Pooya Daravi ELEC321 Asgn 2-1

1

P1

$$P(B|D) = 0.90 \text{ (sensitivity)}, \quad P(B^c|D^c) = 0.95 \text{ (specificity)}$$

$$P(D) = 0.005$$

(a) X_m : Testing cost for pools of size $\underline{\underline{m}}$

$$\rightarrow X_m = \underbrace{b^- \times 10}_{\text{cost of testing 1 group}} + \underbrace{b^+ \times (m \times 10)}_{\text{Cost of testing } m \text{ items}}$$

Where $\begin{cases} B^- = 1 & \text{if } B^c \\ B^- = 0 & \text{if } B \end{cases}$ and $\begin{cases} B^+ = 1 & \text{if } B \\ B^+ = 0 & \text{if } B^c \end{cases}$

$$\mu_m = E(X_m) = 10 \times \overset{*}{E}(B^-) + 10m \times E(B^+)$$

$$= 10 \times P(B^c) + 10m \times P(B)$$

Note: $AB \equiv A \cap B$

But we know:

$$P(B) = P(BD) + P(BD^c) \quad \text{[from 3rd axiom of prob.]}$$

$$= P(B|D)P(D) + P(B|D^c) \cdot P(D^c)$$

$$= P(B|D)P(D) + (1 - P(B^c|D^c)) \cdot (1 - P(D))$$

$$= 0.9 \times 0.005 + 0.05 \cdot 0.995 = 0.05425$$

$$\rightarrow P(B^c) = 1 - P(B) = 0.94575$$

$$\rightarrow \boxed{\mu_m = 9.4575 + 0.5425m}$$

$$* \; E(B^-) = \sum_{b^-} P(b^-) \cdot b^- = 0 \times P(B^- = 0) + 1 P(B^- = 1) = P(B^- = 1) = P(B^c)$$

(2)

$$\sigma_m = SD(X_m) = \sqrt{Var(X_m)} = \sqrt{E(X_m^2) - E(X_m)^2}$$

Where

$$E(X_m^2) = \sum_{x_m} x_m^2 \cdot f(x_m) \quad \text{where } f(x_m) = g(b^-, b^+)$$

$$\rightarrow g(1,0) = P(B_c), \; g(0,1) = P(B), \; g(0,0) = g(1,1) = 0$$

(Since $BB^c = \emptyset$)

$$\rightarrow E(X_m^2) = [X_m(1,0)]^2 \cdot P(B_c) + [X_m(0,1)]^2 \cdot P(B)$$

$$= 100 \times P(B^c) + 100m^2 \times P(B)$$

$$= 94.575 + 5.425\, m^2$$

$$\rightarrow \sigma_m = \sqrt{E(X_m^2) - \mu^2} = \sqrt{5.1307 - 10.2614m + 5.1307m^2}$$

$$\rightarrow \boxed{\sigma_m = 2.2651|m-1|}$$

(b) T_j: Total testing cost for strategy j

$$\rightarrow T_j = (\# \text{ of pools}) \times (\text{cost per pool})$$

$$= k_j \times X_{m_j}$$

$$\rightarrow \mu_j = E(T_j) = k_j \times E(X_{m_j}) = \boxed{k_j \times (9.4575 + 0.5425 m_j)}$$

$$\sigma_j = SD(T_j) = |k_j| \times SD(X_{m_j}) = \boxed{k_j \times 2.2651\,|m_j - 1|}$$

(c) Using the attached code we arrive at the following table:

i	M_i
1	55195.75 ← minimum
2	56141.5
3	58978.75
4	63707.5
5	73165
6	92080
7	101537.5
8	148825
9	172468.8
10	243400

As we see the minimum expected total cost occurs when choosing $\boxed{\text{strategy 1}}$

P2

(a) A': Waiting time for first occurance of A

$\rightarrow \tau_A = E(A')$

We know $P(A'=1) = 0.005$

$P(A'=2) = 0.995 \times 0.005$

$\vdots$

$P(A'=n) = 0.995^{n-1} \times 0.005$

0.995^{n-1} ← does not happen in first $(n-1)$ years

0.005 ← happens in nth year

$\rightarrow E(A') = \sum_{n=1}^{\infty} n \times (0.995^{(n-1)} \times 0.005) = 200$

