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## Heat conduction in a metallic rod with Newtonian losses

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An experimental realization is presented of the problem of heat conduction along a one-dimensional rod subjected to an externally imposed constant temperature difference and allowing for heat losses to the environment. One end of the rod is at a fixed elevated temperature, while, untypically, both the lateral surface and far end of the rod are considered to be at the same fixed lower temperature. Newton's Law of Cooling has been invoked as a boundary condition. The experimental results are used to illustrate some fundamental concepts of nonequilibrium thermodynamics such as the meaning and temporal evolution of the different contributions to the entropy balance equation, including entropy production, and the influence of initial and boundary conditions on the steady state.

### I. INTRODUCTION

The problem of heat conduction along a one-dimensional rod subjected to an externally imposed constant temperature difference is a common example of the evolution of a thermodynamic system toward a nonequilibrium steady state.<sup>1,2</sup> In the most simple situation the rod is assumed to be insulated everywhere from the ambience (except at the two ends) and the steady state is given by a linear temperature distribution joining the two extreme temperatures. This simple situation has been used by Lurié and Wagensberg<sup>3</sup> to simulate the temperature evolution in the rod using a finite-element method, and demonstrate that the entropy production in the system evolves monotonically toward a minimum value, as predicted by Prigogine's theorem of the minimum entropy production.<sup>4</sup> These authors have also used the example to test the accuracy of a finite-difference method in solving the partial differential equation of heat transport.

When trying to implement the problem in a graduate laboratory, however, it comes out that the theoretical requirement of a perfectly insulating one-dimensional rod is impossible to achieve in practice. We present here an alternative simple problem of heat transport in a metallic rod, allowing for heat losses from the rod to the surroundings, which play the role of a thermal bath. This is a slightly more general situation which includes the perfectly

insulating rod as a singular particular case and stresses the importance of boundary conditions on the resolution of partial differential equations.

This work has a double purpose: (i) to present a simple experimental implementation of the problem in a graduate laboratory, using automated data acquisition, and (ii) to use a number of experimental results to illustrate fundamental concepts of heat transport such as the influence of initial and boundary conditions on the steady-state configuration, and also to illustrate a number of concepts of nonequilibrium thermodynamics such as the time evolution of the different terms entering the entropy balance of the system.

### II. THEORY

#### A. Heat transport equation

We consider heat conduction along a cylindrical bar (a rod) of constant cross section, subjected to a fixed temperature difference and exchanging heat with a thermal bath. The left end of the rod is at a constant temperature  $T_L$  while both the right end of the rod and the thermal bath are at a constant temperature  $T_R$ , lower than  $T_L$ .

A local energy balance in the rod leads to

$$\rho c \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{J}_q \quad (1)$$

where  $\rho$  is the density,  $c$  the heat capacity per unit mass, and  $\mathbf{J}_q$  the heat flux. Heat conduction in the rod follows Fourier's law:

$$\mathbf{J}_q = -\lambda \nabla T, \quad (2)$$

with  $\lambda$  the thermal conductivity of the system, so that Eq. (1) becomes

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (\lambda \nabla T). \quad (3)$$

This differential equation governing the evolution of the temperature distribution in the rod must be supplemented with the initial temperature distribution (initial condition), and with boundary conditions. We implement the latter by Newton's law of cooling:

$$\lambda \hat{\rho}_0 \cdot (\nabla T) + H(T - T_R) = 0, \quad (4)$$

which states that the normal heat flow through the surface is proportional to the difference in temperature between the surface and the external bath.  $\hat{\rho}_0$  is the unit vector in the radial direction and  $H$  is the surface conductance, dependent on the thermal and geometric characteristics of the two media.

Newton's law of cooling is particularly useful as a boundary condition because of its simplicity and because it reduces to two of the most common boundary conditions in the following limiting cases: for  $H \rightarrow 0$  it represents a perfectly insulating surface, while for  $H \rightarrow \infty$  it represents a perfectly conducting surface.<sup>5</sup> Newton's law of cooling has often been invoked as the appropriate boundary condition in problems of heat transport in the introductory physics laboratory.<sup>6,7</sup> In our case the use of Newton's law of cooling to represent lateral heat losses relies upon two assumptions: On the one hand, given the high thermal conductivity of the metallic rod, its temperature is assumed to be uniform throughout a given cross section and, in particular, coincides with the temperature  $T$  at the surface. On the other hand, the air in the vicinity of the rod is assumed to behave as a thermal bath at a constant temperature  $T_R$ ; this is a reasonable approximation since the heat capacity of the air is large, its thermal conductivity is very small and convection around the rod tends to keep homogeneous the temperature of the air.

