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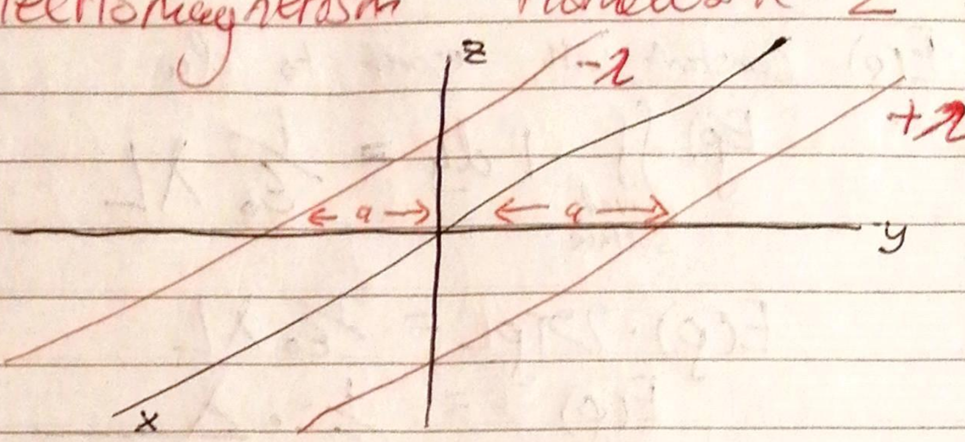
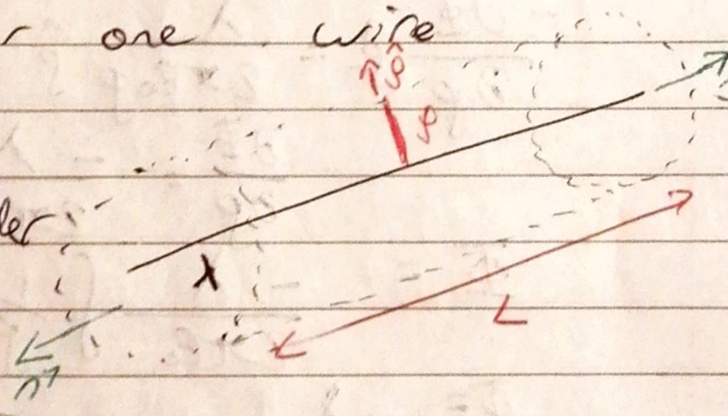
MP366 Electromagnetism Homework 2 30/10/21



Two infinitely long wires running parallel to the x-axis carry uniform charge densities $+\lambda$ & $-\lambda$

(a) Find the Potential Φ at any point (x, y, z) taking Φ to be zero at the origin.

Construct a Gaussian Surface S to find the potential for one wire



ρ = distance from cylinder to wire

$\hat{\rho}$ is unit vector pointing out from wire to surface of cylinder

Can write $\underline{E} = E(\rho)\,\hat{\underline{\rho}}$

Gauss' Law $\oiint_S \underline{E}\cdot\hat{\underline{n}}\,da = \frac{1}{\varepsilon_0}\cdot\begin{pmatrix}\text{enclosed}\\ \text{charge}\end{pmatrix}$

$$\Rightarrow \iint_{\substack{\text{curved}\\ \text{surface}}} E(\rho)\,\hat{\underline{\rho}}\cdot\hat{\underline{\rho}}\,da + \iint_{\text{sides}} E(\rho)\,\hat{\underline{\rho}}\cdot\hat{\underline{n}}\,da$$

$$= \frac{1}{\varepsilon_0}(\lambda L)$$

$\hat{\rho}\cdot\hat{\rho} = 1$, $\hat{\rho}\cdot\hat{n} = 0$

$$\iint_{\substack{\text{curved}\\ \text{surface}}} E(\rho)\,da = \frac{1}{\varepsilon_0}(\lambda L)$$