$\rightarrow \boxed{\tau_A = 200 \text{ years}}$

similarly defined $\rightarrow$

$E(B') = \sum_{n=1}^{\infty} n \times (1-P(B))^{n-1} \times P(B) = 170$

for (c) $\rightarrow$

using attached code we find $P(B)$ to be: $P(B) \simeq 0.005882$

(b) $F_A(\lfloor \tau_A \rfloor) = \sum_{n=1}^{\tau_A - 1} P(A' = n)$

$= \sum_{n=1}^{199} 0.995^{n-1} \times 0.005$

$\simeq \boxed{0.6312}$

(c) We calculated $P(B) \simeq 0.005882$ (numerically)

From attached code we see that probability of B occurance before A is: $\simeq 0.54$

P3
(a) Probability that $\underline{i}$ successes happen before $\underline{j}$ failures occure can be worded as:

→ Probability that $\underline{i}$ successes happen while the number of failures is less than or equal to $\underline{j-1}$, or in other words while the total number of trials (successes + failures) is less than $\underline{i+j-1}$

i.e.

$$p(T_j > S_i) = p(\underbrace{Bin(i+j-1, p) \geqslant i}_{})$$

number of successes out of $\underline{i+j-1}$ is bigger than or equal i → number of failures is less than or equal to $\underline{j-1}$ → $T_j > S_i$

(b) (i) $p(T_j > S_i)\Big|_{(1,2)} = P(Bin(2,p) \geqslant 1)$

$= P(Bin(2,p) = 1) + P(Bin(2,p) = 2)$

$= \binom{2}{1} p(1-p) + \binom{2}{2} p^2 = 2 \times 0.1 \times 0.9 + 0.1^2 = \boxed{0.19}$

(ii) $P(T_j > S_i)\Big|_{(2,1)} = P(Bin(2,p) \geqslant 2) = P(Bin(2,p) = 2) = \boxed{0.01}$

(iii) $P(T_j > S_i)\Big|_{(5,7)} = P(Bin(11,p) \geqslant 5) = \dots = \boxed{2.75 \times 10^{-3}}$

(iv) $P(T_j > S_i)\Big|_{(7,5)} = P(Bin(11,p) \geqslant 7) = \dots = \boxed{2.29 \times 10^{-5}}$

(c) The estimated expected value & standard deviation of the number of failures are as follows: (see attached code)

$\mu \sim 180$ $SD \sim 42.3$

P4

(a) D: accidents in a day s: serious

W: accidents in a week n: not serious

$\lambda_{ds} = 1$ $\lambda_{dn} = 5$

$\lambda_{ws} = 7 \times \lambda_{ds} = 7$, $\lambda_{wn} = 7 \times \lambda_{dn} = 35$

→ We know that $\mu = \sigma^2 = \lambda$ for Poisson distributions: $E(W_s) = Var(W_s) = \lambda_{ws} = \boxed{7}$

$E(W_n) = Var(W_n) = \lambda_{wn} = \boxed{35}$

(b) $P(W > 45) = P(W_s + W_n > 45)$

$= 1 - \sum_{j=0}^{45} P(W_s + W_n = j)$

using Lemma $= 1 - \sum_{j=0}^{45} \left[\frac{(\lambda_{ws} + \lambda_{wn})^j}{j!} \times e^{-(\lambda_{ws} + \lambda_{wn})} \right]^*$

*since s and n are independent

(7)

Lemma: if $Y = X_1 + X_2$ where $\left\{ \begin{array}{l} X_1 \sim \text{Poisson}(\lambda_1) \\ \& \\ X_2 \sim \text{Poisson}(\lambda_2) \\ \text{are independent,} \end{array} \right.$

then $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Proof:

from 3rd axiom of prob. and def^{n} of cond. prob.

$$P(Y=n) = P(X_1+X_2=n)$$

$$= \sum_{j=0}^{n} P(X_1+X_2=n \mid X_2=k)\, P(X_2=k)$$

since X_1 & X_2 are indep.

$$= \sum_{j=0}^{n} P(X_1=n-k)\, P(X_2=k)$$

$$= \sum_{j=0}^{n} \frac{\lambda_1^{n-k}}{(n-k)!} e^{-\lambda_1} \times \frac{\lambda_2^{k}}{k!} e^{-\lambda_2}$$

$$= \frac{1}{n!} \left[\sum_{j=0}^{n} \binom{n}{k} \lambda_1^{n-k} \times \lambda_2^{k} \right] e^{-(\lambda_1+\lambda_2)}$$

$$= \frac{1}{n!} (\lambda_1+\lambda_2)^{n} e^{-(\lambda_1+\lambda_2)} \qquad \text{Q.E.D.}$$

