Electric Circuits

An Introduction

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Contents

Preface

This book is made in reaction to many introductory Electrical Engineering texts, which tend to assume a Sophomore- or even Junior-level understanding of Mathematics. In contrast, we aim our text at Freshmen, who may or may not have completed Calculus I.

1

Why Circuits?

Why do we have electric circuits? This is not a philosophical question but a practical one. Circuits tend to serve one of two purposes: they either power devices (like lights, motors, heaters, etc.) or they condition *signals* (i.e., information of some kind, which in the context of circuits will be encoded in voltage or current). Any electrical or electronic gadget you can think of will do at least one of these two things if not both. Here is a schematic diagram of one of the simplest circuits:

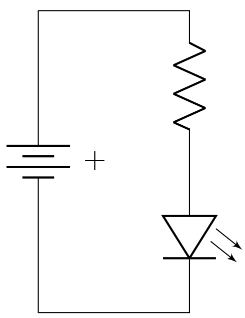


Figure 1.1: A simple circuit with a battery, a resistor, and an LED.

This circuit includes a battery, a *resistor* (the zig-zagged lines), and a light-emitting diode (LED). Although this circuit may look unfamiliar in schematic form, you have certainly encountered circuits like this in your everyday life, because LEDs are everywhere. This circuit is designed to power (and thus light up) the LED. Let's spend some time discussing the properties of this circuit.

First, let's assume the battery, which is represented as four parallel lines, is a standard 9V. That means there is a 9 volt difference in potential energy per unit charge between the positive terminal of the battery and the negative terminal. This electric potential difference will cause electrons to move through the circuit from the negative terminal to the positive terminal. The flow of charge between these terminals is known as *current*, which is expressed in Amperes, amps, or just 'A'. This current is affected by everything in the circuit between the battery terminals—in this case, the resistor and the LED. Also, the current flowing through a single *branch* has a constant value at all points along that branch. In other words, the current flowing out of the positive terminal of the battery into the resistor has the same value as the current that flows out of the resistor and into the LED, which has the same value as the current that flows out of the LED and into the negative battery terminal. Instead of diving in to the physics of semiconductors, let's just assume that the LED has a constant 1.5V difference between its terminals as long as current flows in the direction of the arrow of its schematic symbol. Let's also put a few labels on the diagram to make things a little more clear:

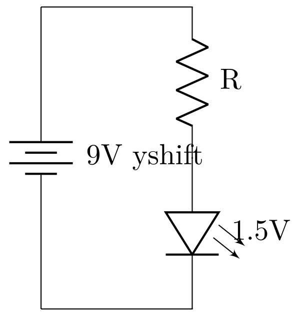


Figure 1.2: A simple circuit with a battery, a resistor, and an LED.

Part I

DC Circuit Analysis

2

The First Laws

2.1 Ohm's Law on a Single Resistor

Ohm's Law describes the amount of energy it takes to push some amount of charge through a resistor. A reasonable analogy is a ball moving quickly as it approaches a patch of mud. It starts off with some amount of kinetic energy, then as it moves through the mud, it starts to slow down, losing some portion of the kinetic energy it has, until it clears the puddle.

Unfortunately, our analogy breaks down a bit, since our electrons are moving at the same speed the entire time. We are not siphoning off the kinetic energy of the electrons, but rather the energy stored in the electrical potential, or voltage.

Ohm's Law states that the voltage drop *over* a resistor is proportional to both the resistance, R, and the current *through* the resistor:

$$\Delta V = iR \tag{2.1}$$

There is emphasis placed on the words *over* and *through* due to how we measure the voltage and current in a resistor for Ohm's Law. We do not care about the voltage at any one place, but rather only the change between the two ends. Likewise, whatever current is present at any point on the resistor will be present for all of it. Neither occurs at a single point, but rather describes the entire resistor.

