PHYC40600 Physics with Astronomy and Space Science Lab 2; Diffraction and Fourier Optics

Daragh Hollman
daragh.hollman@ucdconnect.ie
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The aims of this report were to investigate the links between diffraction and the Fourier transform, along with describing a Fourier filtering system and demonstrating several basic filtering cases. The diffraction pattern for single slits of varying widths and a double-slit example were imaged and compared to computational examples using the Fast Fourier Transform. The process of image filtering through Fourier optics was described and an apparatus for demonstrating simple filtering scenarios was constructed and used to outline four basic use cases.

I. INTRODUCTION AND THEORY

Fourier optics is in essence the description of classical optics through the use of the Fourier transform [1]. The term was first described in the 1930s with some previous work dating as early as the 1860s [2]. The broad uses of optics and mathematical finesse of the Fourier transform results in complex and diverse applications across a number of fields. Such applications include image edge detection algorithms [3], image restoration [4], and adaptive optics [5] to name a few important examples.

A. Relating Diffraction and the Fourier Transform

1. *Diffraction*

Diffraction is the optical process which governs how light behaves as it passes between or around objects. From Huygens' Principle we can treat an advancing wave front as many individual point sources of spherical waves (called wavelets) progressing at the same speed. These wavelets interfere with each other to form the next wavefront at a later time [1]. An example of this for plane and spherical waves can be seen in figure 1 [1]. If we then envision this process - as it would pass through or around an obstacle, for instance a single slit - it is clear how the resulting wavefront will be the superposition of the wavelets passing through the slit, and it becomes obvious how this results in the diffraction we see experimentally [1]. An example of this is shown in figure 2 [6].

When mathematically defining diffraction we are required to make some assumptions for simplicity. One such assumption is known as Fraunhofer (far-field) diffraction, which requires the source of light and the screen on which the output is observed to both be far enough away that they can be treated as effectively infinitely far away, giving a planar wave [1, 7]. An alternative treatment of diffraction is known as Fresnel (near-field) diffraction - used when the wavefront cannot be considered planar, and the curvature is accounted for [1].

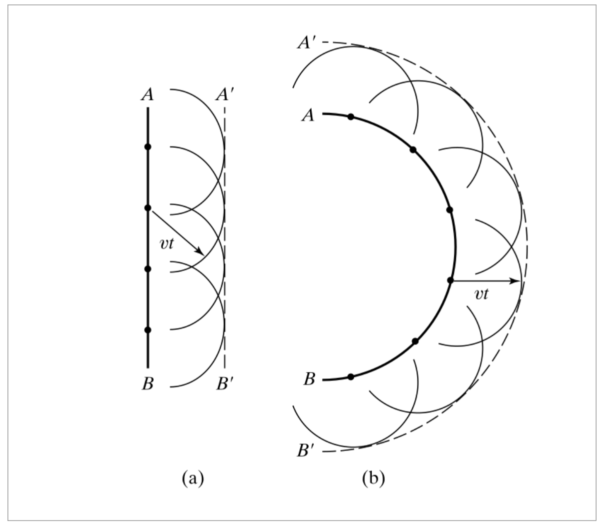


FIG. 1: Sketches of Huygens' Principle for plane waves (a), and spherical waves (b) [1].

2. *Fourier Transform*

The Fourier transform of a function $f(x)$ is defined as:

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ixk}dx \tag{1}$$

which transforms a function in spatial coordinates, x, to a function in frequency coordinates, k [8]. We can use a similar transform to return to the original function [8]:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{ixk}dk \tag{2}$$

These two equations together are known as the one-dimensional Fourier transform pair [1, 9].

It can be found that an analogy can be made between an aperture and its diffraction pattern; This analogous relationship being described by the Fourier transform [9]. We find that Fraunhofer diffraction can be described by the Fourier transform of the electric field amplitude distribution in the object plane [1]. Pedrotti goes into detail showing how the Fraunhofer diffraction pattern - more specifically the amplitude distribution in the far field,

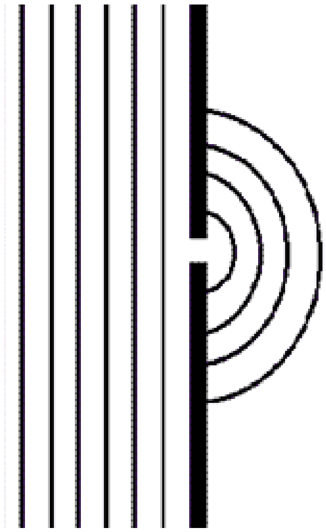


FIG. 2: An example of how Huygens' Principle yields diffraction. Here the slit is, in theory, small enough such that only one wavelet is present in the slit. Hence, we see circular wavefronts emerging from the slit [6].

$A_P(k_x, k_y)$ - is within approximation, simply the two-dimensional Fourier transform of the aperture function $E_A(x, y)$, the source amplitude at the aperture [1].

$$A_P(k_x, k_y) = \iint E_A(x, y) e^{i(xk_x + yk_y)} dxdy \qquad (3)$$

$$E_A(x, y) = \frac{1}{(2\pi)^2} \iint e^{-i(xk_x + yk_y)} dk_x dk_y \qquad (4)$$

B. Single Slit Diffraction Pattern

1. Theory

The single slit aperture is a simple example of a Fraunhofer diffraction pattern [9]. A generalised example is shown in figure 3, a top down view of a slit of with a uniformly illuminated [9]. In Fraunhofer diffraction, the diffraction pattern is theoretically observed at infinity. A lens is placed after the aperture to image the diffraction plane onto the screen [1]. The one dimensional diffraction pattern - along the direction of the slit width - can be determined using Huygens' Principle to yield a sinc function (as derived in full by Steward [9]), however, as discussed earlier this can be done by use of the Fourier transform as follows.