→ using R: 1 - ppois(45, lambda = 42)

→ $\boxed{P(W > 45) = 0.2883}$

(c) $\lambda_{hs} = \lambda_{ds}/24 = 1/24 \rightarrow P(t_{accident} < 4_{hrs}) = P(4\text{Hr} \geq 1)$ (h: hour)

$= 1 - P(4\text{Hr}=0) \overset{\text{using R}}{=} 1 - \text{dpois}(0, 4 \times \frac{1}{24}) = \boxed{0.1535}$

(d) Expected waiting time is whatever time for which λ is equal to 4:

$$\lambda_w = \lambda_{ws} + \lambda_{wn} = 42$$

$$\rightarrow \frac{\lambda_x}{\lambda_w} = \frac{4}{42} = \frac{t_x}{t_w} \rightarrow t_x = \frac{4}{42} \times (7 \times 24)_{hr}$$

$$\boxed{= 16 hrs}$$

P5

(Case A) $X \sim N(\mu, \sigma^2)$

using method of moments we estimate μ as: $\mu = E(X) \approx \bar{X} = \boxed{11.96 \text{ volts}}$

Assuming $X_i = \mu + \sigma Z_i$ where $Z_i \sim N(0,1)$ for all i we have (see lemma): $se = \frac{\sigma}{\sqrt{n}} = \frac{\sigma}{\sqrt{15}}$

$$\rightarrow se \approx \frac{sd}{\sqrt{15}} = \frac{0.21 \text{ volts}}{\sqrt{15}} = \boxed{0.054 \text{ volts}}$$

Proof of Lemma:

Since all $Z_i \sim N(0,1) \rightarrow Var(X_i) = \sigma^2$

$$\Rightarrow Var(\bar{X}) = Var\left(\frac{\Sigma X_i}{n}\right) = \frac{1}{n^2} Var(\Sigma X_i) = \frac{1}{n^2}(n \times \sigma^2) = \frac{\sigma^2}{n}$$

$$\Rightarrow se := \sqrt{Var(\bar{X})} = \frac{\sigma}{\sqrt{n}} \quad Q.E.D.$$

(Case B) $X \sim \text{Gamma}(\alpha, \lambda)$

B.1

$$M(t) := E(e^{tX}) = \int_{-\infty}^{\infty} f(x) \cdot e^{tx}\, dx$$

$$= \int_0^{\infty} \left[\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\right] \cdot e^{tx}\, dx$$

$$= \frac{\lambda^\alpha}{(\lambda-t)^\alpha \Gamma(\alpha)} \int_0^{\infty} [(\lambda-t)x]^{\alpha-1} \cdot e^{-(\lambda-t)x}\, d[(\lambda-t)x]$$

$y = (\lambda - t)x$

(assuming $\lambda - t > 0$) $= \left(\frac{\lambda}{\lambda - t}\right)^\alpha \times \frac{1}{\Gamma(\alpha)} \times \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy = \left(\frac{\lambda}{\lambda - t}\right)^\alpha$

$$\to M(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}, \quad t < \lambda \quad \text{Q.E.D.}$$

B.2

$$\mu = M'(0) = (-\alpha) \times \left(\frac{-1}{\lambda}\right) \times \left(1 - \frac{t}{\lambda}\right)^{-\alpha-1}\Big|_{t=0} = \frac{\alpha}{\lambda}$$

$$\sigma^2 = M''(0) - M'(0)^2 = \left(\frac{-\alpha}{-\lambda}\right)\left(\frac{-\alpha-1}{-\lambda}\right)\left(1 - \frac{t}{\lambda}\right)^{-\alpha-2}\Big|_{t=0} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2} = \frac{\mu}{\lambda}$$

$$\to \lambda = \frac{\mu}{\sigma^2} \to \alpha = \frac{\mu^2}{\sigma^2}$$

$$\to \hat{\lambda} \approx \frac{\hat{\mu}}{\hat{\sigma}^2} \approx \frac{11.96}{0.21^2} \times \frac{1}{\text{volts}} \approx \boxed{271.2 \ \text{1/volts}}$$

$$\hat{\alpha} \approx \frac{\hat{\mu}^2}{\hat{\sigma}^2} \approx \frac{11.96^2}{0.21^2} \approx \boxed{3243.6}$$

