Adaptive and Array Signal Processing

Solution to Homework 3

(1)

a) $$m\left\{\begin{bmatrix} \overset{r}{U_1} & \overset{m-r}{U_2} \end{bmatrix}\right. \; m\left\{\begin{bmatrix} \overset{r}{\Sigma_1} & \overset{n-r}{0} \\ 0 & 0 \end{bmatrix}\right. \; \overset{n}{\begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix}} \begin{matrix} \}\, r \\ \}\, n-r \end{matrix}$$

$U_1 \in \mathbb{C}^{m\times r}$ $\quad V_1 \in \mathbb{C}^{n\times r}$ $\quad \Sigma_1 \in \mathbb{R}_0^{+\, r\times r}$ $\quad A^+ \in \mathbb{C}^{n\times m}$

$U_2 \in \mathbb{C}^{m\times(m-r)}$ $\quad V_2 \in \mathbb{C}^{n\times(n-r)}$ $\quad \Sigma \in \mathbb{R}_0^{+\, m\times n}$

b) $$1 = U^H U = \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} U_1^H U_1 & U_1^H U_2 \\ U_2^H U_1 & U_2^H U_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with $U_1^H U_1 = 1 \in \mathbb{R}^{r\times r}$ and $U_2^H U_2 = 1 \in \mathbb{R}^{(m-r)\times(m-r)}$

To show that $V_1^H V_1 = 1$ the same thing can be done with $V^H V = 1$

c)

- $$A A^+ A = U_1 \Sigma_1 \underbrace{V_1^H V_1}_{1} \Sigma_1^{-1} \underbrace{U_1^H U_1}_{1} \Sigma_1 V_1^H = U_1 \underbrace{\Sigma_1 \Sigma_1^{-1}}_{1} \Sigma_1 V_1^H$$
$$= U_1 \Sigma_1 V_1^H = A$$

- $$A^+ A A^+ = V_1 \Sigma_1^{-1} \underbrace{U_1^H U_1}_{1} \Sigma_1 \underbrace{V_1^H V_1}_{1} \Sigma_1^{-1} U_1^H = V_1 \underbrace{\Sigma_1^{-1} \Sigma_1}_{1} \Sigma_1^{-1} U_1^H$$
$$= V_1 \Sigma_1^{-1} U_1^H = A^+$$

- $$(A A^+)^H = (U_1 \Sigma_1 \underbrace{V_1^H V_1}_{1} \Sigma_1^{-1} U_1^H)^H = (U_1 \underbrace{\Sigma_1 \Sigma_1^{-1}}_{1} U_1^H)^H = (U_1 U_1^H)^H$$
$$= U_1 U_1^H = U_1 \Sigma_1 \Sigma_1^{-1} U_1^H = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = A A^+$$

- $$(A^+ A)^H = (V_1 \Sigma_1^{-1} \underbrace{U_1^H U_1}_{1} \Sigma_1 V_1^H)^H = (V_1 \underbrace{\Sigma_1^{-1} \Sigma_1}_{1} V_1^H)^H = (V_1 V_1^H)^H$$
$$= V_1 V_1^H = V_1 \Sigma_1^{-1} \Sigma_1 V_1^H = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = A^+ A$$

② a) $\underset{\sim}{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\underset{\sim}{A}^+ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $\underset{\sim}{A}\,\underset{\sim}{A}^+\underset{\sim}{A} = \underset{\sim}{A}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b+c+d & a+b+c+d \\ a+b+c+d & a+b+c+d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

then $a+b+c+d = 1$

- $\underset{\sim}{A}\,\underset{\sim}{A}^+ = (\underset{\sim}{A}\,\underset{\sim}{A}^+)^H$

$$\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}^H = \begin{bmatrix} a+c & a+c \\ b+d & b+d \end{bmatrix}$$

then $a+c = b+d \implies a-b+c-d = 0$

- $\underset{\sim}{A}^+\underset{\sim}{A} = (\underset{\sim}{A}^+\underset{\sim}{A})^H$

$$\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}^H = \begin{bmatrix} a+b & c+d \\ a+b & c+d \end{bmatrix}$$

then $a+b = c+d \implies a+b-c-d = 0$

We can write all the equations as follows

$$\underset{\sim}{D}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underset{\sim}{e}$$

b) Solving the above system of equations in terms of d we get

$a = d$ $b = \frac{1}{2} - d$ and $c = \frac{1}{2} - d$

c) $$\underset{\sim}{A}^+\underset{\sim}{A}\,\underset{\sim}{A}^+ = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a(a+b) + c(a+b) & b(a+b) + d(a+b) \\ a(c+d) + c(c+d) & b(c+d) + d(c+d) \end{bmatrix}$$

$$= \begin{bmatrix} (a+b)(a+c) & (a+b)(b+d) \\ (a+c)(c+d) & (c+d)(b+d) \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$(a+b)(a+c) \stackrel{!}{=} a$

$(a+b)(a+c) = (d + \frac{1}{2} - d)(d + \frac{1}{2} - d) = \frac{1}{4} = a = d$

Then $d = \frac{1}{4}$

d) $\underset{\sim}{A}^+ = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$ since $a = d = 1/4$

$c = b = \frac{1}{2} - d = 1/4$

$d = 1/4$

e) $\underset{\sim}{A}^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix} [2]^{-1} [-1 \quad -1] \cdot \frac{1}{\sqrt{2}} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Same solution!