When the length of the rod is much larger than its radial dimensions, the system can be considered as one dimensional. Díaz-Guilera,<sup>5</sup> by proper averaging over the radial dimensions, has recently given the procedure to convert Eq. (3) and the associated boundary condition (4) into its one-dimensional form:

$$\rho c \frac{\partial T(z,t)}{\partial t} = \lambda \frac{\partial^2 T(z,t)}{\partial z^2} - \frac{2H}{r} [T(z,t) - T_R], \quad (5)$$

$r$  being the radius of the rod.

## B. Steady-state solution

Under an externally maintained temperature gradient, the system cannot relax to *equilibrium*. It tends to a final *steady state*, characterized by a nonzero transport of energy through the rod and a nonuniform temperature distribu-

tion. The steady-state solution to Eq. (5) is found by requiring the temperature distribution to be independent of time. Thus

$$0 = \lambda \frac{\partial^2 T(z,t)}{\partial z^2} - \frac{2H}{r} [T(z,t) - T_R]. \quad (6)$$

This is a linear second-order differential equation which, defining:

$$\theta(z,t) = T(z,t) - T_R \quad (7)$$

will admit a general solution:

$$\theta = A e^{-\beta z} \quad (8)$$

assuming that the rod is long enough to associate the temperature  $T_R$  of its right end to the limit  $z \rightarrow \infty$ . Introducing this solution into the differential equation and identifying terms, we obtain the exponentially decreasing steady-state temperature distribution:

$$[T(z, \infty) - T_R] = [T_L - T_R] e^{-\beta z}, \quad (9)$$

with a coefficient:

$$\beta = [2H/\lambda r]^{1/2}. \quad (10)$$

While  $H$  is generally unknown, it is remarkable that the coefficient  $\beta$  in a given system (dependent also on thermal and geometric properties of the system and surroundings) can be determined experimentally from the steady-state temperature distribution.

## C. Entropy balance equation

In classical thermodynamics, entropy is defined only for states of thermodynamic equilibrium. In order to generalize the entropy concept to a nonequilibrium situation, an *assumption of local equilibrium* is adopted: The system can be subdivided in a large number of volume elements such that thermodynamic equilibrium prevails locally within each element. For this assumption to be reasonable, each volume element must be microscopically large enough for intensive variables such as pressure and temperature to be defined, but macroscopically small enough for these quantities not to vary appreciably within the element. Then, the assumption of local equilibrium implies that Gibbs' fundamental relation (combination of the first and second laws) is applicable to each volume element in the system.<sup>1,2</sup> Neglecting volume changes, Gibbs' relation leads to

$$ds = (1/T) du, \quad (11)$$

where  $s$  and  $u$  are entropy and internal energy per unit mass of the volume element. Taking a time derivative:

$$\rho \frac{ds}{dt} = \rho \frac{1}{T} \frac{du}{dt} = -\frac{1}{T} \nabla \cdot \mathbf{J}_q \quad (12)$$

The last equality follows from the energy balance in Eq. (1). Now, the identity

$$\nabla \cdot (f \mathbf{v}) = \nabla \cdot f \mathbf{v} + f \nabla \cdot \mathbf{v} \quad (13)$$

leads to the *entropy balance* equation:

$$\rho \frac{ds}{dt} + \nabla \cdot \mathbf{J}_s = \sigma. \quad (14)$$

In this equation,

$$\mathbf{J}_s \equiv \mathbf{J}_q/T \quad (15)$$

represents the *entropy flux density*, i.e., a *flow of entropy per unit volume and per unit time* across the boundaries of the volume element, and

$$\sigma \equiv \mathbf{J}_q \cdot \nabla(1/T) \quad (16)$$

a *rate of entropy production per unit volume* within this same element. It follows straightforwardly from the second law of thermodynamics that  $\sigma$  must be positive. It is worth noting that the entropy production  $\sigma$  appears as a scalar product between a flux  $\mathbf{J}_q$  and its conjugate force  $\nabla(1/T)$ ; this is a general result.

