RESEARCH STATEMENT

DAPING WENG

My research mainly focuses on cluster algebras and their applications in geometry, representation theory, and mathematical physics.

In the past, I have studied cluster structures and constructed Donaldson-Thomas transformations on various moduli spaces such as Grassmannians [Wen21], double Bruhat cells [Wen20], and double Bott-Samelson cells [SW21]. I also have applied techniques from cluster theory to construct a canonical basis for Grassmannians [SW20] and to give a new geometric proof of Zamolodchikov's periodicity conjecture [SW21].

In recent years, I started to explore a new connection between cluster theory and Legendrian links in contact topology. Below are some results I have obtained so far.

- In [GSW20a] we established a connection between fillings and clusters using Floer theory. Consequently, we applied cluster techniques to obtain an infinitely-many-filling result for rainbow closures of positive braids in [GSW20b].
- In [CW22] we constructed quasi-cluster ensemble structures on the Kashiwara-Schapira moduli spaces of sheaves and gave a new geometric interpretation of cluster variables as microlocal merodromies. This new approach also extended the connection between fillings and clusters beyond rainbow closures of positive braids.

In the near future, part of my research focus is devoted to deepening the connection between cluster theory and contact topology. I would like to introduce Legendrian weaves [CZ22] to other objects of study in cluster theory and investigate their combinatorics and geometry through the lens of contact topology.

- In joint work with R. Casals and I. Le, we discover a recurrence among reduced plabic graphs, whose union naturally produces a Legendrian weave for the corresponding Grassmannian positroid cell. Furthermore, we develop a recipe to turn this Legendrian weave into a spectral network, enabling us to relate fillings of Legendrian links with spectral curves and Stokes phenomenon [GMN13].
- In joint work with R. Casals, H. Gao, and L. Shen, we consider the Fock-Goncharov cluster ensemble structure on higher Teichmüller spaces [FG06, GS19] and associate a Legendrian weave with any N-triangulation of the surface. This also gives rise to new construction of fillings for satellites of links and allows us to relate the Goncharov-Kontsevich non-commutative cluster structures [GK21] with non-commutative augmentation varieties.

Besides the ongoing projects above, I also plan to study a couple of quantization problems related to various topics in cluster theory, contact topology, and representation theory.

- In joint work with R. Casals, H. Gao, L. Ng, L. Shen, and E. Zaslow, we aim to quantize the commutative Chekanov-Eliashberg dga via Floer-theoretical means and prove that the quantization coincides with the cluster quantization in the case where the latter exists. We also plan to investigate the connection between our Floer-theoretical solution and the Poisson-CGL extension method of constructing cluster algebras described in [GY18].

- In joint work with L. Shen and Z. Sun, we study the $\mathfrak{sl}_n$-skein algebra on punctured surfaces and its extension obtained by including the Goncharov-Shen potentials at punctures as invertible elements. We plan to embed the quantum cluster algebra into this extension and use $\mathfrak{sl}_n$-webs to parametrize a canonical basis of the quantum cluster algebra.

1. Cluster Varieties and Donaldson-Thomas Transformation

Cluster algebras were introduced by Fomin and Zelevinsky [FZ99] in the early 2000s. Cluster varieties were introduced by Fock and Goncharov [FG09] soon after as a geometric enrichment of cluster algebras. Cluster varieties come in dual pairs $(\mathcal{A}, \mathcal{X})$ called *cluster ensembles* and they are conjectured to satisfy the Fock-Goncharov cluster duality. Cluster duality conjecture is about the existence of a canonical basis parametrized by tropical integer points of the dual space. One key step in verifying the cluster duality for a particular cluster ensemble is the construction of its cluster Donaldson-Thomas (DT) transformation [GHKK18, GS18], which is a biregular automorphism that encodes the DT-invariant of certain 3d Calabi-Yau category [Kel17].

