where $B = q/(AkT)$, A being the diode quality factor, R_s and R_{sh} the series and shunt resistances, I_{ph} the photocurrent, and I_0 the reverse saturation current. For large values of R_{sh} and small values of R_s, which is the case with good quality solar cells, Eq. (1) gives, with $V = 0$ and $I = 0$, respectively,

$$I_{sc} = I_{ph}, \qquad (2)$$

$$V_{oc} = (AkT/q)\ln(1 + I_{ph}/I_0). \qquad (3)$$

From Eq. (3) it can be seen that the decrease which is observed in practice in V_{oc} with increasing temperature is a consequence of a large increase in I_0, given that $I_{ph} \gg I_0$. Using published results[8-10] for the temperature dependence of the intrinsic carrier density n_i and of the diffusion length D (through the lifetime and mobility), it is found that

$$I_0 = C_0 T^\alpha \exp(-E_g/kT), \qquad (4)$$

with α values between 0.5 and 2.0.

The temperature dependence of I_0 is dominated by the exponential factor, therefore the exact value of α is not critical. With consideration of $\alpha = 1$ for our range of measurements, Eqs. (3) and (4) give

$$V_{oc} = \frac{AE_g}{q} - \frac{AkT}{q}(\ln C_0 + \ln T - \ln I_{sc}). \qquad (5)$$

The variation of the energy gap E_g with temperature above room temperature is -2.3×10^{-4} eV/K (Ref. 11) for silicon. Its contribution is negligible in Eq. (5) and the more so because it participates in a corresponding increase in the short-circuit current I_{sc}. Equation (5) is the equation of the line displayed in Fig. 3. The experiment substantiates the fact that the variations of $\ln T$ and of $\ln I_{sc}$ are not of major significance.

From Fig. 3 it is possible to determine graphically A and C_0 from the ordinate of the point of null-abscissa and from the slope:

$$A = 1.20 \pm 0.08,$$

$$C_0 = (3.0 \pm 0.7) \times 10^7 A/K.$$

Then, through Eq. (4), we can calculate the variation of I_0 with the temperature and, through Eq. (3), predict the values of V_{oc} in different conditions.

V. CONCLUSION

These experiments about photovoltaic cells can be readily carried out in any university physics laboratory. They illustrate the effect of temperature on the cell performance and lead to a first approximation of a theoretical interpretation. They should be considered as an introduction to the fast growing field of solar concentration studies.

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Dispersion relations for waves on liquid surfaces

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The dispersion relations for waves on liquid surfaces are derived in terms of the Lagrange equations of an oscillator, with gravity and surface tension as restoring mechanisms. This derivation only requires the continuity equation and the assumption of irrotational flow, but not the Bernoulli equation.

The following derivation of the dispersion relation for waves on liquid surfaces is based on the Lagrange equation of motion rather than on the Euler–Bernoulli equation.[1,2] It may thus be useful as a simple illustration of the application of the Lagrange equations to waves in continuous systems, for undergraduates without prior knowledge of continuum mechanics. It does require use of the continuity equation for incompressible liquids, as well as the concept

of irrotational flow and velocity potential. It also uses the general principle that if $V = Aq^2$ and $T = B(\dot q)^2$, the system is a harmonic oscillator with angular frequency ω, where $\omega^2 = A/B$.

Consider a single Fourier component of the vertical displacement η of the surface, of wavenumber k, so that

$$\eta(x) = a \cos kx. \qquad (1)$$

We shall treat a as the generalized coordinate. The potential energy due to the surface deformation arises from gravity and from surface tension. The energy of removing fluid to a depth η below the surface over an area $z_0\,dx$, and raising it to a height η above the mean level over an equal area, is

$$V_g = \rho g \eta^2 z_0\,dx \qquad (2)$$

and is associated with a total area $2z_0\,dx$. Here ρ is the density and g the acceleration due to gravity.

