Proportional and Derivative Control in Closed-Loop Systems

James Williams
Department of Electrical Engineering
University of Bristol
jw17202@bristol.ac.uk

Louise Wong
Department of Electrical Engineering
University of Bristol
lw17122@bristol.ac.uk

***Abstract*— A brief analysis of proportional and derivative control systems. Techniques such as Routh-Hurwitz stability analysis and eigenvalue location are used to determine critical values for system step repsonse and stability.**

I. INTRODUCTION

This section introduces some of the key mathematical concepts used in the technical note. A control system can be represented diagrammatically using a block diagram such as in Figure 1.

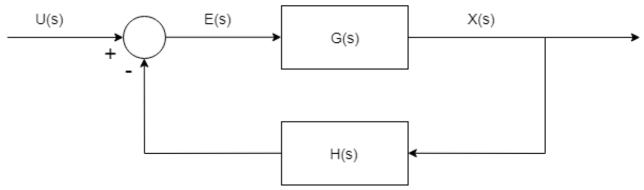

Fig. 1: Closed-loop control system

A control system can also be represented mathematically by a Laplace transform transfer function that is defined to be the ratio of the Laplace transformed output and input. If we consider a time invariant system described by the differential relation

$$a_n \frac{d^n x}{dt^n} + \cdots + a_0 x = b_m \frac{d^m u}{dt^m} + \cdots + b_0 u \tag{1}$$

where $n \geq m$. We can take Laplace transforms throughout to obtain the Laplace transform transfer function

$$G(s) = \frac{X(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{a_n s^n + \cdots + a_0} \tag{2}$$

If we define $P(s)$ and $Q(s)$ as so

$$P(s) = b_m s^m + \cdots + b_0 \tag{3}$$

$$Q(s) = a_n s^n + \cdots + a_0 \tag{4}$$

then $Q(s) = 0$ can tell us a lot about the system, thus it is called the characteristic equation. Its order n is the order of the system and the roots of the equation are the poles. Additionally, the roots of $P(s) = 0$ are the zeros of the system.

The roots of $Q(s)$ indicate whether a system is stable: if all poles lie in the left hand side of the Argand diagram (have negative real parts) then the system is considered stable. It is not necessary to solve the characteristic equation to determine whether a system is stable, the Routh-Hurwitz [1] criterion can provide a sufficient condition for all the roots of the characteristic equation to have negative reals parts if the determinants $\triangle_1, \triangle_2, \cdots, \triangle_n$, are all positive, where

$$\triangle_r = \begin{vmatrix} a_{n-1} & a_n & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-(2r-1)} & a_{n-2r} & a_{n-2r-1} & \cdots & a_{n-r} \end{vmatrix} \tag{5}$$

and any a with a negative subscript or subscript greater than n is replaced by zero. This stability analysis will be used to determine whether or the not system being designed is stable or not.

II. PROPORTIONAL CONTROL

a) The effect of proportional control on step response

The final-value theorem enables us to analytically determine the steady state gain and steady state error of a feedback control system. The final-value theorem states that if both $f(t)$ and $f'(t)$ have Laplace transforms and the limit $\lim_{t\to\infty} f'(t)$ exists then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{6}$$

Figure 2 shows the block diagram representation of the proposed control system.

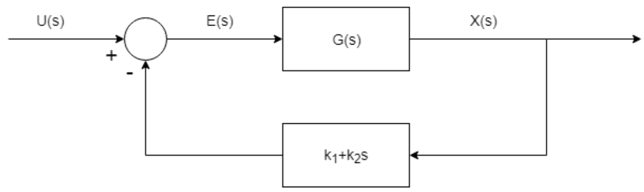

Fig. 2: Closed-loop control system

The transfer function of this system is defined as

$$G_e(s) = \frac{G(s)}{1 + G(s)(k_1 + k_2 s)} \tag{7}$$

where

$$G(s) = \frac{10}{s^2 + 11s + 10} \tag{8}$$

and applying the final-value theorem we can determine the steady state gain as

$$\text{SSG} = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{9}$$

From Figure 2 it is clear that $X(s) = G(s)U(s) - X(s)G(s)(k_1 + k_2 s)$. Solving for $X(s)$ obtains

$$X(s) = \frac{G(s)U(s)}{1 + G(s)(k_1 + k_2 s)} \tag{10}$$

from this the following limit can be evaluated

$$\text{SSG} = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{sG(s)U(s)}{1 + G(s)(k_1 + k_2 s)} \tag{11}$$

substituting Equation 8 into Equation 11 and remembering that $U(s) = 1/s$ gives

$$\text{SSG} = \lim_{s \to 0} \frac{10}{s^2 + 11s + 10 + 10(k_1 + k_2 s)} = \frac{1}{1 + k_1} \tag{12}$$

