1 Hamiltonian Monte Carlo for fuzzy spaces

1.1 Statement of the problem

The fuzzy space action considered here is:

$$S[D] = g_2 \operatorname{Tr} D^2 + \operatorname{Tr} D^4 \tag{1}$$

where $g_2 \in \mathbb{R}$ and D is of the form:

$$D = \sum_i \omega_i \otimes \left(M_i \otimes I + \epsilon_i I \otimes M_i^T \right) \tag{2}$$

for Hermitian ω_i and M_i, and $\epsilon_i = \pm 1$.
The dynamical variables in the Monte Carlo are the $n \times n$ matrices M_i. Hamiltonian Monte Carlo requires to take derivatives such as:

$$\frac{\partial S[M_i]}{\partial M_k} \tag{3}$$

which amounts to finding formulas for terms like:

$$\frac{\partial \operatorname{Tr} D^p}{\partial M_k}. \tag{4}$$

In the following, formulas for $p = 2$ and $p = 4$ are developed.

1.2 Matrix calculus

Let $A \in M_n(\mathbb{C})$ and $f(A)$ be a complex valued function of A. The derivative of f with respect to A is defined in components as the $n \times n$ matrix:

$$\left(\frac{\partial f}{\partial A} \right)_{lm} \equiv \frac{\partial f}{\partial A_{lm}}. \tag{5}$$

The two special cases of interest here are:

$$\frac{\partial \operatorname{Tr} A}{\partial A} = I \tag{6}$$

$$\frac{\partial \operatorname{Tr} AB}{\partial A} = B^T. \tag{7}$$

1.3 The case $p = 2$

When $p = 2$ the M_i matrices are decoupled:

$$\operatorname{Tr} D^2 = \sum_i \operatorname{Tr} \omega_i^2 (2n \operatorname{Tr} M_i^2 + 2\epsilon_i (\operatorname{Tr} M_i)^2). \tag{8}$$

Taking a derivative with respect to M_k yields:

$$\begin{aligned} \frac{\partial}{\partial M_k} \left(\sum_i \operatorname{Tr} \omega_i^2 (2n \operatorname{Tr} M_i^2 + 2\epsilon_i (\operatorname{Tr} M_i)^2) \right) &= \\ \sum_i \delta_{ik} \operatorname{Tr} \omega_i^2 \left(4n M_i^T + 4\epsilon_i (\operatorname{Tr} M_i) I \right) &= \\ 4C \left(n M_k^T + \epsilon_k (\operatorname{Tr} M_k) I \right) & \end{aligned} \tag{9}$$

where $C \equiv \operatorname{Tr} \omega_i^2$ is the dimension of the Clifford module.

1.4 The case $p = 4$

First expand $\operatorname{Tr} D^4$:

$$\begin{aligned} \operatorname{Tr} D^4 = \sum_{i_1, i_2, i_3, i_4} & \operatorname{Tr}(\omega_{i_1} \omega_{i_2} \omega_{i_3} \omega_{i_4}) \cdot \\ \Big(& n[1 + \epsilon\, *] \operatorname{Tr}(M_{i_1} M_{i_2} M_{i_3} M_{i_4}) \;\; + \\ & \epsilon_{i_1} \operatorname{Tr} M_{i_1} [1 + \epsilon\, *] \operatorname{Tr}(M_{i_2} M_{i_3} M_{i_4}) \;\; + \\ & \epsilon_{i_2} \operatorname{Tr} M_{i_2} [1 + \epsilon\, *] \operatorname{Tr}(M_{i_1} M_{i_3} M_{i_4}) \;\; + \\ & \epsilon_{i_3} \operatorname{Tr} M_{i_3} [1 + \epsilon\, *] \operatorname{Tr}(M_{i_1} M_{i_2} M_{i_4}) \;\; + \\ & \epsilon_{i_4} \operatorname{Tr} M_{i_4} [1 + \epsilon\, *] \operatorname{Tr}(M_{i_1} M_{i_2} M_{i_3}) \;\; + \\ & \epsilon_{i_1} \epsilon_{i_2} [1 + \epsilon] \operatorname{Tr}(M_{i_1} M_{i_2}) \operatorname{Tr}(M_{i_3} M_{i_4}) \;\; + \\ & \epsilon_{i_1} \epsilon_{i_3} [1 + \epsilon] \operatorname{Tr}(M_{i_1} M_{i_3}) \operatorname{Tr}(M_{i_2} M_{i_4}) \;\; + \\ & \epsilon_{i_1} \epsilon_{i_4} [1 + \epsilon] \operatorname{Tr}(M_{i_1} M_{i_4}) \operatorname{Tr}(M_{i_2} M_{i_3}) \;\Big) \end{aligned} \tag{10}$$

