

## 1 setup

Let  $U = \{[x] : x \in \mathbb{Z}, x \leq n\}$ ,  $n \in \mathbb{N}$

All functions unless other ways noted maps  $U^2 \rightarrow U$

## 2 gates

$$OP(a, b) = \begin{array}{c|ccccccc} & 0 & 1 & 2 & \dots & b & \dots & n \\ \hline 0 & n & 0 & 0 & \dots & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 2 & 0 & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a & 0 & 0 & 0 & \dots & a-1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & 0 & 0 & \dots & 0 & \dots & n-1 \end{array}$$

$$= \begin{cases} a-1 & a = b \\ 0 & a \neq b \end{cases}$$

$ROT : U \rightarrow U$

$ROT(a) = OP(a, a)$

$\overline{A_0}(a, b) = a$

$\overline{A_m}(a, b) = ROT(\overline{A_{m-1}})$

$\overline{B_0}(a, b) = b$

$\overline{B_m}(a, b) = ROT(\overline{B_{m-1}})$

$$\overline{F_{ab}}(x, y) = \begin{cases} x-1-a-b & x = y \\ -b & x \neq y \end{cases}$$

$\overline{F_{a0}}(x, y) = OP(\overline{A_a}, \overline{B_a})$

$\overline{F_{ab}}(x, y) = ROT(\overline{F_{a(b-a)}})$

$\overline{0} = OP(\overline{A_0}, \overline{A_1})$

$\overline{0}(x, y) = 0$

$\overline{m} = ROT(\overline{m+1})$

$\overline{m}(x, y) = m$

$$\overline{(a, b, c)}(x, y) = \begin{cases} c & x = a \wedge y = b \\ 0 & x \neq a \vee y \neq b \end{cases}$$

$$\overline{(a, b, c)} = \begin{cases} \overline{0} & c = 0 \\ OP(\overline{F_{(-2-c)0}}(\overline{A_a}, \overline{B_b}), \overline{C+1}) & c \neq n-1 \\ OP(\overline{F_{(c-1)1}}(\overline{A_a}, \overline{B_b}), \overline{0}) & c = n-1 \end{cases}$$

$$\overline{S_a}(x, y) = \begin{cases} a & x = a \wedge y = 0 \\ a & x = 0 \wedge y = a \\ 0 & \end{cases}$$

$\overline{S_a} = \overline{F_{(a-1)(-a)}}(\overline{(0, a, a)}, \overline{(a, 0, a)})$

$\overline{AS_{ab}}(x, y) =$

$if(x = 0 \vee y = 0)\{$

$if(x = a \vee y = a)\{$

return b

$\}if(x = a+1 \vee y = a+1)\{$

return b

$$\begin{array}{l}
\} \\
\} else \{ return 0 \} \\
\overline{AS_{ab}} = \overline{F_{(b-1)(-b)}}(\overline{S_a}, \overline{S_{a+1}})
\end{array}$$

	0	1	2	...	a	a+1	a+2...	n
0	0	1	2	...	a	0	0...	0
1	1	0	0	0	...	0	...	0
2	2	0	0	0	...	0	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a	a	0	0	0	...	0	...	0
a+1	0	0	0	0	...	0	...	0
a+2	0	0	0	0	...	0	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	0	0	0	0	...	0	...	0

$$\begin{array}{l}
\overline{PCO_a}(x, y) = \\
\overline{PCO_0} = \overline{0} \\
\overline{PCO_1} = \overline{S_1} \\
\overline{PCO_2} = \overline{F_{0(-1)}}(\overline{S_2}, \overline{AS_{12}})
\end{array}$$