Advanced Algorithms Homework # 3

Due May 18 before class.

Collaboration policy: You can discuss the problem with other students, but you must obtain and write the final solution by yourself. Please specify all of your collaborators (name and student id) for each question. If you solve some problems by yourself, please also specify "no collaborators". Homeworks without collaborator specification will not be graded.

Problem 1 (10%)

Prove that MAX-SAT has integrality gap $\frac{3}{4}$ by constructing an example.

Problem 2 (20%)

In the class, we described an LP relaxation for MAX-SAT. Let the optimal fractional solution be (y^*, z^*). If we set variable x_i to true with probability $4^{y_i^*-1}$, prove that the expected number of satisfied clauses is at least $\frac{3}{4}\sum z_c^*$. (hint: $1-4^{-y} \leq 4^{y-1}$ for all $0 \leq y \leq 1$.)

Problem 3 (15%)

Given a graph $G = (V, E)$. Our goal is to color the vertices into k different colors such that the number of edges in $S = \{(u, v) | u \text{ and } v \text{ have different colors}\}$ is maximized. Design a randomized $(1 - \frac{1}{k})$-approximation algorithm. Derandomize the algorithm into a deterministic algorithm with the same approximation guarantee. Briefly justify the correctness of your algorithm and analyze the running time.

Problem 4 (20%)

Consider the following perfect hashing scheme. Assuming that we need to perform n insertions, but do not know what n is in advance. Starting from two empty hash tables of size $O(1)$. When the i-th insertion arrives, find the corresponding slots in both tables and do the following:

1. If only one of the slots is currently empty, insert the item into the empty slot.
2. If both slots are empty, insert the item into either slot randomly (with equal probability).
3. If collisions happen on both slots, rebuild two new hash tables of sizes $\lceil 4 \cdot i^{1.5} \rceil$ each and move all previous items to the new tables according to steps 1 and 2. Repeat the rebuild process with the same table size $\lceil 4 \cdot i^{1.5} \rceil$ until there is no collision for the first i items.

Supposed that all hash functions are chosen independently from universal sets of hash functions, rebuilding hash tables has 0 cost, and inserting/moving any item has cost 1. Prove that the expected total cost for n insertions is $O(n)$.

Problem 5 (15%)

In the consistent hashing scheme described in class. Prove that the size of the smallest interval is less than $\frac{1}{n^2}$ with probability $\Omega(1)$.