1 Gaussian model for the $(2,0)$ Dirac operator

1.1 Notation

$$D = \gamma_1 \otimes \{H_1, \cdot\} + \gamma_2 \otimes \{H_2, \cdot\}. \tag{1}$$

Define $\gamma_\pm := \frac{1}{2}(\gamma_1 \mp i\gamma_2)$ and $W := H_1 + iH_2$. D becomes:

$$D = \gamma_+ \otimes \{W, \cdot\} + \gamma_- \otimes \{W^\dagger, \cdot\} \tag{2}$$

And the following relations hold:

$$\gamma_\pm^2 = 0 \tag{3}$$

$$\gamma_\pm \gamma_\mp = \frac{1}{2}(I_C \pm i\gamma_1\gamma_2) \tag{4}$$

$$(\gamma_\pm\gamma_\mp)^2 = \gamma_\pm\gamma_\mp \tag{5}$$

where I_C is the identity in the space of γ matrices.

1.2 Action, $\operatorname{Tr} D^2$ part

Because of Eq.(3), only the cross terms survive in D^2:

$$D^2 = \gamma_+\gamma_- \otimes \{W, \cdot\}\{W^\dagger, \cdot\} + \gamma_-\gamma_+ \otimes \{W^\dagger, \cdot\}\{W, \cdot\} \tag{6}$$

Taking the trace yields:

$$\operatorname{Tr} D^2 = 2C(n \operatorname{Tr} WW^\dagger + \operatorname{Tr} W \operatorname{Tr} W^\dagger) \tag{7}$$

where C is the dimension of the γ matrices and n the dimension of the W matrix.

1.3 Action, $\operatorname{Tr} D^4$ part

Because of Eq.(3), the cross terms vanish when squaring D^2:

$$D^4 = \gamma_+\gamma_- \otimes (\{W, \cdot\}\{W^\dagger, \cdot\})^2 + \gamma_-\gamma_+ \otimes (\{W^\dagger, \cdot\}\{W, \cdot\})^2 \tag{8}$$

Taking the trace yields:

$$\begin{aligned}\operatorname{Tr} D^4 = 2C\big[&n \operatorname{Tr}(WW^\dagger)^2 + \operatorname{Tr} W^2 \operatorname{Tr} W^{\dagger 2} + 2(\operatorname{Tr} WW^\dagger)^2 \\ &+ 2\operatorname{Tr} W^2W^\dagger \operatorname{Tr} W^\dagger + 2\operatorname{Tr} W^{\dagger 2}W \operatorname{Tr} W\big].\end{aligned} \tag{9}$$

1.4 Stationary point

Stationary points are solutions to the equation

$$\delta S = S[W+\delta W] - S[W] = 0 \quad \forall\, \delta W \ll 1. \tag{10}$$

To first order in δW the terms $\operatorname{Tr} D^2$ and $\operatorname{Tr} D^4$ give:

$$\delta \operatorname{Tr} D^2 = 2C\big[n \operatorname{Tr} W\delta W^\dagger + n \operatorname{Tr} W^\dagger \delta W + \operatorname{Tr} W \operatorname{Tr} \delta W^\dagger + \operatorname{Tr} W^\dagger \operatorname{Tr} \delta W\big] \tag{11}$$

$$\begin{aligned}\delta \operatorname{Tr} D^4 = 2C\big[&2n \operatorname{Tr}(WW^\dagger W\delta W^\dagger) + 2n \operatorname{Tr}(W^\dagger W W^\dagger \delta W)\\ &+ 2\operatorname{Tr} W^2 \operatorname{Tr} W^\dagger \delta W^\dagger + 2 \operatorname{Tr} W^{\dagger 2} \operatorname{Tr} W\delta W\\ &+ 4\operatorname{Tr} WW^\dagger(\operatorname{Tr} W\delta W^\dagger + \operatorname{Tr} W^\dagger \delta W)\\ &+ 2\operatorname{Tr} W^2 W^\dagger \operatorname{Tr} \delta W^\dagger + 2\operatorname{Tr} W^{\dagger 2} W \operatorname{Tr} \delta W\\ &+ 2\operatorname{Tr} W^\dagger(\operatorname{Tr} W^2 \delta W^\dagger + \operatorname{Tr} WW^\dagger \delta W + \operatorname{Tr} W^\dagger W \delta W)\\ &+ 2\operatorname{Tr} W(\operatorname{Tr} W^{\dagger 2}\delta W + \operatorname{Tr} W^\dagger W \delta W^\dagger + \operatorname{Tr} WW^\dagger \delta W^\dagger)\big].\end{aligned} \tag{12}$$

