On heat conduction in one-dimensional solids

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A simple and intuitive procedure to derive the equation of heat conduction in a one-dimensional solid is presented. A boundary condition at the surface for a three-dimensional solid becomes a new term of heat exchange with an external bath in the equation for the temperature in one dimension.

I. INTRODUCTION

Heat conduction in solids has been studied extensively for many years, and a complete set of solutions involving different kinds of boundary conditions, dimensions, and geometries is available in the literature. One of the best known references is the monograph of Carlslaw and Jaeger.[1] Our aim in this article is not to solve the equation of conduction in a special case, but to provide a procedure by which one can derive an equation for the temperature in a one-dimensional solid in which energy flows longitudinally by means of heat conduction and through the lateral surface by heat exchange with an external bath.

In Sec. II, our point of departure involves the energy balance equation and Fourier's law, from which one is able to derive the equation for the temperature in a solid. The solution of this equation must satisfy certain boundary conditions at the surface of the solid, the most general being Newton's law of cooling. In Sec. III we define an average temperature and, by means of very simple algebra, the boundary condition at the surface then appears as an additional term of heat exchange (with the external bath) in the equation of conduction. The same line of reasoning also may be applied to a two-dimensional problem.

II. HEAT CONDUCTION IN SOLIDS

The law of conservation of internal energy u in a solid reads[2]

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{J}_q, \tag{1}$$

where $\mathbf{J}_q$ is the heat flux. This differential equation is supplemented with the phenomenological equation (Fourier's law)

$$\mathbf{J}_q = -\lambda \nabla T, \tag{2}$$

λ being the thermal conductivity of the solid. In principle, λ depends on $\mathbf{r}$ through the temperature. For a solid, the internal energy depends only on the temperature, $u = u(T)$,

$$du = \left(\frac{\partial u}{\partial T}\right)_v dT \tag{3}$$

and Eq. (1) becomes

$$\rho c \frac{dT}{dt} = \nabla \cdot (\lambda \nabla T), \tag{4}$$

where c is the heat capacity per unit volume.

In solving the previous equation, one must take into account boundary conditions as well as the initial temperature distribution. The most common boundary conditions at the surface are: a fixed temperature, thermal isolation, or a prescribed heat flux. However, there is a boundary condi-

tion, known as Newton's law of cooling or the radiation boundary condition, that has two of the previous boundary conditions as particular cases:

$$\lambda \hat{n} \cdot (\nabla T)_s + H(T_s - T_{\text{ext}}) = 0. \tag{5}$$

Here, H represents the surface conductance, a term that depends on the characteristics of both media. This equation states that the normal heat flow through the surface is proportional to the difference in temperature between the surface and the external bath. In the limiting case $H \to 0$, the surface is perfectly insulating. For $H \to \infty$, the surface is perfectly conducting, the temperature at the surface equals the temperature of the bath, and the boundary condition amounts to a prescribed temperature.

III. ONE-DIMENSIONAL SYSTEMS

Considering a system as one-dimensional is a simplification one can make when the dependence of physical quantities on one particular dimension is much more important than it is on the others, and an average over these less important dimensions represents an acceptable approximation.

Let us consider a rod of constant cross section S (see Fig. 1) in which one defines a temperature $T(x,t)$, x being the direction along the rod axis, as an average over the cross section,

$$T(x,t) = \frac{1}{S} \int_S T(\mathbf{r},t) dS. \tag{6}$$

Since the surface S does not depend on time, one gets from Eq. (4)

$$\frac{\partial T(x,t)}{\partial t} = \frac{\alpha}{S} \int_S \nabla^2 T(\mathbf{r},t) dS, \tag{7}$$

where $\alpha = \lambda/\rho c$ is the thermal diffusivity. Equation (7) can be written in the following way:

$$\frac{\partial T(x,t)}{\partial t} = \frac{\alpha}{S} \int_S \frac{\partial^2 T(\mathbf{r},t)}{\partial x^2} dS + \frac{\alpha}{S} \int_S \nabla_\parallel^2 T(\mathbf{r},t) dS, \tag{8}$$
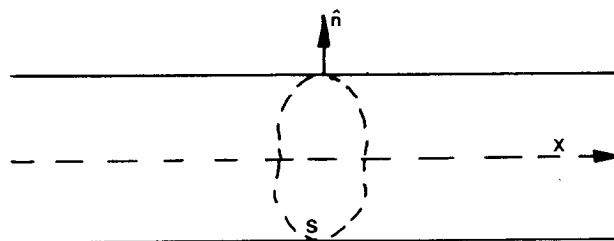

Fig. 1. Scheme of the system.