#### D. Local entropy flow and entropy production in a heat-conducting rod with Newtonian losses

We now wish to derive explicit expressions of entropy flow and entropy production in our particular system. Defining

$$\Phi(\mathbf{r}, t) \equiv \nabla \cdot \mathbf{J}_s \quad (17)$$

and making use of Eqs. (2) and (15), we get for the entropy flow in each point of the rod:

$$\Phi(\mathbf{r}, t) = -\lambda \nabla \cdot \frac{\nabla T}{T}. \quad (18)$$

Equivalently, the local entropy production follows straightforwardly from Eqs. (2) and (16):

$$\sigma(\mathbf{r}, t) = -\lambda \nabla T \cdot \nabla(1/T). \quad (19)$$

Temperature variations in the radial direction of the rod, compared to variations along the axial direction, can be assumed to be negligibly small due to the particular geometry of the system considered and the large thermal conductivity of a metallic rod. For this reason, we average local entropy flow and entropy production over the cross section of the rod, and get local quantities dependent only on temperature variation in the axial direction. Details of the calculation are given in the Appendix. The result for the entropy flow reads

$$\Phi(z, t) = -\frac{\lambda}{T} \frac{\partial^2 T}{\partial z^2} + \frac{\lambda}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 + \frac{2H}{r} \frac{1}{T} (T - T_R) \quad (20)$$

and for the entropy production

$$\sigma(z, t) = \frac{\lambda}{T^2} \left( \frac{\partial T}{\partial z} \right)^2. \quad (21)$$

We find a positive value, as expected. It is worth noting, too, that in the equilibrium situation, with no temperature gradients, entropy production would reduce to zero. Finally, from Eq. (14):

$$\rho \frac{ds}{dt} = \frac{\lambda}{T} \frac{\partial^2 T}{\partial z^2} - \frac{2H}{r} \frac{1}{T} (T - T_R). \quad (22)$$

In the steady state the second member vanishes identically, as follows from Eq. (6). This reflects the condition that in the steady state all thermodynamic potentials, including entropy, become time independent.

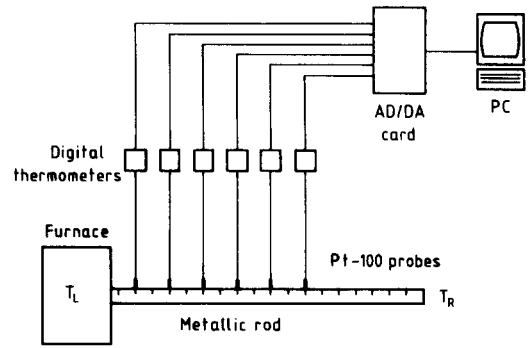


Fig. 1. Schematic drawing of the experimental setup and the data acquisition system.

#### E. Global entropy transfer and entropy production in a heat-conducting rod with Newtonian losses

Next, we examine the implications of the entropy balance equation for the system as a whole. Integrating Eq. (14) over the volume  $V$  of the system:

$$\rho \frac{d}{dt} \int_V s dV = - \int_V \nabla \cdot \mathbf{J}_s dV + \int_V \sigma dV, \quad (23)$$

which is usually rewritten as

$$\frac{dS}{dt} = \frac{d_e S}{dt} + \frac{d_i S}{dt}. \quad (24)$$

the first term in the right-hand member is actually, by the divergence theorem, an integral over the closed surface of the system. Hence, the equation states that the entropy change  $dS$  of a system is composed of two terms: the term  $d_e S$  which derives from the transfer of heat across the boundary of the system, and the term  $d_i S$  due to exchanges inside the system.

In our case, these magnitudes can be computed as

$$\frac{d_e S}{dt} = -A \int_0^L \Phi(z, t) dz \quad (25)$$

and

$$\frac{d_i S}{dt} = A \int_0^L \sigma(z, t) dz, \quad (26)$$

since integration over the cross section was already performed.  $L$  is the length of the bar and  $A = \pi r^2$  its cross-sectional area. Rescaling the result by  $\lambda \lambda$  we finally obtain:

$$\begin{aligned} \frac{d_e S}{dt} \propto \int_0^L \frac{1}{T} \left( \frac{\partial^2 T}{\partial z^2} \right) dz - \int_0^L \frac{1}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 dz \\ - \frac{2H}{\lambda r} \int_0^L \frac{1}{T} (T - T_R) dz \end{aligned} \quad (27)$$

and

$$\frac{d_i S}{dt} \propto \int_0^L \frac{1}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 dz \quad (28)$$

dependent only on experimentally accessible quantities.

