axioms for 3-group:

identity: $a = \left(\begin{smallmatrix} & 0 & \\ a & & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} & 0 & \\ 0 & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ 0 & & 0 \end{smallmatrix}\right)$

inverses: $0 = \left(\begin{smallmatrix} & a^{-1} & \\ a & & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} & 0 & \\ a^{-1} & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ 0 & & a^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} & 0 & \\ a & & a^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a^{-1} & \\ 0 & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ a^{-1} & & 0 \end{smallmatrix}\right)$

associative: $\left(\begin{smallmatrix} & \left(\begin{smallmatrix} & e & \\ d & & f \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & b & \\ a & & c \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & h & \\ g & & i \end{smallmatrix}\right) \end{smallmatrix}\right) = \left(\begin{smallmatrix} & \left(\begin{smallmatrix} & e & \\ 0 & & f \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & d & \\ \left(\begin{smallmatrix} & b & \\ a & & c \end{smallmatrix}\right) & & g \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & h & \\ 0 & & i \end{smallmatrix}\right) \end{smallmatrix}\right) = \left(\begin{smallmatrix} & \left(\begin{smallmatrix} & \left(\begin{smallmatrix} & e & \\ d & & f \end{smallmatrix}\right) & \\ b & & h \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & 0 & \\ a & & c \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & 0 & \\ g & & i \end{smallmatrix}\right) \end{smallmatrix}\right) = \left(\begin{smallmatrix} & \left(\begin{smallmatrix} & e & \\ d & & 0 \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & b & \\ a & & 0 \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & f & \\ c & & \left(\begin{smallmatrix} & h & \\ g & & i \end{smallmatrix}\right) \end{smallmatrix}\right) \end{smallmatrix}\right)$

definitin: boolean N-group An N-group where every element is its own inverse.

Theorem 1 (3-group, communitive 2-group equivalency)**.** *For every 3-group there exists an equivalent communative 2-group.*

lemma 1. $b = \left(\begin{smallmatrix} & a^{-1} & \\ a & & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} & b & \\ a^{-1} & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ b & & a^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} & b & \\ a & & a^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a^{-1} & \\ b & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ a^{-1} & & b \end{smallmatrix}\right)$

Proof.

$$\begin{aligned}
\begin{pmatrix} & a^{-1} & \\ a & & b \end{pmatrix} &= \begin{pmatrix} & \left(\begin{smallmatrix} & 0 & \\ a^{-1} & & 0 \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & 0 & \\ a & & 0 \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & 0 & \\ 0 & & b \end{smallmatrix}\right) \end{pmatrix} && \text{identity} \\
&= \begin{pmatrix} & \left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & a^{-1} & \\ \left(\begin{smallmatrix} & 0 & \\ a & & 0 \end{smallmatrix}\right) & & 0 \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & 0 & \\ 0 & & b \end{smallmatrix}\right) \end{pmatrix} && \text{associative} \\
&= \begin{pmatrix} & 0 & \\ \left(\begin{smallmatrix} & a^{-1} & \\ a & & 0 \end{smallmatrix}\right) & & b \end{pmatrix} && \text{identity} \\
&= \begin{pmatrix} & 0 & \\ 0 & & b \end{pmatrix} && \text{inverses} \\
&= b && \text{identity}
\end{aligned} \tag{1}$$

The others follow similar form $\square$

lemma 2. $\left(\begin{smallmatrix} & b & \\ a & & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} & b & \\ a & & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} & 0 & \\ b & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ 0 & & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} & 0 & \\ a & & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} & b & \\ 0 & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ b & & 0 \end{smallmatrix}\right)$

Proof.

$$\begin{aligned}
\begin{pmatrix} & b & \\ a & & 0 \end{pmatrix} &= \begin{pmatrix} & \left(\begin{smallmatrix} & 0 & \\ 0 & & b \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & a & \\ 0 & & 0 \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \end{smallmatrix}\right) \end{pmatrix} && \text{identity} \\
&= \begin{pmatrix} & \left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \end{smallmatrix}\right) & \\ \left(\begin{smallmatrix} & a & \\ 0 & & 0 \end{smallmatrix}\right) & & \left(\begin{smallmatrix} & b & \\ 0 & & \left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \end{smallmatrix}\right) \end{smallmatrix}\right) \end{pmatrix} && \text{associative} \\
&= \begin{pmatrix} & 0 & \\ a & & \left(\begin{smallmatrix} & b & \\ 0 & & 0 \end{smallmatrix}\right) \end{pmatrix} && \text{identity} \\
&= \begin{pmatrix} & 0 & \\ a & & b \end{pmatrix} && \text{identity}
\end{aligned} \tag{2}$$

