## 1 Gaussian model for the (2,0) Dirac operator

#### 1.1 Notation

$$D = \gamma_1 \otimes \{H_1, \cdot\} + \gamma_2 \otimes \{H_2, \cdot\}. \tag{1}$$

Define  $\gamma_{\pm} := \frac{1}{2}(\gamma_1 \mp i\gamma_2)$  and  $W := H_1 + iH_2$ . D becomes:

$$D = \gamma_{+} \otimes \{W, \cdot\} + \gamma_{-} \otimes \{W^{\dagger}, \cdot\} \tag{2}$$

And the following relations hold:

$$\gamma_+^2 = 0 \tag{3}$$

$$\gamma_{\pm}\gamma_{\mp} = \frac{1}{2}(I_C \pm i\gamma_1\gamma_2) \tag{4}$$

$$(\gamma_{\pm}\gamma_{\mp})^2 = \gamma_{\pm}\gamma_{\mp} \tag{5}$$

where  $I_C$  is the identity in the space of  $\gamma$  matrices.

## 1.2 Action, $\operatorname{Tr} D^2$ part

Because of Eq.(3), only the cross terms survive in  $D^2$ :

$$D^{2} = \gamma_{+}\gamma_{-} \otimes \{W, \cdot\}\{W^{\dagger}, \cdot\} + \gamma_{-}\gamma_{+} \otimes \{W^{\dagger}, \cdot\}\{W, \cdot\}$$
 (6)

Taking the trace yields:

$$\operatorname{Tr} D^2 = 2C(n\operatorname{Tr} WW^{\dagger} + \operatorname{Tr} W\operatorname{Tr} W^{\dagger}) \tag{7}$$

where C is the dimension of the  $\gamma$  matrices and n the dimension of the W matrix.

# 1.3 Action, $\operatorname{Tr} D^4$ part

Because of Eq.(3), the cross terms vanish when squaring  $D^2$ :

$$D^{4} = \gamma_{+}\gamma_{-} \otimes (\{W, \cdot\}\{W^{\dagger}, \cdot\})^{2} + \gamma_{-}\gamma_{+} \otimes (\{W^{\dagger}, \cdot\}\{W, \cdot\})^{2}$$
 (8)

Taking the trace yields:

$$\operatorname{Tr} D^{4} = 2C \left[ n \operatorname{Tr}(WW^{\dagger})^{2} + \operatorname{Tr} W^{2} \operatorname{Tr} W^{\dagger 2} + 2(\operatorname{Tr} WW^{\dagger})^{2} + 2 \operatorname{Tr} W^{2} W^{\dagger} \operatorname{Tr} W^{\dagger} + 2 \operatorname{Tr} W^{\dagger 2} W \operatorname{Tr} W \right]. \tag{9}$$

### 1.4 Stationary point

Stationary points are solutions to the equation

$$\delta S = S[W + \delta W] - S[W] = 0 \quad \forall \ \delta W \ll 1. \tag{10}$$

To first order in  $\delta W$  the terms  $\operatorname{Tr} D^2$  and  $\operatorname{Tr} D^4$  give:

$$\delta \operatorname{Tr} D^{2} = 2C \left[ n \operatorname{Tr} W \delta W^{\dagger} + n \operatorname{Tr} W^{\dagger} \delta W + \operatorname{Tr} W \operatorname{Tr} \delta W^{\dagger} + \operatorname{Tr} W^{\dagger} \operatorname{Tr} \delta W \right]$$
(11)