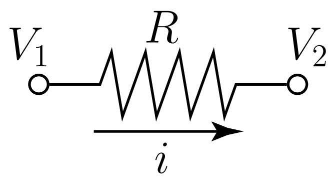


Figure 2.1: $\Delta V = V_1 - V_2$ for the circuit above, since energy is lost as charge passes through the resistor.

Looking at Figure 2.1, you can see that we have labeled the two voltages on either side. When we apply these voltages to ΔV, we should always start on the tail end of our current (that is, the direction our current is coming from) and end on the head.

Looking back to the pipe analogy from section B, ΔV is the loss of energy, i is the flow rate through the thin pipe, and R is how hard it is to get through the pipe. More flow, whether fluid or electric, is more loss of energy, and a smaller pipe or larger resistance also means more loss.

2.2 Ohm's Law on a Simple Circuit

2.3 Kirchoff's Current Law

2.4 Watt's Law

2.5 Incorporating Computation - Plotting

3

Equivalent Circuits

This chapter introduces the concept of an equivalent circuit. Two equivalent circuits share common values of voltage/current at a point of interest, usually the source. In this chapter, we will simplify complex resistive circuits down to the simplest version (such as those analyzed in Section 2.2). Once simplified, we will expand them again through methods known as voltage division and current division to find the current through each element and the voltage at each node.

3.1 Resistors in Series

First, lets go over in a bit more detail what we mean by **series**. Two elements are in series if and only if:

- Those elements share a common node
- No other elements share the same node

In this case, there is exactly one path for current to follow through both resistors. Because of this, we can write the following:

$$\Delta V_1 = V_A - V_B = iR_1 \quad (3.1)$$
$$\Delta V_2 = V_B - V_C = iR_2 \quad (3.2)$$

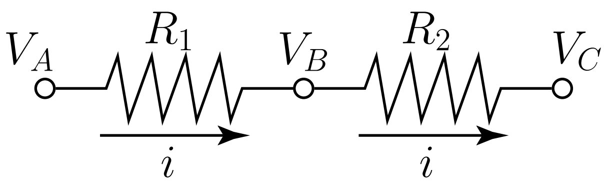


Figure 3.1: The two resistors above are the only circuit elements connected to node B, which means that any current from node A to B will also have to travel from B to C.

Now, our goal is to create simpler circuit which has the same characteristics as seen by nodes A and C. The only way to get simpler is to have a single circuit between nodes A and C with a resistance which we still need to determine.

We want our new circuit to have the following characteristics:

- The current through the new resistor should still be the i that passes through R_1 and R_2
- The voltage difference across the new resistor should be $V_A - V_C$

We can fulfill those two requirements by summing equations 3.1 and 3.2:

$$\Delta V_1 + \Delta V_2 = V_A - V_C = i(R_1 + R_2) \tag{3.3}$$

Equation 3.3 essentially states that we can replace both resistors with a single resistor that has a resistance

$$R_{EQ} = R_1 + R_2. \tag{3.4}$$

3.2 Resistors in Parallel

- Definition of parallel - recap
- Derivation of parallel resistance
  - Voltage constant across both resistors
  - Equivalent resistance should sum current while maintaining voltage
  - End result: $R_{EQ} = R_1 \cdot R_2/(R_1 + R_2)$