where $*$ denotes complex conjugation of everything that appears on the right, ϵ is defined as the product $\epsilon \equiv \epsilon_{i_1} \epsilon_{i_2} \epsilon_{i_3} \epsilon_{i_4}$, and the relation $M^T = M^*$ has been used. Since D is Hermitian, the expression must be real. It is not immediate to see that this is the case because of the $\epsilon = \pm 1$ factor inside the square brackets. Reality nonetheless holds, and becomes manifest by observing that a simultaneous index exchange $i_1 \leftrightarrow i_4$ and $i_2 \leftrightarrow i_3$ is equivalent to taking the complex conjugate (in fact, this is not the only index exchange that amounts to complex conjugation).

Taking a matrix derivative with respect to M_k results in non-vanishing contributions when $k = i_1, k = i_2, k = i_3$ or $k = i_4$:

$$\begin{aligned}\frac{\partial}{\partial M_k} \operatorname{Tr} D^4 = \sum_{i_1,i_2,i_3,i_4} & \operatorname{Tr}(\omega_{i_1}\omega_{i_2}\omega_{i_3}\omega_{i_4}) \cdot \\ & \Big(\delta_{ki_1} A(i_1,i_2,i_3,i_4)^T + \delta_{ki_2} A(i_2,i_3,i_4,i_1)^T + \\ & \delta_{ki_3} A(i_3,i_4,i_1,i_2)^T + \delta_{ki_4} A(i_4,i_1,i_2,i_3)^T\Big)\end{aligned} \tag{11}$$

where $A(a,b,c,d)$ is the following $n \times n$ matrix:

$$\begin{aligned}A(a,b,c,d) \equiv\ & n[1 + \epsilon\ \dagger] M_b M_c M_d + \\ & \epsilon_a I [1 + \epsilon\ *] \operatorname{Tr} M_b M_c M_d + \\ & \epsilon_b \operatorname{Tr} M_b [1 + \epsilon\ \dagger] M_c M_d + \\ & \epsilon_c \operatorname{Tr} M_c [1 + \epsilon\ \dagger] M_b M_d + \\ & \epsilon_d \operatorname{Tr} M_d [1 + \epsilon\ \dagger] M_b M_c + \\ & \epsilon_a \epsilon_b M_b [1 + \epsilon] \operatorname{Tr} M_c M_d + \\ & \epsilon_a \epsilon_c M_c [1 + \epsilon] \operatorname{Tr} M_b M_d + \\ & \epsilon_a \epsilon_d M_d [1 + \epsilon] \operatorname{Tr} M_b M_c\end{aligned} \tag{12}$$

and $\dagger$ denotes Hermitian conjugation of everything that appears on the right. Upon relabeling the indices and cycling the ω matrices in the trace, the equation becomes:

$$\begin{aligned}\frac{\partial}{\partial M_k} \operatorname{Tr} D^4 &= 4 \sum_{i_1,i_2,i_3,i_4} \delta_{ki_1} \operatorname{Tr}(\omega_{i_1}\omega_{i_2}\omega_{i_3}\omega_{i_4}) A(i_1,i_2,i_3,i_4)^T \\ &= 4 \sum_{i_1,i_2,i_3} \operatorname{Tr}(\omega_k \omega_{i_1}\omega_{i_2}\omega_{i_3}) A(k,i_1,i_2,i_3)^T \equiv 4 \sum_{i_1,i_2,i_3} \mathcal{B}_k(i_1,i_2,i_3)\end{aligned} \tag{13}$$

with $\mathcal{B}_k(a,b,c)$ denoting the generic term in the sum.