### III. EXPERIMENTAL

The experimental system (Fig. 1) consists of an iron rod of 1.30 m length, and 0.015 m diameter. One end of the rod

is placed in a furnace at constant temperature ( $T_L$ ), and the rest of the rod is in thermal contact with the ambience, which behaves as a thermal bath at room temperature ( $T_R$ ). The length of the metallic rod has been chosen large enough to ensure that the second end of the rod remains constantly at the temperature  $T_R$ , in spite of its large thermal conductivity which guarantees a uniform temperature in the axial direction. Eighteen narrow holes (1.7 mm  $\varnothing$ ) have been drilled every 0.05 m along the rod, starting at the furnace, each one to house the Pt100 probe (20  $\times$  1.55 mm  $\varnothing$ ) of a resistance thermometer. Vaseline oil has been used to minimize the thermal contact resistance between the Pt100 probes and the rod. Six different resistance thermometers, calibrated against a high precision platinum resistance thermometer (Leeds & Northup 8163-QB), have been used simultaneously in every measurement. The thermometers provide a 5-mV/deg analog signal which is scanned and converted into a digital signal in a 14-bit ad card of a pc compatible computer. For each thermometer the computer takes an average of 50 readings and stores the result at a rate of 1 min. In our experiments a complete measurement lasts 90 min, an interval considered to be long enough for the rod to reach the steady state.

#### IV. RESULTS

We have performed measurements for two different initial conditions (uniform and nonuniform), and two different furnace temperatures (high and low). This results in four different cases to consider:

U.L.: Uniform initial temperature, large temperature gradient.

N.L.: Nonuniform initial temperature, large temperature gradient.

U.S.: Uniform initial temperature, small temperature gradient.

N.S.: Nonuniform initial temperature, small temperature gradient.

An initial condition of uniform temperature is established if the whole rod is at room temperature when one end of the rod is put into the furnace; a nonuniform initial condition is established by heating up a region of the rod above room temperature, using a cylindrical heating coil, before putting one end of the rod into the furnace. For every case we have taken a set of more than 20 complete measurements, which differ in the actual positions of the 6 temperature probes out of the 18 available.

Finally, since the ambient temperature may change slightly between measurements, this temperature has been subtracted from the temperature readings in the rod in every measurement, and the average over the whole set has been computed from the differences. To obtain the absolute temperature again, we have added to the average curve the average of ambient temperatures.

The error in the measurement of absolute temperatures, after calibration of the thermometers, is less than 0.1  $^{\circ}\text{C}$  and can be considered negligible compared to other sources of error. These include lack of reproducibility in the furnace temperature, in the thermal contact between furnace and rod, in the room temperature  $T_R$ , and also in the initial temperatures of the preheated region for nonuniform initial conditions. These sources of error are difficult to evaluate independently, but have been shown to result in temperature uncertainties from measurement to measurement less