3.3 Wye-Delta Transform

3.4 Using Voltage Division

3.5 Using Current Division

3.6 Circuit Shorthand

3.7 Incorporating Computation - $R_1||R_2$ Function

4

Extra Uses for Voltage Dividers

4.1 Maximum Power Transfer

- Sources/loads
- If we know all the information about the source, we can calculate the power drawn by any load
  - Use voltage division to find voltage $V_0 = V_s \frac{R_0}{R_s + R_0}$
  - $P_0 = V_0^2 / R_0$
- When we ask the question "What load will draw the maximum power?", we need to dip into Calculus
  - The maximum of a function $f(x)$ occurs when the derivative $f'(x)$ goes to zero
  - Our function is $P_0(R_0) = \left(V_s \dfrac{R_0}{R_s + R_0}\right)^2 \cdot \dfrac{1}{R_0}$
  - We need to solve $P_0'(R_0) = V_s^2 \cdot \dfrac{(R_s^2 + R_0^2) - R_0 \cdot 2R_0}{(R_s^2 + R_0^2)^2} = 0$
  - A little bit of algebra tells us that this occurs when $R_0 = \pm R_s$
  - Only positive R_0 makes sense, so $R_0 = R_s$ for maximum power
- Once we know where maximum power occurs, it is trivial to find the value of the maximum power in terms of known values V_s and R_s
  - Use voltage division to find $V_0 = V_s/2$
  - Use the power equation to get $P_0 = V_s^2/(4R_s)$
- Other interesting notes:

Figure 4.1: This plot shows the power absorbed by the load resistor as its resistance R_0 is changed. The x-axis has been normalized by R_s, and the y-axis has been normalized by the maximum power, V_s/R_s^2.

- Why doesn't higher voltage across R_0 lead to higher power? If we increase R_0 past the value of R_s, we do increase the value of voltage over R_s, but we decrease the current through both resistors, which ends up decreasing the power absorbed by R_0.
- If that's the case, why don't we decrease R_0? Well, decreasing R_0 does indeed increase the current through both resistors, but only to a point (our current can't go higher than V_s/R_s). Decreasing R_0 also has the effect of decreasing the voltage over the resistor, and the total effect is to decrease the power absorbed by the load.
- It turns out that $R_0 = R_s$ is the sweet spot, as you can see by the plot shown in Figure 4.1

4.2 Nonlinear Circuit Elements

4.3 Incorporating Computation - Graphical Analysis

5

Operational Amplifiers

5.1 What is an Op-Amp?

- Op-Amp stands for Operational Amplifier
- An amplifier is a device that receives an input and sends an output that is proportional to the input

5.2 Golden Rules

Input Current is Zero $(i_+ = i_- = 0)$

Looking back at our simple model of the Op-Amp, this stems from our assumption that R_{in} is very large. Lets assume that the input voltage is in a normal range (something like 5-20 V) and that R_{in} is about 1 GΩ. In that case, the current through the resistor looks something like

$$i_{in} = V_{in}/R_{in} = 20\text{V}/1\text{G}\Omega = 20\text{nA}.$$

That is, admittedly, not zero. However, if we try to account for that error in a normal circuit (one in which the current is measured in mA), the error we get from neglecting it is on the order of 0.01%.

Voltage is Equivalent if Feedback is Present $(V_- = V_+)$

5.3 Analyzing Circuits with Op-Amps

Part II

Alternating Current

6

AC Circuits

- Description of time-varying current/voltage in general
- Specification of sinusoidal time-varying current/voltage
  - Frequency
  - Phase angle
- Definition of lagging/leading
  - Difference between phase angles
  - Recognition that there is no lead/lag for resistive circuits
- Solution of $\Delta V = iR$ for purely resistive circuits

6.1 Phasor Notation

- Callback to Euler's Formula (back to Ch. 1)
- $\cos \omega t + \phi = \Re \left\{ e^{j\omega t} e^{j\phi} \right\}$
- Give an amplitude associated with the circuit: $\Re \left\{ A e^{j\omega t} e^{j\phi} \right\}$
- Hide $e^{j\omega t}$, since we typically only deal with one frequency at a time: $\Re \left\{ A e^{j\phi} \right\}$
- Hide the $\Re$, since we know we want the real part: $A e^{j\phi}$
- Replace $e^{j\phi}$ with the simpler $\angle\phi$: $A\angle\phi$
- Now you have phasor notation

6.2 Capacitors

- Capacitors are energy storage devices that store up electric charge
- Most simply two plates in parallel with a gap filled with some non-conducting material
- Amount of charge stored is proportional to voltage: $q = CV$
- C is the **capacitance** of the capacitor, and is measured in Farads (F)
- Current is change in charge per change in time: $i = \frac{\Delta q}{\Delta t}$
- Can be analyzed through Calculus ($i = \frac{dq}{dt}$)
- Using Euler's Formula and Calculus, it is possible to reduce $q = CV$ to $\Delta V = iZ_C$, where $Z_C = \frac{1}{j\omega C}$
- Again using Calculus, it is possible to find that $E_C = \frac{1}{2}CV^2$
- Full mathematical details in Appendix (not yet!)