To see that Eq.(13) defines a Hermitian matrix, notice that an exchange of indices $i_1 \leftrightarrow i_3$ is equivalent to taking the Hermitian conjugate:

$$\mathcal{B}_k(i_1,i_2,i_3)^\dagger = \mathcal{B}_k(i_3,i_2,i_1) \tag{14}$$

therefore the sum in Eq.(13) reduces to:

$$\sum_{\substack{i_1>i_3\\ i_2}} [1+\dagger]\mathcal{B}_k(i_1,i_2,i_3) + \sum_{i_1,i_2} \mathcal{B}_k(i_1,i_2,i_1). \tag{15}$$

In fact, by looking at the form of $\mathcal{B}$, it is clear that terms in the sum are qualitatively different based on the number of indices that coincide. Therefore it would be computationally convenient to write Eq.(15) in a way that emphazises this difference.

The only terms that contribute when all indices are different are the following:

$$\sum_{i_1>i_2>i_3} [1+\dagger]\Big(\mathcal{B}_k(i_1,i_2,i_3) + \mathcal{B}_k(i_1,i_3,i_2) + \mathcal{B}_k(i_2,i_1,i_3)\Big). \tag{16}$$

The three inequivalent permutations of indices that appear in this formula are based on a group-theoretical argument that will generalize easily to powers of D higher than 4. First consider the symmetric group of order three S_3 acting on the set of indices $\{i_1,i_2,i_3\}$, and the subgroup of permutations that induce a simple change in $\mathcal{B}$, which in this case is $H = \{(),(13)\} \cong S_2$ (the first element being the identical permutation, and the second the exchange $i_1 \leftrightarrow i_3$ which induces $\mathcal{B} \to \mathcal{B}^\dagger$). The idea is then to restrict the sum to $i_1 > i_2 > i_3$ and quotient out the action of H by introducing a suitable pre-factor that accounts for it (in this case $[1+\dagger]$). Practically, the inequivalent permutations of indices that appear in Eq.(16) are found by computing the (left or right) cosets of $H \subset S_3$ and acting on $\{i_1,i_2,i_3\}$ with a representative from each coset. In this case the representatives where chosen to be $(), (23), (12)$.

What is left are terms in which at least two indices are equal. These are:

$$\sum_{i_1>i_2} [1+\dagger]\Big(\mathcal{B}_k(i_1,i_1,i_2) + \mathcal{B}_k(i_1,i_2,i_2)\Big) + \\ \sum_{i_1\neq i_2} \mathcal{B}_k(i_1,i_2,i_1) + \sum_i \mathcal{B}_k(i,i,i). \tag{17}$$

At this point, a useful property of the ω matrices can be exploited to simplify both Eq.(16) and Eq.(17):

$$\mathrm{Tr}(\omega_{\sigma(i_1)}\omega_{\sigma(i_2)}\omega_{\sigma(j)}\omega_{\sigma(k)}) \propto \mathrm{Tr}(\omega_j\omega_k) = 0 \qquad \text{if } \ i_1 = i_2 \text{ and } j \neq k \tag{18}$$

for any permutation σ acting on $\{i_1, i_2, j, k\}$. In other words, if two indices are the same and the other two are different, the trace on the ω matrices vanishes.