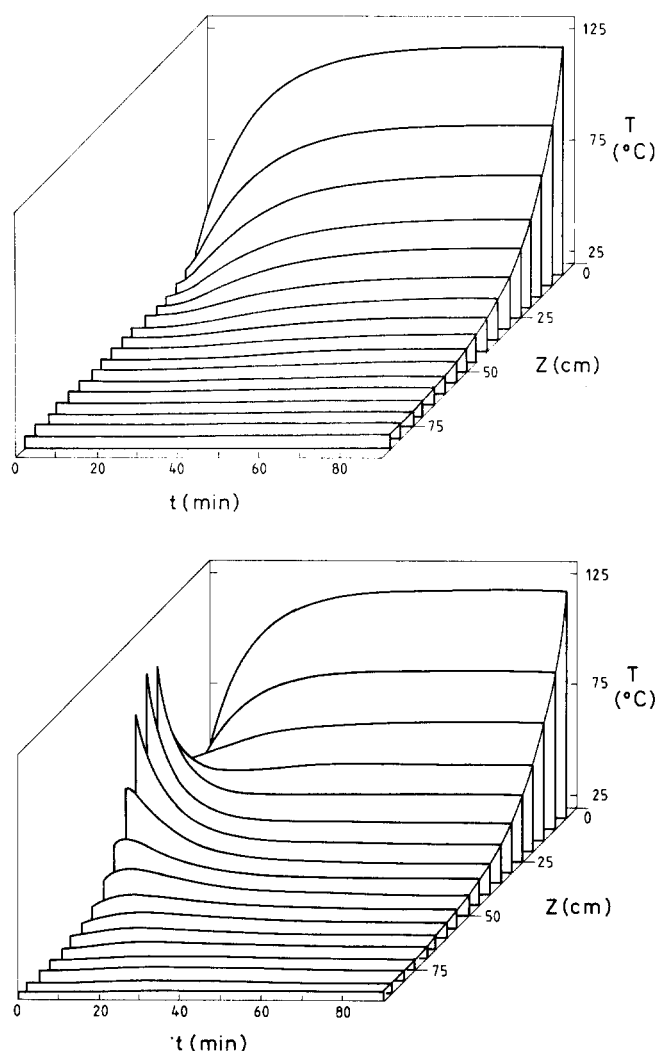


Fig. 2. Two examples of temporal evolution of the temperature distribution in the rod. Since the system is subjected to the same boundary conditions in both cases, the distributions evolve from two different initial conditions to the same steady state. Top: U.S., bottom: N.S. See text for details.

than  $\pm 3^{\circ}\text{C}$  near the furnace and  $\pm 0.5^{\circ}\text{C}$  at the other end of the rod.

#### A. Temperature distribution

Figure 2 shows the temporal evolution of the temperature distribution in the rod for the cases U.S. and N.S., corresponding to the same temperature gradient but two

Table I. Summary of experimental conditions and results.

Initial conditions	Boundary conditions	Steady State			
		$\beta(\text{m}^{-1})$	$d_i S/dt$ (arb. units)		
			analytic	calculated	
U.L. Uniform T	$T_L(K)$ $T_R(K)$	$7.0 \pm 0.4$	0.742	0.738	
N.L. Nonuniform T	494 300	$6.7 \pm 0.4$	0.765	0.768	
U.S. Uniform T	496 296	$6.5 \pm 0.3$	0.380	0.380	
N.S. Nonuniform T	428 299	$6.5 \pm 0.3$	0.389	0.392	

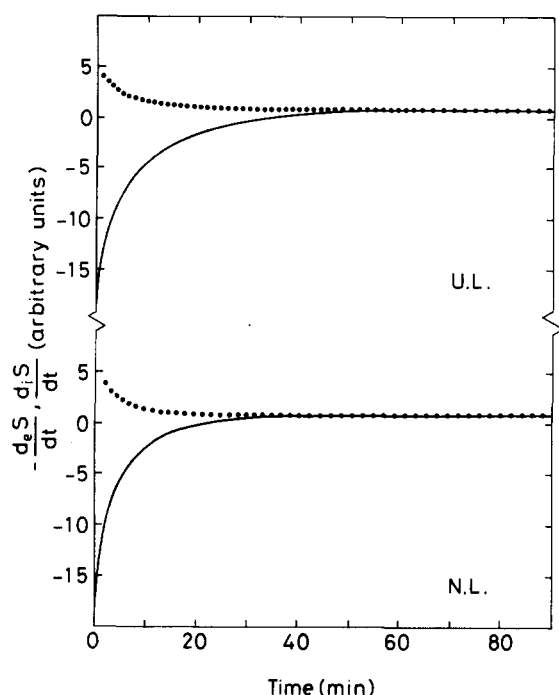


Fig. 3. Evolution of entropy production (dotted line) and entropy transfer (solid line) in two experimental situations with the same thermodynamic force but different initial conditions.

different initial conditions. It is shown how the temperature distribution progressively loses memory of the initial condition and, for long times, reaches a steady state which is independent of the initial conditions. Comparison with the other two cases U.L. and N.L. would reveal that the steady-state distribution depends only on the temperatures imposed at the boundaries, i.e., on the thermodynamic force applied to the rod.