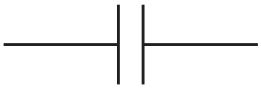
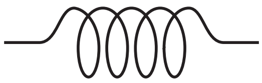
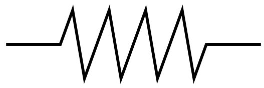
6.3 Inductors

- Inductors are energy storage devices that create a magnetic field
- Typically composed of a coil of wire surrounding a ferromagnetic core
- Magnetic fields resist change, and the change in the magnetic field per change in time results in a voltage difference (Faraday's law): $\Delta V = -\frac{d\Phi}{dt}$
- The magnetic field generated is proportional to the current through the inductor: $\Phi = Li$
- L is the inducatance of the inductor, and is measured in Henrys (H)
- Through the same methods as capacitors, it is possible to reduce $\Delta V = -\frac{d\Phi}{dt}$ to $\Delta V = iZ_L$, where $Z_L = j\omega L$
- The energy stored in the inductor is $E_L = \frac{1}{2}Li^2$
- Full mathematical details in Appendix (not yet!)

6.4 Equivalent Impedance and Ohm's Law

- Ohm's Law works in AC exactly like it does in DC, just replace R with Z: $\Delta V = IZ$
- Impedances of each component summarized in Table 6.1
- Equivalent impedance evaluated exactly as equivalent resistance: $Z_{series} = Z_1 + Z_2$, $Z_{parallel} = \frac{Z_1 \cdot Z_2}{Z_1 + Z_2}$
- Several shortcuts can be used for like elements:
  - $C_{parallel} = C_1 + C_2$, $C_{series} = \frac{C_1 \cdot C_2}{C_1 + C_2}$
  - $L_{series} = L_1 + L_2$, $L_{parallel} = \frac{L_1 \cdot L_2}{L_1 + L_2}$

Table 6.1: A Summary of Elements Which Provide an Impedance

	Capacitor	Inductor	Resistor
Symbol			
Unit	Farad (F)	Henry (H)	Ohm (Ω)
Typical Values	0.1 pF – 0.5 mF[a]	0.1 μH – 10 mH	10 Ω – 1 GΩ
Impedance	$Z_C = \frac{1}{j\omega C}$	$Z_L = j\omega L$	$Z_R = R$
Energy Storage	$E_C = \frac{1}{2}CV^2$	$E_L = \frac{1}{2}Li^2$	N/A

[a] Typical ceramic capacitors. Electrolytic capacitors have larger capacitance.

- Simple RL circuit (series)
  - $V = 10\cos(500t + \pi/4)$ V
  - $R = 100\Omega$
  - $L = 50mH$
  - Find I
- Simple RLC circuit (RL parallel, C series)
  - $I = 40\cos(2400t)$ mA
  - $R = 1000\Omega$
  - $L = 100mH$
  - $C = 15\mu F$
  - Find V

7

AC Power

7.1 Watt's Law for AC

- Power for AC circuits looks a little different than for DC
- $p(t) = v(t) * i(t)$
- Plot voltage, current, and power for a resistive load
- Do the same for a capacitive, inductive load
- Again for resistive + capacitive
- Find average power through integration