Finding the steady state error will require a little extra work as the system represented in Figure 2 is a non-unity feedback control system. By introducing a positive unity feedback loop and a negative unity feedback loop we can convert the system in Figure 2 into a unity feedback control system represented by the block diagram in Figure 3.

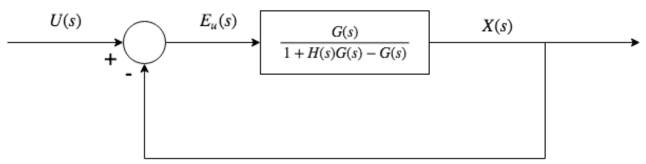

Fig. 3: Unity Closed-loop control system

Now applying the final-value theorem to this new unity feedback control system we can determine the steady state error as

$$\text{SSE} = \lim_{t \to \infty} e_u(t) = \lim_{s \to 0} sE_u(s) \tag{13}$$

where the subscript u indicates this is the error for the unity system. From Figure 3 it is clear that

$$E_u(s) = U(s) - X(s) \tag{14}$$

$$X(s) = \frac{E_u(s)G(s)}{1 + H(s)G(s) - G(s)} \tag{15}$$

Combining Equation 14 and Equation 15 gives

$$E_u(s) = \frac{U(s)\left[1 + H(s)G(s) - G(s)\right]}{1 + H(s)G(s)} \tag{16}$$

We can now define the limit that will determine the steady state error as

$$\text{SSE} = \lim_{s \to 0} \frac{sU(s)\left[1 + H(s)G(s) - G(s)\right]}{1 + H(s)G(s)} \tag{17}$$

$$= \lim_{s \to 0} \frac{s^2 + 11s + 10(k_1 + k_2 s)}{s^2 + 11s + 10 + 10(k_1 + k_2 s)} \tag{18}$$

$$= \frac{k_1}{1 + k_1} = \frac{1}{1 + \frac{1}{k_1}} \tag{19}$$

Figure 4 shows the step response of the system for a range of k_1 values between 10 and 100. As predicted by Equation 12, the steady state gain decreases, as the value of k_1 increases.

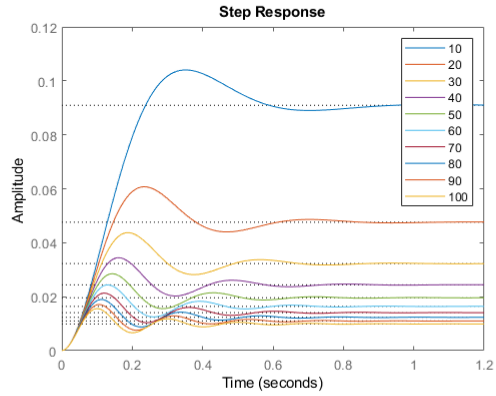

Fig. 4: Step response with $10 \leq k_1 \leq 100$

Figure 5 shows the magnitude and phase plot of the Bode diagram. It is clear that as k_1 increases, the frequency of the peak in the magnitude plot increases.

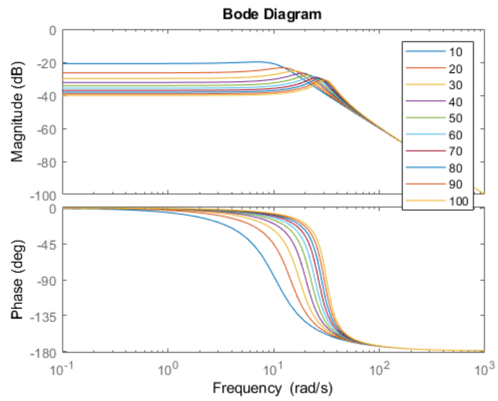

Fig. 5: Bode magnitude and phase plot with $10 \leq k_1 \leq 100$

b) Over, under and critically damped systems

Damping reduces oscillations within a system. There are many types of repsonse, including: undamped, overdamped, underdamped and critically damped.

