Bayes theorem with Gaussian mixtures

NCEO Intensive Course on Data Assimilation

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1 Review of Theory

1.1 Bayes theorem

Let $\mathbf{x} \in \mathcal{R}^{N_x}$ be a random variable with a prior pdf $p(\mathbf{x})$. Let $\mathbf{y} \in \mathcal{R}^{N_y}$ be an observation, which is related to the state variable by the likelihood pdf $p(\mathbf{y}|\mathbf{x})$. The posterior pdf (i.e. the conditional pdf of the state variables *given* the observations) can be obtained using Bayes theorem:

$$p\left(\mathbf{x}|\mathbf{y}\right) = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{p(\mathbf{y})} = \frac{p(\mathbf{y}|\mathbf{x})p(\mathbf{x})}{\int_{\mathcal{X}} p(\mathbf{y}|\mathbf{x})p(\mathbf{x})d\mathbf{x}} \tag{1}$$

where $\mathcal{X}$ is the statistical support of the variable $\mathbf{x}$. The marginal pdf $p(\mathbf{y})$ for the observations does not depend on the values of the state variables, and therefore can be considered a normalisation constant.

1.2 Gaussian mixtures

Let us work in a univariate setup. For brevity, let us denote the pdf of a Gaussian random variable x as:

$$\phi\left(x;\mu,\sigma\right) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{2}$$

where $-\infty < \mu < \infty$ is the population mean, and $\sigma > 0$ is the population standard deviation. If x has a Gaussian mixture distribution, its pdf can then be written as:

$$p(x) = \sum_{n=1}^{N} \alpha_n \phi\left(x;\mu_n,\sigma_n\right) \tag{3}$$

i.e. it is a weighted sum of individual Gaussian components. This sum needs to be convex, i.e.

$$\alpha_n \geq 0 \;\forall\; n, \quad \sum_{n=1}^{N} \alpha_n = 1 \tag{4}$$