Frequency dependence of the skin depth in a metal cylinder

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I. INTRODUCTION

Eddy currents play a very important role in modern research and technology.[1–3] Yet it is surprising to find that this topic represents an area which is generally neglected in undergraduate curricula. One of the reasons for this, we believe, is the lack of simple laboratory experiments, performed at relatively low frequencies, designed to illustrate quantitatively and realistically the concepts discussed in the classroom. As a result, most treatments of the skin effect tend to be mathematical in nature and the physical origin of the effect is often left unclear for the student.

We believe that students would benefit from an earlier and mathematically simpler treatment of the skin effect, provided they could develop a more concrete feeling for the ideas involved, in the laboratory. It is with this objective in mind that we have designed the following experiment, which permits the determination of the frequency dependence of the skin depth in a long metallic cylinder. This experiment complements those described by MacDougall[4] and by Gosselin *et al.*[5]

The method presented in this note exploits the existence of a strong skin effect in a high conductivity cylinder placed with its axis parallel to an otherwise homogeneous magnetic field of frequency ν and amplitude B_0. Let a pick-up coil be wound around the midsection of the cylinder, and let the frequency be high enough that the skin depth is much smaller than the cylinder radius R but much larger than the thickness of the air gap between the cylinder surface and the mean radius R' of the pick-up coil, i.e., $R' - R \ll \delta \ll R$. Under these conditions of operation, one expects the rms pick-up voltage V to contain two measurable terms. The first should originate from the field in the air gap and be proportional to the factor $(2\pi\nu)[B_0\pi(R'^2 - R^2)]$. The second should originate from the field contained in a layer of thickness δ below the surface of the cylinder, and be proportional to the factor $(2\pi\nu)\{B_0\pi[R^2-(R-\delta)^2]\} \simeq (2\pi\nu)(B_0 2\pi R\delta)$. A study of V/ν as a function of frequency should therefore provide direct information about the frequency dependence of the skin depth, at least in the stated range.

We have done measurements on three different samples: brass, aluminum, and copper. We find that the rms amplitude of the pick-up voltage may be related to frequency by the simple relationship

$$V\nu^{-1} = \alpha + \beta\nu^{-1/2} \tag{1}$$

in the frequency range discussed earlier. This conforms to the expectation that δ varies like $\nu^{-1/2}$; furthermore, the constant parameters α and β compare very well with the calculated values (see Sec. III).

II. EXPERIMENTAL SETUP

The applied induction field was provided by a long solenoid (length $L_1 = 37.786 \pm 0.002$ cm and diameter $D_1 = 2.377 \pm 0.002$ cm), consisting of $N_1 = 1013$ turns of 28-gauge copper wire wound on a hollow bakelite cylinder. The actual field provided by this arrangement deviates from the field expected from an infinitely long solenoid by less than 0.2% on the axis, near the center. A constant current of rms amplitude $I_1 = 19.00 \pm 0.05$ mA was fed through the solenoid by a WAVETEK Model 113 voltage-controlled generator. The current level was controlled by monitoring the voltage drop across a 49-Ω metal film resistor, placed in series with the solenoid, and which showed no appreciable skin effect in the frequency range of interest. Frequencies were measured to within 5 Hz using a Dynascan B & K Precision Model 1801 frequency counter. Voltages were measured with a Hewlett–Packard Model 3435A digital voltmeter with a bandwidth of 100 kHz. Secondary voltages were measured to within 0.1 mV, off a short solenoid ($N_2 = 100$ turns) wound on the midsection of the metallic sample. Finally, the axes and midsections of the solenoid and sample were made to coincide. The radius R of each sample used is given in the accompanying figure caption.

III. THEORY AND DATA ANALYSIS

The purpose of the present section is to justify Eq. (1) and to obtain expressions for the parameters α and β.

Under the frequency conditions discussed in Sec. I, magnetic flux only penetrates a small distance, of order δ, below the surface of the specimen. As a result, penetration behaves in essentially the same way as it would in a flat sample. One can therefore view the field $B(r,t)$ as an attenuated traveling wave with spatial damping length δ due to the absorption that takes place in the metal. If one denotes the field at the sample surface by $B_0 \cos(\omega t)$, then one expects the result

$$B(r,t) = B_0 e^{-(R-r)/\delta} \cos[\omega t - (R-r)/\delta] \tag{2}$$

at a distance r from the cylinder axis. By definition, $\omega \equiv 2\pi\nu$ is the circular frequency of the field; also,

$$B_0 = \sqrt{2}\mu_0 N_1 I_1 / L_1 \tag{3}$$

is the amplitude of the field inside the primary coil, in the absence of the sample. A factor $\sqrt{2}$ has been included here because I_1 is the rms current, by definition (see Sec. III); μ_0 is the vacuum permeability. The field of Eq. (2) is seen to be the real part of the complex expression $B_0 \exp[i\omega t - \sqrt{2i}(R-r)/\delta]$; this form will now be used to obtain the complex pick-up voltage $V(t)$.

