$$E_0 = -\frac{(4Z-1)^2}{8}\times 13.6\text{ eV}. \quad (6)$$

We calculate the ground state energies of He, Li^+, and Be^{++} from Eq. (6). Table I compares these calculated values (E_{calc}) with experimental values[4] (E_{exp}).

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[1]See, for example, D. Halliday and R. Resnick, *Fundamentals of Physics* (Wiley, New York, 1986), p. 850.

[2]R. S. Shankland, *Atomic and Nuclear Physics* (Macmillan, New York, 1955), p. 67; see also, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. I, I-38-6.

[3]See, for example, A. P. French, *Newtonian Mechanics* (Norton, New York, 1965), p. 561.

[4]A. Messiah, *Quantum Mechanics* (Wiley, New York, 1962), Vol. II, p. 691.

Two-dimensional fast Fourier transform and pattern processing with IBM PC

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The history of fast Fourier transforms and the use of digital computers for performing one-dimensional fast Fourier transforms are discussed in detail elsewhere.[1–4] This note introduces the two-dimensional Fourier transform, inverse Fourier transform, fast Fourier transform, spatial filtering at the Fourier plane, and pattern subtraction and addition at the Fourier plane. All these are done on an IBM PC. The object pattern, spectrum pattern, and image pattern can be displayed on the monitor or the printer.

I. TWO-DIMENSIONAL FOURIER TRANSFORM AND ITS INVERSE TRANSFORM

The Fourier transform of any continuous and integrable function of two variables can be expressed as

$$F(\mu,\nu) = \iint_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(\mu x + \nu y)]dx\,dy,$$

and its inverse Fourier transform can be expressed as

$$f(x,y) = \iint_{-\infty}^{\infty} F(\mu,\nu)\exp[j2\pi(\mu x + \nu y)]d\mu\,d\nu,$$

where x, y are coordinate on the object plane or the image plane, and μ,ν are coordinates on the spectrum plane. In order to use the computer to do the work, the discrete Fourier transform is adopted. Assuming that there are $M\times N$ sampling points with Δx and Δy intervals in the x and y directions, the two-dimensional discrete Fourier transform can be written as

$$F(\mu,\nu) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \times\exp\left[-j2\pi\left(\frac{\mu x}{M}+\frac{\nu y}{N}\right)\right]. \quad (1)$$

And its corresponding inverse Fourier transform is

$$f(x,y) = \sum_{\mu=0}^{M-1}\sum_{\nu=0}^{N-1} F(\mu,\nu)\exp\left[j2\pi\left(\frac{\mu x}{M}+\frac{\nu y}{N}\right)\right]. \quad (2)$$

In these formulas x is 0, 1, 2,..., $M-1$, and y is 0, 1, 2,...,$N-1$, respectively.

The spatial and spectrum intervals are related by

$$\Delta\mu = 1/M\,\Delta x, \quad (3a)$$

and

$$\Delta\nu = 1/N\,\Delta y. \quad (3b)$$

In discrete Fourier transforms, the origin is not at the center for the values taken between the intervals $(0,M-1)$ and $(0,N-1)$; therefore, the spectrum pattern obtained will show up on four corners of the monitor. In order to avoid this, the origin of the spectrum is shifted to $M/2$, $N/2$ relative to μ, ν.[5]

II. FAST FOURIER TRANSFORM (FFT)

The algorithm used here for FFT is called successive doubling.[5] Now let us introduce the method with an example in one dimension.

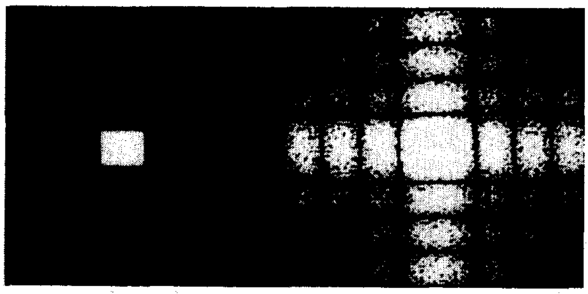

Fig. 1. A square pattern and its Fourier transform.