2. Derivation from the Fourier Transformation of the Aperture

We can define a top-hat function in a piece-wise manner as follows:

$$\sqcap\left(\frac{x}{a}\right) = \begin{cases} 1, & |x| \leq \frac{a}{2} \\ 0, & |x| > \frac{a}{2} \end{cases} \qquad (5)$$

where the top-hat function is centred around $x = 0$ with width a. The Fourier transform is then given as follows:

$$F(k) = \mathcal{F}\left[\sqcap\left(\frac{x}{a}\right)\right] = \int_{-\infty}^{\infty} \sqcap\left(\frac{x}{a}\right) e^{-ikx} dx \qquad (6)$$

Outside of $-\frac{a}{2} \leq x \leq \frac{a}{2}$ this integral equates to zero, and we can hence reduce it to the following where the top-hat is equal to one:

$$\begin{aligned} F(k) &= \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-ikx} dx \\ &= \frac{-1}{ik} \left[e^{-ikx}\right]_{-\frac{a}{2}}^{\frac{a}{2}} \\ &= \frac{1}{ik} \left(e^{\frac{ikx}{2}} - e^{\frac{-ikx}{2}}\right) \\ &= \frac{2}{k} \sin\left(\frac{ak}{2}\right) \\ &= a \,\mathrm{sinc}\left(\frac{ak}{2}\right) \end{aligned}$$

We can represent our single slit as a thin rectangular aperture centred on the origin with dimensions $(\Delta x, \Delta y) = (a, b)$. For simplicity, we can calculate the one dimensional Fourier transforms along each direction, x and y, and only later combine them. Treating the slit as a normalised top-hat function along each direction (see figure 3b) we can determine the Fourier transform as follows:

$$X: \quad \mathcal{F}_x\left[\sqcap\left(\frac{x}{a}\right)\right] = a \,\mathrm{sinc}\left(\frac{ak_x}{2}\right) \qquad (7)$$

$$Y: \quad \mathcal{F}_y\left[\sqcap\left(\frac{y}{b}\right)\right] = b \,\mathrm{sinc}\left(\frac{bk_y}{2}\right) \qquad (8)$$

We know from above that the diffraction pattern is related to the Fourier transform of the aperture. Then intensity is hence the square of the Fourier transform of the aperture [10]. Again splitting into x and y we have:

$$I_x = \left(\mathcal{F}_x\left[\sqcap\left(\frac{x}{a}\right)\right]\right)^2 = a^2 \,\mathrm{sinc}^2\left(\frac{ak_x}{2}\right) \qquad (9)$$

$$I_y = \left(\mathcal{F}_y\left[\sqcap\left(\frac{y}{a}\right)\right]\right)^2 = b^2 \,\mathrm{sinc}^2\left(\frac{bk_y}{2}\right) \qquad (10)$$

We can as such expect to see the single slit diffraction pattern taking the form of a sinc function along both axes. An example sinc function is plotted in figure 4 for visual reference. The two dimensional intensity is then given by the product of these terms [10]:

$$I(x, y) = I_x I_y = a^2 b^2 \,\mathrm{sinc}^2\left(\frac{ak_x}{2}\right) \mathrm{sinc}^2\left(\frac{bk_y}{2}\right) \qquad (11)$$

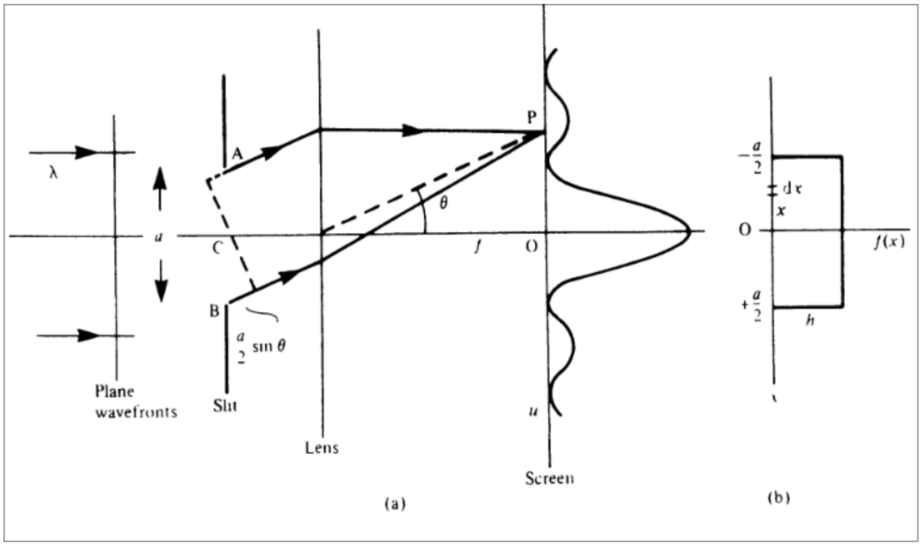


FIG. 3: A generalised example of the single slit set-up and diffraction pattern [9]. The sketch shows the traditional schematic for determining the diffraction pattern using Huygens' Principle (a), but also shows the top-hat function for which the Fourier transform yields the same sinc function (b).

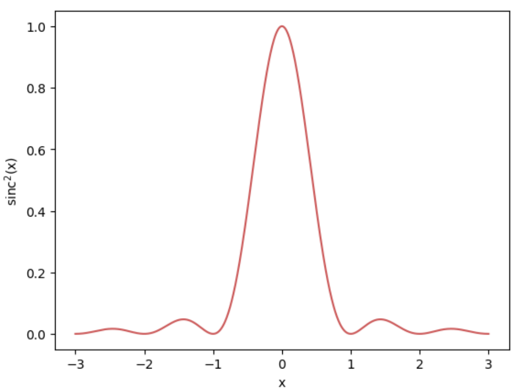


FIG. 4: An example of an arbitrary sinc^2 function. Where $\text{sinc}(x) = \frac{\sin(x)}{x}$

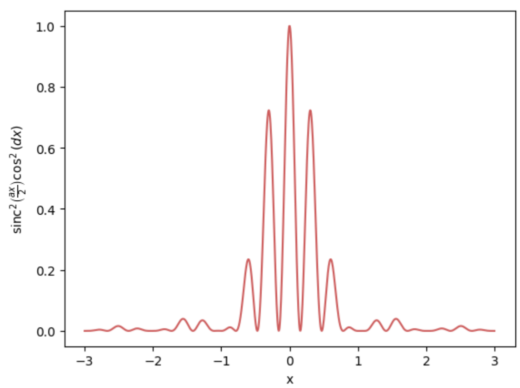


FIG. 5: A normalised example of the form of the double slit diffraction pattern output (equation 13) with parameters $a = 2$, $d = 10$