By Faraday's law of induction, the time-dependent pick-up voltage across the secondary coil is

$$V(t) = -i\omega N_2 \left(\int_0^R B(r)\cdot 2\pi r\, dr + \int_R^{R'} B_0 \cdot 2\pi r\, dr \right) e^{i\omega t}. \tag{4}$$

The first integral on the right-hand side represents the flux *inside* the sample and it is the high-frequency limit of that term which will eventually be proportional to the skin depth δ. The second integral represents *external* flux and will give us the linear term $\alpha\nu$ in the pickup. R' is the mean radius to be ascribed to the secondary loops circling the specimen; however, in spite of the fact that $R'\simeq R$, this term cannot be omitted at the higher frequencies used here.

The integrals involved in Eq. (4) are straightforward and the result may be written in the form

$$V(t) = [-ia\nu - (1+i)b\delta\nu + \gamma(\nu)]e^{i\omega t}. \quad (5)$$

In this expression,

$$a \equiv 2\pi^2 N_2 B_0 (R'^2 - R^2) \quad (6)$$

and

$$b \equiv 2\pi^2 N_2 B_0 R, \quad (7)$$

with B_0 given by Eq. (3). The term $\gamma(\nu)$ contains two contributions: one is frequency independent and the second decays exponentially with δ^{-1}. Both are negligible when compared to the terms in ν and $\nu\delta$, in the frequency range of interest here. Finally, because the external pickup is smaller than the internal one, by assumption ($R' - R \ll \delta \ll R$), one may expand the amplitude of $V(t)$ and find, for the rms voltage,

$$V/\nu = a/2 + b\delta. \quad (8)$$

Writing $\delta = (\pi\sigma\mu_0\nu)^{-1/2}$ and identifying α with $a/2$ and β with $b(\pi\sigma\mu_0)^{-1/2}$ then relates Eq. (8) to Eq. (1); σ is the conductivity of the sample.

IV. DISCUSSION AND CONCLUSIONS

Figure 1 presents results for a pure (99.999%) polycrystalline copper sample of conductivity $\sigma = 5.6\times10^7$ S m^{-1} at room temperature. We also used an aluminum sample of conductivity $\sigma = 2.6\times10^7$ S m^{-1} and a brass sample of conductivity $\sigma = 1.3\times10^7$ S m^{-1}. The radius of each sample is given in the figure caption. The values of α and β obtained in each case are in agreement with the predictions of Eq. (8). Deviations from linearity can be seen near the low-frequency end of the curves; they arise because the ratio δ/R starts being too large for the simple theory proposed here to hold. Indeed, from Eq. (2), the field on the cylinder axis is equal to 1% of its surface value when $\delta/R = 0.2$. For δ/R above that value, flux expulsion is incomplete; for δ/R below 0.2, expulsion is important. Points which lie at frequencies below this threshold were not included in a least-squares-fit analysis of our results; consequently, the correlation factor was above 0.99 in all cases. Our results clearly indicate that conductivity affects the skin depth, at a given frequency; flux expulsion is indeed seen to occur more rapidly in frequency when the conductivity is high.

To conclude, we have proposed a simple method for determining directly the frequency dependence of the skin

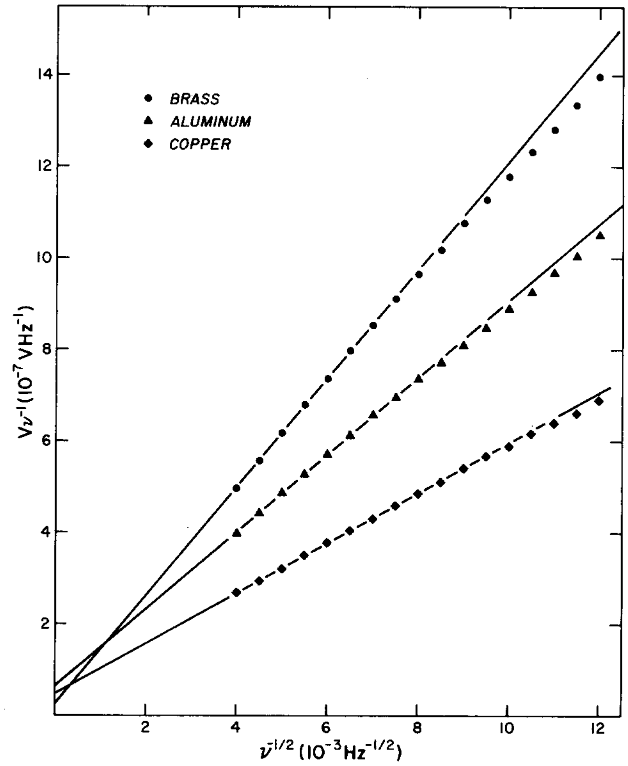

Fig. 1. Plot of the rms pick-up voltage divided by the frequency as a function of the inverse square root of the frequency. (Sample radii: copper, 0.4565 cm; aluminum, 0.4765 cm; brass, 0.4779 cm, all $\pm$ 0.0002 cm). The points represent experimental data; the lines were obtained by least-squares fits of the data in the high-frequency regime according to Eq. (1).

depth in a metal. We have shown by comparison of theory to experiment that the method gives excellent results. We have argued that the data analysis is straightforward and that the experiment could be performed simply, at an early undergraduate level, as an aid to classroom discussions of the skin effect in metals. We hope that our results will be helpful to others in their discussion of the skin effect.

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