Theorem 1. *We have constructed cluster DT transformation in the following cases.*
- [Wen21] *Grassmannians* $\mathrm{Gr}_{m,n}$.
- [Wen20] *Double Bruhat cells, which are intersections of opposite Bruhat cells.*
- [SW21] *Double Bott Samelson cells, which are configuration spaces of flags in Kac-Moody groups satisfying relative position conditions imposed by a pair of positive braids.*

In the case of Grassmannian $\mathrm{Gr}_{m,n}$, we constructed its dual cluster $\mathcal{X}$-variety as a configuration space of n lines in $\mathbb{C}^m$ related by isomorphisms, together with a superpotential function that is dual to an anticanonical divisor of $\mathrm{Gr}_{m,n}$ [SW20]. This cluster dual is equivalent to the mirror Landau-Ginzburg model considered in [EHX97, MR20, RW19]. In addition, we also gave an explicit bijection between the tropical integer points of this cluster dual and the set of plane partitions. As an application, we proved the following cyclic sieving phenomenon.

Theorem 2 (Shen-W. [SW20])**.** *The cyclic action on Grassmannian induces a toggling sequence on plane partitions. The number of fixed points under this toggling sequence can be computed by evaluating the MacMahon function at roots of unity.*

During our study on double Bott-Samelson cells in [SW21], we related the Zamolodchikov mutation sequence [FZ03] to a cyclic action on double Bott-Samelson cells, which in turn gave a new geometric proof of the Zamolodchikov periodicity conjecture in the case of $\Delta \times A_n$ for any Dynkin diagram Δ.

We also described a Deodhar stratification of double Bott-Samelson cells in [SW21], which enabled us to develop an algorithm to count their $\mathbb{F}_q$-points. Under an isomorphism between double Bott-Samelson cells of SL_n and augmentation varieties of rainbow closures of positive braids, the Deodhar stratification on the former coincides with the ruling stratification of the latter, and hence their $\mathbb{F}_q$-point count, in turn, gives the leading coefficient of the HOMFLY-PT polynomials of rainbow closures of positive braids. [Rut06, STZ17].

2. Cluster Theory for Legendrian Links

A Legendrian link is a link in $\mathbb{R}^3$ satisfying some specific tangent condition given by the standard contact structure on $\mathbb{R}^3$. One important invariant of a Legendrian link is the Chekanov-Eliashberg (CE) dga $\mathcal{A}_\Lambda$ [Che02]. By viewing the standard contact $\mathbb{R}^3$ as horizontal slices of a symplectic $\mathbb{R}^4$, one can consider exact Lagrangian cobordisms between Legendrian links. An exact Lagrangian cobordism $L : \Lambda_- \to \Lambda_+$ gives rise to a functorial morphism $\Phi_L^* : \mathcal{A}_{\Lambda_+} \to \mathcal{A}_{\Lambda_-}$ [EHK16].

The augmentation variety Aug_Λ is defined to be the moduli space of augmentations of the CE dga $\mathcal{A}_\Lambda$. In [GSW20a], we equipped the augmentation variety Aug_Λ with a cluster $\mathcal{A}$-structure for any rainbow closure Λ of a positive braid, and proved that the functorial morphism $\Phi_L^* : \mathcal{A}_{\Lambda_+} \to \mathcal{A}_{\Lambda_-}$ induces a cluster localization on augmentation varieties.[1] As a corollary, we deduced that many decomposable exact Lagrangian fillings (cobordisms from the empty set) give rise to clusters on augmentation varieties, and hence clusters can be used to distinguish non-Hamiltonian isotopic exact Lagrangian fillings.

Furthermore, by relating the cluster DT transformation on augmentation varieties with a generalized version of the Kálmán loop, we proved the following infinitely-many-filling result.

Theorem 3 (Gao-Shen-W. [GSW20b])**.** *With the exception of positive braids of finite type, all rainbow closures of positive braids admit infinitely many non-Hamiltonian isotopic fillings.*

Augmentation varieties are known to be closely related to another important invariant of Legendrian links, the Kashiwara-Shapira sheaf moduli spaces Sh^1_Λ [KS94, GKS12, STZ17, NRSSZ15]. Cluster $\mathcal{X}$-structures have been observed on the sheaf moduli spaces of some Legendrian links [STWZ19, SW21, CW22, CGG$^+$22]. In all known cases, these cluster structures ultimately come from open embeddings of the form $\Phi_L : \mathrm{Loc}_1(L) \to \mathrm{Sh}^1_\Lambda$, where $\mathrm{Loc}_1(L) \cong H^1(L, \mathbb{C}^\times)$ denotes the moduli space of rank-1 local systems on an exact Lagrangian filling L. In [CW22], we laid down a set of principles on how to capture these cluster structures through symplectic means.