Putting together Eq.(16), Eq.(17) and Eq.(18), the final formula for $\partial_k \,\mathrm{Tr}\, D^4$ reads:

$$\begin{aligned}\frac{\partial}{\partial M_k}\,\mathrm{Tr}\, D^4 = 4\Bigg[\sum_{\substack{i_1>i_2>i_3\\ i_1,i_2,i_3\neq k}} [1+\dagger]\Big(\mathcal{B}_k(i_1,i_2,i_3) + \mathcal{B}_k(i_1,i_3,i_2) + \mathcal{B}_k(i_2,i_1,i_3)\Big) \\ + \sum_{\substack{i\\ i\neq k}} \Big([1+\dagger]\mathcal{B}_k(i,i,k) + \mathcal{B}_k(i,k,i)\Big) + \mathcal{B}_k(k,k,k)\Bigg].\end{aligned} \tag{19}$$

1.5 A general formula for every p

The first problem is to write $\mathrm{Tr}\, D^p$ in a useful form, along the lines of Eq.(10). $\mathrm{Tr}\, D^p$ expands to:

$$\begin{aligned}\mathrm{Tr}\, D^p = \sum_{i_1\ldots i_p} \mathrm{Tr}\,\omega_{i_1}\ldots\omega_{i_p}\cdot \\ \mathrm{Tr}\left((M_{i_1}\otimes I + \epsilon_{i_1} I \otimes M_{i_1}^T)\ldots(M_{i_p}\otimes I + \epsilon_{i_p} I\otimes M_{i_p}^T)\right)\end{aligned} \tag{20}$$

Ignoring (for now) the trace over the ω matrices, a typical term in the sum is:

$$\mathrm{Tr}\left(\epsilon_B A\otimes B^* + \epsilon_A B\otimes A^*\right) \tag{21}$$

where A and B are related to the product $M_{i_1}\ldots M_{i_p}$ in the following way:

1. pick $r \geq 0$ numbers $k_1 < \ldots < k_r$ from $\{1,\ldots,p\}$ and call the remaining $p-r$ numbers $j_1 < \ldots < j_{p-r}$;

2. define $A = M_{i_{k_1}}\ldots M_{i_{k_r}}$ and $B = M_{i_{j_1}}\ldots M_{i_{j_{p-r}}}$ (if $r=0$, $A=I$);

3. define $\epsilon_A = \epsilon_{i_{k_1}} \dots \epsilon_{i_{k_r}}$ and $\epsilon_B = \epsilon_{i_{j_1}} \dots \epsilon_{i_{j_{p-r}}} = \epsilon_A \epsilon_{i_1} \dots \epsilon_{i_p}$.

In particular, a choice of A completely characterizes B.
By varying r from 0 to $\left[\frac{p}{2}\right]$ and summing over all possible choices of $k_1 \dots k_r$, every term in $\operatorname{Tr} D^p$ is generated.
One can verify that every term in Eq.(10) ($p = 4$) is of that type. For example:

$$
\begin{aligned}
&\operatorname{Tr} M_{i_1}[\epsilon_{i_1} + \epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} *] \operatorname{Tr}(M_{i_2}M_{i_3}M_{i_4}) = \\
&\qquad \epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} \operatorname{Tr} M_{i_1} \operatorname{Tr}(M_{i_2}M_{i_3}M_{i_4})^* + \epsilon_{i_1} \operatorname{Tr} M_{i_1} \operatorname{Tr}(M_{i_2}M_{i_3}M_{i_4}) = \\
&\qquad \epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} \operatorname{Tr} M_{i_1} \operatorname{Tr}(M_{i_2}M_{i_3}M_{i_4})^* + \epsilon_{i_1} \operatorname{Tr}(M_{i_1})^* \operatorname{Tr}(M_{i_2}M_{i_3}M_{i_4}) = \\
&\qquad \operatorname{Tr}\big(\epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} M_{i_1} \otimes (M_{i_2}M_{i_3}M_{i_4})^* + \epsilon_{i_1} M_{i_2}M_{i_3}M_{i_4} \otimes M_{i_1}^*\big)
\end{aligned} \tag{22}
$$