3. Double Slit Diffraction Pattern

The double slit (sometimes referred to as Young slits [11]) process follows the same principles as outlined for the single slit. The spatial pattern output from each slit is the same as described previously, but the combined result of the two for any direction depends on the path length difference between both in that direction [9]. The same method of determining this output can be used as in the single slit, however this time, the aperture function must contain two top-hat functions - the Fourier transform of which, can be computed by use of the shift theorem [9, 10]:

$$\mathcal{F}\left[f(x - x_0)\right] = F(k)e^{ikx_0} \tag{12}$$

where the transform of the shifted function is related to the stationary function. The resulting intensity in the slit width direction takes the form:

$$I_x = 4a^2 \,\text{sinc}^2\left(\frac{k_x a}{2}\right)\cos^2(k_x d) \tag{13}$$

where a is the width of the two slits, and d is the distance between them [10, 12]. An example of this function is shown in figure 5. We see in the peaks of the regular sinc function oscillations between 0 and the sinc function.

C. Optical Filtering

In the previous section we see the use of a lens to image the far field diffraction pattern onto a screen, and in doing so, it performs a Fourier transform of the object plane. [1, 13]. It is hence this process - which, in an imaging system - allows the manipulation (using masks or filters) of the Fourier transform of the image

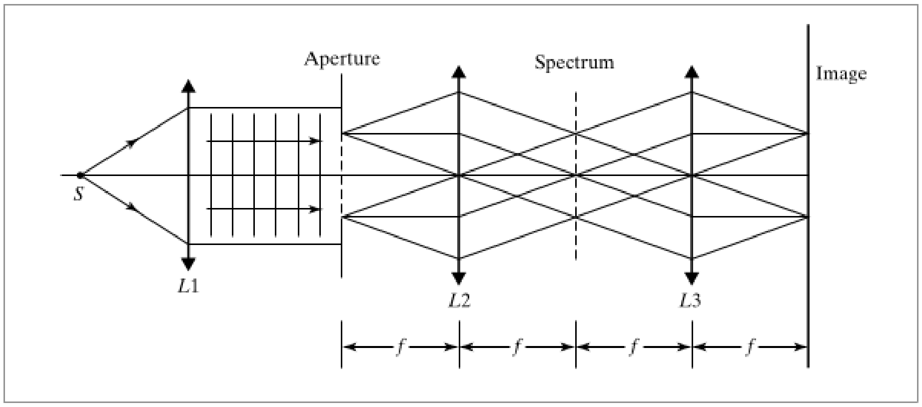


FIG. 6: An example system used for spatial filtering [1]. A source (S) is collimated by a lens (L1) such that parallel rays illuminate at the aperture. A second lens (L2) - one focal length away from the aperture - forms the Fourier transform at the spectrum plane. A third lens (L3) receives the modified Fourier transform from the spectrum plane and, as L2 transformed the aperture, L3 performs a Fourier transform on the transform of the original aperture, and hence returns the output to this original aperture (with the addition of the spatial filtering performed) [1]

to modify the final image. This as a whole is known as optical filtering [1]. An example configuration of lenses to achieve this is shown in figure 6 [1]. A source (S) is collimated by a lens (L1) such that parallel rays illuminate at the aperture. A second lens (L2) - one focal length away from the aperture - forms the Fourier transform at the spectrum plane. It is this spectrum plane where modifications to the Fourier transform to affect the image can be made. A third lens (L3) receives the modified Fourier transform from the spectrum plane and, as L2 transformed the aperture, L3 performs a Fourier transform on the transform of the original aperture, and hence returns the output to this original aperture (with the addition of the spatial filtering performed) [1]. We find optical filtering to be an incredibly useful means for optical image processing, with many use cases including the removal or isolation of periodic structures [10] and edge detection [10, 13]. Optical filters are typically masks placed to block specific information in the Fourier plane to create a desired effect on the output image [13].

This report aims to demonstrate the ease and effectiveness of simple Fourier optics. Therefore, an apparatus will be set up to demonstrate the principles of diffraction in the simple cases of the single slit and double slit examples, with comparisons made computationally using Python and the Fast Fourier Transform (FFT). A second apparatus will then be set up to show how the lens performs the Fourier transform and how masking this plane results in effects on the final image. Different apertures and differing Fourier transform masks will be used to show some use cases of optical filtering.

II. SINGLE AND DOUBLE SLIT DIFFRACTION PATTERNS

A. Methodology

A selection of slits were chosen to demonstrate slit diffraction patterns. Each slit was a section of a small (and otherwise opaque) card, which was held in place by an adjustable XY mount (Thorlabs XYF1(/M)). Three single slits were illuminated in turn, each with increasing widths 40 nm, 80 nm, and 160 nm. One double slit was illuminated with a slit width of 40 nm for both slits, and a distance of 250 nm between the slits. A green laser ($\lambda = 515\,\text{nm}$) was chosen to illuminate the slits as monochromatic and collimated light was required for a clear diffraction pattern. A tube lens ($f = 150\,\text{mm}$) was positioned after the slits to place the diffraction pattern onto the sensor of a CMOS Color Camera (Thorlabs CS165CU(/M)). The image from the camera was displayed live using Thorlabs software *ThorCam* where the exposure time and other camera settings could be adjusted.