Theorem 4 (Casals-W. [CW22])**.** *For a cluster that comes from a filling L:*

- *The vertices of the quiver are in bijection with special 1-cycles on L.*
- *The arrows in the quiver are given by intersection pairings among the special 1-cycles.*
- *The $\mathcal{X}$-coordinates are given by microlocal monodromies along these special 1-cycles.*
- *The $\mathcal{A}$-coordinates are given by microlocal merodromies (parallel transport) along relative 1-cycles dual to these special 1-cycles under the Poincaré duality on L.*
- *A mutation at a mutable quiver vertex corresponds to performing a Polterovich surgery* [Pol91] *along the corresponding 1-cycle on L.*

The geometry of some fillings L and their special 1-cycles can be combinatorialized using *Legendrian weaves* [CZ22]. A Legendrian weave is a graph with colored edges on the unit disk $\mathbb{D}$ describing the singular locus of the front projection of the Legendrian lift Σ of L.

For example, any 3-regular tree $\mathfrak{w}$ with n outgoing edges defines a Legendrian weave for a $(2, n-2)$-torus link. The corresponding front projection is a $2:1$ cover of $\mathbb{D}$, branching along the edges of $\mathfrak{w}$, with trivalent vertices being D_4^- singularities [Arn90]. Moreover, each internal edge defines a 1-cycle on the filling L, and these are the special 1-cycles that correspond to quiver vertices. A mutation (Polterovich surgery) taking place at any such 1-cycle corresponds to a contraction-expansion move on the internal edge of $\mathfrak{w}$. On the other hand, the dual graph of $\mathfrak{w}$ is a triangulation of an n-gon, whose diagonals can be seen as relative 1-cycles on L dual to the internal edges of $\mathfrak{w}$. Moreover, it is not hard to see that

[1] I have written a javascript program to implement such a computation explicitly on my personal website.

a mutation along an internal edge of $\mathfrak{w}$ induces a diagonal flip of the triangulation. It is well-known that triangulations of an n-gon describe the cluster structure of Dynkin type A_{n-3}. Thus, this example shows that the cluster ensemble of a $(2, n-2)$-torus link is of Dynkin type A_{n-3}.

FIGURE 1. Left: a Legendrian weave and its quiver. Right: a weave mutation.

3. Applications of Legendrian Weaves in Cluster Theory

Theorem 4 not only reveals the cluster nature of fillings of Legendrian links but also builds a new bridge between Legendrian links and cluster theory. A few of my ongoing projects are attempts to bring techniques and constructions from one side to the other across this bridge.

3.1. **Positroids.** Positroids were first introduced by Postnikov [Pos06] as strata of the totally non-negative Grassmannian. Positroids are parametrized by decorated permutations, and their cluster structures can be captured by reduced planar bicolor (plabic) graphs [GL19]. In a joint work-in-progress with R. Casals and I. Le, we discover an interesting recursion among reduced plabic graphs: roughly speaking, it takes in a reduced plabic graph $\mathbb{G}$ with decorated permutation π and outputs a reduced plabic graph $\mathbb{G}'$ with decorated permutation $c\circ\pi$ where c is a cyclic permutation that maps i to $i-1$ modulo n. This recursion seems to be closely related to the T-duality in the study of amplitudehedron [LPW20], and we would like to further investigate on this observation.

On the other hand, for a positroid in a Grassmannian $\mathrm{Gr}_{m,n}$, we show that if we fix a reduced plabic graph $\mathbb{G}_m$ and define $\mathbb{G}_k := \mathbb{G}'_{k+1}$ using the recursion, then the recursion terminates exactly at $\mathbb{G}_1$, and the union $\bigcup_{k=1}^{m-1}\mathbb{G}_k$ (with appropriate coloring) is a Legendrian weave $\mathfrak{w}$. We call such weaves *positroid weaves*. We show that equivalent reduced plabic graphs give rise to equivalent positroid weaves and a square move on a reduced plabic graph corresponds to a weave mutation (right picture in Figure 1). By combining this construction with Theorem 4 we can now interpret Plücker coordinates geometrically as microlocal merodromies. Moreover, since positroid weaves allow us to mutate at quiver vertices that do not correspond to square faces in a plabic graph, we can extend this new geometric interpretation to cluster variables beyond Plücker coordinates as well. This new formalism will greatly increase the accessibility of cluster structures on positroids.

