SciComp with Py

Linear Algebra: Part 2

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Outline

Identity Matrix

An **identity** matrix is a square matrix (i.e., the number of rows is equal to the number of columns) where all diagonal elements are equal to 1 and the other elements are equal to 0. It is typically denoted as $\mathbf{I}_m$.

$$\mathbf{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{I_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Matrix

An $n \times n$ matrix $\mathbf{A}$ is invertible if there exists an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the $n \times n$ identity matrix. The matrix $\mathbf{A}^{-1}$ is called $\mathbf{A}$'s inverse.

Inverse Matrix in Numpy

```
import numpy as np
A = np.matrix([
    [2, 1],
    [6, 5]
    ])
I2 = np.matrix(np.eye(2))
## compute inverse of A.
print A.I
assert np.array_equal(I2, np.dot(A, A.I))
assert np.array_equal(np.dot(A, A.I), I2)
print 'assertions passed'
```

Augmented Matrix

An augmented matrix is the matrix in which rows or columns of another matrix of the appropriate order are appended to the original matrix. If $\mathbf{A}$ is augmented on the right with $\mathbf{B}$, the resultant matrix is denoted as $[\mathbf{A}|\mathbf{B}]$. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 5 & 6 \end{bmatrix} \; \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \; \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $[\mathbf{A}|\mathbf{B}] = \begin{bmatrix} 1 & 4 & | & 3 \\ 5 & 6 & | & 1 \end{bmatrix}$ and $[\mathbf{A}|\mathbf{I}] = \begin{bmatrix} 1 & 4 & | & 1 & 0 \\ 5 & 6 & | & 0 & 1 \end{bmatrix}$.

Generic Linear System

A generic linear systems with m equations in n unknowns is written as follows:

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
. \\
. \\
. \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m
\end{gathered}
$$

Since the system is determined by its $m \times n$ coefficient matrix $\mathbf{A} = [a_{ij}]$ and its column vector $\mathbf{b}$, it can be written as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x}$ is a column vector $(x_1, x_2, ..., x_n)$.

Generic Linear System as Augmented Matrix

A generic linear systems with m equations in n can be expressed with the following augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

The above matrix is typically shorthanded as $[\mathbf{A}|\mathbf{b}]$.

Example

Here is a concrete linear system:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned}$$

Here is the corresponding linear system:

$$\begin{bmatrix} 3 & 2 & 1 & | & 39 \\ 2 & 3 & 1 & | & 34 \\ 1 & 2 & 3 & | & 26 \end{bmatrix}$$

Elementary Row Operations on Augmented Matrix

Computing all solutions to a linear system is done with the three elementary row operations on the corresponding augmented matrix.

- **R1** (Row interchange): Interchange any two rows in a matrix.
- **R2** (Row scaling): Multiply any row in the matrix by a nonzero scalar.
- **R3** (Row addition): Replace any row in the matrix with the sum of that row and a multiple of another row in the matrix.

Accessing Rows and Columns in Numpy

```
import numpy as np
A = np.array([
        [7, 1],
        [4, -3],
        [2, 0]
        ])

for r in xrange(A.shape[0]):
    print 'row', r, A[r,:]

for c in xrange(A.shape[1]):
    print 'col', c, A[:,c]
```

Row Interchange

```
import numpy as np
A = np.array([
        [7, 1],
        [4, -3],
        [2, 0]
        ])
## Interchanging two rows r1, r2 in a matrix M, do
## M[[r1, r2]] = M[[r2, r1]].
print A
A[[1, 2]] = A[[2, 1]]
print A
```

Row and Column Scaling

```
## To multiply a row r by a scalar s in a 2D matrix
## M, do s*M[r,:].
print 4.0*A[1,:]

## To multiply a colum c by a scalar s in a 2D matrix
## M, do s*M[:,c].
print 4.0*A[:,0]
```

Row Addition

```
## To replace row r1 in 2D matrix M with the sum
## of r and a multiple of row r2,
## do M[r1,:] = M[r1,:] + s*M[r2,:]
A2 = A.copy()
print A2
A2[1,:] = A2[1,:] + 2*A2[2,:]
print A2
```

Row Equivalence

- If a matrix $\mathbf{B}$ can be obtained from a matrix $\mathbf{A}$ by a sequence of elementary row operations, then $\mathbf{B}$ is **row equivalent** to $\mathbf{A}$.
- Since each elementary row operation can be undone (reversed), if $\mathbf{B}$ is row equivalent to $\mathbf{A}$, denoted as $\mathbf{B} \sim \mathbf{A}$, then $\mathbf{A}$ is row equivalent to $\mathbf{B}$, i.e., $\mathbf{A} \sim \mathbf{B}$.
- The elementary row operations do not change the solution set of an augmented matrix.