B. Results and Analysis

1. *Single Slit Patterns*

The diffraction patterns for the three single slits were imaged and saved as shown in figure 7 (a, b, and c). We see clearly that the pattern resembles what was calculated from the Fourier transform of a rectangular aperture as shown in figure 4 for slits (b) and (c). We don't see the diffraction pattern for slit (a), and instead only see the central peak of the pattern due to the narrowness of the slit. Similar slit apertures were plotted using Python, and NUMPY's Fast Fourier Transform (FFT) (`https://numpy.org/doc/stable/reference/routines.fft.html`) was used to compute the Fourier transform of each aperture function. A set of three with width ratios similar to the physical slits used are shown in figure 8. We see clearly the same inverse relationship between slit width and pattern width which gives rise to only seeing the central peak in figure 4a. While we do see a diffraction pattern along the y axis in the computed examples, it is not nearly to the same extent as in the imaged diffraction patterns. This could be due to the physical slits being shorter in length than the aperture functions used, but could also be due to errors in focusing the pattern onto the camera.

2. Double Slit Patterns

This process was repeated for the double slit. The diffraction pattern was imaged and saved as shown in figure 7 (d). We see clearly that the pattern resembles what was calculated from the Fourier transform of two rectangular apertures as shown in figure 5. Only the central overall 'peak' can be seen in this image due to the slit geometry as discussed for the single slit above. Again, a similar aperture set-up was plotted using Python, along with the corresponding Fourier transform (see figure 9). A similar pattern is observed.

III. FOURIER OPTICS AND OPTICAL FILTERING

A. Methodology

A different apparatus was required to demonstrate optical filtering. A set-up similar to figure 6 was used. This was set up following the Thorlabs Fourier Optics Kit manual [13]. Figure 18 in appendix A shows a diagram of the apparatus in full. The basic filtering system - akin to figure 6 - is comprised of the target, objective lens, masks, tube lens, and camera only. Prior to these components, the apparatus serves to illuminate the target with collimated and monochromatic light. The field lens and condenser lens work to negatively magnify the light from the LED, reducing the area of the light arriving at the target thus increasing the intensity [13]. The irises are placed to control the intensity and size of the light incident on the target. The field iris controls the illuminated area on the target and the aperture iris controls the intensity of the light passing through the condenser lens [13]. A variable slit (with variable width and orientation) was placed in the Fourier plane of the system (the *spectrum* plane in figure 6) to mask the Fourier transform, after which a beam splitter is used before the tube lens to show the masked Fourier transform onto a screen. The rest of the beam passes through the tube lens forming the image on the camera. A selection of target apertures were chosen to best demonstrate simple Fourier filtering. Each target was illuminated as described above and the Fourier transform observed on the screen. The relevant mask to achieve the wanted effect was applied and the resulting image at the camera was saved.

B. Showcase of Fourier Filtering and Comparison with a Computed FFT

Due to a lack of equipment to provide consistent and clear images of the Fourier transforms projected onto the screen, in the following figures the filtered and unfiltered Fourier transforms shown are taken from the Thorlabs Fourier Optics Kit Manual [13], which contains filtered and unfiltered Fourier transforms for each target used in this demonstration. Each collection of four images is in the format: *top-left:* target, *top-right:* Fourier transform, *bottom-left:* filtered/masked Fourier transform, *bottom-right:* output image.

1. Grid

The first example demonstrated begins with a simple grid composed of horizontal and vertical lines. Due to the repeating structures in the vertical and horizontal line patterns, the Fourier transform can be masked in such a way to remove information from the image. Figure 10 shows how the Fourier transform can be masked with a horizontal slit to output only the vertical lines in the resulting image. Similarly, figure 11 demonstrates the corresponding mask required to only output the horizontal component of the grid. An unexpected effect is observed when using a slit mask at an angle. In figure 12, a slit mask at 45 degrees to the target is used to remove both the horizontal and the vertical periodic information from the Fourier transform, leaving only the information on the diagonal of the Fourier transform. The result is an image now instead with repeating diagonal lines.

2. Letters

In this second example, our target consists of a letter 'A' composed of horizontal line segments, and a letter 'B' composed of vertical line segments. Both letters are superimposed on one another, each obscuring the other letter. The principles of Fourier optics discussed prior can be used to view either of the letters in isolation. In figure 13, a vertical slit is used to mask the Fourier transform yielding the letter 'A'. Similarly in figure 14, a horizontal slit is used to mask the Fourier transform, yielding the letter 'B'. In contrast to the previous example, we note that the letter we are trying to remove doesn't disappear completely, it remains as a dim evenly illuminated area. This is due to the geometric information about the shape of the letters being contained in the central maximum of the Fourier transform, and hence is not removed by either masking process.