7.2 Average Power

- Helpful to introduce some additional ways of talking about amplitude
  - The amplitude we've been using is the **peak** amplitude. As an example, the peak voltage would be written as V_p.
  - Another useful amplitude is the **peak-to-peak** amplitude. This measures the total distance between peaks, and is typically written as V_{pp}. The peak-to-peak voltage is twice the peak amplitude.
  - Finally, we have the **RMS** amplitude. RMS stands for root-mean-square, which refers to the method used to obtain it (find more information in the Appendix). For sinusoidal voltage, $V_{RMS} = \frac{1}{\sqrt{2}}V_p$. For non-sinusoids, the RMS voltage will be different.
- The RMS voltage and current are directly used in finding the power. Watt's law for AC gives that the average power is the product of the root-mean-square voltage and current multiplied by the cosine of the phase offset between the two: $P_{avg} = V_{RMS}I_{RMS}\cos(\phi_V -$

ϕ_I). Translated to peak amplitudes (which we normally use), $P_{avg} = \frac{1}{2} V_p I_p \cos(\phi_V - \phi_I)$

7.3 Complex Power

- Definition: $S = VI^*$
- What does Imaginary Power mean? Why does it matter?
- Power factor: $pf = \cos(\phi_Z)$
- Calculating complex power for simple circuits

7.4 Power Factor Correction

7.5 Maximum Power for AC Circuits

8

Passive Filters

8.1 Frequency Response

- General circuit (right now a highpass)
- Plotting (linear -> semilog -> loglog)
- RC and RL circuits (analysis and design)
  - Highpass filter
  - Lowpass filter
- RLC circuits (analysis and design)
  - Bandpass
  - Bandstop/notch
  - Second-order Highpass
  - Second-order Lowpass
- RCRC circuits (Commentary on ease of construction)

When we talk about analyzing filters, what we mean is how does the output voltage respond to different frequencies. We call this the **freqeuncy response** of the circuit. In practice, this results in solving for the output voltage while keeping ω a variable. Figure 8.1 shows an AC source connected to a capacitor and resistor in series.

Our goal here is to find the frequency response of the output voltage. We know that $Z_C = \frac{1}{j\omega C}$ and $Z_R = R$. From voltage division, we obtain the following:

$$V_{out} = V_s \frac{R}{R + \frac{1}{j\omega C}} \tag{8.1}$$

Figure 8.1: A simple circuit composed of a source and two components.

Simplified, we can rewrite this as:

$$\frac{V_{out}}{V_s} = \frac{j\omega RC}{1 + j\omega RC} \tag{8.2}$$

At this point, we either feed this into a program, which can calculate the response over a wide variety of ω values, or we look into some limits. Specifically, we'd like to find the limit as ω approaches zero and infinity.

$$\lim_{\omega \to 0} \frac{j\omega RC}{1 + j\omega RC} = \frac{j(0)RC}{1 + j(0)RC} = 0 \tag{8.3}$$

$$\lim_{\omega \to \infty} \frac{j\omega RC}{1 + j\omega RC} = \frac{jRC}{jRC} = 1 \tag{8.4}$$

If we want to plot the response by hand, we need to look at one more value, typically called the **breakpoint frequency**, ω_b. This is defined as the freqency at which the magnitude of impedance of the two components are equal ($|Z_R| = |Z_C|$):

$$\left|\frac{1}{j\omega_b C}\right| = R \qquad \frac{1}{\omega_b C} = R \qquad \omega_b = \frac{1}{RC}$$

If we plug this back into our frequency response, we find the following:

$$\frac{V_{out}}{V_s} = \frac{jRC\left(\frac{1}{RC}\right)}{1 + jRC\left(\frac{1}{RC}\right)}$$

$$\frac{V_{out}}{V_s} = \frac{j}{1 + j}$$

$$\left|\frac{V_{out}}{V_s}\right| = \frac{|j|}{|1 + j|} = \frac{1}{\sqrt{2}}$$

In fact, this is an equivalent definition of the breakpoint frequency. Our breakpoint occurs when the magnitude of our frequency response is exactly $\sqrt{2}$.

Let's do the other option. If we have access to a computer, we can plot the amplitude and phase angle of the frequency response as seen in Figure 8.2. This plot is useful, but we can change our axes to see things a little cleaner.

