Adaptive and Array Signal Processing
Homework 03

1. Given the SVD of a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} = \mathbf{U}_1\mathbf{\Sigma}_1\mathbf{V}_1^H \tag{1}$$

with the unitary matrices $\mathbf{U} \in \mathbb{C}^{m \times m}$, $\mathbf{V} \in \mathbb{C}^{n \times n}$ and the positive definite diagonal matrix $\mathbf{\Sigma}_1 \in \mathbb{R}^{r \times r}$, where $r = \text{rank}(\mathbf{A})$, the Moore-Penrose pseudo inverse $\mathbf{A}^+$ of $\mathbf{A}$ is defined as

$$\mathbf{A}^+ = \mathbf{V}_1\mathbf{\Sigma}_1^{-1}\mathbf{U}_1^H \tag{2}$$

   (a) What are the dimensions of the matrices $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{\Sigma}$ and $\mathbf{A}^+$?

   (b) Show that $\mathbf{U}_1^H\mathbf{U}_1 = \mathbf{1} \in \mathbb{R}^{r \times r}$ and $\mathbf{V}_1^H\mathbf{V}_1 = \mathbf{1} \in \mathbb{R}^{r \times r}$

   (c) Show that (2) satisfies the four Moore-Penrose conditions for a pseudo inverse

$$\begin{aligned} \mathbf{A}\mathbf{A}^+\mathbf{A} &= \mathbf{A} && (3) \\ \mathbf{A}^+\mathbf{A}\mathbf{A}^+ &= \mathbf{A}^+ && (4) \\ \mathbf{A}\mathbf{A}^+ &= \left(\mathbf{A}\mathbf{A}^+\right)^H && (5) \\ \mathbf{A}^+\mathbf{A} &= \left(\mathbf{A}^+\mathbf{A}\right)^H && (6) \end{aligned}$$

2. Look at the matrix $\mathbf{A}$ and its pseudo inverse $\mathbf{A}^+$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \mathbf{A}^+ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in terms of four real numbers $a, b, c, d \in \mathbb{R}$.

   (a) Write down the three linear equations of the variables a, b, c, d that follow from (3), (5) and (6) in the form:

$$\mathbf{D} \cdot \begin{bmatrix} a & b & c & d \end{bmatrix}^T = \mathbf{e} \tag{7}$$

   (Hint: since we are dealing with real numbers, the $(\bullet)^H$ operator becomes a pure transposition)

   (b) Compute the general solution of the underdetermined system (7) in terms of d.

   (c) Now determine the unique value d, that will also satisify the last Moore-Penrose condition (4)

   (d) Write down the pseudo inverse $\mathbf{A}^+$ you have obtained in this way.

(e) A SVD of $\mathbf{A}$ is given as

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

Compute $\mathbf{A}^+$ by the definition in (2) and compare to the previous result.

3. Consider the matrix

$$\mathbf{P} = \frac{1}{9} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \tag{8}$$

(a) Show that $\mathbf{P}$ is a projector onto a vector space.

(b) This vector space $\mathcal{S} = \text{range}(\mathbf{P})$ is a subspace of $\mathbb{C}^3$. What is its dimension?

(c) What is the dimension of the orthogonal complement $\mathcal{S}^\perp \subset \mathbb{C}^3$ of $\mathcal{S}$ in $\mathbf{C}^3$?

(d) Compute the projector $\mathbf{P}^\perp$ onto $\mathcal{S}^\perp$

(e) Using this result compute an orthonormal base of $\mathcal{S}^\perp$

(f) Have a look at the following subspace

$$\mathcal{S}_2 = \text{range}\left(\frac{1}{\sqrt{18}} \begin{bmatrix} 3 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix} \right) \tag{9}$$

Is $\mathcal{S}_2 = \mathcal{S}$? (Hint: Compute the projector onto $\mathcal{S}_2$ and compare to (8).)

4. The system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{b} \in \mathbb{C}^m$ has an exact solution only if $b \in \text{range}(\mathbf{A})$. If $m > n$ (i.e. more equations than unknowns) there is usually no exact solution, since $b \notin \text{range}(\mathbf{A})$ in most cases. The best we can do is modify the righthand side of the system such that $\mathbf{b}$ is replaced by $\widehat{\mathbf{b}}$, its projection onto the range of $\mathbf{A}$. Let $\mathbf{A}$ have a SVD as in (1) and define a projector $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{U}_1\mathbf{U}_1^H \tag{10}$$

Show that

(a) the least-squares solution obtained by the pseudo inverse, i.e. $\mathbf{x}_{LS} = \mathbf{A}^+\mathbf{b}$, is really an *exact* solution to the modified system $\mathbf{Ax} = \widehat{\mathbf{b}} = \mathbf{Pb}$

(b) the error $(\mathbf{b} - \widehat{\mathbf{b}})$ of the righthand sides is orthogonal to $\widehat{\mathbf{b}}$.