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PDEs CA 2 8/10/21

Q1

$$x\frac{\partial u}{\partial x} + \underbrace{(x+y)}_{b}\frac{\partial u}{\partial y} = 1$$

$\uparrow a$ $\qquad\qquad \uparrow c_1$

$$u(x, 2x) = 2$$

Using method of characteristics:

$$\left.\begin{aligned} \frac{dx}{dt} &= a \longrightarrow \frac{dx}{dt} = x \\ \frac{dy}{dt} &= b \longrightarrow \frac{dy}{dt} = x+y \\ \frac{du}{dt} &= c_0 u + c_1 \longrightarrow \frac{du}{dt} = 1 \end{aligned}\right\} \quad (1)$$

With initial conditions:

$$x(t=0,s) = x_0(s), \quad y(t=0,s) = y_0(s), \quad u(t=0,s) = u_0(s)$$

Solving (1):

$$\int \frac{1}{x}\,dx = \int dt$$

$$\ln x = t + f_1(s) \qquad \hat{f}_1(s) = e^{f_1}$$

$$x = e^{t + f_1(s)} \Rightarrow x = \hat{f}_1(s)e^t$$

$$\frac{dy}{dt} = x+y$$

$$\frac{dy}{dt} - y = x \quad \rightarrow \text{use integration factor } IF = e^{\int -1\,dt} = e^{-t}$$

$$e^{-t}\frac{dy}{dt} - e^{-t}y = e^{-t}x$$

$$\frac{d}{dt}(e^{-t}y) = e^{-t}x$$

$$\int \frac{d}{dt}(e^{-t}y)\,dt = \int e^{-t}x\,dt$$

$$e^{-t}y = \cancel{-e^{-t}x} + c \quad \leftarrow x \text{ is fn of } s \,\&\, t$$

$$\cancel{y = -x + ce^t}$$

$$\cancel{y = -x + f_2(s)e^t}$$

$$\frac{du}{dt} = 1 \rightarrow \int du = \int dt$$

$$u = t + f_3(s)$$

initial condition: $u(x, 2x) = 2$

$$x = s$$
$$y = 2s$$
$$u = 2$$

$$x(t=0,s) = X_0(s) = \hat{f}_1(s) e^0 = \hat{f}_1(s) = s$$

$$x(t,s) = se^t$$

$$\cancel{y(t=0,s) = y_0(s) = -x + f_2(s) e^0}$$
$$\cancel{\Rightarrow -se^0 + f_2(s) = 2s}$$
$$\cancel{f_2(s) = 3s}$$
$$\cancel{y(t,s) = -se^t + 3se^t}$$

$$ye^{-t} = \int e^{-t} x \, dt$$
$$= \int e^{-t} se^t = \int s \, dt$$
$$ye^{-t} = st + c$$
$$y = e^t(st + c)$$
$$y = ste^t + ce^t$$

$$y_0(s) = y(t=0,s) = 0 + ce^0 = c = 2s$$

$$y(t,s) = 2se^t + ste^t$$
$$= (2s + st)e^t$$

$$u = t + f_3(s) \qquad u(t=0,s) = 2$$
$$u = 0 + f_3(s) = 2$$
$$u = t + 2$$

$$x(t,s) = se^t$$
$$y(t,s) = (2s + st)e^t$$
$$u(t,s) = t + 2$$

(b) $x\frac{\partial u}{\partial x} + (x+y)\frac{\partial u}{\partial y} = 1$, $u(x,2x) = 2$

Check solution satisfies the equation & initial condition

$$y/x = \frac{2\cancel{se^t} + \cancel{s}t\cancel{e^t}}{\cancel{se^t}}$$
$$y/x = t+2 = u$$

$u = y/x$ → substitute into eqn.

$$x\frac{\partial}{\partial x}\left(\frac{y}{x}\right) + (x+y)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = 1$$

$$xy\left(\frac{\partial x^{-1}}{\partial x}\right) + \frac{(x+y)}{x}\frac{\partial y}{\partial y} = 1$$
$$xy \cdot -x^{-2} + \frac{x+y}{x} \cdot 1 = 1$$
$$\cancel{-\frac{y}{x}} + 1 + \cancel{\frac{y}{x}} = 1$$
$$1 = 1 \checkmark$$

check initial condition $u(x,2x) = 2$

$u = y/x$ $\quad u(x,2x) = \frac{2\cancel{x}}{\cancel{x}} = 2 \checkmark$.