③ a) $\underset{\sim}{P}$ is an orthogonal projector if it satisfies

$$\underset{\sim}{P} \cdot \underset{\sim}{P} = \underset{\sim}{P} \quad \text{and} \quad \underset{\sim}{P}^H = \underset{\sim}{P}$$

- $\underset{\sim}{P}\,\underset{\sim}{P} = \frac{1}{9^2} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = \underset{\sim}{P}$ and
- $\underset{\sim}{P}^H = \underset{\sim}{P}$ then $\underset{\sim}{P}$ is an orthogonal projector onto a vector space

b) $\dim S = \operatorname{rank} \underset{\sim}{P}$

Reducing $\underset{\sim}{P}$ by Gaussian elimination

$$\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2/5 & 4/5 \\ 0 & 36/5 & 18/5 \\ 0 & 18/5 & 9/5 \end{bmatrix} \to \begin{bmatrix} 1 & -2/5 & 4/5 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$ Two pivots

So $\underset{\sim}{P}$ has only two independent columns (or rows) so the rank $\underset{\sim}{P} = 2$ and $\dim S = 2$

c) $\dim S^\perp = 3 - \dim S = 1$ since $\underset{\sim}{P} \in \mathbb{C}^3$

d) The projector $\underset{\sim}{P}^\perp$ onto $S^\perp$ is given by $\underset{\sim}{P}^\perp = \underset{\sim}{1} - \underset{\sim}{P}$

$$\underset{\sim}{P}^\perp = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix}$$

e) Notice that the dimension of $S^{\perp}$ is 1, therefore we would have that all of the columns of $\underset{\sim}{P}^{\perp}$ are multiples of one vector. Through Gaussian elimination we get

$$\begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ which indicates that the $\mathrm{rank}(\underset{\sim}{P}^{\perp}) = 1$

We can take any column of $\underset{\sim}{P}^{\perp}$ and from it we can construct a basis for $\underset{\sim}{P}^{\perp}$. Taking the first column:

$$\frac{1}{9}\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$$ and normalizing it $\longrightarrow \frac{1}{9}\sqrt{4^2 + 2^2 + (-4)^2} = \frac{2}{3}$

we then can get a basis $\underset{\sim}{B}$

$$\underset{\sim}{B} = \frac{1}{9}\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} \div \frac{2}{3} = \frac{1}{6}\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

f) $S_2' = S'$ if they have the same projector $\underset{\sim}{P}$.

First we need the basis B_2 of S_2'. The vectors given in (9) are already a basis for S_2 because they are orthonormal:

$$\underset{\sim}{B}_2^H \underset{\sim}{B}_2 = \frac{1}{\sqrt{18}}\begin{bmatrix} 3 & 0 & 3 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}\frac{1}{\sqrt{18}} = \frac{1}{18}\begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore $\underset{\sim}{B}_2$ is a basis for S_2'. Now let us compute the projector onto S_2'

$$\underset{\sim}{P}_2 = \underset{\sim}{B}_2 \underset{\sim}{B}_2^H = \frac{1}{18}\begin{bmatrix} 3 & -1 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 3 \\ -1 & 4 & 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 10 & -4 & 8 \\ -4 & 16 & 4 \\ 8 & 4 & 10 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

So since $\underset{\sim}{P}_2 = \underset{\sim}{P}$ and the projector onto the same subspace is unique, then $S_2' = S'$

④ a) $\hat{\underline{b}}$ is the projection of $\underline{b}$ onto the range of $\underset{\sim}{A}$.
A basis for the range of $\underset{\sim}{A}$ is $\underset{\sim}{U}_1$ so the projector can be written as

$$\underset{\sim}{P} = \underset{\sim}{U}_1 \underset{\sim}{U}_1^H \qquad \text{and we have} \quad \hat{\underline{b}} = \underset{\sim}{P}\underline{b}$$

Then let us substitute $\underline{x}_{LS}$ into $\underline{x}$ in $\underset{\sim}{A}\underline{x} = \hat{\underline{b}}$ to see if the equality holds:

$$\underset{\sim}{A}\,\underline{x} = \underset{\sim}{A}\,(\underset{\sim}{A}^+\underline{b}) = \underset{\sim}{U}_1 \underset{\sim}{\Sigma}_1 \underbrace{\underset{\sim}{V}_1^H \underset{\sim}{V}_1}_{\underset{\sim}{1}} \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{U}_1^H \underline{b}$$

$$= \underset{\sim}{U}_1 \underbrace{\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_1^{-1}}_{\underset{\sim}{1}} \underset{\sim}{U}_1^H \underline{b} = \underset{\sim}{U}_1 \underset{\sim}{U}_1^H \underline{b} = \underset{\sim}{P}\,\underline{b} = \hat{\underline{b}}$$

b) $\hat{\underline{b}}$ and the error $(\underline{b} - \hat{\underline{b}})$ are orthogonal if

$$\hat{\underline{b}}^H (\underline{b} - \hat{\underline{b}}) = 0$$

$$\hat{\underline{b}}^H (\underline{b} - \hat{\underline{b}}) = (\underset{\sim}{P}\,\underline{b})^H (\underline{b} - \underset{\sim}{P}\,\underline{b})$$

$$= \underline{b}^H \underset{\sim}{P}^H (\underset{\sim}{1} - \underset{\sim}{P})\,\underline{b}$$

$$= \underline{b}^H \underset{\sim}{P} (\underset{\sim}{1} - \underset{\sim}{P})\,\underline{b} \qquad \text{since } \underset{\sim}{P}^H = \underset{\sim}{P}$$

$$= \underline{b}^H (\underset{\sim}{P} - \underset{\sim}{P}^2)\,\underline{b}$$

$$= \underline{b}^H (\underset{\sim}{P} - \underset{\sim}{P})\,\underline{b} \qquad \text{since } \underset{\sim}{P}^2 = \underset{\sim}{P}$$

$$= \underline{b}^H \underset{\sim}{0}\,\underline{b}$$

$$= 0$$