where $\nabla^2_{\parallel}$ corresponds to the Laplace operator in coordinates parallel to the cross section (i.e., perpendicular to the x axis). Since the cross section does not depend on x,

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{\alpha}{S} \int_S \nabla^2_{\parallel} T(\mathbf{r},t) dS. \qquad (9)$$

The last integral can be evaluated with the aid of the divergence theorem

$$\int_S \nabla^2_{\parallel} T(\mathbf{r},t) dS = \oint \hat{n} \cdot \nabla T(\mathbf{r},t) dl. \qquad (10)$$

The right-hand side of Eq. (10) must be computed along the contour defined by the cross section S. The unit vector $\hat{n}$ is perpendicular to this contour and to the x axis. By means of Newton's law of cooling [cf. Eq. (5)] one obtains

$$\oint \hat{n} \cdot \nabla T(\mathbf{r},t) dl = -\frac{H}{\lambda} \oint [T(\mathbf{r},t) - T_{\text{ext}}] dl. \qquad (11)$$

Approximating the average temperature over the contour by the average temperature over the cross section, which is a good approximation since one of the assumptions is that the temperature varies slowly over the cross section, we obtain the following identity:

$$\frac{H}{\lambda} \oint [T(\mathbf{r},t) - T_{\text{ext}}] dl = \frac{H}{\lambda} p[T(x,t) - T_{\text{ext}}], \qquad (12)$$

where we have introduced p as the perimeter of the cross section. This procedure enables us to write the equation for the temperature of a one-dimensional solid as

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} - \beta [T(x,t) - T_{\text{ext}}]. \qquad (13)$$

$\beta = Hp/S\rho c$ is the heat transfer coefficient at the surface of the conductor. It contains not only the physical properties of both the solid and the external bath, but also the geometric characteristics of the medium through its cross section and perimeter.

Equation (13) can be solved exactly only in certain special cases, since it involves boundary conditions as well as the initial temperature distribution along the conductor. One of those cases corresponds to the propagation of heat from a point source at the origin; the solution of this problem can be obtained straightforwardly by means of Green's functions.

The situation changes when one deals with steady-state temperature distributions since Eq. (13) then becomes an ordinary differential equation:

$$0 = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} - \beta [T(x,t) - T_{\text{ext}}]. \qquad (14)$$

This equation is not, in general, exactly solvable because nonlinearities in the transport coefficients may arise. If one assumes that the variations in temperature are sufficiently small, these nonlinearities are eliminated. In this case, Eq. (14) may be solved exactly, the result depending only on the bath temperature and the boundary conditions at the ends of the conductor.

To extract some information about this simpler problem, let us consider a conductor of length $2L$ with the origin of coordinates at its center. For fixed temperatures at the ends of the conductor and in the external bath, the temperature distribution is $T_s(x) \sim \cosh(x/\mu)$, where $\mu = \sqrt{\alpha/\beta}$ plays the role of a characteristic length accounting for both of the processes taking place in the system. At this point, it is interesting to note that all of the information contained in this equation can be written in terms of a dimensionless coordinate x/μ.

Another case of interest involves a heat source located at the origin and maintained at a fixed temperature. Such a configuration allows one to study long rods without ignoring the effects due to boundary conditions. Clearly, this problem presents a discontinuity that can be avoided by solving Eq. (14) separately for positive and negative x. The steady-state temperature distribution then becomes $T_s(x) \sim \sinh[(L - |x|)/\mu]$, assuming that the temperature of the rod ends is equal to that of the bath, a good approximation for long conductors.

A spatially varying bath temperature will give rise to a solution for the steady-state temperature that reflects the character of the bath variations. For example, an oscillating bath temperature will lead to an oscillating variation in the temperature along the rod. When the heat transfer coefficient at the surface vanishes, the effect of the bath is lost and only heat conduction along the rod must be considered. In this simplest case, the steady-state temperature varies linearly with the spatial coordinate (depending on the boundary conditions, it could be constant).

Finally, we would like to remark that although Eq. (13) cannot be solved analytically in general, there exist numerical methods to deal with this kind of equation. From our point of view, the use of these numerical tools in computers can be of great pedagogical value for the understanding of heat conduction in one-dimensional solids. They can be useful, for example, to investigate the influence of different coefficients, initial temperature distributions and boundary conditions, and also to see how a system relaxes to an equilibrium state or to a nonequilibrium stationary state. Even a random initial distribution of temperatures would be of interest in understanding how statistical fluctuations decay in a thermodynamic system.

ACKNOWLEDGMENTS

[1] H. S. Carlslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford U.P., Oxford, 1959).

[2] S. R. de Groot and P. Mazur, *Non-Equilibrium Thermodynamics* (Dover, New York, 1984).