# Práctica 1. Dynamic model of a robot manipulator with the Lagrange method.

Consider the robot manipulator represented in Figure 1, which moves in a vertical plane.

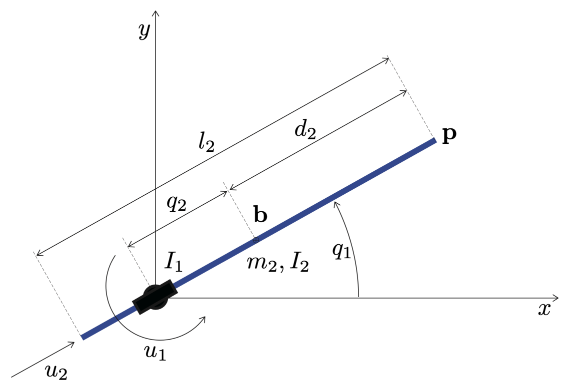


Figure 1: Planar robot manipulator that moves in a vertical plane.

The dynamic model of this robotic system is represented by the second order differential equation

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{N}(\mathbf{q}) = \mathbf{u},$$

where the two matrices $\mathbf{B}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ have the following expressions

$$\begin{aligned}
\mathbf{B}(\mathbf{q}) &= \begin{bmatrix} I_1 + I_2 + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix}, \\
\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) &= \begin{bmatrix} 2m_2 q_2 \dot{q}_1 \dot{q}_2 \\ -m_2 q_2 \dot{q}_1^2 \end{bmatrix}, \\
\mathbf{N}(\mathbf{q}) &= m_2 g \begin{bmatrix} q_2 \cos q_1 \\ \sin q_1 \end{bmatrix}.
\end{aligned}$$

$\mathbf{q} = (q_1, q_2)^T$ is the vector of configuration variables, where $q_1$ is the angular position of the link with respect to the $x$ axis of the reference frame $\{x, y\}$ and $q_2$ is the linear position of the center of mass $\mathbf{b}$ of the link with respect to the origin of the reference frame. The vector $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2)^T$ is the vector of joint velocities, where $\dot{q}_1$ is an angular velocity and $\dot{q}_2$ is a linear velovity. The vector $\ddot{\mathbf{q}} = (\ddot{q}_1, \ddot{q}_2)^T$ is the vector of accelerations, where $\ddot{q}_1$ is an angular acceleration and $\ddot{q}_2$ is a linear acceleration. The control inputs of the system are $\mathbf{u} = (u_1, u_2)^T$, where $u_1$ is the torque applied by the angular actuator to the link and $u_2$ is the force applied by the linear actuator to the link. $I_1$ is the barycentric moment of inertia of the angular and linear actuators, $I_2$ is the barycentric moment of inertia of the link, and $m_2$ is the mass of the link.

a. Demonstrate the equations of the dynamic model using the Lagrange method.

b. Compute the state space representation of the dynamics of the manipulator in which $\mathbf{x} = (x_1, x_2, x_3, x_4)^T = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$. Take the coordinates of point $\mathbf{p}$ as the output variables.

Write a detailed report answering each question in a different section. Originality and completeness of the answers will be the aspects that will be taken into account in the grading of the report.

## Solution of Práctica 1

a. Demonstrate the equations of the dynamic model using the Lagrange method.

Mediante el metodo de Lagrange hallamos las ecuaciones de estado del modelo dinamico, sabiendo que: L = T - V

La energía cinética T se descompone en la suma de la energía cinética angular y lineal:

$T_{ang} = \frac{1}{2}(I_1 + I_2)\dot{q_1}^2$,

$T_{lin} = \frac{1}{2}m_2\dot{q_1}^2 q_2^2 + \frac{1}{2}m_2\dot{q_2}^2$

La energía potencial es $V = m_2gh$, siendo $h = q_{2y} = sen(q_1)q_2$.

Por tanto,

$L = \frac{1}{2}(I_1 + I_2 + m_2q_2^2)\dot{q_1}^2 + \frac{1}{2}m_2\dot{q_2}^2 - m_2gsen(q_1)q_2$

De este modo, obtenemos las dos siguientes ecuaciones de Lagrange,
la primera:

$\frac{\partial L}{\partial \dot{q_1}} = (I_1 + I_2)\dot{q_1} + m_2q_2^2\dot{q_1}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} = (I_1 + I_2 + m_2q_2^2)\ddot{q_1} + 2mq_1\dot{q_1}\dot{q_2}$

$\frac{\partial L}{\partial q_1} = -m_2gcos(q_1)q_2$

$(I_1 + I_2 + m_2q_2^2)\ddot{q_1} + 2mq_2\dot{q_1}\dot{q_2} + m_2gcos(q_1)q_2 = u_1$

y la segunda:

$\frac{\partial L}{\partial \dot{q_2}} = m_2\dot{q_2}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_2}} = m_2\ddot{q_2}$

$\frac{\partial L}{\partial q_2} = m_2q_2\dot{q_1}^2 - m_2gsen(q_1)$

$m_2\ddot{q_2} - m_2q_2\dot{q_1}^2 + m_2gsen(q_1) = u_2$

Por lo tanto, escribiendo las ecuaciones en forma matricial obtenemos:

$$\begin{bmatrix} I_1 + I_2 + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{q} + \begin{bmatrix} 2m_2 q_2 \dot{q}_1 \dot{q}_2 \\ -m_2 q_2 \dot{q}_1^2 \end{bmatrix} + \begin{bmatrix} m_2 g cos(q_1) q_2 \\ m_2 g sen(q_1) \end{bmatrix} = u$$

Finalmente,

$$\ddot{q} = \begin{pmatrix} \frac{2\dot{q}_1 \dot{q}_2 m_2 q_2 - u_1 + g m_2 q_2 cos(q_1))}{m_2 q_2^2 + I_1 + I_2} \\ \frac{m_2 q_2 \dot{q_1}^2 + u_2 - g m_2 sin(q_1)}{m_2} \end{pmatrix}$$

b. Compute the state space representation of the dynamics of the manipulator in which $\mathbf{x} = (x_1, x_2, x_3, x_4)^T = (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$. Take the coordinates of point $\mathbf{p}$ as the output variables.

Asumiendo que:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix},$$

podemos escribir:

$$\frac{d}{dt} \begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{2\dot{q}_1 \dot{q}_2 m_2 q_2 - u_1 + g m_2 q_2 cos(q_1))}{m_2 q_2^2 + I_1 + I_2} \\ \frac{m_2 q_2 \dot{q_1}^2 + u_2 - g m_2 sin(q_1)}{m_2} \end{pmatrix}.$$

Por tanto, la representacion del espacio de estados es:

$$\begin{aligned} \dot{x_1} &= x_3, \\ \dot{x_2} &= x_4, \\ \dot{x_3} &= \frac{2\dot{q}_1 \dot{q}_2 m_2 q_2 - u_1 + g m_2 q_2 cos(q_1))}{m_2 q_2^2 + I_1 + I_2}, \\ \dot{x_4} &= \frac{m_2 q_2 \dot{q_1}^2 + u_2 - g m_2 sin(q_1)}{m_2}. \end{aligned}$$

Tomando las coordenadas del punto P como outputs obtenemos:

$$\begin{aligned} P_x &= cos(q_1)(q_2 + d_2) \\ P_y &= sen(q_1)(q_2 + d_2). \end{aligned}$$