3.2. **Spectral Networks.** Spectral networks were introduced by Gaiotto, Moore, and Neitzke as a combinatorial tool to capture the geometry of an $\mathcal{N}=2$ supersymmetric theory [GMN13]. Similar to Legendrian weaves, spectral networks also describe branched covers of the base surface, which can be viewed as spectral curves in the canonical bundle (whereas Legendrian weaves describe Lagrangian surfaces in the real cotangent bundle). Spectral networks are closely related to Stokes phenomenon of flat connections on Higgs bundles. Spectral networks are also conjectured to be describing certain cluster structures, with cluster $\mathcal{X}$-variables being monodromies along special 1-cycles. Moreover, spectral networks come in families $\{\mathcal{W}_\theta\}$ parametrized by $\theta\in S^1$, and it is conjectured that rotation in θ is related to the DT transformation of the corresponding cluster structure. However, only spectral networks of rank 2 are relatively well-understood so far, and very few examples of spectral networks of rank higher than 2 have been constructed.

On the other hand, Casals, Gorsky, Gorsky, and Simental introduced a special family of Legendrian weaves called *Demazure weaves*, which are modeled on the Demazure product on the braid group [CGGS20]. In particular, edges in a Demazure weave are oriented in a way such that the Demazure weave can be arranged into a position where all edges are pointing downward. In a joint work-in-progress with R. Casals and I. Le, we develop a recipe to construct a spectral network from a Demazure weave, which unifies the monodromy interpretations of cluster $\mathcal{X}$-variables in both spectral networks and Legendrian weaves. Moreover, by using Postnikov's perfect orientation on reduced plabic graphs, we show that all positroid weaves are Demazure weaves, from which we can produce an abundant amount of new spectral networks that were not previously known to exist.

3.3. **Higher Teichmüller Spaces.** Higher Teichmüller spaces were introduced by Fock and Goncharov as a moduli space of decorated G-local systems on a surface Σ [FG06]. Goncharov and Shen [GS19] proved that higher Teichmüller spaces form cluster ensembles, whose cluster structures can be described by ideal triangulations of Σ. Goncharov and Kontsevich [GK21] further generalized the cluster structure in the case of GL_n to be defined over a non-commutative ring.

In a joint work-in-progress with R. Casals, H. Gao, L. Ng, L. Shen, and E. Zaslow, we adopt the Legendrian weave approach of cluster structures to higher Teichmüller spaces and give a construction of Legendrian weaves from ideal triangulations. Like the case of positroids, this new approach permits more mutations and can give us access to many more cluster seeds in the structure. On the other hand, the Legendrian weave approach also sheds new light on contact geometry: from a Legendrian weave on Σ we obtain an exact Lagrangian surface L on $T^*\Sigma$ which fills a Legendrian link in $J^1(\partial\Sigma)$. If we identify Σ with an exact Lagrangian filling of a Legendrian link Λ (which is topologically $\partial\Sigma$), then by the neighborhood theorem, we can lift L to an exact Lagrangian in a neighborhood of Σ, and it fills a satellite of Λ defined by some braid data along $\partial\Sigma$. This allows us to study the relationship between the moduli space of microlocal sheaves of higher rank on Λ and the moduli space of microlocal sheaves of rank 1 on a satellite of Λ.

Another interesting topic of study is the non-commutative augmentation variety of the Chekanov-Eliashberg (CE) dga. In the commutative case, it is known that given a filling L, a Reeb chord (generator of the CE dga) gives rise to a collection of special relative 1-cycles L whose microlocal merodromies define the value of its augementation. By identifying homotopy classes of paths on a Legendrian weave corresponding to Goncharov and Kontsevich's generators of their non-commutative cluster structure in [GK21], we hope to obtain a non-commutative cluster structure on the non-commutative augmentation variety as well.