The energy associated with surface tension S is given in terms of the increase of surface area as

$$V_s = S z_0\,dx\left\{\left[1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right]^{1/2} - 1\right\}$$
$$\simeq \frac{1}{2}S\left(\frac{\partial\eta}{\partial x}\right)^2 z_0\,dx. \qquad (3)$$

This is an approximation, which neglects terms of order $(\eta k)^4$. Substituting the waveform of Eq. (1), adding it to the term V_g and integrating over a finite length l, equal to an integral number of wavelengths, one obtains for the potential energy due to the surface disturbance over an area lz_0

$$V = \frac{1}{4}lz_0(\rho g + Sk^2)a^2. \qquad (4)$$

To find the kinetic energy, we must integrate over the entire velocity field, which arises below the surface in consequence of the surface displacement $\eta(x)$. As in the conventional treatment one assumes the flow to be irrotational, so that the velocity $\mathbf{v}$ can be derived from a velocity potential Φ, i.e., $\mathbf{v} = -\operatorname{grad}\Phi$. The equation of continuity for an incompressible liquid is $\operatorname{div}\mathbf{v} = 0$, which thus becomes

$$\operatorname{grad}\operatorname{div}\Phi = \nabla^2\Phi = 0 \qquad (5)$$

at any instant of time. Letting Φ have the same x dependence as η, and letting it depend on y, the depth below the surface, the functional form of $\Phi(x,y)$ consistent with (4) is

$$\Phi(x,y) = (C_1 e^{-ky} + C_2 e^{ky})\cos kx. \qquad (6)$$

In the case of an infinitely deep liquid, $C_2 = 0$, but for a finite depth D the boundary condition at the bottom $(y = D)$ is that $v_y = 0$ or that $\partial\Phi/\partial y]_{y=D} = 0$. This enables one to write (5) in the form

$$\Phi(x,y) = A\cosh[(y-D)k]\cos kx. \qquad (7)$$

The velocity potential Φ and the wave displacement η are related by the condition that, at the surface,

$$\frac{\partial\eta}{\partial t} = \frac{\partial\Phi}{\partial y}, \qquad (8)$$

since both are a measure of the vertical component of the velocity at the surface. Strictly speaking this holds at the surface ($y = -\eta$), but for small amplitudes it can be taken to hold at $y = 0$.

The kinetic energy of the fluid of depth D is given by

$$T = \frac{1}{2}z_0\rho\int dx\int_0^D dy\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right]. \qquad (9)$$

Again, the lower limit of the integral should be $y = -\eta$. The error thus introduced is cubic in η, and hence more important than the error incurred in V_s of Eq. (3).

Writing $k(y - D) = \alpha$ and $kx = \beta$, one finds

$$\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 = A^2 k^2(\cosh^2\alpha\sin^2\beta + \sinh^2\alpha\cos^2\beta)$$
$$= A^2 k^2(e^{2\alpha} + e^{-2\alpha}$$
$$+ 2\sin^2\beta - 2\cos^2\beta). \qquad (10)$$

The two terms in β cancel when integrating over x or β. Integrating over y

$$(e^{2\alpha} + e^{-2\alpha})dy = \frac{1}{2}k(e^{2kD} - e^{-2kD}). \qquad (11)$$

Also, A can be expressed in terms of $\dot{a}$ through (8), i.e.,

$$\dot{a} = Ak\sinh(-kD), \qquad (12)$$

so that finally one obtains for the total kinetic energy

$$T = (1/16)\rho lz_0 k(e^{2kD} - e^{-2kD})4k^{-2}(e^{kD} - e^{-kD})^{-2}\dot{a}^2$$
$$= \rho lz_0[k\tanh(kD)]^{-1}\dot{a}^2. \qquad (13)$$

Now V of Eq. (3) and T of Eq. (13) describe a harmonic oscillator of the dynamical variable a, with angular frequency $\pm\omega$ where

$$\omega^2 = (gk + Sk^3/\rho)\tanh(kD), \qquad (14)$$

which is the general dispersion relation for surface waves on liquids of any depth with both gravity and surface tension as restoring forces. It leads to the deep and shallow limits when $kD \gg 1$ and $kD \ll 1$, respectively.

Similar arguments can be used to derive the velocity of elastic waves, but they do require a knowledge of the strain energy in terms of the elastic moduli and the strain components. The case of a linear chain of equal masses and equal springs is also conveniently done by means of the Lagrange equation. Yet the case of surface waves discussed here requires a minimum of prior knowledge of continuum mechanics, and is conceptually at least as simple as the case of the stretched string, sometimes used as an illustration of a continuous system,[3] since in that last mentioned case the expression for the potential energy is more difficult to comprehend.

The introduction of surface waves also permits the instructor to discuss a wide range of physical topics such as waves on beaches, tsunamis, tidal bores, and related topics,[4] as well as surface tension measurements by capillary waves on liquid metals and liquid helium. Finally, since ripple tanks are frequently used to illustrate wave propagation and diffraction, it is appropriate for students to have some appreciation of the dispersion properties of these waves and of the consequent limitations on the design of ripple tanks.

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