(c) is the solution defined for all values (x, y)?

no, since $u = y/x$, the solution is undefined for $x = 0$

Q2 (a) given PDE

(1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \rho u$ (φ is a real no.)

$-\infty < x < \infty$

$-\infty < y < \infty$

Find characteristic curves

eqn of form $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c_0 u + c_1$
has the following equations

$$\frac{dx(t)}{dt} = a(x(t), y(t))$$

$$\frac{dy(t)}{dt} = b(x(t), y(t))$$

$$\frac{du(t)}{dt} = c_0(x(t), y(t))u(t) + c_1(x(t), y(t))$$

applying to (1):

$$\left.\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= y \\ \frac{du}{dt} &= \rho u \end{aligned}\right\} (2)$$

with initial conditions:

$x(t=0,s) = x_0(s)$

$y(t=0,s) = y_0(s)$

$u(t=0,s) = u_0(s)$

integrate (2) $\int \frac{1}{x}dx = \int dt$

$$\ln x = t + f_1(s)$$

$$x = e^{t+f_1(s)} = e^t e^{f_1(s)}$$

$$x(t=0,s) = x_0(s)$$

$$x(t) = x_0(s)e^t$$

$$\int \frac{1}{y}\, dy = \int dt$$

$$\ln y = t + f_2(s)$$

$$y = e^t e^{f_2(s)} = \hat{f}_2(s)\, e^t$$

$$y(t=0,s) = y_0(s)$$

$$y = y_0(s)\, e^t$$

$$\int \frac{1}{u}\, du = \int \rho\, dt$$

$$\ln u = \rho t + f_3(s)$$

$$u = e^{\rho t} \cdot e^{f_3(s)} = e^{\rho t} \hat{f}_3(s)$$

$$u(t) = u_0(s)\, e^{\rho t}$$

$$\begin{cases} x(t) = x_0(s) e^t \\ y(t) = y_0(s) e^t \\ u(t) = u_0(s) e^{\rho t} \end{cases} \quad (3)$$

for each given s, (3) determines a characteristic curve

$$\frac{y}{x} = \frac{y_0(s)\, \cancel{e^t}}{x_0(s)\, \cancel{e^t}} = \alpha(s)$$

$$\alpha(s) = \frac{y_0(s)}{x_0(s)}$$

$$y = \alpha(s)\, x$$

gives the characteristic curves for each value of s

(b) Let $\rho = 4$, find explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$

parameterize the surface of the circle

$$\begin{cases} x_0(s) = \cos(s) \\ y_0(s) = \sin(s) \\ u(\cos(s), \sin(s)) = 1 \end{cases} \qquad 0 \leq s \leq 2\pi$$

From the last part we have

$x(t,s) = X_0(s)e^t \quad = \cos(s)\, e^t$

$y(t,s) = y_0(s)\, e^t \quad = \sin(s)\, e^t$

$u(t,s) = U_0(s) e^{\rho t} \quad = ? \cdot e^{4t}$

$$x^2 + y^2 = \cos^2(s)\, e^{2t} + \sin^2(s)\, e^{2t}$$
$$= e^{2t}$$

$$u = e^{4t} = (e^{2t})^2$$
$$u = (x^2+y^2)^2 \quad \leftarrow \text{solution}$$

(C) let $\rho = 2$

Find two solutions that satisfy $u(x,0) = x^2$ for every $x > 0$

$X(t=0,s) = s \qquad y(t=0,s) = 0, \qquad u(t=0,s) = s^2$

$X(t,s) = X_0(s)e^t \quad = se^t$

$y(t,s) = 0 \cdot e^t = 0$

$u(t,s) = s^2$

Since $y(t,s) = 0$, method of characteristics won't work here. $\cancel{\ }(e^t = x/s)$ $(t = \ln(x/s))$ $(s = xe^{-t})$