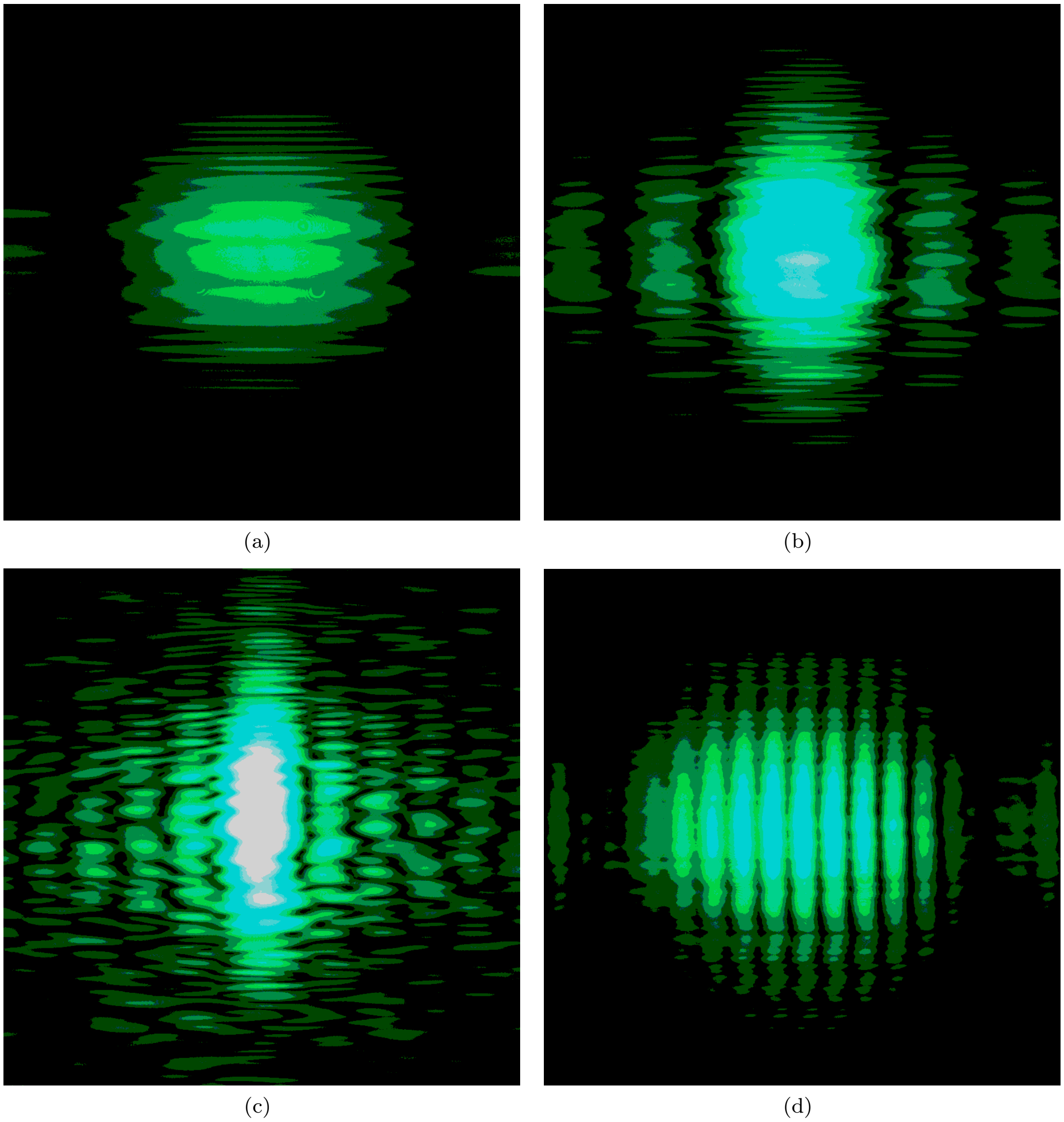


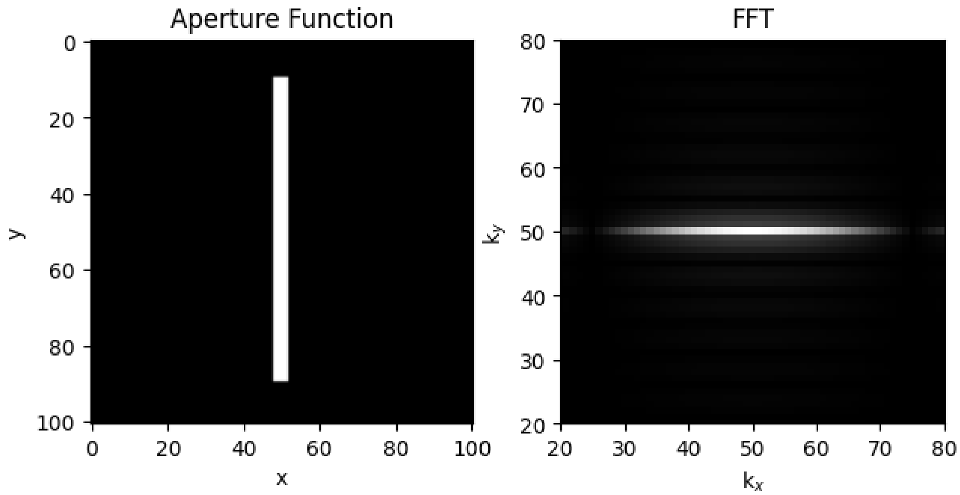
FIG. 7: Single slit diffraction patterns using the apparatus described with varying slit widths: 40 nm (a), 80 nm (b), and 160 nm (c). A double slit diffraction pattern with slit width 40 nm and slit spacing 250 nm is also shown (d). The patterns observed along the vertical axis are due to diffraction along that axis - due to a finite slit height - as well as the slit not being lit uniformly in the axis because of constraints on the geometry of the laser source illuminating the slit. This could also be a focusing issue.

3. Face

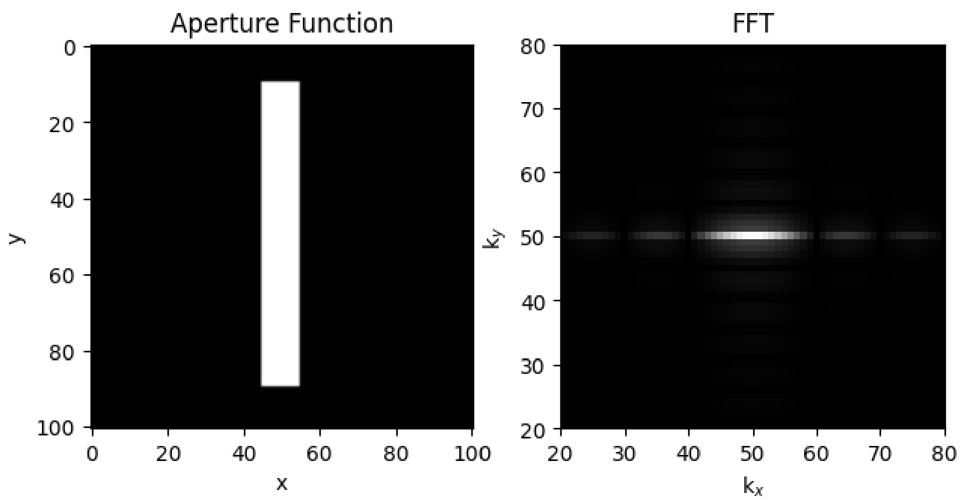
The third example demonstrates a more obviously practical use for Fourier optics. In figure 15, a smiley face is obscured by periodic vertical lines, however, by employing simple Fourier optics techniques, a simple mask in the Fourier plane can be used to remove the vertical lines and isolate the face behind. Note that we see the parts of the lines of smiley face which are vertical are also removed, showing the method is effective but not without flaws. A similar use-case of Fourier optics is removing the background lines of ruled paper (or perhaps a fabric material such as canvas) from a fingerprint, see figure 16 [10].