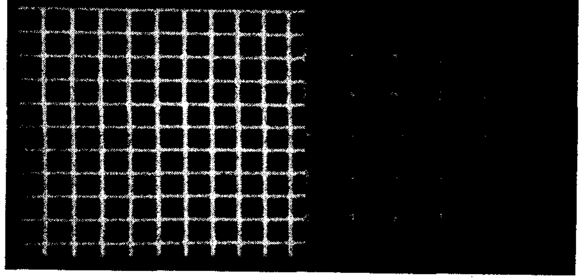

Fig. 2. A finer grid pattern and its Fourier transform.

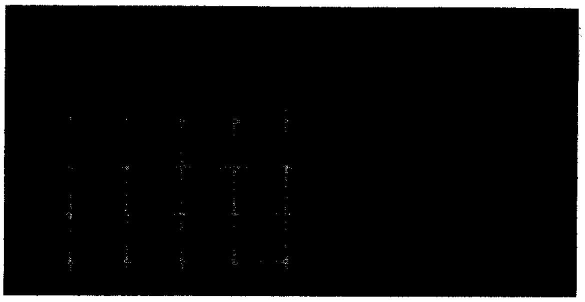

Fig. 3. A coarser grid pattern and its Fourier transform.

$$F(\mu) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp\left(-j2\pi\frac{\mu x}{N}\right)$$
$$= \frac{1}{N}\sum_{x=0}^{N-1} f(x)W_N^{\mu x}, \tag{4}$$

where $W_N = \exp(-j2\pi/N)$.

Let N satisfy $N = 2^n$ and $N = 2M$, where both n and M are positive integers. Then

$$F(\mu) = \frac{1}{2M}\sum_{x=0}^{2M-1} f(x)W_{2M}^{\mu x}$$
$$= \frac{1}{2}\left(\frac{1}{M}\sum_{x=0}^{M-1} f(2x)W_{2M}^{\mu(2x)} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)W_{2M}^{\mu(2x+1)}\right)$$
$$= \frac{1}{2}\left(\frac{1}{M}\sum_{x=0}^{M-1} f(2x)W_M^{\mu x} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)W_M^{\mu x}W_{2M}^{\mu}\right). \tag{5}$$

Let us define

$$F_{\text{even}}(\mu) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)W_M^{\mu x},$$

and

$$F_{\text{odd}}(\mu) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)W_M^{\mu x},$$

where $\mu = 0, 1, 2,\ldots, M-1$. Hence,

$$F(\mu) = \tfrac{1}{2}\left[F_{\text{even}}(\mu) + F_{\text{odd}}(\mu)W_{2M}^{\mu}\right]. \tag{6}$$

Because $W_M^{\mu+M} = W_M^{\mu}$ and $W_{2M}^{\mu+M} = -W_{2M}^{\mu}$, Eq. (5) can be written as

$$F(\mu+M) = \tfrac{1}{2}\left[F_{\text{even}}(\mu) - F_{\text{odd}}(\mu)W_{2M}^{\mu}\right]. \tag{7}$$

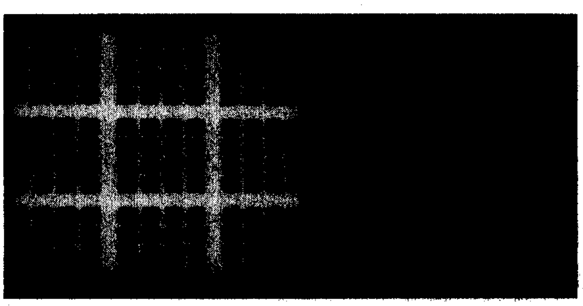

Fig. 4. A fine-coarse combination grid pattern and its Fourier transform.

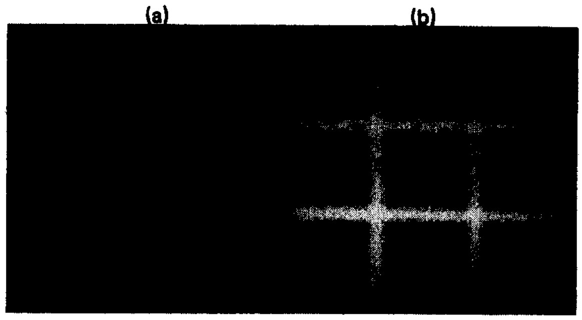

Fig. 5. (a) A low-pass spatial frequency pattern and (b) its corresponding image.

