + \dfrac{e^v}{e^v} = \dfrac{2}{v} + 1$

5. $\dfrac{d}{dv}e^{ve^v} =$ _______________

   Let $f(v) = v$ and $g(v) = e^v$.

   $$e^{ve^v}(ve^v + e^v * 1) = e^{ve^v}(ve^v + e^v)$$

6. $\dfrac{d}{dx}sin(x^2) =$ _______________

   Apply the chain rule to the trigonometric functions.
   Let $f(x) = sin(x^2)$ and $g(x) = x^2$.

   $$\frac{d}{du}(f(g(u))) = f'(g(u)) * g'(u) * du = cos(x^2) * 2x = 2xcos(x^2)$$

Differentiation (Part III)

1. Partial derivatives:
   Consider $f(x,y) = x^2y^3$.

   - $\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x. [Hint: treat y as constant; take derivative.]

   $$\frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial x}x^2y^3 = 2x^{2-1} * y^3 = 2xy^3$$

   - $\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y. [Hint: treat x as constant; take derivative.]

   $$\frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial y}x^2y^3 = x^2 * 3y^{3-1} = 3x^2y^2$$

Examples:

1. Let $f(x,y) = x^2y + e^x + sin(xy)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
   $\frac{\partial f}{\partial x} = 2 * x^{2-1}y + e^x + cos(xy) * y = 2xy + e^x + ycos(xy)$
   $\frac{\partial f}{\partial y} = x^2 + cos(xy) * x = x^2 + xcos(xy)$

2. Let $f(x,y) = 5x^3y^2$. Calculate $\frac{\partial}{\partial x}f(1,2)$.
   $\frac{\partial f}{\partial x} = 5 * 3 * x^{3-1}y^2 = 15x^2y^2$
   $\frac{\partial}{\partial x}f(1,2) = 15 * 1^2 * 2^2 = 60$

Integral (Part I)

1. General Ideas:
   - The integral of a sum is the sum of the integrals.
   $$\int \left(u^n + \frac{1}{u}\right) du = \int u^n du + \int \frac{1}{u} du$$
   - The integral of u^{-1}:
   $$\int \frac{1}{u} du = log(u) + c$$
   - The integral of u to a constant power:
   $$\int u^n du = \frac{1}{n+1} u^{n+1} + c \qquad n \neq -1$$
2. Special Cases:
   - The integral of e:
   $$\int e^u du = e^u + c$$
   - The integral of the derivative:
   $$\int f'(u) du = f(u) + c$$
   - Integral with bounds:
   $$\int_a^b f'(x) dx = f(x)\Big|_a^b = f(b) - f(a)$$

Examples:

1. $\int (a+5) da =$ ______________ $\int a da + \int 5 da$
2. $\int \frac{1}{2t} dt =$ ______________ $\frac{1}{2} log(t) + c$
3. $\int_1^3 y^3 dy ==$ ______________ $\frac{1}{3+1} y^{3+1}\Big|_1^3 = \frac{1}{4} y^4\Big|_1^3$

4. $\int_{-1}^{1} e^t dt =$ ______________ $e^t\Big|_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$

5. $\int_{-0}^{2}(6y^5+3y)dy =$ ______________ $\int_{-0}^{2} 6y^5 dy + \int_{-0}^{2} 3y dy = y^6\Big|_{-0}^{2} + \frac{3}{2}y^2\Big|_{-0}^{2} = 70$

Integral (Part II)

1. Useful Rules:

   - Integration by parts:

$$\int u(x)v(x)dx = u(x)\int v(x)dx - \int u'(x)\left(\int v(x)dx\right)dx$$

   - U-du substitution:

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) = F(g(x))$$

     Let $u = g(x)$ and $du = g'(x)dx$.

   - Double integrals:
     If we want to compute the double integral of $f(x, y)$ over the domain D, where D has boundaries with $a < x < b$ and $c < y < d$, we could estimate the integral:

$$\int\int f(x, y)dA = \int_c^d \left(\int_a^b f(x, y)dx\right)dy$$

2. Integrals of Trigonometric Functions

$\int sin(u)du = -cos(u) + c$	$\int cos(u) = sin(u) + c$
$\int sec^2(u)du = tan(u) + c$	$\int csc^2(u)du = -cot(u) + c$
$\int sec(u)tan(u)du = sec(u) + c$	$\int csc(u)cot(u)du = -csc(u) + c$

Examples:

1. $\int \frac{ln(x)}{x^2} =$ ______________
   Let $u(x) = ln(x)$ and $v(x) = \frac{1}{x^2}$
   Differentiate $u(x)$: $ln(x)' = \frac{1}{x}$
   Integrate $v(x)$: $\int \frac{1}{x^2} = \int x^{-2}dx = -x^{-1} = -\frac{1}{x}$

