Adaptive and Array Signal Processing

Homework 01

1. A complex valued function $f(z) \in \mathbb{C}$ of a complex valued argument $z \in \mathbb{C}$ can always be expressed in terms of two real valued functions $u(x,y), v(x,y) \in \mathbb{R}$ of two real-valued variables $x, y \in \mathbb{R}$:

$$f(z) = f(x + \mathrm{j} \cdot y) = u(x,y) + \mathrm{j} \cdot v(x,y).$$

In the following $u(x,y), v(x,y)$ are to be continuously differentiable with respect to x and y in an arbitrarily small region around z. The complex derivative of $f(z)$ with respect to z is defined as

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{1}$$

   (a) Write (1) in terms of $\partial u/\partial x$ and $\partial v/\partial x$ by using $\Delta z = \Delta x$, i.e. by moving parallel to the real axis to the point z.

   (b) Repeat the exercise using $\Delta z = \mathrm{j} \cdot \Delta y$, i.e. by moving parallel to the imaginary axis to the point z.

   (c) In order for (1) to be uniquely defined, these two results must be the same. What constraint does this impose on $u(x,y)$ and $v(x,y)$?

   (d) Compare this result to the Cauchy-Riemann equations.

2. Let $g(\mathbf{z}, \mathbf{z}^*) = f(\mathbf{x}, \mathbf{y}) \in \mathbb{C}$ be a function of a complex vector $\mathbf{z} = \mathbf{x} + \mathrm{j} \cdot \mathbf{y} \in \mathbb{C}^n$ and its complex conjugate $\mathbf{z}^* = \mathbf{x} - \mathrm{j} \cdot \mathbf{y} \in \mathbb{C}^n$ with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We have that the total differential of g and f, respectively, is

$$\mathrm{d}g = \left(\frac{\partial g}{\partial \mathbf{z}}\right)^{\mathrm{T}} \mathrm{d}\mathbf{z} + \left(\frac{\partial g}{\partial \mathbf{z}^*}\right)^{\mathrm{T}} \mathrm{d}\mathbf{z}^* \tag{2}$$

$$\mathrm{d}f = \left(\frac{\partial f}{\partial \mathbf{x}}\right)^{\mathrm{T}} \mathrm{d}\mathbf{x} + \left(\frac{\partial f}{\partial \mathbf{y}}\right)^{\mathrm{T}} \mathrm{d}\mathbf{y}. \tag{3}$$

   (a) By using the fact that $\mathrm{d}g = \mathrm{d}f$, show that

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial g}{\partial \mathbf{z}} + \frac{\partial g}{\partial \mathbf{z}^*} \tag{4}$$

$$\frac{\partial f}{\partial \mathbf{y}} = \mathrm{j} \cdot \left(\frac{\partial g}{\partial \mathbf{z}} - \frac{\partial g}{\partial \mathbf{z}^*}\right). \tag{5}$$

(b) From the previous result show that

$$\frac{\partial g}{\partial \mathbf{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial \mathbf{x}} - \mathrm{j}\cdot\frac{\partial f}{\partial \mathbf{y}}\right) \tag{6}$$

$$\frac{\partial g}{\partial \mathbf{z}^*} = \frac{1}{2}\left(\frac{\partial f}{\partial \mathbf{x}} + \mathrm{j}\cdot\frac{\partial f}{\partial \mathbf{y}}\right). \tag{7}$$

(c) If $f(x,y) = u(x,y) + \mathrm{j}\cdot v(x,y)$, where $u(x,y), v(x,y) \in \mathbb{R}$ show that the differential dg does not depend on the differential d$\mathbf{z}^*$ if $g(\mathbf{z},\mathbf{z}^*) = f(\mathbf{x},\mathbf{y})$ is analytic, i.e. show that $\frac{\partial g}{\partial \mathbf{z}^*} = \mathbf{0}$.

3. Consider the function

$$I(\mathbf{w},\mathbf{w}^*) = \mathbf{w}^H\mathbf{R}\mathbf{w} - 2\cdot\mathrm{Re}\left\{\mathbf{w}^H\mathbf{p}\right\},$$

with $\mathbf{w},\mathbf{p} \in \mathbb{C}^n$ and $\mathbf{R} = \mathbf{R}^H \in \mathbb{C}^{n\times n}$.

(a) Is $I(\mathbf{w},\mathbf{w}^*)$ a real valued function?

(b) Find a $\mathbf{w}$ that minimizes $I(\mathbf{w},\mathbf{w}^*)$ by solving $\frac{\partial I}{\partial \mathbf{w}^*} = \mathbf{0}$.

(c) Find a $\mathbf{w}$ that minimizes $I(\mathbf{w},\mathbf{w}^*)$ by solving $\frac{\partial I}{\partial \mathbf{w}} = \mathbf{0}$.

(d) Compare the results of 3b and 3c.

4. Solve the following constrained real-valued minimization problem

$$\text{minimize} \quad f(x_1,x_2) = 1 + 2x_1x_2 + x_1^2 + 3x_2^2 \tag{8}$$

$$\text{subject to} \quad g(x_1,x_2) = 1 + x_1 - 2x_2 = 0 \tag{9}$$

$$x_1, x_2, f, g \in \mathbb{R},$$

(a) by solving (9) for x_2 in terms of x_1 and then minimizing (8).

(b) by means of (real) Lagrangian multipliers.

5. Solve the following constrained complex minimization problem:

$$\text{minimize} \quad f(\mathbf{w}) = \mathbf{w}^H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{w} \tag{10}$$

$$\text{subject to} \quad \mathbf{g}(\mathbf{w}) = \begin{pmatrix} 1 & -\mathrm{j} \\ \mathrm{j} & 2 \\ 1 & \mathrm{j} \end{pmatrix}^{\mathrm{H}} \mathbf{w} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{0}, \tag{11}$$

with $\mathbf{w} \in \mathbb{C}^3$, $f \in \mathbb{R}$, $\mathbf{g} \in \mathbb{C}^2$ by means of complex Lagrangian multipliers.