An undamped system will have two simple poles $\pm j\beta$ with a natural response

$$y(t) = Acos(\beta t - \phi) \tag{20}$$

here the output will not return to equilibrium as $t \to \infty$, thus its limit is undefined.

An underdamped system has two simple poles at $-\alpha \pm j\omega$, as can be seen in Figure 6 and has the natural response

$$y(t) = Ae^{-\alpha t}cos(\omega t - \phi), \tag{21}$$

this system will return to equilibrium exponentially but the presence of the cosine factor represents oscillations that

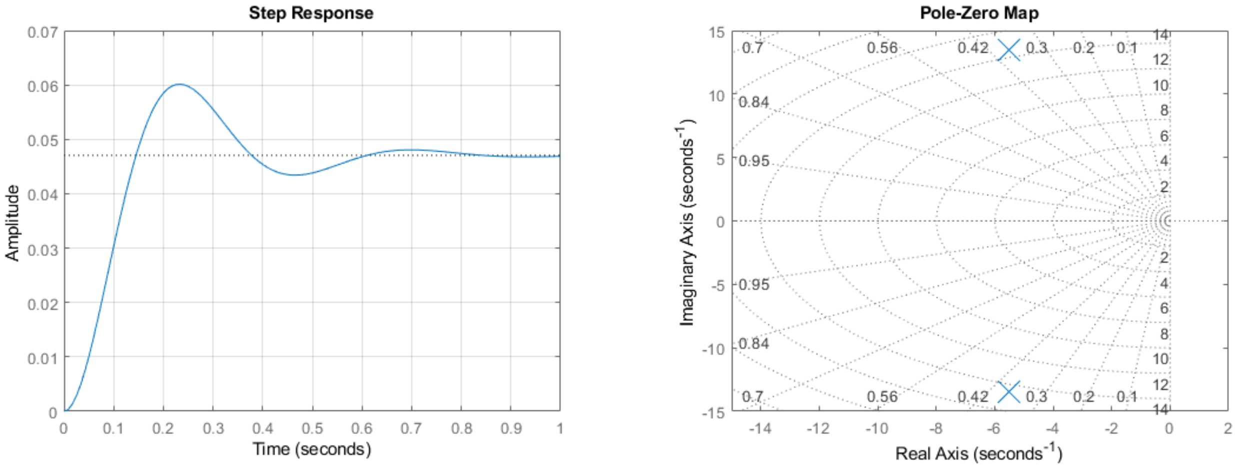

Fig. 6: Underdamped step response with $k_1 > 81/40$

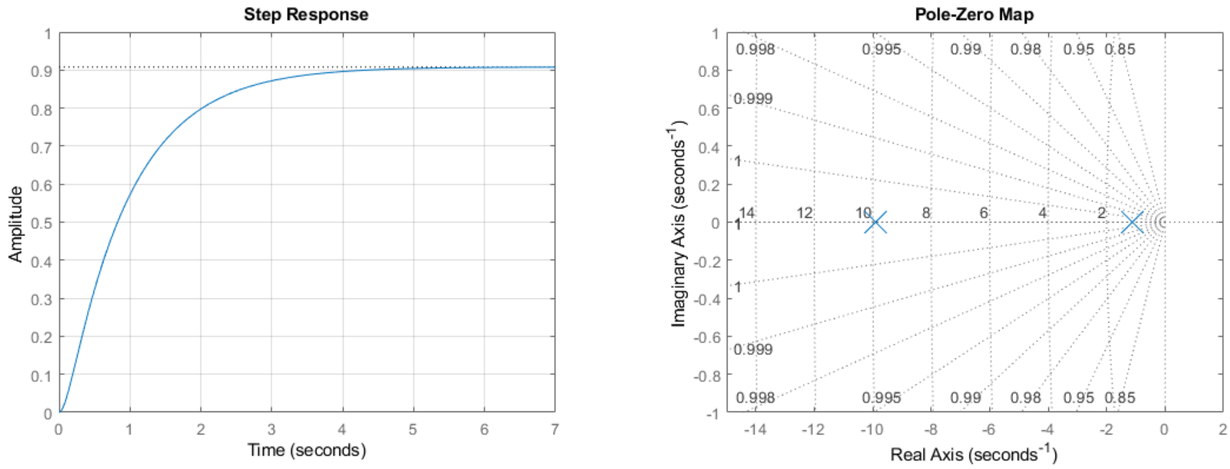

Fig. 7: Overdamped step response with $k_1 < 81/40$

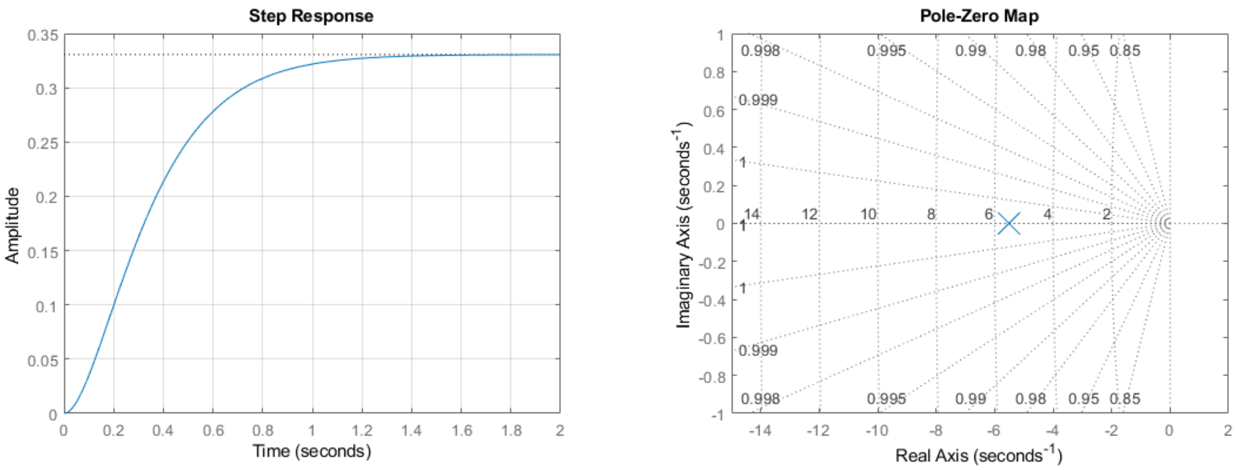

Fig. 8: Critically damped step response with $k_1 = 81/40$

remain in the system. The effect of underdamping on the step response can be seen with $k_1 > 81/40$ in Figure 6.

An overdamped system will have two real poles at $-\alpha_1$

and $-\alpha_2$, shown in Figure 7 and has the natural response

$$y(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} \tag{22}$$

and the output will return to equilibrium exponentially. Figure 7 shows effect of overdamping on the step response with $k_1 < 81/40$.

A critically damped system has a double pole at $-\alpha$ and has a natural response defined by

$$y(t) = (A_1 + A_2 t)e^{-\alpha t}, \tag{23}$$