Figure 8.2: The frequency response of the circuit using linear axes.

Figure 8.3: The frequency response of the circuit using logarithmic axes.

Figure 8.1 shows a log-log plot of the frequency response. In this plot, ω is replaced in the x-axis by $\log(\omega)$ and $\left|\frac{V_{out}}{V_s}\right|$ is replaced by $\log\left(\left|\frac{V_{out}}{V_s}\right|\right)$. Doing so, the behaviour of the circuit stands out very clearly. Note that the phase angle ϕ did not change between Figures 8.2 and .

So what's happening? Starting at high frequencies, the response is fairly constant until you hit the break point, and then it drops off logarithmically as you decrease the frequency. To describe this, we say that we allow high frequencies to *pass through*. In shorter terms, this is a **highpass filter**.

8.2 Two-element Filters

To start off, let's analyze some filters that we can create with two elements. The first of these will combine a resistor and a capacitor, as shown in Figure 8.4.

8.3 Second Order Filters

Figure 8.4: A simple circuit composed of a source and two components.

9

Active Filters

9.1 Op-Amps in AC

- Recap inverting amplifier: $G(\omega) = -Z_f/Z_s$
- Recap non-inverting amplifier: $G(\omega) = 1 + Z_f/Z_s$
- General construction rules (avoiding $Z_f \to \infty$ and $Z_s \to 0$)

9.2 First Order Filters

- Highpass filter
  - Add a capacitor in series with R_s or an inductor in parallel with R_f
  - Maximum amplitude will be R_f/R_s
- Lowpass filter
  - Add an inductor in series with R_s or a capacitor in parallel with R_f
  - Maximum amplitude will be R_f/R_s
- Redrawing circuits with equivalent components at $\omega \to 0$ and $\omega \to \infty$ limits

9.3 Second Order Filters

- Highpass filter
- Lowpass filter
- Bandpass filter

- Bandstop filter
- Composite filters

Part III

Analysis of Circuit Networks

10

Superposition

11

Node Voltage Method

11.1 Kirchoff's Current Law Revisited

11.2 Using the Node Voltage Method

Writing KCL

Converting to Voltage with Ohm's Law

Dealing with Voltage Sources

Dealing with Dependent Sources

11.3 Incorporating Computation - Linear Algebra

12

Mesh Current Method

12.1 Mesh Currents vs. Branch Currents

12.2 Kirchoff's Voltage Law

12.3 Incorporating Computation - Linear Algebra

13

Thevenin and Norton Equivalent Circuits

13.1 Circuit Loads

13.2 Determining Thevenin Resistance

13.3 Determining Thevenin Voltage

13.4 Determining Norton Current

Appendix A

Review of Algebra

The things in this section are parts of mathematics that are critical to success in an introductory Circuits course, but that we assume you already know. In that vein, this section will not be an in-depth coverage of the topic of Algebra, but rather an overview of the specific topics that you will need.

Isolating Variables

Simple Systems of Equations

Trigonometry

Sine and Cosine Curves

Appendix B

Units

What is electric charge?

- Fundamental property of matter
- Most matter is neutral on a macroscopic level
- Atoms are composed of:
  - protons, which have a positive charge
  - electrons, which have a negative charge
  - neutrons, which do not possess a net charge
- The charge of an electron is constant and the most fundamental unit of charge. However, it is very small
- A more useful unit of charge is the Coulomb, which is equivalent to roughly 6.24×10^{18} electrons.

Current, Voltage, Resistance

The three concepts we will be most interested in throughout this book are current, voltage, and resistance.