B3

See attached code. A sample run:

$$se(\hat{\alpha}) \approx sd(\hat{\alpha}_1, \dots, \hat{\alpha}_{1000}) \simeq \boxed{1690}$$

$$se(\hat{\lambda}) \approx sd(\hat{\lambda}_1, \dots, \hat{\lambda}_{1000}) \simeq \boxed{142}$$

Note: Valid to the extend that the estimated distribution be a close representation of the actual distribution

P6

$$U \sim Unif(0,1) \to \begin{cases} f_U(u) = 1 \\ F_U(u) = u \end{cases}$$

$$X = -\ln(1-U)/\lambda \quad (\lambda > 0)$$

(a) $F_X(x) = p(X < x) = p\left(\frac{-\ln(1-U)}{\lambda} < x\right)$

$$= p(U < 1 - e^{-\lambda x}) = F(1 - e^{-\lambda x}) = \boxed{1 - e^{-\lambda x}} \text{ Q.E.D.}$$

$x(u=0) = 0$, $x(u=1) = \infty$

$$E(X) = \int_0^\infty F'(x)\, x\, dx = \int_0^\infty \lambda e^{-\lambda x} x\, dx = \frac{1}{\lambda}\int_0^\infty (\lambda x) e^{-\lambda x} d\lambda x$$

$y = \lambda x$

$$= \frac{1}{\lambda}\int_0^\infty y e^{-y} dy = \boxed{\frac{1}{\lambda}} \text{ Q.E.D.}$$

← integration by parts (IBP)

$$Var(X) = \int_0^\infty F'(x) . x^2 dx - \frac{1}{\lambda^2} \overset{IBP}{=} \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}} \text{ Q.E.D.}$$

since: $\int_0^\infty \lambda e^{-\lambda x} x^2 dx = -\int_0^\infty x^2 d e^{-\lambda x} \overset{IBP}{=} -\cancel{x^2 e^{-\lambda x}\Big|_0^\infty} + \int_0^\infty e^{-\lambda x}(2x) dx$

$$= -\frac{2}{\lambda}\int_0^\infty x\, d e^{-\lambda x} \overset{IBP}{=} -\frac{2}{\lambda}\cancel{x e^{-\lambda x}\Big|_0^\infty} + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x} dx$$

$$= -\frac{2}{\lambda^2} e^{-\lambda x}\Big|_0^\infty = \frac{2}{\lambda^2}$$

(b) $Y = (X - 1/2)^2$

As shown in part (a), $Range(X) = (0, \infty)$

Since $Y \geq 0$ and as X moves from $1/2$ to ∞

y moves from 0 to ∞ we have: $\boxed{Range(Y) = [0, \infty)}$

$$F_Y(y) = P(Y < y) = P[(X - 1/2)^2 < y]$$

$$= P(X < \tfrac{1}{2} + \sqrt{y}) = F_X(\tfrac{1}{2} + \sqrt{y})$$

$$= \boxed{1 - e^{-\lambda(\frac{1}{2} + \sqrt{y})}} \text{ cdf}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -e^{-\lambda(\frac{1}{2} + \sqrt{y})} \times (-\lambda) \times \frac{1}{2\sqrt{y}}$$

$$= \boxed{\frac{\lambda}{2\sqrt{y}} e^{-\lambda(\frac{1}{2} + \sqrt{y})}} \text{ pdf}$$

mean: $\mu = E(Y) = E[(X - 1/2)^2] = Var(X - 1/2) + E(X - 1/2)^2$

$$= Var(X) + [E(X) - 1/2]^2 = \frac{1}{\lambda^2} + [\frac{1}{\lambda} - \frac{1}{2}]^2 \leftarrow \text{using results from part (a)}$$

$$= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4} - \frac{1}{\lambda} = \boxed{\frac{2}{\lambda^2} - \frac{1}{\lambda} + \frac{1}{4}}$$

median: $F_Y(y_{med}) = 0.5 \rightarrow y_{med} = \boxed{[0.5 + \frac{\ln(0.5)}{\lambda}]^2}$