try find solution of the form:
$$u(x,y) = x^2 + f(y)$$

PDE: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \qquad , u(x,0) = x^2$

$$x(2x) + y(f'(y)) = 2(x^2 + f(y))$$
$$2x^2 + yf'(y) = 2x^2 + 2f(y)$$
$$yf'(y) = 2f(y)$$

$$\frac{f'(y)}{f(y)} = \frac{2}{y}$$

$$\frac{\frac{df}{dy}}{f(y)} = \frac{2}{y}$$

$$\frac{df}{f(y)} = \frac{2}{y}\, dy$$

$$\int \frac{1}{f}\, df = 2\int \frac{1}{y}\, dy$$

$$\ln f = 2\ln y + c$$

$$\ln f = \ln y^2 + c$$

$$f = e^{\ln y^2 + c}$$

$$f = \cancel{\;} e^{\ln y^2} \cdot e^c$$

$$f = y^2 e^c$$

where c is an arbitrary const. of integration

$f = \hat{c} y^2$ where $\hat{c} = e^c$

Since $\hat{c}$ is an arbitrary constant, there are an infinity of solutions

i.e. 2 solutions ~~are~~ that satisfy the initial conditions are:

1) $u = x^2 + y^2$ 2) $u = x^2 + 4y^2$

Q2 (d) $J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{vmatrix} = \begin{vmatrix} se^t & e^t \\ 0 & 0 \end{vmatrix} = 0$

$x(t,s) = se^t$
$y(t,s) = 0$
$u(t,s) = s^2$

Since $J = 0$, ~~the~~ the result in (c) does not contradict the existence-uniqueness theorem.

Q3 Solve $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4u$

$-\infty < x, y < \infty$

$u(x, -1) = x^2 \quad , -\infty < x < \infty$

$p = \frac{\partial u}{\partial x} \qquad q = \frac{\partial u}{\partial y}$

$F = p^2 + q^2 - 4u = 0$

(i) $\frac{dx}{dt} = \frac{\partial F}{\partial p} \rightarrow \frac{dx}{dt} = 2p$

(ii) $dy/dt = \partial F/\partial q \rightarrow dy/dt = 2q$

(iii) $du/dt = p\frac{\partial F}{\partial p} + q\,\partial F/\partial q = 2p^2 + 2q^2$
$= 2(p^2+q^2)$
$= 2(4u) = 8u$

(iv) $dp/dt = -\partial F/\partial x - p\,\partial F/\partial u$
$dp/dt = 0 + 4p$

(v) $dq/dt = -\partial F/\partial y - q\,\partial F/\partial u \qquad \partial q/\partial t = 0 + 4q$

$x(t=0,s) = x_0(s),\ y(t=0,s) = y_0(s),\ u(t=0,s) = u_0(s)$

$x(t=0,s) = s \quad = x_0$
$y(t=0,s) = -1 \quad = y_0$
$u(t=0,s) = s^2 \quad = u_0$

$F(x_0, y_0, u_0, p(t=0,s), q(t=0,s))$

$p^2(t=0,s) + q^2(t=0,s) - 4u_0 = 0$

$p^2(t=0,s) + q^2(t=0,s) - 4s^2 = 0$ (vi)