A Fundamental Theorem of Linear Algebra

If $[\mathbf{A}|\mathbf{b}] \sim [\mathbf{H}|\mathbf{c}]$, then the corresponding linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{H}\mathbf{x} = \mathbf{c}$ have the same solution set.

Outline

Row Echelon Form and Pivot

A matrix is in **row echelon form** if:

- All rows containing only zeros appear below the rows containing nonzero entries
- The first nonzero entry in any row appears in a column to the right of the first nonzero entry in any preceding row

The first nonzero entry in a row of a row echelon form matrix is called the **pivot**

Row Echelon Form Quiz

Which matrices are in row echelon form?

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Back Substitution: Solving $\mathbf{Hx} = \mathbf{c}$

Before outlining a generic algorithm for solving linear systems, let's do several examples to determine all solutions of the system in row echelon form. Let's find all solutions of $\mathbf{Hx} = \mathbf{c}$, where

$$[\mathbf{H}|\mathbf{c}] = \left[\begin{array}{ccc|c} -5 & -1 & 3 & 3 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

The equations corresponding to $[\mathbf{H}|\mathbf{c}]$ are

1. $-5x_1 - x_2 + 3x_3 = 3.$
2. $0x_1 + 3x_2 + 5x_3 = 8.$
3. $0x_1 + 0x_2 + 2x_3 = -4.$

We solve for $x_3 = -2$ in 3, substitute x_3 in 2 to get $x_2 = 6$, substitute x_2 and x_3 into 1 to get $x_1 = -3$. This method is **back substitution**.

Back Substitution: No Solution

Use back substitution to find all solutions of $\mathbf{Hx} = \mathbf{c}$, where

$$[\mathbf{H}|\mathbf{c}] = \left[\begin{array}{ccc|c} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The equations corresponding to $[\mathbf{H}|\mathbf{c}]$ are

1. $1x_1 - 3x_2 + 5x_3 = 3.$
2. $0x_1 + 1x_2 + 2x_3 = 2.$
3. $0x_1 + 0x_2 + 0x_3 = -1.$

We cannot solve for x_3 in equation 3, thus the system has no solution, i.e., **inconsistent**.

Back Substitution: Multiple Solutions

Use back substitution to find all solutions of $\mathbf{Hx} = \mathbf{c}$, where

$$[\mathbf{H}|\mathbf{c}] = \left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that the 2nd and 4th columns have no pivots. The equations corresponding to $[\mathbf{H}|\mathbf{c}]$ are

1. $x_1 - 3x_2 + 5x_4 = 4$.
2. $x_3 + 2x_4 = -7$.
3. $x_5 = 1$.

We obtain the following solutions:

1. $x_1 = 3x_2 - 5x_4 + 4$.
2. $x_3 = -2x_4 - 7$.
3. $x_5 = 1$.

Back Substitution: Multiple Solutions

What does it mean to have a solution like this?

1. $x_1 = 3x_2 - 5x_4 + 4$.
2. $x_3 = -2x_4 - 7$.
3. $x_5 = 1$.

It means that we can take any real values for the values of x_2 and x_4, say r and s, and get a solution to the system. In other words, the solution vector looks as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 5s + 4 \\ r \\ -2s - 7 \\ s \\ 1 \end{bmatrix}$$

Outline

A Generic Algorithm for Solving Linear Systems

Given a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$,

1. Obtain $[\mathbf{A}|\mathbf{b}]$;
2. Row-reduce $[\mathbf{A}|\mathbf{b}]$ to $[\mathbf{H}|\mathbf{c}]$, i.e., $[\mathbf{A}|\mathbf{b}] \sim [\mathbf{H}|\mathbf{c}]$;
3. Use back substitution to find a solution to $[\mathbf{H}|\mathbf{c}]$ (or not in case of inconsistency).