4. Babinet's Principle

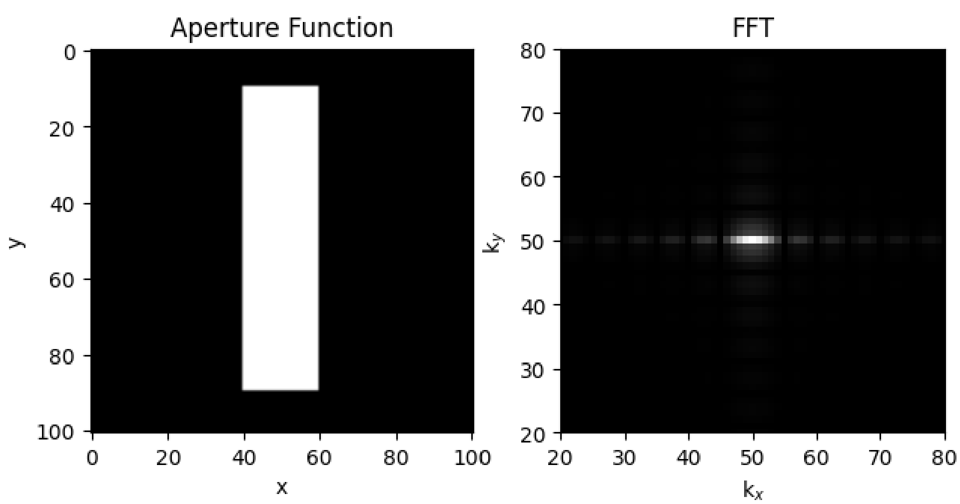
In optics, Babinet's Principle describes the process in which the sum of the output of two complimentary apertures (apertures which relate to each other by inverting the passable and opaque regions) will result in the unobstructed output, i.e. the amplitude if no apertures were present [1]. A consequence of this, is that complimentary apertures give identical diffraction patterns aside from the central maximum [1]. This can be demonstrated by the masking of the central maximum of the Fourier transform. A set of complimentary apertures is shown in figure 17, where vertical strips of isophase light are out of phase from its adjacent strips. We mask the central maximum of the Fourier transform which results in the



(a)



(b)



(c)

FIG. 8: Aperture functions and corresponding Fourier transforms (computed using NUMPY's FFT) with geometry chosen to emulate the slits used to create the diffraction patterns plotted in figure 7.

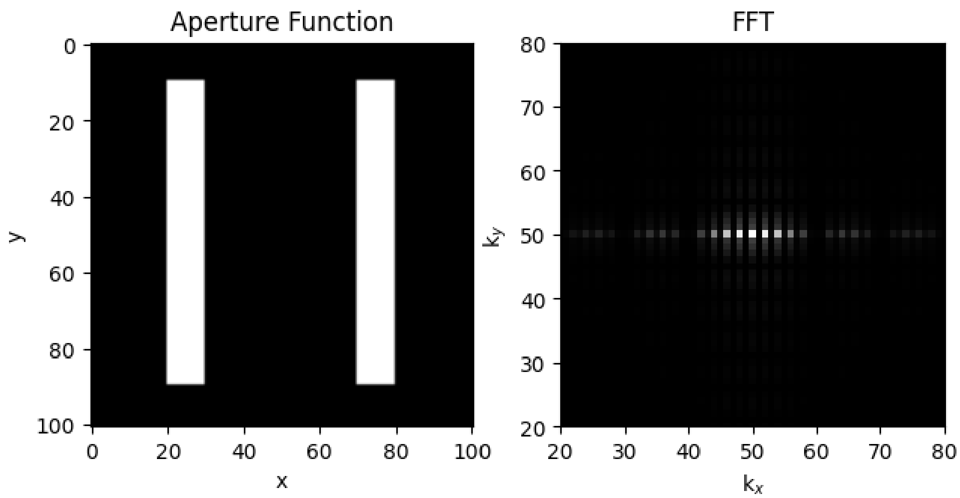


FIG. 9: Aperture function and corresponding Fourier transform (computed using NUMPY's FFT) with geometry chosen to emulate the slits used to create the diffraction pattern plotted in figure **??**.

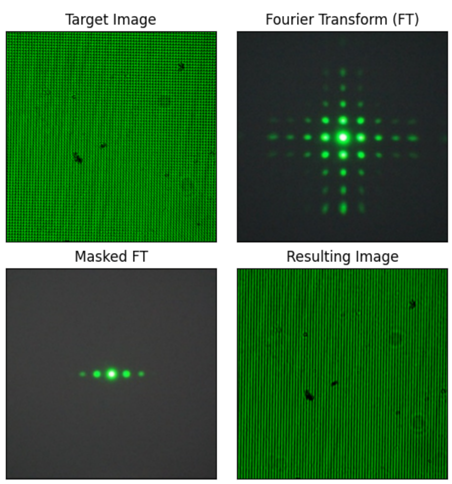


FIG. 10: An example of optical filtering: removing periodic horizontal lines.

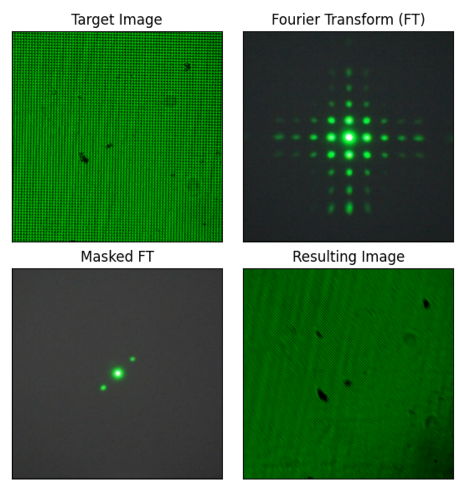


FIG. 12: An example of optical filtering: creating a diagonal pattern from initially horizontal and vertical information in the original image.