$$\frac{du_0}{ds} = p(t=0,s)\frac{dx_0}{ds} + q(t=0,s)\frac{dy_0}{ds}$$

$$2s = p(t=0,s)(1) + q(t=0,s)(0)$$

$$\boxed{2s = p(t=0,s)} \quad \text{vii}$$

vi

$$p^2(t=0,s) + q^2(t=0,s) - 4s^2 = 0$$

$$(2s)^2 + q^2 - 4s^2 = 0$$

$$\cancel{4s^2} + q^2 - \cancel{4s^2} = 0$$

$$q^2 = 0 \rightarrow q = 0$$

$$\boxed{q(t=0,s) = 0}$$

(iii) $\frac{du}{dt} = 8u$

$$\int \frac{1}{8u}du = \int dt$$

$$\frac{1}{8}\int \frac{1}{u}du = dt$$

$$\frac{1}{8}\ln u = t + c \qquad c = \text{const}$$

$$\ln u = 8(t+c)$$

$$u = e^{8(t+c_1)} = e^{8t}e^{8c_1} \qquad e^{8c_1} = \hat{C}_1$$

$$u = \hat{C}_1 e^{8t}$$

$$u_0 = s^2$$

$$u(t,s) = s^2 e^{8t}$$

(iv) $\frac{dp}{dt} = 4p$

$$\frac{1}{4}\int \frac{1}{p}dt = \int dt$$

$$\frac{1}{4}\ln p = t + C_2$$

$$\ln p = 4(t + C_2)$$

$$p = e^{4(t+C_2)} = e^{4t}e^{4C_2} = e^{4t}\hat{C}_2$$

$$p = \hat{C}_2 e^{4t}$$

$$2s = p(t=0,s) \rightarrow p(t,s) = 2s\,e^{4t}$$

(v) $dq/dt = 4q$

$$\frac{1}{4}\int \frac{1}{q}\, dq = \int dt$$

$$\frac{1}{4} \ln q = t + c_3$$

$$\ln q = 4(t + c_3)$$

$$q = e^{4(t+c_3)} = \hat{c}_3 e^{4t}$$

$$q(t=0, s) = 0 \quad \rightarrow \quad q(t,s) = 0$$

(i) $\frac{dx}{dt} = 2p \rightarrow \int \frac{1}{2p}\, dx = \int dt$

$$\frac{1}{2} \frac{1}{p} x = t + c_4$$

$$\frac{1}{p} x = 2t + 2c_4$$

$$x = (2t + 2c_4) p$$

$2c_4 = \hat{c}_4$ $\quad x = (2t + \hat{c}_4) p$

$$p = 2se^{4t}$$

$$x = (2t + \hat{c}_4) \cdot 2se^{4t}$$

$$x_0 = s \quad \rightarrow \quad \hat{c}_4 \cdot 2s = s \rightarrow \hat{c}_4 = 1/2$$

$$x(t,s) = (2t + 1/2)\, 2se^{4t}$$

$$x(t,s) = (4t + 1) se^{4t} = 4ste^{4t} + se^{4t}$$

(ii) $dy/dt = 2q \rightarrow \quad q = 0$

$$\frac{dy}{dt} = 0$$

$$\therefore \quad y(t,s) = y_0$$

$y_0 = -1$

$$y(t,s) = -1$$

$u = s^2 e^{8t}$, $\cancel{x = (4t+1)se^{4t}}$, $y = -1$

$\frac{dx}{dt} = 2p$

$$\int dx = \int 2p\,dt$$

$$\cancel{x = 2pt + c_4}$$

$$\cancel{2se^{4t}}$$

$$\cancel{x = 4se^{4t} \cdot t}$$

$$x = \int 2 \cdot 2se^{4t}\,dt$$

$$x = \frac{4}{4} se^{4t} + c_4$$

$$x = se^{4t} + c_4$$

$x_0 = s \rightarrow c_4 = 0$ $\boxed{x(t,s) = se^{4t}}$

$\boxed{u = s^2 e^{8t} \,,\, x = se^{4t} \,,\, y = -1}$ Solution

$$u = x^2$$

Check. $\rightarrow \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 4u$

$$(2x)^2 + (0)^2 = 4 \cdot x^2$$

$$4x^2 = 4x^2 \quad ✓$$

$u(x,-1) = x^2$

$$x^2 = x^2 \quad ✓$$