Gauss Reduction of $\mathbf{Ax} = \mathbf{b}$ to $\mathbf{Hx} = \mathbf{c}$

- Place all zero rows to the bottom of the augmented matrix $[\mathbf{A}|\mathbf{b}]$.
- If the first column of $\mathbf{A}$ contains only 0's, cross it off. Keep crossing off zero columns until the left column has a non-zero entry (or the matrix is exhausted).
- Use the row interchange, if necessary, to obtain the pivot entry in the top row of the first non-zero column.
- Use the scalar multiplication to set the value of the pivot entry in the top row of the first non-zero column to 1.
- Use row addition and scalar multiplication to create 0's below the pivot.
- Cross off the left column and recurse to step 1 on the smaller matrix.

Problem 1

Reduce to row echolon form the following matrix and make all pivots equal to 1.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{bmatrix}$$

Answer 1

Reduce to row echolon form the following matrix and make all pivots equal to 1.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -7/2 & 5/2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 2

Solve the following linear system:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \\ x_1 + x_2 - 6x_3 &= 0 \end{aligned}$$

Answer 2

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \\ x_1 + x_2 - 6x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 1 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Problem 3

Reduce to row echolon form the following matrix and make all pivots equal to 1.

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Answer 3

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4

Solve the following linear system:

$$\begin{aligned} x_2 - 3x_3 &= -5 \\ 2x_1 + 3x_2 - x_3 &= 7 \\ 4x_1 + 5x_2 - 2x_3 &= 10 \end{aligned}$$

Answer 4

$$\mathbf{A} = \left[\begin{array}{ccc|c} 0 & 1 & -3 & -5 \\ 2 & 3 & -1 & 7 \\ 4 & 5 & -2 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So the solutions are $x_1 = -1$, $x_2 = 4$, $x_3 = 3$.

Computation of $\mathbf{A}^{-1}$

The inverse of a matrix $\mathbf{A}$ is denoted as $\mathbf{A}^{-1}$. To find $\mathbf{A}^{-1}$, if it exists, proceed as follows:

1. Form the augmented matrix $[\mathbf{A}|\mathbf{I}]$.
2. Apply the Gauss method to attempt to reduce $[\mathbf{A}|\mathbf{I}]$ to $[\mathbf{I}|\mathbf{C}]$. If the reduction can be carried out, then $\mathbf{A}^{-1} = \mathbf{C}$. Otherwise, $\mathbf{A}^{-1}$ does not exist.

Problem 5

Compute $\mathbf{A}^{-1}$ for the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

Answer 5

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

$$A \times A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \times \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} \times A = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using $\mathbf{A}^{-1}$ to solve $\mathbf{Ax} = \mathbf{b}$

- Compute $\mathbf{A}^{-1}$, if possible.
- Then $\mathbf{A^{-1}Ax} = \mathbf{A^{-1}b} \Rightarrow \mathbf{Ix} = \mathbf{A^{-1}b} \Rightarrow \mathbf{x} = \mathbf{A^{-1}b}$.
- In other words, if $\mathbf{A}^{-1}$ exists, then $\mathbf{x} = \mathbf{A^{-1}b}$.

Example of Using $\mathbf{A}^{-1}$ to Solve Linear Systems

Solve the linear system:

$$\begin{aligned} 2x_1 + 9x_2 &= -5 \\ x_1 + 4x_2 &= 7 \end{aligned}$$

Then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \ \mathbf{b} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}; \ \mathbf{A} = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$$

Answer to Example

$$[\mathbf{A}|\mathbf{I}] = \begin{bmatrix} 2 & 9 & | & 1 & 0 \\ 1 & 4 & | & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & -4 & 9 \\ 0 & 1 & | & 1 & -2 \end{bmatrix} \Rightarrow$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}.$$ Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 83 \\ -19 \end{bmatrix}$$

Conditions for $\mathbf{A}^{-1}$ to Exist

1. $\mathbf{A}$ is invertible.
2. $\mathbf{A}$ is row equivalent to the identify matrix $\mathbf{I}$.