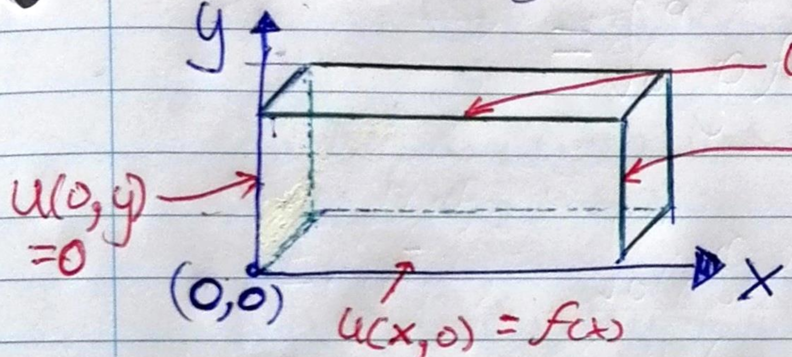
Maths Methods II Assignment 3

BVP: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad , \quad \begin{matrix} 0<x<a \\ 0<y<b \end{matrix}$ ①

$u(x,0) = f(x) \qquad , \; 0<x<a$ ②
$u(x,b) = g(x) \qquad , \; 0<x<a$ ③

$u(0,y) = 0 \qquad ; \; 0<y<b$
$u(a,y) = 0 \qquad ; \; 0<y<b$

for rectangle $0 \leq x \leq a \quad , \quad 0 \leq y \leq b$



1) seek solution of the form $u(x,y) = X(x)Y(y)$

① $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$\dfrac{\partial^2}{\partial x^2} X(x)Y(y) + \dfrac{\partial^2}{\partial y^2} X(x)Y(y) = 0$

$Y(y)X''(x) + X(x)Y''(y) = 0$

$Y(y)X''(x) = -X(x)Y''(y)$

÷ accross by $Y(y)$ & $X(x)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$f(x) = g(y) \iff f(x) = g(y) = $ a constant

$X''(x) + \lambda X(x) = 0 \quad , \quad Y'' - \lambda Y(y) = 0$

for constant $-\lambda$

2)
$$X'' + \lambda X = 0$$
$$Y'' - \lambda Y = 0$$

$\boxed{\lambda < 0}$: $Y = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y$
$X = C_3 \cosh(\sqrt{-\lambda}\, x) + C_4 \sinh(\sqrt{-\lambda}\, x)$

BCs: (4) $u(0,y) = 0$, $0 < y < b$
(5) $u(a,y) = 0$, $0 < y < b$

(4) $X(0)\,Y(y) = 0$, $0 < y < b$
(5) $X(a)\,Y(y) = 0$, $0 < y < b$

$X(0) = C_3 \cosh(\sqrt{-\lambda}\, 0) + C_4 \sinh(\sqrt{-\lambda}\,(0))$
$X(0) = C_3 + 0$

$Y(y) = C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y \neq 0$ (assuming $C_1, C_2 > 0$)
$\therefore C_3 = 0$

$X(a) = C_4 \sinh(\sqrt{-\lambda}\, a)$

$u(a,y) = X(a)Y(y) = C_4 \sinh(\sqrt{-\lambda}\, a)\left[C_1 \cos\sqrt{\lambda}\, y + C_2 \sin\sqrt{\lambda}\, y\right] = 0$

this statement can only hold if C_1 & $C_2 = 0$ OR $C_4 = 0$
giving the trivial solution $u(x,y) = 0$ for all x, y
this solution is discarded.

$\boxed{\lambda = 0}$ $X'' + \lambda X = 0 \xrightarrow{\text{becomes}} X'' = 0$
$Y'' - \lambda Y = 0 \rightarrow Y'' = 0$

$X = Ax + B$ $\quad Y = Cy + D$
$X(0) = B \rightarrow X(0)Y(y) = B(Cy + D) = 0$
take $C, D > 0$, $B = 0$

$X(a) = Aa \qquad X(a)Y(y) = Aa(Cy+D) = 0$

either $A=0$ or $C \& D = 0$ for this statement to hold, both give the trivial solution of $u(x,y) = 0$ for all x,y

