1 A general formula for every p

The first problem is to write $\mathrm{Tr}\, D^p$ in a useful form, along the lines of Eq.(**??**). $\mathrm{Tr}\, D^p$ expands to:

$$\mathrm{Tr}\, D^p = \sum_{i_1 \dots i_p} \mathrm{Tr}\, \omega_{i_1} \dots \omega_{i_p} \cdot$$
$$\mathrm{Tr}\left((M_{i_1} \otimes I + \epsilon_{i_1} I \otimes M_{i_1}^T) \dots (M_{i_p} \otimes I + \epsilon_{i_p} I \otimes M_{i_p}^T) \right) \tag{1}$$

Ignoring (for now) the trace over the ω matrices, a typical term in the sum is:

$$\mathrm{Tr}\left(\epsilon_B A \otimes B^* + \epsilon_A B \otimes A^* \right) \tag{2}$$

where A and B are related to the product $M_{i_1} \dots M_{i_p}$ in the following way:

1. pick $r \geq 0$ numbers $k_1 < \dots < k_r$ from $\{1, \dots, p\}$ and call the remaining $p - r$ numbers $j_1 < \dots < j_{p-r}$;
2. define $A = M_{i_{k_1}} \dots M_{i_{k_r}}$ and $B = M_{i_{j_1}} \dots M_{i_{j_{p-r}}}$ (if $r = 0$, $A = I$);
3. define $\epsilon_A = \epsilon_{i_{k_1}} \dots \epsilon_{i_{k_r}}$ and $\epsilon_B = \epsilon_{i_{j_1}} \dots \epsilon_{i_{j_{p-r}}} = \epsilon_A \epsilon_{i_1} \dots \epsilon_{i_p}$.

In particular, a choice of A completely characterizes B.
By varying r from 0 to $\left[\frac{p}{2}\right]$ and summing over all possible choices of $k_1 \dots k_r$, every term in $\mathrm{Tr}\, D^p$ is generated.
One can verify that every term in Eq.(**??**) ($p = 4$) is of that type. For example:

$$\begin{aligned}
&\mathrm{Tr}\, M_{i_1} [\epsilon_{i_1} + \epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} *] \,\mathrm{Tr}(M_{i_2} M_{i_3} M_{i_4}) = \\
&\qquad \epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} \,\mathrm{Tr}\, M_{i_1} \,\mathrm{Tr}(M_{i_2} M_{i_3} M_{i_4})^* + \epsilon_{i_1} \,\mathrm{Tr}\, M_{i_1} \,\mathrm{Tr}(M_{i_2} M_{i_3} M_{i_4}) = \\
&\qquad \epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} \,\mathrm{Tr}\, M_{i_1} \,\mathrm{Tr}(M_{i_2} M_{i_3} M_{i_4})^* + \epsilon_{i_1} \,\mathrm{Tr}(M_{i_1})^* \,\mathrm{Tr}(M_{i_2} M_{i_3} M_{i_4}) = \\
&\qquad \mathrm{Tr}\left(\epsilon_{i_2}\epsilon_{i_3}\epsilon_{i_4} M_{i_1} \otimes (M_{i_2} M_{i_3} M_{i_4})^* + \epsilon_{i_1} M_{i_2} M_{i_3} M_{i_4} \otimes M_{i_1}^* \right)
\end{aligned} \tag{3}$$

which is of the form of Eq.(**??**) upon identifying M_{i_1} with A and $M_{i_2} M_{i_3} M_{i_4}$ with B (in the second equality the reality of $\mathrm{Tr}\, M_{i_1}$ has been used).
A way of expressing $\mathrm{Tr}\, B$ given A is using a modified derivative operator D_i defined as:

$$\mathrm{D}_i \equiv \mathrm{Tr} \circ \frac{\partial}{\partial M_i} \tag{4}$$

which allows to write:

$$A = M_{i_{k_1}} \dots M_{i_{k_r}} \implies \operatorname{Tr} B = \mathrm{D}_{i_{k_r}} \dots \mathrm{D}_{i_{k_1}} \operatorname{Tr}(M_{i_1} \dots M_{i_p}). \tag{5}$$

Therefore Eq.(**??**) becomes:

$$\begin{aligned}&\epsilon_A[1+\epsilon_{i_1}\dots\epsilon_{i_p}*](\operatorname{Tr} A)^* \operatorname{Tr} B = \\ &\quad \epsilon_{i_{k_1}}\dots\epsilon_{i_{k_r}}[1+\epsilon_{i_1}\dots\epsilon_{i_p}*](\operatorname{Tr} M_{i_{k_1}}\dots M_{i_{k_r}})^*\big(\mathrm{D}_{i_{k_r}}\dots\mathrm{D}_{i_{k_1}}\operatorname{Tr}(M_{i_1}\dots M_{i_p})\big).\end{aligned} \tag{6}$$

There are some special cases that make the expression simpler, namely:

1. $r=0$ gives a factor $\operatorname{Tr} I = n$;
2. $r=1,2$ make $\operatorname{Tr} A$ real;
3. $p-r=1,2$ (which can only occur for $p=2,4$) make $\operatorname{Tr} B$ real.

Putting everything together, $\operatorname{Tr} D^p$ can be written as:

$$\begin{aligned}\operatorname{Tr} D^p = \sum_{i_1\dots i_p} \operatorname{Tr}\, \omega_{i_1}\dots\omega_{i_p}\Bigg[\sum_{r=0}^{\left[\frac{p}{2}\right]}\ \sum_{k_1<\dots<k_r=1}^{p} \epsilon_{i_{k_1}}\dots\epsilon_{i_{k_r}}[1+\epsilon_{i_1}\dots\epsilon_{i_p}*] \\ (\operatorname{Tr} M_{i_{k_1}}\dots M_{i_{k_r}})^*\big(\mathrm{D}_{i_{k_r}}\dots\mathrm{D}_{i_{k_1}}\operatorname{Tr}(M_{i_1}\dots M_{i_p})\big)\Bigg].\end{aligned} \tag{7}$$

where:

$$r=0 \quad \longrightarrow \quad n[1+\epsilon_{i_1}\dots\epsilon_{i_p}*]\operatorname{Tr}(M_{i_1}\dots M_{i_p}) \tag{8}$$

$$r=1 \quad \longrightarrow \quad \sum_{k_1=1}^{p}\epsilon_{i_{k_1}}\operatorname{Tr}(M_{i_{k_1}})[1+\epsilon_{i_1}\dots\epsilon_{i_p}*]\,\mathrm{D}_{i_{k_1}}\operatorname{Tr}(M_{i_1}\dots M_{i_p}) \tag{9}$$

$$r=2 \quad \longrightarrow \quad \sum_{k_1<k_2=1}^{p}\epsilon_{i_{k_1}}\epsilon_{i_{k_2}}\operatorname{Tr}(M_{i_{k_1}}M_{i_{k_2}})[1+\epsilon_{i_1}\dots\epsilon_{i_p}*]\,\mathrm{D}_{i_{k_2}}\mathrm{D}_{i_{k_1}}\operatorname{Tr}(M_{i_1}\dots M_{i_p}). \tag{10}$$