We check for solutions of the form:

$$W = I\rho e^{i\theta} \tag{13}$$

with $\rho > 0$ and $\theta \in [0, 2\pi)$.

$$\operatorname{Tr} D^2 : \quad 4Cn\rho\ (e^{i\theta} \operatorname{Tr} \delta W^\dagger + e^{-i\theta} \operatorname{Tr} \delta W) \tag{14}$$

$$\operatorname{Tr} D^4 : \quad 4Cn\rho\ (8\rho^2 e^{i\theta} \operatorname{Tr} \delta W^\dagger + 8\rho^2 e^{-i\theta} \operatorname{Tr} \delta W) \tag{15}$$

putting the coefficient of δW or $\delta W^\dagger$ to zero gives:

$$\rho^2 = 0 \quad \text{or} \quad \rho^2 = -\frac{g}{8}. \tag{16}$$

1.5 Quadratic action in δW

Write $W = \rho e^{i\theta} + \epsilon V$ for small ϵ. We will separate the contributions to the action in orders of ϵ.

$$\operatorname{Tr} D^2$$

$$O(1) : \quad 4Cn^2\rho^2 \tag{17}$$

$$O(\epsilon) : \quad 4Cn\rho(e^{i\theta} \operatorname{Tr} V^\dagger + e^{-i\theta} \operatorname{Tr} V) \tag{18}$$

$$O(\epsilon^2) : \quad 2C(n \operatorname{Tr} VV^\dagger + \operatorname{Tr} V \operatorname{Tr} V^\dagger) \tag{19}$$

$\mathrm{Tr}\, D^4$

$$\begin{aligned} O(1): &\quad 16Cn^2\rho^4 && (20)\\ O(\epsilon): &\quad 24Cn\rho^3 e^{-i\theta}\,\mathrm{Tr}\, V + \text{c.c.} && (21)\\ O(\epsilon^2): &\quad 8C\rho^2(2n\,\mathrm{Tr}\, VV^\dagger + ne^{-2i\theta}\,\mathrm{Tr}\, VV && (22)\\ &\quad + 2\,\mathrm{Tr}\, V\,\mathrm{Tr}\, V^\dagger + e^{-2i\theta}\,\mathrm{Tr}\, V\,\mathrm{Tr}\, V + \text{c.c.}) \end{aligned}$$

Replacing $\mathrm{Tr}\, V$ with $\langle \mathrm{Tr}\, V\rangle$ (?does this makes sense?), the action to second order in ϵ reads:

$$\begin{aligned} S = {} & 2Cn^2(g\rho^2 + 4\rho^4) + Cn\epsilon^2\,\mathrm{Tr}\, VV^\dagger(g + 16\rho^2)\\ & + 8Cn\epsilon^2\rho^2 e^{-2i\theta}\,\mathrm{Tr}\, VV + \text{c.c.} \end{aligned} \tag{23}$$

Or:

$$S = 4Cn^2(g\rho^2 + 4\rho^4) + Cn\epsilon^2\,\mathrm{Tr}(\vec{V}^T M \vec{V}) \tag{24}$$

with $\vec{V}^T = (V, V^\dagger)$ and:

$$M = \begin{bmatrix} 8\rho^2 e^{-2i\theta} & g + 16\rho^2 \\ g + 16\rho^2 & 8\rho^2 e^{2i\theta} \end{bmatrix} \tag{25}$$