Can discard this solution

Therefore BC's 4 & 5 can only be satisfied by the trivial solution when $\lambda < 0$ & $\lambda = 0$

3) $\boxed{\lambda > 0}$ $X'' + \lambda X = 0$
$Y'' - \lambda Y = 0$

$\lambda = \mu^2, \ \mu > 0$

$X(x) = A\cos\mu x + B\sin\mu x$

$Y(y) = C\cosh\mu y + D\sinh\mu y$

BCs
(4) $u(0,y) = 0$, $0 < y < b$
(5) $u(a,y) = 0$, $0 < y < b$

$X(0) = A + 0$
$u(0,y) = X(0)Y(y) = A[C\cosh\mu y + D\sinh\mu y] = 0$

take $A = 0$ assume $C, D > 0$

$X(a) = B\sin\mu a$
$u(a,y) = X(a)Y(y) = B\sin\mu a[C\cosh\mu y + D\sinh\mu y] = 0$

taking $B = 0$ or C & $D = 0$ gives trivial soln.

$u(a,y) = 0$ is satisfied for $\sin\mu a = 0$

$\mu a = n\pi$ since $\sin(n\pi) = 0$
$\mu = \frac{n\pi}{a}$ $n = 1, 2, 3, \ldots$

$\boxed{\mu = \sqrt{\lambda}}$ $\boxed{\lambda = \lambda_n = \frac{n^2\pi^2}{a^2}}$ eigenvalues

$\hat{C} = BC$ redefine constants
$\hat{D} = BD$

$$X_nY_n = \boxed{\underbrace{\sin\left(\frac{n\pi x}{a}\right)}_{X(x)}\underbrace{\left[\hat{C}\cosh\left(\frac{n\pi}{a}y\right) + \hat{D}\sinh\left(\frac{n\pi}{a}y\right)\right]}_{Y(y)}}$$

eigensolutions

$$X_n = \sin\left(\frac{n\pi}{a}x\right) \qquad n=1,2,3\ldots$$

$$Y_n = \hat{C}\cosh\left(\frac{n\pi}{a}y\right) + \hat{D}\sinh\left(\frac{n\pi}{a}y\right) \qquad n=1,2,3\ldots$$

4) $X_n Y_n = \sin\left(\frac{n\pi}{a}x\right)\left[\hat{C}\cosh\left(\frac{n\pi}{a}y\right) + \hat{D}\sinh\left(\frac{n\pi}{a}y\right)\right]$

$n = 1, 2, 3 \ldots$

Superpose all Solutions:

$$u(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\left(\hat{C}\cosh\left(\frac{n\pi}{a}y\right) + \hat{D}\sinh\left(\frac{n\pi}{a}y\right)\right)$$

BC's

(2) $u(x,0) = f(x)$, $0 < x < a$

(3) $u(x,b) = g(x)$, $0 < x < a$

$$u(x,0) = \sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{a}x\right)\left(\hat{C}\cosh(0) + \hat{D}\sinh(0)\right) = f(x)$$

$$= \sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{a}x\right)\left(\hat{C}\right) = f(x)$$

multiply both sides by $\sin\left(\frac{m\pi}{a}x\right)$ & integrate wrt x

$$\int_0^a \hat{C}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)dx = \int_0^a \sin\left(\frac{m\pi}{a}x\right) f(x)\,dx$$

integral $= 0$ unless $m = n \rightarrow$ orthogonality.

$$\hat{C}\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)dx = \int_0^a \sin\left(\frac{m\pi}{a}x\right) f(x)\,dx$$

$$\hat{C}\int_0^a \tfrac{1}{2}\left(1 - \cos\left(\frac{2n\pi}{a}x\right)\right)dx$$

$$\frac{\hat{C}}{2}\int_0^a 1 - \cos\left(\frac{2n\pi}{a}x\right)dx = \frac{\hat{C}}{2}\left[1 - \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right)\right]\Big|_0^a$$

$$= \frac{\hat{C}}{2}\left([a-0] - [0-0]\right) = \frac{\hat{C}}{2}a$$

$$\frac{\hat{C}}{2}a = \int_0^a \sin\left(\frac{n\pi}{a}x\right) f(x)\,dx$$

$$A_n = \hat{C} = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi}{a}x\right) f(x)\,dx$$

