Dafni Nikolaidou
10546
dafninikol@ece.auth.gr

Homework Set 1
Advanced Signal Processing

April 23, 2024

1 Exercise 1

$X(k) = W(k) - W(k-1)$, $k = \pm -1, \pm -2, ...$, where W(k) is stationary stochastic with i.i.d. stochastic variables and $E[W(k)] = 0, E[W(k)^2] = 1, E[W(k)^3] = 1$

Covariance of X(k): $c_2^x = m_2^x = E[X(k)X(k+\tau)] = 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1)$ where $\delta(\tau)$ is the delta Kroenecker function; hence,

$$c_2^x(\tau) = \begin{cases} 2 & \text{if } \tau = 0 \\ -1 & \text{if } \tau = -1, \tau = 1 \\ 0 & \text{elsewhere} \end{cases}$$

Power Spectrum: $C_2^x(\omega) = \sum_{\tau=-1}^{1} c_2^x(\tau) e^(- j\omega\tau) = 2 - 2cos(\omega)$

1.1 Computing c_3^x

We compute the 3rd-order cumulants of X(k) as shown.

$c_3^x(\tau1, \tau2) = m_3^x(\tau1, \tau2) - m_1^x[m_2^x(\tau1) + m_2^x(\tau2) + m_2^x(\tau2 - \tau1)] + 2(m_1^x)^3$
$= m_3^x(\tau1, \tau2)$ because $m_1^x = E[X(k)] = E[W(k)] - E[W(k-1)] = 0$

Hence,

$c_3^x(\tau1, \tau2) = m_3^x(\tau1, \tau2) = E(X(k)X(k+\tau1)X(k+\tau2))$
$= E((W(k) - W(k-1))(W(k+\tau1) - W(k+\tau1-1))(W(k+\tau2) - W(k+\tau2-1)))$
$= E(W(k)W(k+\tau1)W(k+\tau2) - W(k)W(k+\tau1)W(k+\tau2-1)$
$- W(k)W(k+\tau1-1)W(k+\tau2) + W(k)W(k+\tau1-1)W(k+\tau2-1)$
$- W(k-1)W(k+\tau1)W(k+\tau2) + W(k-1)W(k+\tau1)W(k+\tau2-1)$
$+ W(k-1)W(k+\tau1-1)W(k+\tau2) - W(k-1)W(k+\tau1-1)W(k+\tau2-1)$

As $W(k), W(k+\tau)$ are independent stochastic processes,
$E(W(k)W(k+\tau)) = E(W(k)E(W(k+\tau))$

Therefore, for all possible combinations of $\tau1$ and $\tau2$

$$\begin{aligned} c_3^x(\tau1, \tau2) &= -\delta(\tau1, \tau2-1) - \delta(\tau1-1, \tau2) \\ &+ \delta(\tau1-1, \tau2-1) - \delta(\tau1+1, \tau2+1) \\ &+ \delta(\tau1+1, \tau2-1) + \delta(\tau1, \tau2+1) \\ &= \begin{cases} -1, & \tau1 = 0, \tau2 = 1 \\ -1, & \tau1 = 1, \tau2 = 0 \\ +1, & \tau1 = 1, \tau2 = 1 \\ -1, & \tau1 = -1, \tau2 = -1 \\ +1, & \tau1 = -1, \tau2 = 0 \\ +1 & \tau1 = 0, \tau2 = -1 \end{cases} \end{aligned}$$

We notice that the values of $c_3^x(\tau 1, \tau 2)$ are anti-symmetrical

1.2 Calculating the skewness γ_3^x

We can find the skewness by observing the 3rd order cumulants in $(\tau 1, \tau 2) = (0, 0)$ as,

$$\gamma_3^x = c_3^x(0, 0) = 0$$

We can, therefore, conclude that the stochastic process X(k) has a probability distribution in which the observations are perfectly symmetrical around the mean. So, $(m_1^x = 0)$ and the observations are symmetrical around (0,0).

1.3 Finding the bispectrum $C_3^x(\omega 1, \omega 2)$

$$\begin{aligned}
C_3^x(\omega 1, \omega 2) &= \sum_{\tau 1=-1}^{\tau 1=+1} \sum_{\tau 2=-1}^{\tau 2=+1} c_3^x(\tau 1, \tau 2) e^{-j(\omega 1 \tau 1 + \omega 2 \tau 2)} \text{ where, } |\omega 1| \leq \pi, |\omega 2| \leq \pi, |\omega 1 + \omega 2| \leq \pi \\
&= e^{-j(\omega 1 + \omega 2)} - e^{-j(-\omega 1 - \omega 2)} \\
&+ e^{j\omega 1} + e^{-j\omega 2} - e^{-j\omega 1} - e^{-j\omega 2} \\
&= -2j(-\sin \omega 1) - 2j(-\sin \omega 2) - 2j(\sin(\omega 1 + \omega 2)) \\
&= 2j(\sin \omega 1 + \sin \omega 2 - \sin(\omega 1 + \omega 2))
\end{aligned}$$

We notice that the bispectrum $C_3^x(\omega 1, \omega 2)$ is imaginary.

1.4 General Comments

If $c_3^x(\tau 1, \tau 2) \neq 0$, apart from the imaginary part of the bispectrum, there would also be a real part in it. We can, therefore, assume that the skewness of X(k) affects the real part of the bispectrum.