the output will return to equilibrium exponentially even with the second term being multiplied by t, as $f(t) = t$ is of exponential order the second term will tend to a defined value as $t \to \infty$. Figure 8 shows the effect of critical damping on the step response with $k_1 = 81/40$.

The Matlab code in Code Listing 1 is used to plot both the step responses and pole-zero maps seen in Figures 6 through 8.

Code Listing 1: Plotting code

```
% Code for plotting step response
num = [0 0 10];
den = [1 11 10];
k1 = 81/40; % Choose diff values for k1
k2 = 0;
[numc,denc] = feedback(num,den, ...
    [k2 k1],[0 1]);
step(numc,denc);
grid on

% code for plotting pole-zero maps
sys = tf(numc, denc);
h = pzplot(sys);
p = getoptions(h);
p.Xlim = [-15, 2];
setoptions(h, p);
set(h.allaxes.Children(1).Children, ...
    'MarkerSize', 20);
grid on
```

The `num` and `den` arrays define the transfer function of the forward path open-loop system. The `feedback()` function returns the parameters that define the closed-loop system including the feedback system.

c) *Pole placement for peak at* 10*Hz*

Figures 9 and 10 show the step response and bode diagrams respectively, when $k_1 = 400$. With k being much greater than the critical value 81/40 this is a case of underdamping, this is shown by the oscillatory artefacts in the step response. It has a peak very close to 20π rad/s or 10Hz.