In the two cases where the initial temperature distribution is not uniform along the rod, the steady state is reached earlier; the reason is that the central part of the rod is from the start at temperatures higher than the rest, what results in an additional heat flow which contributes to rise the temperature in positions far from the furnace.

The steady state reached in the four cases considered is very well described by an exponential distribution of temperatures. This fact, consistent with the prediction based on the heat transport equation, supports the choice of modeling heat losses to the environment with Newton's law of cooling. The coefficient  $\beta$  in the decreasing exponential, calculated by a logarithmic regression and shown in Table I, is practically the same in all cases with a slight tendency to be higher for large thermal gradients. This tendency could be related to the decrease of thermal conductivity with temperature in iron.<sup>8</sup> The regression coefficient is always better than 0.998 and the uncertainty reflects the range of values obtained using different ensembles of data in the regression.

## B. Entropy production and entropy transfer

To compute the time evolution of entropy production and entropy transfer in the system, we use a discrete version of Eqs. (27) and (28) equivalent to the one proposed by Lurié and Wagensberg.<sup>3</sup> In this way we avoid numeric

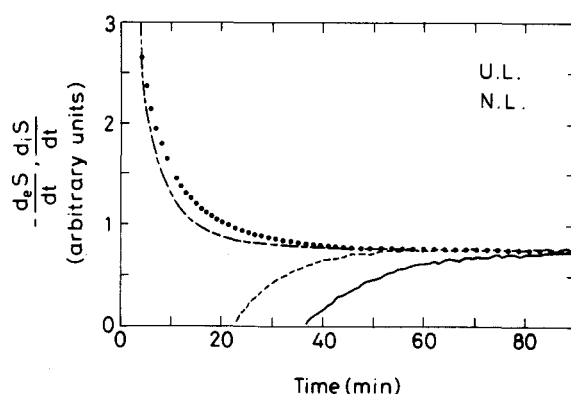


Fig. 4. Magnified view of the temporal evolution of entropy production and entropy transfer in two different cases: U.L., where  $d_e S/dt$  is represented by (···) and  $-d_e S/dt$  by (—), and, N.L., where  $d_e S/dt$  is represented by (---) and  $-d_e S/dt$  by (-.-).

computation of a second spatial derivative of the measured temperature distribution, which has been shown to be unreliable due to the low density of experimental points along the  $z$  axis. The discrete equations read:

$$\frac{d_e S}{dt} \propto -\frac{1}{l} \sum_{i=0}^{n-1} (T_{i+1} - T_i) \left( \frac{1}{T_{i+1}} - \frac{1}{T_i} \right), \quad (29)$$

$$\frac{d_e S}{dt} \propto -\frac{1}{l} \frac{1}{T_0} (T_1 - T_0) + \frac{1}{l} \frac{1}{T_n} (T_n - T_{n-1}) - \beta^2 l \sum_{i=1}^{n-1} \left( 1 - \frac{T_n}{T_i} \right). \quad (30)$$

Here,  $T_0$  represents the furnace temperature  $T_L$  and  $T_n$  the ambience temperature  $T_R$ ;  $l$  represents the length of the volume element in the discrete model of the rod. The equations have been rescaled in such a way that both geometric and thermal properties of the rod appear only through the length  $l$  and the coefficient  $\beta$  of the exponential distribution of temperatures in the steady state, both available from the experiments.

In order to improve the computation of differences along the  $z$  axis, where only 18 experimental points are available, we have interpolated 20 points between every two experimental ones. Thus  $l=0.05/20$  m. Nevertheless, in the first minutes the only relevant term in Eqs. (29) and (30) is the temperature gradient near the furnace; since this is determined by no more than three experimental points, the interpolated curve will never be very realistic, no matter the algorithm used. Hence, our calculation of  $d_e S/dt$  and  $-d_e S/dt$  cannot be considered very reliable for the first 4 or 5 mins.