Inspection of the above equations shows that the original n point transformation decomposes into two parts (6) and (7), each of which is an $N/2$ point transformation. This is why it is called successive doubling.[1,5]

III. THE MAKING OF THE PATTERN AND ITS DISPLAY

The patterns on the object plane, spectrum plane, and image plane are formed into a square array through sampling. There are 64×64 sampling points for each pattern. Every sampling point is made by 10 pixels on the monitor, with two rows and five columns. Thus, there are 11 different grayness levels from 0 to 10. The brightness of the sampling point is equal to 0 if all pixels are dark, and equal to 10 when all are bright. For any intermediate brightness from 1 to 9 for the sampling point, the position of the bright pixel is randomly distributed.

For each pattern, there are $2\times 64\times 64$ pieces of data inputs. We normally create a set of data files according to the desired format of the object pattern and data files are called from the FFT programs.

Figure 1 shows a square pattern and its Fourier transform. Figures 2 and 3 show the grid patterns and their corresponding Fourier transforms. The object pattern of Fig. 2 is finer than that of Fig. 3, and the frequency spectrum of Fig. 3 is finer than that of Fig. 2. Performing an inverse Fourier transform on the frequency spectrum pattern will result in an image similar to the object pattern.

IV. PATTERN PROCESSING

The four programs related to this note are: (1) Fourier transform program; (2) object pattern display and Fourier spectrum display program; (3) inverse Fourier transform program; and (4) filtered spectrum display and image pattern display program. Through proper use of these programs, spatial filtering of the frequency spectrum and pattern subtraction and addition can be performed.

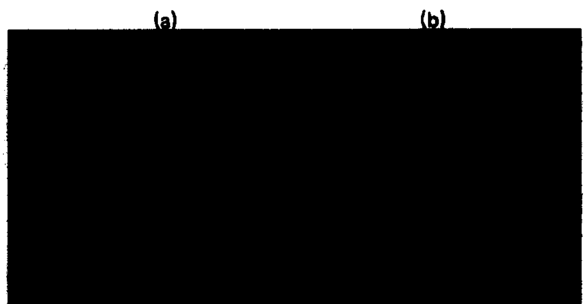

Fig. 6. (a) A high-pass spatial frequency pattern and (b) its corresponding image.

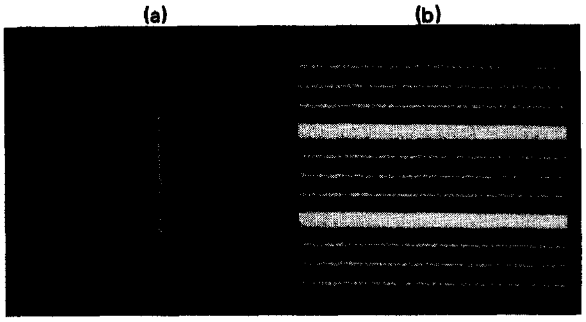

Fig. 7. (a) A vertical-pass spatial frequency pattern and (b) its corresponding image.

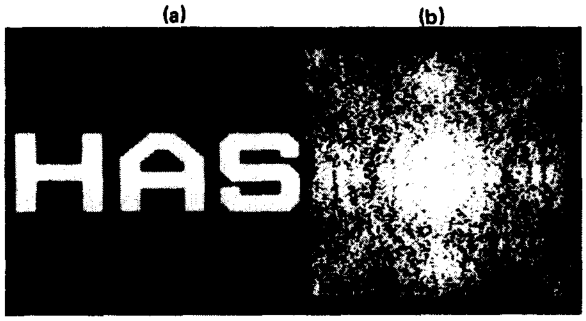

Fig. 8. (a) Three letters "HAS" and (b) its Fourier transform.

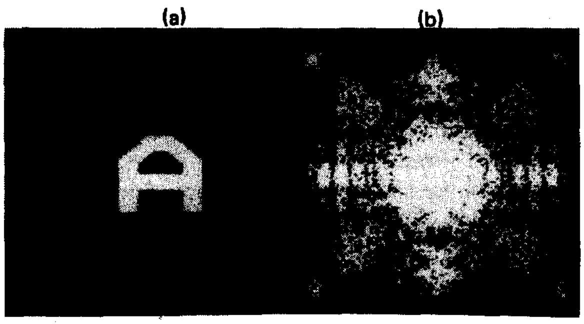

Fig. 9. (a) A letter "A" and (b) its Fourier transform.

