MTH 310– Midterm Exam II
Abstract Algebra and Number Theory I

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Name: Solution

This exam contains 8 pages (including this cover page) and 6 questions. You have 50 minutes to complete the exam. Total of points is 100.

No textbooks, lecture notes, cell phones, calculators, or computers are allowed.

Good luck!

Distribution of Points

Question	Points	Score
1	10	
2	18	
3	21	
4	20	
5	10	
6	21	
Total:	100	

1. (10 points) Suppose A and B are two commutative rings with identity and suppose $f : A \to B$ is a surjective homomorphism. Prove that if u is a unit in A, then $f(u)$ is a unit in B.

Proof. Since f is surjective, $f(1) = 1$.

Since u is a unit, $\exists\, u^{-1} \in A$ s.t.

$$uu^{-1} = u^{-1}u = 1.$$

Then

$$f(u)f(u^{-1}) = f(uu^{-1}) = f(1) = 1$$

$$f(u^{-1})f(u) = f(u^{-1}u) = f(1) = 1$$

(Only need one of the above two equalities since B is assumed to be commutative.)

$\Rightarrow$ $f(u^{-1})$ is the multiplicative inverse of $f(u)$.

$\Rightarrow$ $f(u)$ is a unit in B. $\square$

2. Determine whether each of the following polynomials are irreducible. Please write a brief explanation of your reasoning for each of them.

   (a) (6 points) $2x^3 + 4x + 1$ in $\mathbb{Z}_5[x]$.

Let $f(x) := 2x^3 + 4x + 1$.

$f(2) = 2 \cdot 2^5 + 4 \cdot 2 + 1 = 16 + 8 + 1 = 25 = 0$.

$f(4) = f(-1) = 2 \cdot (-1)^3 + 4 \cdot (-1) + 1 = -2 - 4 + 1 = 0$

$\Rightarrow$ f has roots in $\mathbb{Z}_5$. $\Rightarrow$ f is reducible.

   (b) (6 points) $2x^6 - 13x^4 + 39x + 26$ in $\mathbb{Q}[x]$.

Let $p = 13 \leftarrow$ prime

$p \nmid 2$. $p \mid 13, 39, 26$. $p^2 \nmid 26$.

$\Rightarrow$ $2x^6 - 13x^4 + 39x + 26$ is irreducible by Eisenstein's criterion

   (c) (6 points) $x^2 - 2$ in $\mathbb{R}[x]$.

$x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$

reducible.

3. Consider the ring $R = \mathbb{Z}_3[x]/\, x^3 + x + 1$

(a) (4 points) How many elements are there in R?

$3^3 = 27$

(b) (7 points) Compute $(2x + 1)(x^2 + 2)$ in R and express your final answer using a representative of degree less than 3.

$(2x+1)(x^2+2)$

$= 2x^3 + x^2 + 4x + 2$

$= x^2 + 2x$ in R

$$\begin{array}{r|l} & 2 \\ \hline x^3+0x^2+x+1 & 2x^3+x^2+4x+2 \\ & 2x^3+0x^2+2x+2 \\ \hline & x^2+2x \end{array}$$

(c) (5 points) Is R an integral domain? Why or why not?

No. This is because x^3+x+1 is reducible in $\mathbb{Z}_3[x]$

(Note that $1^3+1+1=0$ in $\mathbb{Z}_3$. So x^3+x+1 has a root.)

(d) (5 points) Is $x^3 + x$ a unit in R? Why or why not?

Yes. Since $(x^3+x+1) - (x^3+x) = 1$

$\Rightarrow$ x^3+x+1 and x^3+x are relatively prime.

$\Rightarrow$ x^3+x is a unit in R.

4. Consider the polynomial $f(x) = 9x^7 - 15x^3 + 10x - 1$ in $\mathbb{Q}[x]$.

   (a) (5 points) Please list all possible candidates of roots of f in $\mathbb{Q}$ according to the rational root test.

   Divisors of 9: ± 1, ± 3, ± 9

   Divisors of -1: ± 1

   Possible candidates: $\pm \frac{1}{9}$, $\pm \frac{1}{3}$, ± 1.

   (b) (5 points) Please write down the monic associate of f.

   $$\frac{1}{9}(9x^7 - 15x^3 + 10x - 1)$$

   $$= x^7 - \frac{5}{3}x^3 + \frac{10}{9}x - \frac{1}{9}.$$

   (c) (5 points) Please write down $\overline{f}(x) \in \mathbb{Z}_5[x]$ after reducing mod 5.

   $4x^7 - 1$ (or $4x^7 + 4$).

   (d) (5 points) What is the remainder of f when divided by $x + 1$?

   The remainder Thm tells us that the remainder is

   $$f(-1) = 9\cdot(-1)^7 - 15\cdot(-1)^3 + 10\cdot(-1) - 1$$

   $$= -9 + 15 - 10 - 1$$

   $$= -5.$$

5. (10 points) Suppose R is a commutative ring and suppose I is an ideal in R. Prove that

$$I \times I := \{(a, b) \mid a, b \in I\}$$

is an ideal in $R \times R$.

Proof.

(Non-empty)

Since $0 \in I$, $(0,0) \in I \times I$

This shows that $I \times I$ is non-empty.

(closed under $-$)

Let $(a,b), (c,d) \in I \times I$.

Then $(a,b) - (c,d) = (a-c, b-d)$.

Since I is an ideal, $a-c, b-d \in I$.

$\Rightarrow (a-c, b-d) \in I \times I$.

(Absorbing property under $\cdot$)

$\forall (r,s) \in R \times R$. $\forall (a,b) \in I \times I$.

$(r,s)(a,b) = (ra, sb)$

Since I is an ideal, $ra, sb \in I$.

$\Rightarrow (ra, sb) \in I \times I$ $\square$

6. (21 points) Please determine whether the following statements are true or false based on your best judgement. In the parentheses in front of each statement, write "T" if you think it is true, and write "F" if you think it is false. NO justification is needed.

   (a) (T) For elements $f, g, h \in \mathbb{Q}[x]$, if $f|g$ and $g|h$, then $f|h$.

   (b) (T) A ring isomorphism is a bijective ring homomorphism.

   (c) (F) The Chinese remainder theorem implies that $\mathbb{Z}_{48} \cong \mathbb{Z}_6 \times \mathbb{Z}_8$.

   (d) (F) Let $\overline{f}$ be the reduction of a polynomial $f \in \mathbb{Z}[x]$ mod some prime p. If $\overline{f}$ is reducible in $\mathbb{Z}_p[x]$, then f is also reducible.

   (e) (T) Suppose R is a commutative ring. If $\deg(fg) < \deg(f) + \deg(g)$ for some non-zero polynomials $f, g \in R[x]$, then R is not an integral domain.

   (f) (T) If $f, g \in \mathbb{R}[x]$ induce the same function on $\mathbb{R}$, then $f = g$ in $\mathbb{R}[x]$.

   (g) (F) If a polynomial $f \in \mathbb{Q}[x]$ is irreducible in $\mathbb{Q}[x]$, then it is also irreducible in $\mathbb{R}[x]$.

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