$$\delta \operatorname{Tr} D^{4} = 2C \left[ 2n \operatorname{Tr}(WW^{\dagger}W\delta W^{\dagger}) + 2n \operatorname{Tr}(W^{\dagger}WW^{\dagger}\delta W) \right. \\ + 2 \operatorname{Tr} W^{2} \operatorname{Tr} W^{\dagger}\delta W^{\dagger} + 2 \operatorname{Tr} W^{\dagger 2} \operatorname{Tr} W\delta W \\ + 4 \operatorname{Tr} WW^{\dagger} (\operatorname{Tr} W\delta W^{\dagger} + \operatorname{Tr} W^{\dagger}\delta W) \\ + 2 \operatorname{Tr} W^{2}W^{\dagger} \operatorname{Tr} \delta W^{\dagger} + 2 \operatorname{Tr} W^{\dagger 2}W \operatorname{Tr} \delta W \\ + 2 \operatorname{Tr} W^{\dagger} (\operatorname{Tr} W^{2}\delta W^{\dagger} + \operatorname{Tr} WW^{\dagger}\delta W + \operatorname{Tr} W^{\dagger}W\delta W) \\ + 2 \operatorname{Tr} W (\operatorname{Tr} W^{\dagger 2}\delta W + \operatorname{Tr} W^{\dagger}W\delta W^{\dagger} + \operatorname{Tr} WW^{\dagger}\delta W^{\dagger}) \right]. \tag{12}$$

We check for solutions of the form:

$$W = I\rho e^{i\theta} \tag{13}$$

with  $\rho > 0$  and  $\theta \in [0, 2\pi)$ .

Tr 
$$D^2$$
:  $4Cn\rho \left(e^{i\theta} \operatorname{Tr} \delta W^{\dagger} + e^{-i\theta} \operatorname{Tr} \delta W\right)$  (14)

Tr 
$$D^4$$
:  $4Cn\rho \left(8\rho^2 e^{i\theta} \operatorname{Tr} \delta W^{\dagger} + 8\rho^2 e^{-i\theta} \operatorname{Tr} \delta W\right)$  (15)

putting the coefficient of  $\delta W$  or  $\delta W^{\dagger}$  to zero gives:

$$\rho^2 = 0 \quad \text{or} \quad \rho^2 = -\frac{g}{8}.$$
(16)

## 1.5 Quadratic action in $\delta W$

Write  $W = e^{i\theta}(\rho + \epsilon V)$  for small  $\epsilon$  and traceless V. We will separate the contributions to the action in orders of  $\epsilon$ .

$$\operatorname{Tr} D^2$$

$$O(1): \quad 4Cn^2\rho^2 \tag{17}$$

$$O(\epsilon): 4Cn\rho(\operatorname{Tr} V^{\dagger} + \operatorname{Tr} V)$$
 (18)

$$O(\epsilon^2): \quad 2C(n\operatorname{Tr} VV^{\dagger} + \operatorname{Tr} V\operatorname{Tr} V^{\dagger})$$
 (19)

$${\rm Tr}\, D^4$$

$$O(1): 16Cn^2\rho^4$$
 (20)

$$O(\epsilon): 24Cn\rho^3 \operatorname{Tr} V + \text{c.c.}$$
 (21)

$$O(\epsilon^{2}): 8C\rho^{2}(2n\operatorname{Tr}VV^{\dagger} + n\operatorname{Tr}VV + 2\operatorname{Tr}V\operatorname{Tr}V^{\dagger} + \operatorname{Tr}V\operatorname{Tr}V + c.c.)$$

$$(22)$$

Since V is traceless, the action to second order in  $\epsilon$  reads:

$$S = 2Cn^{2}(g\rho^{2} + 4\rho^{4}) + Cn\epsilon^{2} \operatorname{Tr} VV^{\dagger}(g + 16\rho^{2})$$
$$+ 8Cn\epsilon^{2}\rho^{2} \operatorname{Tr} VV + \text{c.c.}$$
(23)

Or:

$$S = 4Cn^2(g\rho^2 + 4\rho^4) + Cn\epsilon^2 \operatorname{Tr}(\vec{V}^T M \vec{V})$$
(24)

with  $\vec{V}^T = (V, V^{\dagger})$  and:

$$M = \begin{bmatrix} 8\rho^2 & g + 16\rho^2 \\ g + 16\rho^2 & 8\rho^2 \end{bmatrix}$$
 (25)