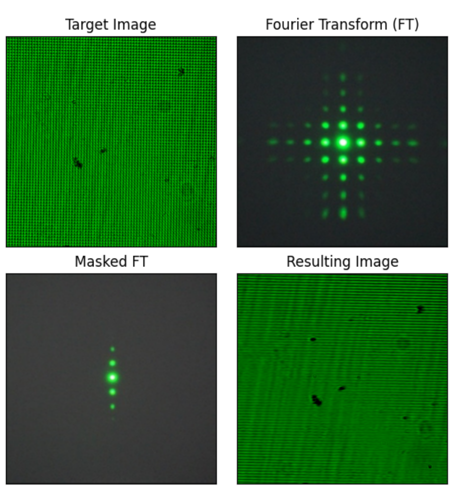


FIG. 11: An example of optical filtering: removing periodic vertical lines.

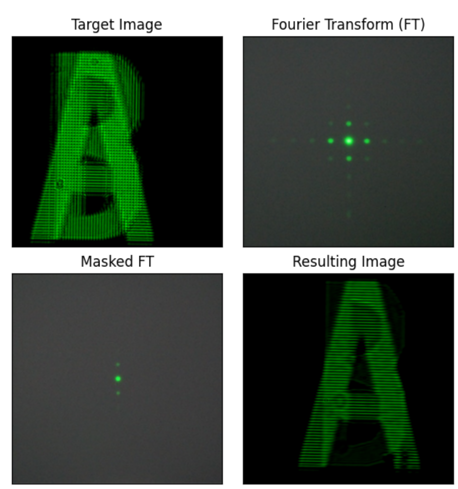


FIG. 13: Fourier optics allows the removal of the vertical lines constructing the letter 'b' revealing solely the letter 'a'.

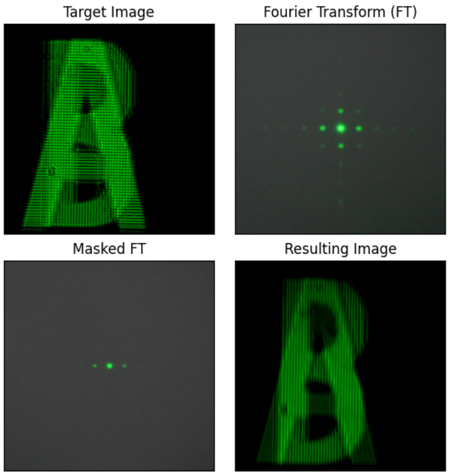


FIG. 14: Fourier optics allows the removal of the horizontal lines constructing the letter 'a' revealing solely the letter 'b'.

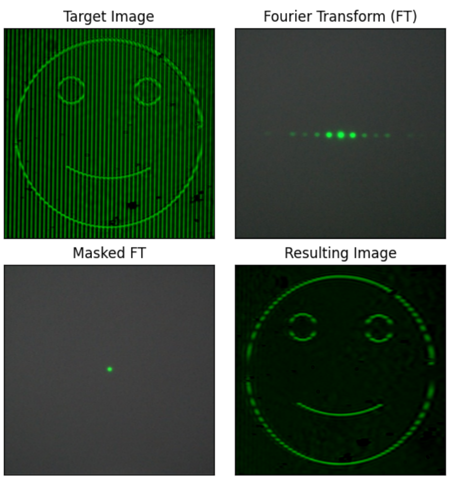


FIG. 15: Here a face is constructed using horizontal lines, but is blocked by a vertical line pattern. These vertical lines can be removed as done previously to reveal solely the smiley face.

image output still having vertical strips of isophase light, yet all vertical strips are now in phase with one another, showcasing Babinet's Principle.

IV. CONCLUSION

The aim of this report was to investigate the links between diffraction and the Fourier transform, along with describing a method of optical filtering making use of this link through the altering of the Fourier transform of the object image to produce a desired result in the output image. The relation between diffraction and the Fourier transform was explained, and the mathematical description of the diffraction pattern via the Fourier transform derived for the simple scenario of a single slit. The diffraction pattern for single slits of varying widths and a double-slit example were imaged and compared to computational examples using the Fast Fourier Transform. The process of image filtering through Fourier optics was described and an apparatus for demonstrating simple filtering scenarios was constructed and used to outline four basic use cases. This work only scratches the surface of what is possible through Fourier optics. As discussed in the introduction, there are many complex and modern applications of these principles across many fields.

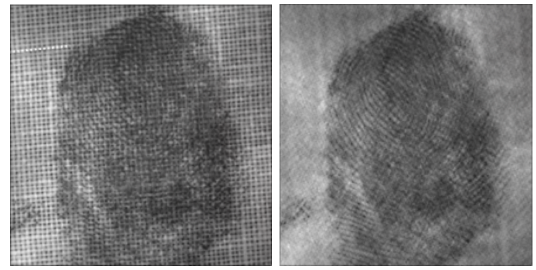


FIG. 16: A practical use-case for Fourier optics, removing periodic background lines from a fingerprint [10].

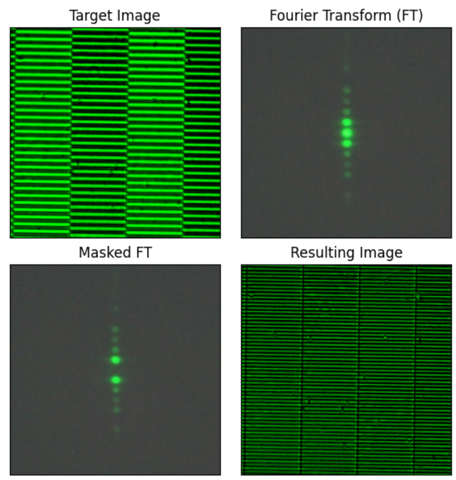


FIG. 17: Babinet's Principle is reproduced. We see the complementary apertures result in the same pattern with the masking of the zeroth order peak of the Fourier transform.

[1] F.L. Pedrotti and L.S. Pedrotti. *Introduction to Optics*. Prentice Hall, 1993.

[2] B.T. Dejene. Fourier optics. Swarthmore College.