③ $u(x,b) = g(x)$

$$u(x,b) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right)\right) = g(x)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a}x\right)\left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right) = g(x)\right.$$

$$\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)\left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right)\right)dx = \int_0^a g(x)\sin\left(\frac{m\pi}{a}x\right)dx$$

$$\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right)\right)dx = \int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$\left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right)\right)\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = \int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$a\cdot\frac{1}{2}\left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right)\right) = \int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$C_n = \left(\hat{C}\cosh\left(\frac{n\pi}{a}b\right) + \hat{D}\sinh\left(\frac{n\pi}{a}b\right)\right) = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$A_n = \hat{C}$$

$$C_n = \hat{C}\,\text{Cosh}\left(\frac{n\pi}{a}b\right) + \hat{D}\,\text{Sinh}\left(\frac{n\pi}{a}b\right)$$

$$= \hat{C}\,\text{Cosh}(\sqrt{\lambda_n}\,b) + \hat{D}\,\text{Sinh}(\sqrt{\lambda_n}\,b)$$

$$D = \frac{C_n}{\text{Sinh}(\sqrt{\lambda_n}\,b)} - \frac{A_n\,\text{Cosh}(\sqrt{\lambda_n}\,b)}{\text{Sinh}(\sqrt{\lambda_n}\,b)}$$

$$u(x,y) = \sum_{n=1}^{\infty}\left[\hat{C}\,\text{Cosh}\left(\frac{n\pi}{a}y\right) + \hat{D}\,\text{sinh}\left(\frac{n\pi}{a}y\right)\right]\cdot \text{Sin}\sqrt{\lambda_n}\,x$$

$$= \sum_{n=1}^{\infty}\left(\left[A_n\,\text{Cosh}(\sqrt{\lambda_n}\,y) + \left(\frac{C_n}{\text{Sinh}(\sqrt{\lambda_n}\,b)} - \frac{A_n\,\text{Cosh}(\sqrt{\lambda_n}\,b)}{\text{Sinh}(\sqrt{\lambda_n}\,b)}\right)\text{Sinh}(\sqrt{\lambda_n}\,y)\right]\cdot \text{Sin}\sqrt{\lambda_n}\,x\right)$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty}\left\{A_n\left[\text{Cosh}(\sqrt{\lambda_n}\,y) - \frac{\text{Cosh}(\sqrt{\lambda_n}\,b)}{\text{Sinh}(\sqrt{\lambda_n}\,b)}\text{Sinh}(\sqrt{\lambda_n}\,y)\right] + C_n\,\frac{\text{Sinh}(\sqrt{\lambda_n}\,y)}{\text{Sinh}(\sqrt{\lambda_n}\,b)}\right\}\cdot \text{Sin}\sqrt{\lambda_n}\,x}$$

5). $$u(x,y) = \sum_{n=1}^{\infty} \left\{ a_n \left[\cosh(\sqrt{\lambda_n} y) - \frac{\cosh(\sqrt{\lambda_n} b)}{\sinh(\sqrt{\lambda_n} b)} \sinh(\sqrt{\lambda_n} y) \right] + C_n \frac{\sinh(\sqrt{\lambda_n} y)}{\sinh(\sqrt{\lambda_n} b)} \right\} \cdot \sin\sqrt{\lambda_n} x$$

$$\left[\cosh(\sqrt{\lambda_n} y) - \frac{\cosh(\sqrt{\lambda_n} b)}{\sinh(\sqrt{\lambda_n} b)} \sinh(\sqrt{\lambda_n} y) \right]$$

multiply accross by $\sinh(\sqrt{\lambda_n} b)$

$$\left[\sinh(\sqrt{\lambda_n} b) \cosh(\sqrt{\lambda_n} y) - \cosh(\sqrt{\lambda_n} b) \sinh(\sqrt{\lambda_n} y) \right]$$

using hyperbolic trig ~~exp~~ identity:

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

expression becomes:

$$\sinh(\sqrt{\lambda_n}(b-y))$$

divide by $\sinh(\sqrt{\lambda_n} b)$ to reverse the multiplication

expression in squared brackets now reads:

$$\frac{\sinh(\sqrt{\lambda_n}(b-y))}{\sinh(\sqrt{\lambda_n} b)}$$

Solution becomes:

$$u(x,y) = \sum_{n=1}^{\infty} \left\{ a_n \left[\frac{\sinh(\sqrt{\lambda_n}(b-y))}{\sinh(\sqrt{\lambda_n} b)} \right] + C_n \frac{\sinh(\sqrt{\lambda_n} y)}{\sinh(\sqrt{\lambda_n} b)} \right\} \cdot \sin\sqrt{\lambda_n} x$$

6) $a = b = 1$

$$f(x) = g(t) = \begin{cases} 2x, & 0 < x < 1/2 \\ 2(1-x) & 1/2 < x < 1 \end{cases}$$

$$a_n = 2/a \int_0^a \sin\left(\frac{n\pi}{a}x\right) f(x)\, dx$$

$$a_n = \frac{2}{1}\left[\int_0^{1/2} \sin(n\pi x) \cdot 2x\, dx + \int_{1/2}^{1} \sin(n\pi x) \cdot 2(1-x)\, dx\right]$$

$$= 2\left[2\int_0^{1/2} x \sin(n\pi x)\, dx + \int_{1/2}^{1} 2\sin(n\pi x) - \int_{1/2}^{1} 2x \sin(n\pi x)\, dx\right]$$

$\int_0^{1/2} x \sin(n\pi x)\, dx \to$ by parts

$u = x \quad v' = \sin(n\pi x)$

$u' = 1 \quad v = -\frac{1}{n\pi}\cos(n\pi x)$

$$\int uv' = uv - \int vu'$$

$$= -\frac{x}{n\pi}\cos(n\pi x)\Big|_0^{1/2} + \int_0^{1/2} \frac{1}{n\pi}\cos(n\pi x)\, dx$$

$$= \left[-\frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + 0\right] + \left[\frac{1}{n^2\pi^2}\sin(n\pi x)\right]_0^{1/2}$$

$$= -\frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{1/2}^{1} x\sin(n\pi x)\, dx \to -\frac{x}{n\pi}\cos(n\pi x)\Big|_{1/2}^{1} + \left[\frac{1}{n^2\pi^2}\sin(n\pi x)\right]\Big|_{1/2}^{1}$$

$$-\left[\frac{1}{n\pi}\cos(n\pi) - \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right)\right] + \left[\overset{0}{\sin(n\pi)} - \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$2\int_{1/2}^{1} \sin(n\pi x) = \cancel{-2\int_{1/2}^{1}\cos(n\pi}$$

$$-2\left[\frac{1}{n\pi}\cos(n\pi x)\right]\Big|_{1/2}^{1}$$

$$= \frac{-2}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos(n\pi)\right)$$

$$a_n = 2\left[2\left(\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right)\right)\right.$$

$$+\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos(n\pi)\right)$$

$$\left.-2\left(-\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\cos(n\pi)\right)\right]$$

$$a_n = 2\left[4\left(\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right)\right)\right.$$

$$\left.+\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right] + \cancel{\frac{2}{n\pi}\cos(n\pi)} - \cancel{\frac{2}{n\pi}\cos(n\pi)}$$

$$\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) = \frac{4}{2n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$a_n = 2\left[4\left(\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{1}{2n\pi}\cos\frac{n\pi}{2}} - \cancel{\frac{1}{2n\pi}\cos\frac{n\pi}{2}}\right]\right.$$

$$a_n = 8\left[\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$a_n = \frac{8}{(n\pi)^2}\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

$$C_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi}{a}x\right) g(x)\, dx$$

since $f(x) = g(x)$, $a_n = C_n$

$a = b = 1$

Solution:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty}\left\{a_n\left[\frac{\sinh(\sqrt{\lambda_n}(1-y))}{\sinh(\sqrt{\lambda_n})}\right] + c_n\frac{\sinh(\sqrt{\lambda_n}\, y)}{\sinh(\sqrt{\lambda_n})}\right\}\cdot \sin(\sqrt{\lambda_n}\, x)}$$

where $\lambda_n = \frac{n^2\pi^2}{a^2}$, $a=b=1$, $\lambda_n = n^2\pi^2$

$\sqrt{\lambda_n} = n\pi$

$$a_n = C_n = \frac{8}{(n\pi)^2}\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