Figure 3 shows the temporal evolution of these two quantities in two different cases: U.L. and N.L. The entropy production evolves monotonously toward a minimum corresponding to the steady state, as expected from Prigogine's theorem of minimum entropy production.<sup>4</sup> Since the entropy of the system becomes independent of time at the steady state, the entropy transfer equates the entropy production of the system (with opposite sign). In

addition, the asymptotic value of both entropy production and entropy transfer at large times is shown to be the same in the two cases U.L. and N.L., a direct consequence of the fact mentioned earlier that the steady state is independent of the initial conditions. The figure, finally, shows again that the final value of entropy production is reached earlier when the system is initially prepared with the nonuniform temperature distribution; this is consistent with the fact that the steady state is approached earlier, too.

The values reached by the entropy production at the steady state, calculated from Eq. (29), are listed in Table I under the title "calculated." It is worth noting that values corresponding to a large thermal gradient almost double values corresponding to a small gradient, giving a measure of the departure of the steady state from the condition of thermodynamic equilibrium in the system (uniform temperature). Values of the entropy production at the steady state can also be obtained from Eq. (28) substituting the steady-state solution given in Eq. (9) and entering the coefficient of the exponential distribution given in Table I. Results are listed in Table I under the title "analytic." Comparison to "calculated" results shows a remarkable agreement and demonstrates that the numerical procedure is highly reliable at least for the latest stages of evolution toward the steady state.

The curves in Fig. 3 are magnified and put together in Fig. 4. The calculation of  $d_i S/dt$  has turned out to be practically insensitive to the particular numeric procedure used and displays almost no fluctuations. On the contrary, calculated  $d_e S/dt$  curves are appreciably noisy and we have verified that they are influenced by the numeric procedure adopted. This is due to the strong dependence of  $d_e S/dt$  on the exponential coefficient  $\beta$  and particularly the temperature distribution near the furnace. It turns out that  $d_e S/dt$  is very sensitive to any change of these experimental factors, even within their error bars.

## V. CONCLUSIONS

The simple problem of heat conduction in a metallic rod subjected to a fixed temperature difference, with heat losses modeled by Newton's law of cooling, illustrates a number of concepts in the thermodynamics of nonequilibrium processes:

(i) The system evolves toward a steady state, characterized by the exponential temperature distribution predicted theoretically.

(ii) The system does not retain memory of the initial conditions; the steady state depends only on boundary conditions, i.e., on the thermodynamic force applied. Nevertheless, the initial conditions determine that a longer or shorter transient takes place before the system reaches the steady state.

(iii) The evolution toward the steady state is accompanied by a monotonic evolution of the entropy production toward a minimum value. In particular, the value of the entropy production is a measure of the separation of the steady state from thermodynamic equilibrium.

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## APPENDIX

We want to compute here the averages of  $\Phi(\mathbf{r}, t)$  and  $\sigma(\mathbf{r}, t)$  over the cross section of the rod. To this purpose, it is useful to introduce cylindrical coordinates  $(\rho, \varphi, z)$  with the  $z$  axis running along the axis of the rod. The gradient of a scalar magnitude  $T$  in these coordinates is given by

$$\nabla T = \hat{\rho}_0 \frac{\partial T}{\partial \rho} + \hat{\varphi}_0 \frac{1}{\rho} \frac{\partial T}{\partial \varphi} + \hat{k} \frac{\partial T}{\partial z}, \quad (\text{A1})$$

where  $\hat{\rho}_0$ ,  $\hat{\varphi}_0$ , and  $\hat{k}$  are the unit vectors along the radial, azimuthal, and axial directions, respectively. When there is cylindrical symmetry, as in our problem,  $T$  does not depend on the azimuthal angle  $\varphi$  and the second term of the right-hand member vanishes. It is also interesting to introduce the notation

$$\nabla_{\parallel} T = \nabla T - \hat{k} \frac{\partial T}{\partial z} \quad (\text{A2})$$

representing the contribution to the gradient from directions normal to the axial direction.

