1 Powers of D

For many applications it is useful to write $\mathrm{Tr}\, D^p$ in a meaningful form. $\mathrm{Tr}\, D^p$ expands to:

$$\mathrm{Tr}\, D^p = \sum_{i_1,\ldots,i_p} \mathrm{Tr}(\omega_{i_1} \cdots \omega_{i_p}) \cdot \\ \mathrm{Tr}\left((M_{i_1} \otimes I + \epsilon_{i_1} I \otimes M_{i_1}^T) \cdots (M_{i_p} \otimes I + \epsilon_{i_p} I \otimes M_{i_p}^T)\right). \quad (1)$$

Focus first on the product over (anti-)commutators:

$$(M_{i_1} \otimes I + \epsilon_{i_1} I \otimes M_{i_1}^T) \cdots (M_{i_p} \otimes I + \epsilon_{i_p} I \otimes M_{i_p}^T). \quad (2)$$

The following observations are useful for writing the product explicitly.

1. The terms appearing in the product are given by all possible ways to distribute r matrices $(0 \leq r \leq p)$ on the left side of the tensor product, and the remaining $s = p - r$ matrices on the right, but keeping the order of the indices (on each side separately). This means that for example:

   $$M_{i_1} M_{i_p} \otimes M_{i_2}^T \cdots M_{i_{p-1}}^T$$

   is a valid term with $r = 2$ and $s = p - 2$, but:

   $$M_{i_p} M_{i_1} \otimes M_{i_2}^T \cdots M_{i_{p-1}}^T$$

   is not, because on the left side the index i_p appears before i_1. When $r = 0$ (respectively $s = 0$) it is intended that there is an identity matrix on the left (right) side of the tensor product.

2. Each term is multiplied by a product of ϵ_i factors given by the matrices that appear on the right. In the case of the previous example:

   $$\epsilon_{i_2} \cdots \epsilon_{i_{p-1}} M_{i_1} M_{i_p} \otimes M_{i_2}^T \cdots M_{i_{p-1}}^T.$$

3. For each term with r matrices on the left, there is a corresponding one with the same r matrices on the right. Again using the same example:

   $$\epsilon_{i_1} \epsilon_{i_p} M_{i_2} \cdots M_{i_{p-1}} \otimes M_{i_1}^T M_{i_p}^T.$$

It follows that the product (2) can be written as:

$$\sum_{r=0}^{\lfloor \frac{p}{2} \rfloor} \sum_{j_1<\ldots<j_r=1}^{p} \Big[\epsilon_{i_{k_1}} \cdots \epsilon_{i_{k_s}} (M_{i_{j_1}} \cdots M_{i_{j_r}}) \otimes (M^T_{i_{k_1}} \cdots M^T_{i_{k_s}}) + \\ + \epsilon_{i_{j_1}} \cdots \epsilon_{i_{j_r}} (M_{i_{k_1}} \cdots M_{i_{k_s}}) \otimes (M^T_{i_{j_1}} \cdots M^T_{i_{j_r}})\Big] \tag{3}$$

where $\lfloor\ \rfloor$ denotes the floor function (i.e., the sum in r runs from 0 to the greatest integer less than or equal to $p/2$), $j_1, \ldots, j_r$ are a set of r indices picked from $\{1, \ldots, p\}$ in increasing order, and $k_1, \ldots, k_s$ are the remaining ones, also in increasing order.
Recall now that $\epsilon_i^{-1} = \epsilon_i$ and that $M_i^\dagger = M_i$ for all i, so in particular $M_i^T = M_i^*$. Tracing over the product then gives:

$$\sum_{r=0}^{\lfloor \frac{p}{2} \rfloor} \sum_{j_1<\ldots<j_r=1}^{p} \epsilon_{i_{j_1}} \cdots \epsilon_{i_{j_r}} [1 + \epsilon^p *] \operatorname{Tr}(M_{i_{j_1}} \cdots M_{i_{j_r}})^* \operatorname{Tr}(M_{i_{k_1}} \cdots M_{i_{k_s}}) \tag{4}$$

where ϵ^p is defined to be the product $\epsilon_{i_1} \cdots \epsilon_{i_p}$ and the in-line operator $*$ means complex conjugation of everything that appears on the right.
Notice that the generic term in the sum only depends on the choice of $j_1, \ldots, j_r$. Denote the geneirc term $T(j_1, \ldots, j_r)$. The formula for $\operatorname{Tr} D^p$ can finally be written coincisely as:

$$\operatorname{Tr} D^p = \sum_{i_1,\ldots,i_p} \operatorname{Tr}(\omega_{i_1} \cdots \omega_{i_p}) \left[\sum_{r=0}^{\lfloor \frac{p}{2} \rfloor} \sum_{j_1<\ldots<j_r=1}^{p} T(j_1, \ldots, j_r)\right] \tag{5}$$