$u(x)\int v(x)dx - \int u'(x)(\int v(x)dx)dx$
$= ln(x)\frac{-1}{x} - \int \frac{1}{x}(\frac{-1}{x})dx$
$= \frac{-ln(x)}{x} - \int \frac{-1}{x^2}dx = \frac{-(ln(x)+1)}{x} + c$

2. $\int_0^2 -2x\sqrt{4-x^2}dx =$ ______________
   Let $u = 4 - x^2$, then we have $du = -2xdx$.
   $\int_0^2 -2x\sqrt{4-x^2}dx = -\int_{u_1}^{u_2}\sqrt{u}du = (\frac{2}{3}u^{\frac{3}{2}})\Big|_{u_1}^{u_2} = -(\frac{2}{3}(4-x^2)^{\frac{3}{2}}\Big|_0^2 = \frac{-16}{3}$

3. $\int e^{2v}dv =$ ______________ Let $\frac{d}{dv}2v = 2dv$. $\frac{1}{2}\int e^{2v}(2dv) = \frac{1}{2}e^{2v} + c$

4. Compute $\int_0^8\int_0^{16}(12 - \frac{x}{4} - \frac{y}{8})dydx$.

$$\int_0^8\int_0^{16}\left(12 - \frac{x}{4} - \frac{y}{8}\right)dydx$$
$$= \int_0^8\left(\int_0^{16}\left(12 - \frac{x}{4} - \frac{y}{8}\right)dy\right)dx$$
$$= \int_0^8 12y - \frac{xy}{4} - \frac{y^2}{16}\Big|_0^{16}dx$$
$$= \int_0^8(176 - 4x)dx$$
$$= 176x - 2x^2\Big|_0^8$$
$$= 1280$$

5. $\int_0^{\frac{\pi}{2}} cos(x)dx =$ ______________ $sin(x)\Big|_0^{\frac{\pi}{2}} = sin(\frac{\pi}{2}) - sin(0) = 1 - 0 = 1$

Practice:

1. Find the derivative of $f(x) = 4x^5 + 3x^2 + x^{\frac{1}{3}}$.
2. Find the derivative of $f(x) = (x^4 + 3x^2 + 8)cos(x)$.
3. Find the derivative of $f(x) = log(1 - x^2)$.
4. Find the derivative of $f(x) = log(4x) - log(2x)$.
5. Find the derivative of $f(x) = e^{\frac{-x^2}{2}}$.
6. Find the following integral: $\int_0^1(x^2 + 2x + 1)^2dx$.
7. Find the following integral: $\int_0^1\frac{x}{1+x^2}dx$.

8. Find the following integral: $\int_0^2 xcos(3x^2)dx$.

9. Find the following integral: $\int_{-4}^4 (x^3 + 6x^2 - 2x - 3)dx$.

10. Let $f(x, y) = e^{xy} - log(x^2 + y^2)$. Find $\partial f/\partial x$ and $\partial f/\partial y$.

Answers:

1. $f'(x) = 5 * 4x^{5-1} + 2 * 3x^{2-1} + \frac{1}{3} * x^{1-\frac{1}{3}} = 20x^4 + 6x + \frac{1}{3}x^{\frac{-2}{3}}$

2. $f'(x) = (4x^3 + 6x)cos(x) - sin(x)(x^4 + 3x^2 + 8)$

3. $f'(x) = \dfrac{1}{1 - x^2} * (-2x) = \dfrac{-2x}{1 - x^2}$

4. $f(x) = log(\dfrac{4x}{2x}) = log(2)$; $f'(x) = 0$

5. $f'(x) = e^{\frac{-x^2}{2}} * \dfrac{-2x}{2} = -x \cdot e^{\frac{-x^2}{2}}$

6. $\dfrac{1}{5}(x + 1)^5\Big|_0^1 = \dfrac{1}{5}(2^5 - 1) = \frac{31}{5} = 6.2$

7. Let $u = 1 + x^2$ and $du = 2xdx$.

$$\frac{1}{2}\int_1^2 \frac{1}{u}du = \frac{1}{2}log(u)\Big|_1^2 = \frac{log(2)}{2}$$

8. Let $u = 3x^2$ and $du = 6xdx$.

$$\frac{1}{6}\int_0^{12} cos(u)du = \frac{1}{6}sin(u)\Big|_0^{12} = \frac{sin(12)}{6} - \frac{sin(0)}{6} = \frac{sin(12)}{6}$$

9. $\dfrac{1}{4}x^4 + \dfrac{6}{3}x^3 - \dfrac{2}{2}x^2 - 3x\Big|_{-4}^4 = 2 * \left(\dfrac{6}{3} * 4^3 - 3(4)\right) = 232$

10. $\partial f/\partial x = y * e^{xy} - \dfrac{1}{x^2 + y^2} * 2x = ye^{xy} - \dfrac{2x}{x^2 + y^2}$

    $\partial f/\partial y = x * e^{xy} - \dfrac{1}{x^2 + y^2} * 2y = xe^{xy} - \dfrac{2y}{x^2 + y^2}$