[3] Qing-Yuan Wu, Jia-Zhi Yang, Jin-Yang Hong, Zhe Meng, and An-Ning Zhang. An edge detail enhancement strategy based on fourier single-pixel imaging. *Optics and Lasers in Engineering*, 172:107828, 2024.

[4] Ronald R. Parenti. Fourier-space image restoration. *The Lincoln Laboratory journal*, 8(1):29–47, 1995.

[5] Pierre Janin-Potiron, Vincent Chambouleyron, Lauren Schatz, Olivier Fauvarque, Charlotte Z. Bond, Yannick Abautret, Eduard Muslimov, Kacem El-Hadi, Jean-François Sauvage, Kjetil Dohlen, Benoît Neichel, Carlos M. Correia, and Thierry Fusco. Adaptive optics with programmable Fourier-based wavefront sensors: a spatial light modulator approach to the LAM/ONERA on-sky pyramid sensor testbed, July 2019.

[6] The University of Tennessee, Department of Physics and Astronomy. Introduction to physics ii: Single slit diffraction. `http://labman.phys.utk.edu/phys136core/modules/m9/diffraction.html`. Accessed: 2024/01/09.

[7] R Nave. Hyperphysics: Fraunhofer diffraction. `http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/fraunhofcon.html`. Accessed: 2024/01/08.

[8] R.N. Bracewell. *The Fourier Transform and Its Applications*. Circuits and systems. McGraw Hill, 2000.

[9] E.G. Steward. *Fourier Optics: An Introduction*. Ballard CREOL collection. E. Horwood, 1983.

[10] I Mercer. Fourier transform in optics and lasers. PHYC30080: Optics and Lasers, School of Physics, University College Dublin. Accessed: 2024/01/09.

[11] T. Young. The bakerian lecture; experiments and calculations relative to physical optics. *Classical and modern diffraction theory*, 29:123–128, 2016.

[12] Bevel. The fourier transform at work: Young's experiment. `https://www.thefouriertransform.com/applications/diffraction3.php`. Accessed: 2024/01/18.

[13] THORLABS Discovery. *EDU-FOP2 Fourier Optics Kit Manual, Rev A*, 2020.

Appendix A: Apparatus

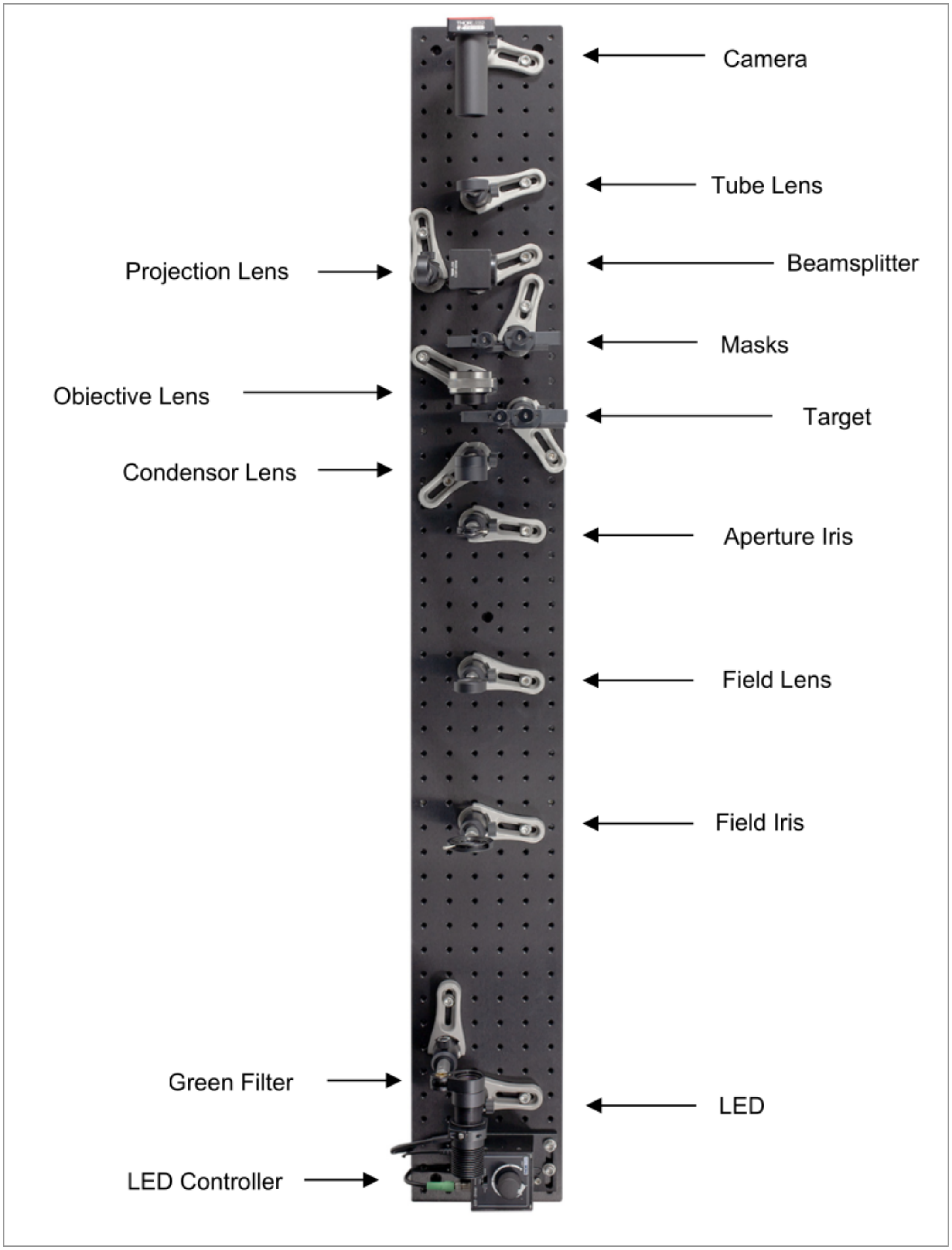


FIG. 18: A diagram of the apparatus used to showcase optical filtering [13].

Appendix B: Python Notebook