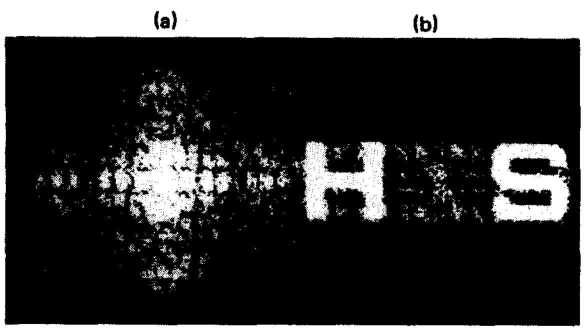

Fig. 10. (a) The spatial spectrum of the subtraction of Fig. 8(b) and Fig. 9(b) and (b) its corresponding inverse Fourier transform.

A. Spatial filtering of the frequency spectrum

Let the transfer function of the spatial filter be $H(\mu,\nu)$. The filtered frequency spectrum $F^*(\mu,\nu)$ can be expressed by the following formula:

$$F^*(\mu,\nu) = H(\mu,\nu)F(\mu,\nu)\,. \tag{8}$$

A low-pass spatial filter can be easily expressed as

$$H(\mu,\nu) = \begin{cases} 1, & \text{when } \sqrt{\mu^2+\nu^2} < D_0\,, \\ 0, & \text{when } \sqrt{\mu^2+\nu^2} > D_0\,, \end{cases} \tag{9}$$

and a high-pass spatial filter as

$$H(\mu,\nu) = \begin{cases} 0, & \text{when } \sqrt{\mu^2+\nu^2} < D_0\,, \\ 1, & \text{when } \sqrt{\mu^2+\nu^2} > D_0\,. \end{cases} \tag{10}$$

Other filters can be expressed by a similar method.

Figure 4 shows a fine–coarse combination grid pattern and its Fourier transform.

Figure 5(a) is a low-pass frequency pattern, and Fig. 5(b) is the corresponding image after inverse Fourier transform. Obviously, it can be observed that the fine grids, i.e., details that correspond to the high spatial frequency, have been filtered out.

Figure 6(a) is a high-pass frequency pattern. The center or the part of the pattern that corresponds to the low spatial frequency is lost. After inverse Fourier transform, the fine grids are preserved and the coarse grids are eliminated, as shown in Fig. 6(b). This means that the low spatial frequencies have been filtered out and high spatial frequencies are kept intact.

Figure 7(a) shows that the central vertical part of Fig. 4 is allowed to be passed. After an inverse Fourier transform on this filtered frequency pattern, as shown in Fig. 7(b), the grid image maintains the horizontal lines and loses all the vertical lines.[6]

B. Pattern subtraction and addition

Figure 8 shows the pattern of three letters "HAS" and its Fourier transform. Fig. 9 shows a single letter "A" and its Fourier transform; Fig. 10(a) is the result of subtraction of Fig. 9(b) from Fig. 8(b), and Fig. 10(b) is the inverse Fourier transform of Fig. 10(a). The final result is the pattern "H S," which is the result of subtracting "A" from "HAS." Similarly, pattern addition can be processed by this method.[7]

Copies of the corresponding BASIC programs are available from the authors.

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[1] E. O. Brigham, *The Fast Fourier Transform* (Prentice-Hall, Englewood Cliffs, NJ, 1974).

[2] R. J. Higgins, Am. J. Phys. **44**, 766 (1976).

[3] D. R. Matthys and F. L. Pedrotti, Am. J. Phys. **50**, 990 (1982).

[4] X. Chen, J. Huang, and E. Loh, Am. J. Phys. **55**, 1129 (1987).

[5] R. C. Gonzalez and P. Wintz, *Digital Image Processing* (Addison-Wesley, Reading, MA, 1977).

[6] J. W. Goodman, *Introduction to Fourier Optics* (McGraw-Hill, New York, 1968).

[7] F. T. S. Yu, *Optical Information Processing* (Wiley, New York, 1983).