Ingeniería de Control
Problem 1

Aircraft vertical takeoff and landing

Consider the simplified planar model of the system for vertical takeoff and landing of an aircraft represented in Figure 1, in which the aircraft is represented by a bar. The position of the center of mass of the aircraft, $\mathbf{c} = (x, y)^T$, the roll angle of the aircraft, θ, and their time derivatives are the state variables of the system. The thrust force S, applied to the center of mass of the aircraft, and the forces F, applied to the wing tips, are the control inputs u_1 and u_2 of the system, respectively. The thrust force S keeps the aircraft flying. The forces F, which always act in opposite directions, control the roll of the aircraft.

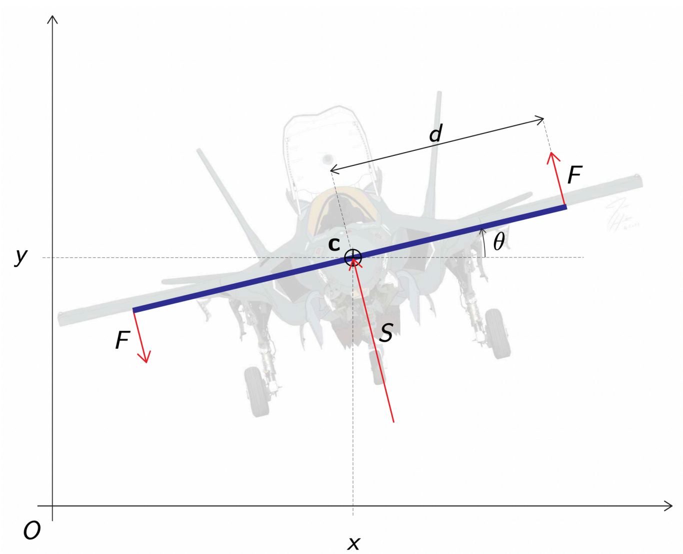

Figure 1: Sketch of the system for aircraft vertical takeoff and landing.

The dynamic model of this system is

$$\begin{aligned}
\ddot{x} &= -\frac{1}{m}\sin(\theta)S, \\
\ddot{y} &= -g + \frac{1}{m}\cos(\theta)S, \\
\ddot{\theta} &= \frac{2d}{J}F,
\end{aligned}$$

with the following parameters

- barycentric moment of inertia of the aircraft: $J = 10000$ [kg m^2],
- mass of the aircraft: $m = 30000$ [kg],
- $d = 5.5$ [m],
- gravity acceleration: $g = 9.81$ [m/s^2].

1) Demonstrate the equations of the dynamic model using the Lagrange method.
2) Calculate the state space representation of the system, assuming that $\mathbf{x} = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})^T = (x_1, x_2, x_3, x_4, x_5, x_6)^T$, where distances are measured in [m], angles in [rad], linear velocities in [m/s], and angular velocities in [rad/s].

Write a detailed report answering each question in a different section. Originality and completeness of the answers will be the aspects that will be taken into account in the grading of the report.

Solution

1) Demonstrate the equations of the dynamic model using the Lagrange method.

   Mediante el metodo de Lagrange hallamos las ecuaciones de estado del modelo dinamico, sabiendo que: L = T - V

   $L = \frac{1}{2}(m(\dot{x}^2 + \dot{y}^2) + J\dot{\theta}^2) - mgy$

   Por tanto, las ecuaciones de Lagrange son las siguientes:

   $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = S$

   $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = S$

   $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = F$

   Tanto en la primera ecuacion como en la segunda, se iguala a la fuerza de empuje aplicada al centro de masas, mientras que en la tercera, se iguala a la fuerza aplicada en las puntas de las alas.

   Resolvemos la primera ecuacion

   $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

   $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$

$\frac{\partial L}{\partial x} = 0$

obteniendo la primera ecuacion de Lagrange:

$m\ddot{x} = -S_x sen(\theta).$

De la misma manera

$\frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y}$

$\frac{\partial L}{\partial y} = -mg$

obtenemos la segunda ecuacion de Lagrange:

$m(\ddot{y} + g) = S_y cos(\theta).$

Por último

$\frac{\partial L}{\partial \dot{\theta}} = J\dot{\theta}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = J\ddot{\theta}$

$\frac{\partial L}{\partial \theta} = 0$

obtenemos la tecera y última ecuacion de Lagrange:

$J\ddot{\theta} = F2d.$

En esta última multiplicamos la fuerza por 2d dado que es la distancia total entre la punta de una ala y la otra.

Tal como podemos comprobar, dichas ecuaciones coinciden con las que teniamos que demostrar.

2) Calculate the state space representation of the system, assuming that $\mathbf{x} = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})^T = (x_1, x_2, x_3, x_4, x_5, x_6)^T$, where distances are measured in [m], angles in [rad], linear velocities in [m/s], and angular velocities in [rad/s].

Asumiendo que:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix},$$

podemos escribir:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ -\frac{1}{m}\sin(\theta)S_x \\ -g + \frac{1}{m}\cos(\theta)S_y \\ \frac{2d}{J}F \end{pmatrix}.$$

Por tanto, la representacion del espacio de estados es:

$$\begin{aligned}
\dot{x_1} &= x_4, \\
\dot{x_2} &= x_5, \\
\dot{x_3} &= x_6, \\
\dot{x_4} &= -\frac{1}{m}\sin(\theta)S_x, \\
\dot{x_5} &= -g + \frac{1}{m}\cos(\theta)S_y, \\
\dot{x_6} &= \frac{2d}{J}F
\end{aligned}$$