4. Quantizations of Cluster-Related Algebras

It is known that cluster $\mathcal{X}$-varieties admit a canonical quantization [FG09], which can be lifted to cluster algebras [BZ14]. Many commutative algebras with geometric origins are closely related to cluster algebras, and hence it is natural to ask whether they admit a compatible quantization as well.

4.1. **Quantization of Chekanov-Eliashberg DGA's.**

In the case of rainbow closures of positive braids, the coordinate ring of an augmentation variety is $H_0(\mathcal{A}_\Lambda)$, the 0th homology of the CE dga. As a result of our previous work [GSW20a], $H_0(\mathcal{A}_\Lambda)$ is a cluster algebra and admits a holomorphic symplectic structure. In a joint work-in-progress with R. Casals, H. Gao, L. Ng, L. Shen, and E. Zaslow, we show that this holomorphic symplectic structure coincides with the symplectic structure from Lie theory [CGGS20] and another symplectic structure from Sabloff duality [Sab06]. As suggested by cluster theory, this holomorphic symplectic structure on $H_0(\mathcal{A}_\Lambda)$ can be quantized. Thus, our next goal of the project is to extend this quantization to the whole commutative CE dga $\mathcal{A}_\Lambda$ in general, obtaining a $\mathbb{C}[q^{\pm 1}]$ algebra $\mathcal{A}_\Lambda^q$ whose semi-classical limit recovers the Poisson algebra structure on $\mathcal{A}_\Lambda$ defined by Sabloff duality.

One way to visualize the Sabloff duality is to perform a push off on the Legendrian link, and the Poisson bracket between Reeb chords (generators of the CE dga) can be computed Floer theoretically by counting immersed holomorphic disks. Based on our preliminary computation, we observe that these disk counting can be divided into two families, and they miraculously match up with the two terms in Goodearl and Yakimov's Poisson-CGL extension formula [GY18]. We would like to explore this correspondence further and hope to provide a more geometric interpretation of the Poisson-CGL extension method of constructing of cluster algebras and their quantization.

4.2. **Skein Algebras on Punctured Surfaces.**

Skein algebras (or $\mathfrak{sl}_2$-skein algebras) first appeared in knot theory as a $\mathbb{Z}[q^{\pm 1}]$-algebra of strand diagrams modulo the skein relation. Since then, generalizations of skein algebras have been made by replacing $\mathfrak{sl}_2$ with other Lie algebras [Kup96, CKM14] and by replacing disk as the base surface with other surfaces in the cases of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ [RY14, Mul16, IY21]. In particular, the Roger-Yang $\mathfrak{sl}_2$-skein algebra introduced a formal invertible element for each puncture on the surface, and we observe that this invertible element can be interpreted as the Goncharov-Shen potential for decorated SL_2-local systems [GS15]. Thus, in a joint work-in-progress with L. Shen and Z. Sun, we plan to generalize the Roger-Yang approach to $\mathfrak{sl}_n$-skein algebras on punctured surfaces by including the $n-1$ Goncharov-Shen potentials (one for each fundamental weight) at each puncture as invertible elements. So far we have successfully produced a full list of relations for the $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$ cases, and our next goal is to design a recursive construction that would solve the $\mathfrak{sl}_n$ case in general. We also expect an embedding of the quantum PGL_n-Teichmüller space into the $\mathfrak{sl}_n$-skein algebra as a subalgebra, which will greatly increase the accessibility of elements of the former.

Generators of the $\mathfrak{sl}_n$-skein algebra on a surface are called $\mathfrak{sl}_n$-webs. Modeling on cluster ensembles, we define two subfamilies of $\mathfrak{sl}_n$-webs called $\mathcal{A}$-webs and $\mathcal{X}$-webs. Following the cluster duality conjecture, one expects the $\mathcal{A}$-webs to parameterize a canonical basis of the PGL_n-Teichmüller space. In the case of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, this conjecture has been verified [FG06, DS20, Kim20], but we would like to generalize this further to the $\mathfrak{sl}_n$ case. We also observe that there is a well-defined intersection number between an $\mathcal{A}$-web and an $\mathcal{X}$-web, and this intersection number resembles the Fock-Goncharov duality pairing map $\mathbb{I}$ [FG09]. We would like to further investigate this connection and hope that it would give us a better understanding of the geometry hidden behind cluster duality.

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