Following Eq. (18), the entropy flow for the one-dimensional system can be computed as

$$\Phi(z, t) = \frac{1}{A} \int_A \Phi(\mathbf{r}, t) dA, \quad (\text{A3})$$

where  $A = \pi r^2$  is the cross-sectional area of the rod. Thus

$$\Phi(z, t) = -\frac{\lambda}{A} \int_A \nabla \cdot \frac{\nabla T}{T} dA. \quad (\text{A4})$$

Dividing the scalar product into axial and radial contributions,

$$\nabla \cdot \frac{\nabla T}{T} = \frac{1}{T} \frac{\partial^2 T}{\partial z^2} - \frac{1}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 + \nabla_{\parallel} \cdot \frac{\nabla_{\parallel} T}{T} \quad (\text{A5})$$

and using the divergence theorem for the later, we are led to

$$\Phi(z, t) = -\lambda \left[ \frac{1}{T} \frac{\partial^2 T}{\partial z^2} - \frac{1}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 \right] - \frac{\lambda}{A} \oint_C \frac{\nabla_{\parallel} T}{T} dC, \quad (\text{A6})$$

where  $C$  represents the circular contour of the cross section and we have used the assumption that both  $T$  and its axial derivative are constant over the cross section of the rod at a given  $z$ . The integral in the last term can be computed using the boundary condition given by Newton's law of cooling [Eq. (4)] and taking the temperature constant around the contour  $C$ :

$$\oint_C \frac{\nabla_{\parallel} T}{T} dC = \oint_C -\frac{H}{\lambda} \frac{T - T_R}{T} dC \simeq -\frac{H}{\lambda} p \frac{T - T_R}{T}, \quad (\text{A7})$$

where  $p = 2\pi r$  is the contour length. The final result for the entropy flow reads

$$\Phi(z, t) = -\frac{\lambda}{T} \frac{\partial^2 T}{\partial z^2} + \frac{\lambda}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 + \frac{2H}{r} \frac{T - T_R}{T}. \quad (\text{A8})$$

On its turn, the entropy production for the one-dimensional system can be computed from Eq. (19) as

$$\sigma(z, t) = \frac{1}{A} \int_A \sigma(\mathbf{r}, t) dA. \quad (\text{A9})$$

Using Eq. (A1) we get:

$$\sigma(z,t) = \frac{\lambda}{A} \int_A \frac{1}{T^2} \left( \frac{\partial T}{\partial z} \right)^2 dA + \frac{\lambda}{A} \int_A \frac{1}{T^2} \left( \frac{\partial T}{\partial \rho} \right)^2 dA. \quad (\text{A10})$$

The first integral is easily evaluated taking, as before, the assumption that both  $T$  and its axial derivative are constant over the cross section of the rod at a given  $z$ . The second integral is zero since the radial derivative of the temperature is null everywhere except at a single point  $\rho=r$ . Hence,

$$\sigma(z,t) = \frac{\lambda}{T^2} \left( \frac{\partial T}{\partial z} \right)^2. \quad (\text{A11})$$

<sup>1</sup>S. R. de Groot and P. Mazur, *Nonequilibrium Thermodynamics* (Do-

ver, New York, 1984).

<sup>2</sup>W. Yourgrau, A. van der Merwe, and G. Raw, *Treatise on Irreversible and Statistical Physics* (Dover, New York, 1982).

<sup>3</sup>D. Lurié and J. Wagensberg, "Concepts of nonequilibrium thermodynamics in discrete model of heat conduction," *Am. J. Phys.* **48**, 868–872 (1980).

<sup>4</sup>I. Prigogine, *Étude thermodynamique des phénomènes irréversibles* (Éditions Desoer, Liège, France, 1947).

<sup>5</sup>A. Díaz-Guilera, "On heat conduction in one-dimensional solids," *Am. J. Phys.* **58**, 779–780 (1990).

<sup>6</sup>J. L. Horst and M. Weber, "Joule's experiment modified by Newton's law of cooling," *Am. J. Phys.* **52**, 259–261 (1984).

<sup>7</sup>P. A. Maurone and C. Shiomos, "Newton's law of cooling with finite reservoirs," *Am. J. Phys.* **51**, 857–859 (1983).

<sup>8</sup>N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Holt, Rinehart and Winston, New York, 1976), p. 21.

### WEISSKOPF AND WIGNER

Eugene Wigner, my doctor father, and I, we talked about some paper that just appeared in one of the periodicals, and it had a very interesting title, and I showed that to Wigner, and Wigner read the title and said "Oh, that's an interesting problem," and then he took a piece of paper and showed me how he calculates what the title seems to have promised, and so I said I am going to go and read that paper, and he said "Oh, no, don't read that paper because either it's what I have just written down, and you don't *need* to read that paper, or it is *not* what I have I written down, and then it's wrong."

Victor F. Weisskopf (1976).