$n = 1, 2, 3, \ldots$ $\qquad \sqrt{\lambda_n} = \frac{n\pi}{a} = n\pi$

$0 \leq x \leq 1$

$0 \leq y \leq 1$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty}\left\{\frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\left[\frac{\sinh(n\pi(1-y))}{\sinh(n\pi)}\right] + \frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\cdot\left(\frac{\sinh(n\pi y)}{\sinh(n\pi)}\right)\right\}\cdot \sin(n\pi x)}$$

```
> a[n] := 8/((n)^2·(Pi)^2) sin(n·Pi/2);
```

$$a_n := \frac{8 \sin\left(\frac{n\,\pi}{2}\right)}{n^2\,\pi^2} \qquad (1)$$

```
>
>
> u(N, x, y) := sum(a[n]·((sinh(n·Pi·(1 - y))/sinh(n·Pi)) + ((sinh(n·Pi·y))/sinh(n·Pi)))·sin(n·Pi·x), n = 1 .. N);
```

$$u := (N, x, y) \rightarrow \sum_{n=1}^{N} a_n \left(\frac{\sinh(n\,\pi\,(1-y))}{\sinh(n\,\pi)} + \frac{\sinh(n\,\pi\,y)}{\sinh(n\,\pi)} \right) \sin(n\,\pi\,x) \qquad (2)$$

```
> plot3d(u(100, x, y), x = 0 .. 1, y = 0 .. 1);
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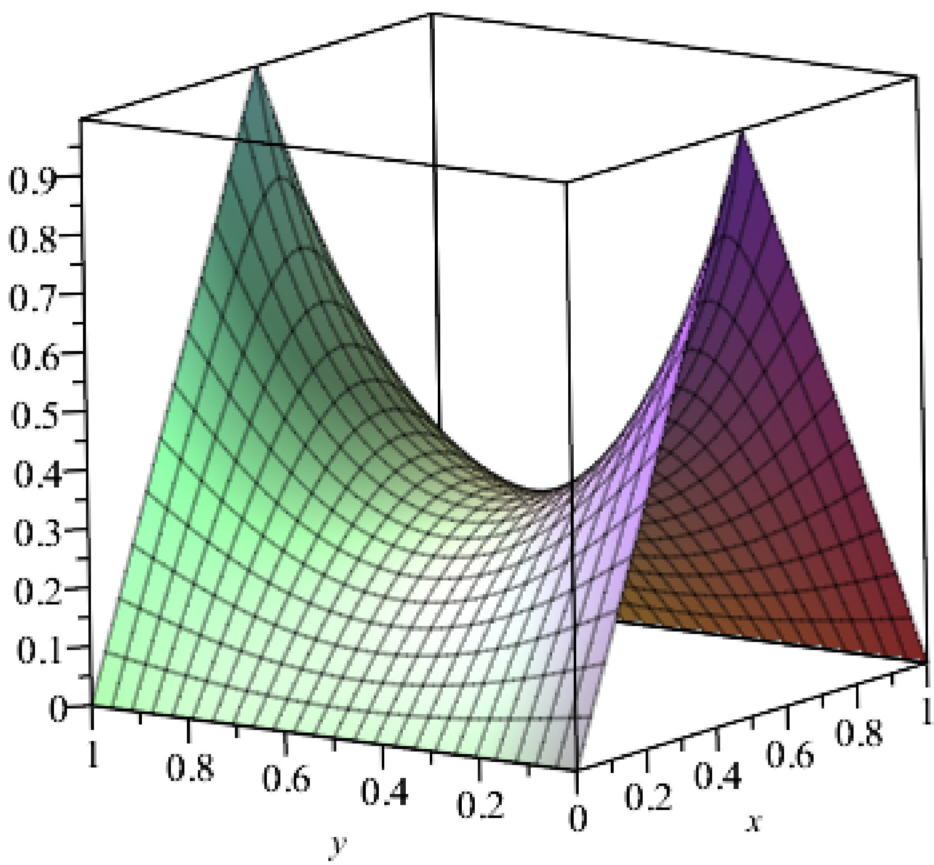


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