To achieve a peak at a frequency of exactly 10Hz the poles will need to placed at $\alpha \pm j20\pi$. In the control system proposed we know $\alpha = -5.5$. To carry out the method of pole placement we will introduce the state space representation [1]. Consider the single-input single-output system represented by

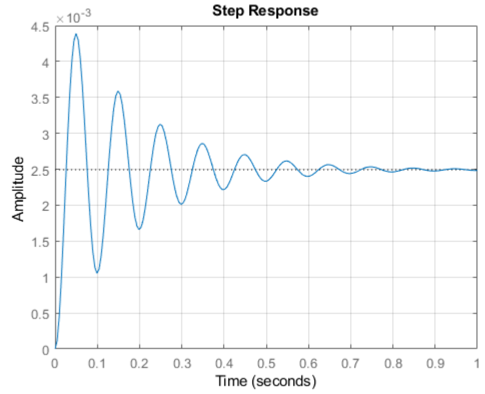

Fig. 9: Step response with $k_1 = 400$

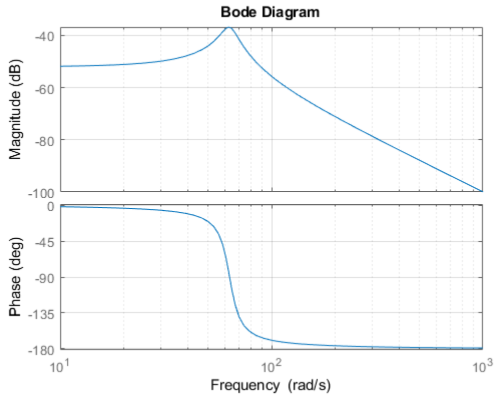

Fig. 10: Bode diagram with $k_1 = 400$

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_0 x = u, \tag{24}$$

to represent this in state space from we define n new variables as

$$y_1(t) = x(t),\ y_2(t) = \dot{x}(t),\ y_3 = \ddot{x}(t),\ \cdots. \tag{25}$$

By taking the newly introduced equations in Equation 25 and substituting them into Equation 24 we get

$$a_n \dot{y}_n + a_{n-1} y_n + \cdots + a_0 y_1 = u, \tag{26}$$

and solving for $\dot{y}$ gives

$$\dot{y} = -\frac{a_{n-1}}{a_n} y_n - \frac{a_{n-2}}{a_n} y_{n-1} - \cdots - \frac{a_0}{a_n} y_1 + \frac{1}{a_n} u. \tag{27}$$

This enables us to represent a differential equation in vector-matrix form such as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{-a_0}{a_n} & \frac{-a_1}{a_n} & \frac{-a_2}{a_n} & \cdots & \frac{-a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{a_n} \end{bmatrix} u \tag{28}$$

or in algebraic notation as

$$\dot{\boldsymbol{y}} = \mathbf{A}\boldsymbol{x} + \boldsymbol{b}u. \tag{29}$$

For the system to have a peak at 10Hz we need the eigenvalues of the control matrix $\mathbf{A}$ to be $-5.5 \pm j20\pi$, where $\mathbf{A}$ is defined as such

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10(1+k_1) & -(11+10k_2) \end{bmatrix}. \tag{30}$$

Solving the following eigenvalue problem will determine the required values of k_1 and k_2

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = \begin{vmatrix} \lambda & -1 \\ -10(1+k_1) & \lambda-(11+10k_2) \end{vmatrix} \tag{31}$$

$$= \lambda^2 + (11+10k_2)\lambda + 10(1+k_1) \tag{32}$$

For the peak to be at 10Hz the coefficients of Equation 32 must match the coefficients of the following equation

$$(\lambda+5.5-j20\pi)(\lambda+5.5+j20\pi) = \lambda^2+11\lambda+(5.5^2+400\pi^2), \tag{33}$$

this gives the following equalities

$$11 + 10k_2 = 11 \to k_2 = 0 \tag{34}$$

$$10(1+k_1) = 5.5^2 + 400\pi^2 \to k_1 \approx 397.709... \tag{35}$$

The state space representation is not *needed* in this use case, it would have been just as correct to construct the equalities in Equations 44 and 45 directly from the characteristic equation $s^2 + (11+10k_2)s + 10(1+k_1)$. Although, in reality it may not always be possible to choose any value for k_1 or k_2, there may be physical limitations restricting the range of values we can choose from. By using the state space representation we can position the poles by introducing state feedback. By defining an input in the following way

$$u = \boldsymbol{p}^T\boldsymbol{x} + u_{ext} \tag{36}$$

where $\boldsymbol{p} = [p_1\ p_2]$ and u_{ext} is the external input. The state space representation becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ p_1 + \frac{a_0}{a_n} & p_2 + \frac{a_1}{a_n} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{\text{ext}} \tag{37}$$