sd: $\sigma^2 = Var(Y) = Var[(X-1/2)^2] = E[(X-1/2)^4] - \mu^2$

$$= E(X^4 - 2X^3 + \frac{3}{2}X^2 - \frac{X}{2} + \frac{1}{16}) - \mu^2$$

$$= E(X^4) - 2E(X^3) + \frac{3}{2}E(X^2) - \frac{1}{2}E(X) - \mu^2$$

Lemma: the moment generating function for the exponential distribution X is $M_X(t) = \frac{\lambda}{\lambda - t}$

$$\rightarrow \sigma^2 = \frac{24}{\lambda^4} - \frac{12}{\lambda^3} + \frac{3}{\lambda^2} - \frac{1}{2\lambda} + \frac{1}{16} - \left(\frac{2}{\lambda^2} - \frac{1}{\lambda} + \frac{1}{4}\right)^2$$

$$= \frac{\lambda^2 - 8\lambda + 20}{\lambda^4} \rightarrow sd = \boxed{\frac{\sqrt{\lambda^2 - 8\lambda + 20}}{\lambda^2}}$$

Proof of lemma:

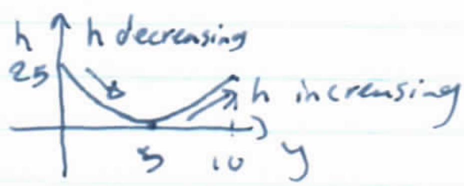
$$M_X(t) = \int_0^\infty e^{tx} \times \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx$$ only finite for $t < \lambda$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x} \Big|_0^\infty = \frac{\lambda}{\lambda - t}$$ Q.E.D. (ok since $\lambda > 0$ and we evaluate $M_X^{(k)}(t)$ at $t=0$)

P7

The system first improves with time from $y=0$ to $y=5$.

(a) It then wears out after $y=5$:



(b) $F_Y(y) = 1 - e^{-\int_0^y h(t)\,dt} = 1 - e^{-\int_0^y (t-5)^2\,dt}$

$= 1 - e^{-\frac{1}{3}y(y^2 - 15y + 75)}$

$f_Y(y) = \dfrac{dF_Y(y)}{dy} = (y-5)^2 e^{-1/3\, y(y^2 - 15y + 75)}$

$F_Y(m) = 1 - e^{-1/3\, m(m^2 - 15m + 75)} = 1/2$

$\rightarrow m(m^2 - 15m + 75) = -3\ln(0.5)$

numerical solution $\rightarrow m \approx 0.02788$

P2 $\bar{x} = 70$, $sd = 10$, $X \sim N(70, 100)$

using R

(a) $P(X > 80) = 1 - F_X(80) = 1 - \text{pnorm}(80, 70, 10)$

$= 15.9\%$

(b) $P(X \geq 60) = 1 - F_X(60) = 0.841\%$

(c) $P(X < 60) = F_X(60) = F_X(\mu - 10) = 1 - F_X(\mu + 10)$

$= 1 - F_X(80) = 15.9\%$

P9 We know:

$$\begin{cases} P(X>105) = 1-F(105) = 0.3 \\ \quad \to F(105) = 0.7 \\ \quad \to z_1 = 0.5244 \\ P(X>110) = 1-F(110) = 0.1 \\ \quad \to F(110) = 0.9 \\ \quad \to z_2 = 1.2816 \end{cases}$$

$$z = \frac{x-\mu}{\sigma}$$

$$\begin{cases} 0.5244 = \dfrac{105-\mu}{\sigma} \\ 1.2816 = \dfrac{110-\mu}{\sigma} \end{cases} \Rightarrow \boxed{\mu = 101.54} \quad \sigma = 6.604 \to \boxed{\sigma^2 = 43.61}$$

P10 $X \sim N(\mu = 1_{mm}, \sigma = 0.1_{mm})$

(a) $P(0.85 < X < 1.1) \overset{\text{using } R}{=} \underbrace{pnorm(1.1, 1, 0.1)}_{F(1.1)} - \underbrace{pnorm(0.85, 1, 0.1)}_{F(0.85)}$

$= \boxed{0.7745}$

(b) A_i: i^{th} wafer is acceptable

Define $Y \sim Bin(200, P(0.85 < X < 1.1))$

$P(140\text{-}160 \text{ acceptable}) = F_Y(160) - F_Y(139)$

$\overset{\text{using } R}{=}$ pbinom(160, 200, 0.7745) - pbinom(140, 200, 0.7745)

$= \boxed{0.8188}$