which is of the form of Eq.(21) upon identifying M_{i_1} with A and $M_{i_2}M_{i_3}M_{i_4}$ with B (in the second equality the reality of $\operatorname{Tr} M_{i_1}$ has been used).
A way of expressing $\operatorname{Tr} B$ given A is using a modified derivative operator D_i defined as:

$$\mathrm{D}_i \equiv \operatorname{Tr} \circ \frac{\partial}{\partial M_i} \tag{23}$$

which allows to write:

$$A = M_{i_{k_1}} \dots M_{i_{k_r}} \implies \operatorname{Tr} B = \mathrm{D}_{i_{k_r}} \dots \mathrm{D}_{i_{k_1}} \operatorname{Tr}(M_{i_1} \dots M_{i_p}). \tag{24}$$

Therefore Eq.(21) becomes:

$$
\begin{aligned}
&\epsilon_A[1 + \epsilon_{i_1} \dots \epsilon_{i_p} *](\operatorname{Tr} A)^* \operatorname{Tr} B = \\
&\quad \epsilon_{i_{k_1}} \dots \epsilon_{i_{k_r}}[1 + \epsilon_{i_1} \dots \epsilon_{i_p} *](\operatorname{Tr} M_{i_{k_1}} \dots M_{i_{k_r}})^* \big(\mathrm{D}_{i_{k_r}} \dots \mathrm{D}_{i_{k_1}} \operatorname{Tr}(M_{i_1} \dots M_{i_p})\big).
\end{aligned} \tag{25}
$$

There are some special cases that make the expression simpler, namely:

1. $r = 0$ gives a factor $\operatorname{Tr} I = n$;

2. $r = 1, 2$ make $\operatorname{Tr} A$ real;

3. $p - r = 1, 2$ (which can only occur for $p = 2, 4$) make $\operatorname{Tr} B$ real.

Putting everything together, $\mathrm{Tr}\, D^p$ can be written as:

$$\mathrm{Tr}\, D^p = \sum_{i_1 \dots i_p} \mathrm{Tr}\ \omega_{i_1} \dots \omega_{i_p} \Bigg[\sum_{r=0}^{\left[\frac{p}{2}\right]} \sum_{k_1 < \dots < k_r = 1}^{p} \epsilon_{i_{k_1}} \dots \epsilon_{i_{k_r}} [1 + \epsilon_{i_1} \dots \epsilon_{i_p} *] \\ (\mathrm{Tr}\, M_{i_{k_1}} \dots M_{i_{k_r}})^* \big(\mathrm{D}_{i_{k_r}} \dots \mathrm{D}_{i_{k_1}}\ \mathrm{Tr}(M_{i_1} \dots M_{i_p}) \big) \Bigg]. \tag{26}$$

where:

$$r = 0 \quad \longrightarrow \quad n[1 + \epsilon_{i_1} \dots \epsilon_{i_p} *]\, \mathrm{Tr}(M_{i_1} \dots M_{i_p}) \tag{27}$$

$$r = 1 \quad \longrightarrow \quad \sum_{k_1=1}^{p} \epsilon_{i_{k_1}}\ \mathrm{Tr}(M_{i_{k_1}})[1 + \epsilon_{i_1} \dots \epsilon_{i_p} *]\, \mathrm{D}_{i_{k_1}}\ \mathrm{Tr}(M_{i_1} \dots M_{i_p}) \tag{28}$$

$$r = 2 \quad \longrightarrow \quad \sum_{k_1 < k_2 = 1}^{p} \epsilon_{i_{k_1}} \epsilon_{i_{k_2}}\ \mathrm{Tr}(M_{i_{k_1}} M_{i_{k_2}})[1 + \epsilon_{i_1} \dots \epsilon_{i_p} *]\, \mathrm{D}_{i_{k_2}}\, \mathrm{D}_{i_{k_1}}\ \mathrm{Tr}(M_{i_1} \dots M_{i_p}). \tag{29}$$