Current is:

- The amount of charge moving through a region (usually a wire) in a given amount of time.
- Measured in Amperes – Coulombs per second (1 A = 1 C/s)

Voltage is:

- The amount of energy present in a given amount of charge
- Also called "electric potential"

- Think about a ball on a hill, which has some potential energy. In this analogy, the mass of the ball is like the charge of our particle, the height of the hill is the voltage, and the potential energy is the electric energy of the particle.
- Measured in Volts – Joules per Coulomb (1 V = 1 J/C)

Resistance is:

- A measure of the amount of energy it takes to get some charge through a circuit element in a certain amount of time
- Measured in Ohms, which does not have a useful conversion to more basic units
- Think of a very thin pipe. In order to get water through the pipe, you will need to push harder and harder as the pipe gets thinner (or, alternatively, you will have to push harder in order to get more water through the same pipe). The pipe has some fluid resistance, just as our circuit elements could have some electric resistance.
- We will cover this more when we get to Ohm's Law

Appendix C

A Word on Graphs

The first thing you probably think of when you hear the word graph is plugging $y = mx + b$ into your calculator. For the rest of this book, we'll refer to that concept as a plot.

A graph in a purely mathematical sense is a collection of **nodes** and **branches**. Nodes are like locations or specific points in space, and branches are the connections between the nodes (similar to pathways between the various locations). You can draw a graph of this type as shown in Figure C.1.

We will be using the concept of a graph to talk about electric circuits.

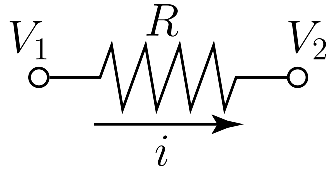


Figure C.1: Pictured is a graph, which consists of a number of nodes and branches connecting those nodes.

Appendix D

What is a Circuit?

A circuit is like a graph in that it contains nodes and branches. For a circuit, a node is any set of locations that is connected by an **ideal wire**. Charges can pass through ideal wires without any resistance, which means that they aren't losing any energy. Another way of phrasing it is that the voltage at every point in the wire will be the same: every charge is going to effectively have the same amount of energy in the node.

Branches are any circuit element that is not an ideal wire. We'll talk later in detail about resistors, voltage sources, and current sources, which are the most common elements used in this class. Charges that move through these sources will lose or gain energy, and therefore change their voltage.

The final detail about a circuit is that they usually contain a complete loop. We very rarely have charges accumulate in specific nodes, which means that for charges to move around, they need to move around in a complete circle. We occasionally have **open circuits**, which do not make a complete loop. These circuits do not allow any movement of charge, and therefore have no current at all.

Appendix E

Vector Mathematics

Appendix F

Complex Numbers

- Complex numbers builds on our understanding of vectors
- If all real numbers can be plotted on a number line, we can draw another number line orthogonal[1] to the first to represent imaginary numbers. A complex number is a point within the plane we created.
- Complex numbers can be represented in two ways: Cartesian form and polar form
  - In Cartesian form, the real and imaginary parts are simply added together, with the imaginary part multiplied with j, which is the Circuits name for $\sqrt{1}$. For instance, $1 + j2$ would be a number which is one unit to the right on the Real number line, and two units up on the imaginary number line.
  - In polar form, a line connecting the point to the origin is defined, and the point is then described using the length of that line and the angle it makes with the Real number line. $1 + j2$ would have a length of $\sqrt{5}$ and an angle of about 63 degrees or 1.1 rad. This is commonly represented as $\sqrt{5}e^{j1.1}$ or just $\sqrt{5}\angle 1.1$
  - If a number C can be described as $C = x + jy$, it can be converted to polar coordinates ($C = A\angle\theta$) through the following formulas:
    * $A = \sqrt{x^2 + y^2}$
    * $\theta = \tan^{-1} \frac{y}{x}$
  - Likewise, the number can be converted back using these formulas:
    * $x = A \cos\theta$
    * $y = A \sin\theta$
- Addition is only feasible in Cartesian coordinates. If you need to add two imaginary numbers in polar form, you should convert both to Cartesian first.