The eigenvalues of the control matrix are given by the solution of

$$\lambda^2 - \left(p_2 + \frac{a_1}{a_n}\right)\lambda - \left(p_1 + \frac{a_0}{a_n}\right) = 0 \tag{38}$$

and by equating coefficients with

$$(\lambda+c_1)(\lambda+c_2) = \lambda^2 + (c_1+c_2)\lambda + c_1c_2 \tag{39}$$

where c_1 and c_2 are the desired pole locations, the following equalities can be formed

$$c_1 + c_2 = -\left(p_2 + \frac{a_1}{a_n}\right) \to p_2 = \frac{a_1}{a_n} - (c_1+c_2) \tag{40}$$

$$c_1c_2 = -\left(p_1 + \frac{a_0}{a_n}\right) \to p_1 = \frac{a_0}{a_n} - c_1c_2 \tag{41}$$

The values of p_1 and p_2 define the gains of the state feedback loops that are to be introduced such as in Figure 11. This process can be generalised to higher order control systems.

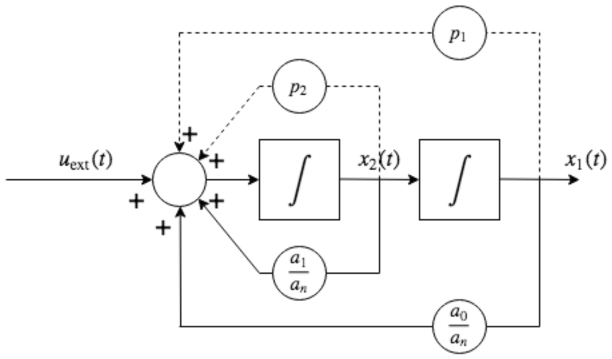

Fig. 11: Time-domain block diagram with state feedback

d) Pole placement for peak at 5Hz

The method used in section c can be applied here, for this case we shall use the direct method as opposed to the state space method. The characteristic equation of the control system is

$$Q(s) = s^2 + (11+10k_2)s + 10(1+k_1) \tag{42}$$

For the peak to be at 5Hz the coefficients of Equation 42 must match the coefficients of the following equation

$$(s+5.5-j10\pi)(s+5.5+j10\pi) = s^2+11s+(5.5^2+100\pi^2), \tag{43}$$

this gives the following equalities

$$11 + 10k_2 = 11 \to k_2 = 0 \tag{44}$$

$$10(1+k_1) = 5.5^2 + 100\pi^2 \to k_1 \approx 101.621... \tag{45}$$

The step response and bode plots can be seen in Figure 12.

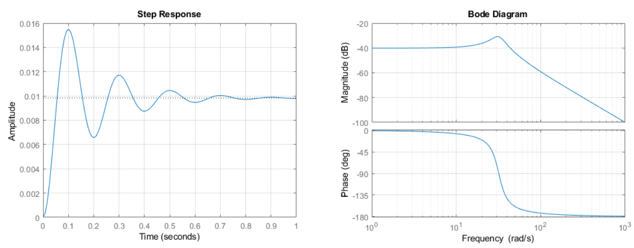

Fig. 12: Bode diagram with $k_1 = 101.621$

III. DERIVATIVE CONTROL

e) The effect of derivative control on step response

Figure 13 shows the effect of derivative control on the step response. The value of k_2 effects the transient response of the system. Higher values for k_2 cause a higher attenuation to the transient response.

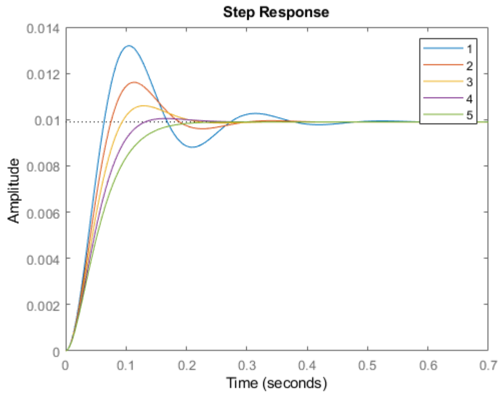

Fig. 13: Step response with $1 \leq k_2 \leq 5$

By plotting the poles of the transfer function for a range of k_2 values we can see that it is possible to compromise the system stability. Figure 14 shows the real part of each conjugate pole pair decreasing as k_2 decreases, we know that for a system to be stable the poles must reside in the left hand side of the complex plane.

