Unit 2 Homework: Characterizing Random Variables

w203: Statistics for Data Science

1. Processing Pasta

A certain manufacturing process creates pieces of pasta that vary by length. Suppose that the length of a particular piece, L, is a continuous random variable with the following probability density function.

$$f(l) = \begin{cases} 0, & l \leq 0 \\ c \cdot l, & 0 < l \leq 2 \\ 0, & 2 < l \end{cases}$$

Where c is a constant.

a. Compute the constant c.
b. Write down a complete expression for the cumulative probability function of L.
c. Compute the median value of L. That is, compute l such that $P(L \leq l) = 1/2$.

Answer a:

$f(l) = \int_0^2 cf(l)\, dl$

2. Broken Rulers

You have a ruler of length 2 and you choose a place to break it using a uniform probability distribution. Let random variable X represent the length of the left piece of the ruler. X is distributed uniformly in $[0, 2]$. You take the left piece of the ruler and once again choose a place to break it using a uniform probability distribution. Let random variable Y be the length of the left piece from the second break.

a. Draw a picture of the region in the X-Y plane for which the joint density of X and Y is non-zero.

b. Compute the joint density function for X and Y. (As always, make sure you write a complete expression.)

c. Compute the marginal probability density for Y, $f_Y(y)$.

d. Compute the conditional probability density of X, conditional on $Y = y$, $f_{X|Y}(x|y)$. (Make sure you state the values of y for which this exists.)

3. Post-Processing and Independence

Suppose X and Y are discrete random variables, and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions. Prove that if X and Y are independent then $f(X)$ and $g(Y)$ are independent.

Challenge Exercise: Characterizing a Function of a Random Variable

Let X be a continuous random variable with probability density function $f(x)$, and let h be an invertible function where h^{-1} is differentiable. Recall that $Y = h(X)$ is itself a continuous random variable.

(Bonus + 3 points) Prove that the probability density function of Y is

$$g(y) = f(h^{-1}(y)) \cdot \left| \frac{d}{dy} h^{-1}(y) \right|$$

Note: The Homework Maximum is 100%