[1] perpendicular

  - Take two complex numbers in Cartesian form: $C_1 = x_1 + jy_1$ and $C_2 = x_2 + jy_2$
  - The sum of those two numbers is defined as $C_1 + C_2 = (x_1 + x_2) + j(y_1 + y_2)$
  - To subtract C_2 from the C_1, simply negate both x_2 and y_2: $C_1 - C_2 = (x_1 - x_2) + j(y_1 - y_2)$

- Multiplication is feasible in either Cartesian or polar coordinates
  - Take two complex numbers in Cartesian form: $C_1 = x_1 + jy_1$ and $C_2 = x_2 + jy_2$
  - The product of those two can be written through the FOIL (First, Outer, Inner, Last) method: $C_1 \cdot C_2 = x_1x_2 + jx_1y_2 + jx_2y_1 + j^2y_1y_2$
  - Recognize that $j^2 = -1$: $C_1 \cdot C_2 = x_1x_2 + jx_1y_2 + jx_2y_1 - y_1y_2$
  - Alternatively, if you have two complex numbers in polar form: $C_1 = A_1\angle\theta_1$ and $C_2 = A_2\angle\theta_2$
  - The new amplitude is simply the product of the original amplitudes, and the new angle is the sum of the original angles: $C_1 \cdot C_2 = C_1C_2\angle(\theta_1 + \theta_2)$

- Division is possible in Cartesian coordinates, but difficult enough that it is much easier to simply convert to polar
  - Take two numbers in polar form: $C_1 = A_1\angle\theta_1$ and $C_2 = A_2\angle\theta_2$
  - The new amplitude is the result of the division of the original amplitudes, and the new angle is the result of subtraction of the original amplitudes: $C_1/C_2 = A_1/A_2\angle(\theta_1 - \theta_2)$

Appendix G

Linear Algebra

The first thing to note is that linear algebra is a huge subject with applications not only in circuit analysis, but also in artificial intelligence, simulation and modeling, signal analysis, and computer graphics, just to name a few. We will only be scratching the surface by covering matrix/vector algebra and matrix inversion.

What is a Matrix?

A matrix is simply a collection of numbers in an array of a specific size. For instance, a 2x3 matrix could be written as follows:

$$A = \begin{bmatrix} 2 & 1 & \text{-4} \\ \pi & 2/9 & 14 \end{bmatrix}$$

Note that the first number in the size refers to the width or the number of columns of the matrix. The second refers to the height or the number of rows.

Vectors, by contrast, will always have one of their dimensions that has a length of 1. A **row vector** would have a size of Nx1, while a **column vector** would have a size of 1xN.

$$x_{row} = \left\{ \begin{array}{ccc} 20 & \sqrt{2} & 0.2 \end{array} \right\} \qquad\qquad x_{col} = \left\{ \begin{array}{c} 5 \\ e^2 \\ 2 \end{array} \right\}$$

A **scalar** is a normal number that we are used to working with. It could be defined as a matrix with a size of 1x1.

Matrix/Vector Algebra

Once we have some matrices and vectors defined, we can start performing some mathematical operations on them. First, we have addition/subtraction.

The most important thing to note is that the matrices we use MUST have the same size for us to add or subtract them.

Since matrices and vectors have some size and shape to them, we have the ability to move around the numbers inside them. The operator that does this is known as the **transpose** operator. Transposing the matrix A from before results in:

$$A^T = \begin{bmatrix} 2 & \pi \\ 1 & 2/9 \\ \text{-4} & 14 \end{bmatrix}$$

Looking at each column, the values come from the original rows.

Multiplication is a bit trickier. In order to multiply two matrices, the number of rows of the first matrix must match the number of columns for the second matrix. Also, some of the normal rules of scalar multiplication ($ab = ba$) are slightly modified ($AB = B^T A$)

Writing Systems of Equations Using Linear Algebra

Matrix Inversion

Appendix H

Computer Resources - Matlab

Setting up Matlab/Octave

Using Matlab to Solve Problems

Appendix I

Computer Resources - Python

Setting up Python

Using Python to Solve Problems