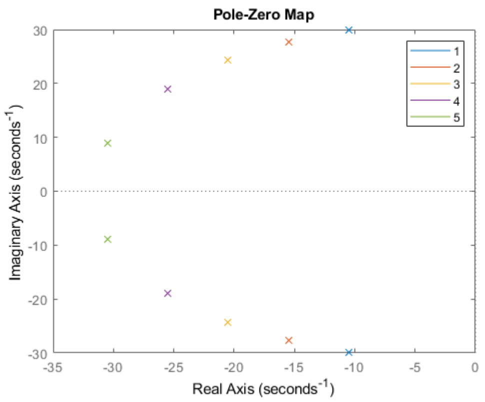

Fig. 14: Pole-zero map with $1 \leq k_2 \leq 5$

We can impose a condition on k_2 using the Routh-Hurwitz condition from Equation 5. For the characteristic equation $Q(s) = s^2 + (11 + 10k_2)s + (k_1 + 1)$ we can define the following determinants (where $n = 2$, the order of the system)

$$\triangle_1 = \left|a_{n-1}\right| = \left|a_1\right| = \left|11 + 10k_2\right| \quad (46)$$

$$\triangle_2 = \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ a_{-1} & a_0 \end{vmatrix} = \begin{vmatrix} 11 + 10k_2 & 1 \\ 0 & 10(1 + k_1) \end{vmatrix} \quad (47)$$

Using Equations 46 and 47 we can place a constraint on k_2 like so

$$\left|11 + 10k_2\right| > 0 \rightarrow k_2 > -\frac{11}{10} \quad (48)$$

$$\begin{vmatrix} 11 + 10k_2 & 1 \\ 0 & 10(1 + k_1) \end{vmatrix} = (11 + 10k_2)(k_1 + 1) > 0 \quad (49)$$

With $k_1 = 100$, Equation 49 gives $k_2 > -1111/1011$, as $-11/10 < -1111/1011$ the constraint placed on k_2 is therefore $k_2 > -11/10$. Figure 15 shows the step response for three values of k_2: $k_2 > -11/10$ this system is stable, $k_2 = -11/10$ this system is marginally stable and $k_2 < -11/10$ this system is unstable.

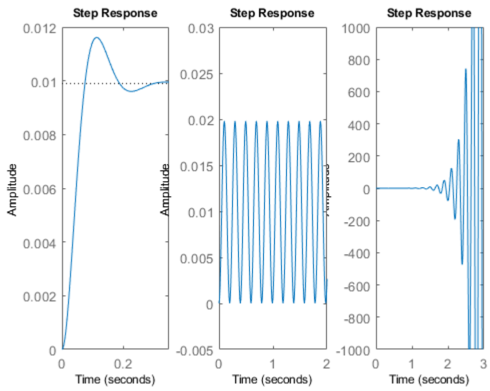

Fig. 15: Stable, marginally stable and unstable step responses

f) Summary of the effects of proportional and derivative control

Proportional control effects the level of damping within a system and damping attenuates oscillations. In section a we saw four different levels of damping: undamped, underdamped, overdamped and critically damped. Increasing the amount of proportional control caused the poles to move further away from the real axis and thus increasing the frequency of the oscillation in the transient response and also decreasing the steady state gain, which was found using the final-value theorem.

Derivative control effects the amplitude of the transient response. By increasing the derivative control the amplitude of the transient response decreased and approached the steady state gain more quickly. Additionally, decreasing derivative control caused the poles to move right in the complex plane. If the derivative control was decreased sufficiently the poles would be located in the right half of the complex plane, resulting in an unstable system. The Routh-Hurwitz condition was used to determine the minimum permitted amount of derivative control that could be applied before the system became unstable.

REFERENCES

[1] Advanced Modern Engineering Mathematics 4th ed. Glyn James
[2] TutorialsPoint - Control Systems - Steady State Error