The others follow similar form $\square$

lemma 3. $\left(\begin{smallmatrix} & b & \\ a & & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} & b & \\ a & & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} & c & \\ b & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ c & & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} & c & \\ a & & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} & b & \\ c & & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} & a & \\ b & & c \end{smallmatrix}\right)$

Proof.

$$
\begin{aligned}
\begin{pmatrix} & b & \\ a & & c \end{pmatrix} &= \begin{pmatrix} & \begin{pmatrix} & 0 & \\ b & & 0 \end{pmatrix} & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & \begin{pmatrix} & c & \\ 0 & & 0 \end{pmatrix} \end{pmatrix} && \text{identity} \\
&= \begin{pmatrix} & \begin{pmatrix} & 0 & \\ 0 & & 0 \end{pmatrix} & \\ \begin{pmatrix} & b & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & 0 \end{pmatrix} & & \begin{pmatrix} & c & \\ 0 & & 0 \end{pmatrix} \end{pmatrix} && \text{associative} \\
&= \begin{pmatrix} & \begin{pmatrix} & 0 & \\ 0 & & 0 \end{pmatrix} & \\ \begin{pmatrix} & 0 & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & b \end{pmatrix} & & \begin{pmatrix} & c & \\ 0 & & 0 \end{pmatrix} \end{pmatrix} && \text{lemma 2} \\
&= \begin{pmatrix} & \begin{pmatrix} & c & \\ 0 & & 0 \end{pmatrix} & \\ \begin{pmatrix} & 0 & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & b \end{pmatrix} & & \begin{pmatrix} & 0 & \\ 0 & & 0 \end{pmatrix} \end{pmatrix} && \text{lemma 2} \\
&= \begin{pmatrix} & \begin{pmatrix} & c & \\ 0 & & 0 \end{pmatrix} & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & \begin{pmatrix} & 0 & \\ b & & 0 \end{pmatrix} \end{pmatrix} && \text{associative} \\
&= \begin{pmatrix} & c & \\ a & & b \end{pmatrix} && \text{identity}
\end{aligned} \tag{3}
$$

The others follow similar form ☐

lemma 4 (linalizability). $\begin{pmatrix} & b & \\ a & & c \end{pmatrix} = \begin{pmatrix} & 0 & \\ \begin{pmatrix} & 0 & \\ a & & b \end{pmatrix} & & c \end{pmatrix}$

Proof.

$$
\begin{aligned}
\begin{pmatrix} & b & \\ a & & c \end{pmatrix} &= \begin{pmatrix} & \begin{pmatrix} & 0 & \\ b & & 0 \end{pmatrix} & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & \begin{pmatrix} & 0 & \\ 0 & & c \end{pmatrix} \end{pmatrix} && \text{identity} \\
&= \begin{pmatrix} & \begin{pmatrix} & 0 & \\ 0 & & 0 \end{pmatrix} & \\ \begin{pmatrix} & b & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & 0 \end{pmatrix} & & \begin{pmatrix} & 0 & \\ 0 & & c \end{pmatrix} \end{pmatrix} && \text{associative} \\
&= \begin{pmatrix} & \begin{pmatrix} & 0 & \\ 0 & & 0 \end{pmatrix} & \\ \begin{pmatrix} & 0 & \\ \begin{pmatrix} & 0 & \\ a & & 0 \end{pmatrix} & & b \end{pmatrix} & & \begin{pmatrix} & 0 & \\ 0 & & c \end{pmatrix} \end{pmatrix} && \text{lemma 2} \\
&= \begin{pmatrix} & 0 & \\ \begin{pmatrix} & 0 & \\ a & & b \end{pmatrix} & & c \end{pmatrix} && \text{identity}
\end{aligned} \tag{4}
$$

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