$E(\rho)$ constant with respect to da

$$E(\rho)\iint_{\substack{\text{curved}\\\text{surface}}} 1\,d\underline{a} = \frac{1}{\varepsilon_0}\lambda L$$

$$E(\rho)\cdot 2\pi\rho L = \frac{1}{\varepsilon_0}\lambda L$$

$$E(\rho) = \frac{1}{\varepsilon_0}\cdot\frac{\lambda}{2\pi\rho}$$

$$E(\rho) = \frac{\lambda}{2\pi\varepsilon_0\rho}$$

$$\underline{E} = E(\rho)\hat{\rho} \quad \longrightarrow \quad \underline{E} = \frac{\lambda}{2\pi\varepsilon_0\rho}\hat{\rho}$$

we know that $\underline{E} = -\nabla\Phi$

$$-\nabla\Phi = \frac{-\partial\Phi}{\partial\rho}\hat{\rho} \qquad (\text{no } \hat{\varphi}, \hat{z} \text{ components here})$$

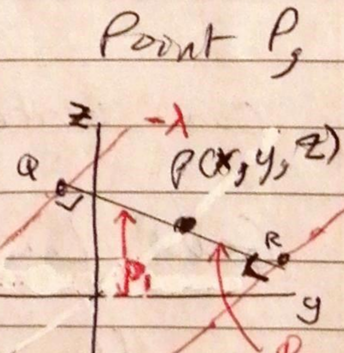
$$\frac{-\partial\Phi}{\partial\rho}\hat{\rho} = \frac{\lambda}{2\pi\varepsilon_0\rho}\hat{\rho}$$

$$\frac{\partial\Phi}{\partial\rho} = \frac{-\lambda}{2\pi\varepsilon_0\rho}$$

$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0}\int\frac{1}{\rho}\,d\rho$$

$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0}\ln\rho + C \qquad \text{for one wire}$$

Consider the potential of a point P, placed between the 2 wires



$$\Phi(P) = \begin{pmatrix}\text{Potential from line}\\ \text{through } Q\end{pmatrix} + \begin{pmatrix}\text{Potential from}\\ \text{line through } R\end{pmatrix}$$

$$\Phi(P) = \frac{|-\lambda|}{2\pi\varepsilon_0}\ln(\rho_1) + \frac{-\lambda}{2\pi\varepsilon_0}\ln(\rho_2) + C$$

$$\Phi(P) = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln(\rho_1) - \ln(\rho_2)\right) + C$$

$$\Phi(P) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\rho_1}{\rho_2}\right) + C$$

$Q:(x, -a, 0) \quad R:(x, a, 0)$

$$\rho_1 = |PQ| = \sqrt{(y+a)^2 + z^2}$$

$$\rho_2 = |PR| = \sqrt{(y-a)^2 + z^2}$$

$$\Phi(x,y,z) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right)$$

$$\Phi(x,y,z) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\sqrt{\frac{(y+a)^2+z^2}{(y-a)^2+z^2}}\right) + C$$

Choose $\Phi = 0$ at the origin $(0,0,0)$

$y = z = 0$

$$\Phi(0,0,0) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\sqrt{\frac{(0+a)^2+0}{(0-a)^2+0}}\right) + C$$

$$\Phi(0) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\sqrt{\frac{a^2}{a^2}}\right) = \frac{\lambda}{2\pi\varepsilon_0} \ln(1) = 0$$

$\therefore C = 0$

$$\Phi(x,y,z) = \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\left(\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right)^{1/2}\right)$$

$$\Phi(x,y,z) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right)$$

(b) Show that the equipotential surfaces are circular cylinders. Equipotential surface satisfied by $\Phi = \Phi_0 = \text{const.}$ Locate the axis & radius of the cylinder for a given potential Φ_0:

$$\Phi(x, y, z) = \Phi_0 = \text{const.}$$

$$\frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right) = \Phi_0$$

$$\exp\left(\frac{\lambda}{4\pi\varepsilon_0}\right) \cdot \left(\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right) = \Phi_0$$

$$\Phi_0 = \left(\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right) = \underbrace{\exp\left(\frac{4\pi\varepsilon_0}{\lambda}\right)}_{\text{call } \alpha}$$

$$\left(\frac{(y+a)^2 + z^2}{(y-a)^2 + z^2}\right) = \alpha$$

$$(y+a^2) + z^2 = \alpha\left[(y-a)^2 + z^2\right]$$

$$(y+a)^2 + z^2 = \alpha(y-a)^2 + \alpha z^2$$

$$(y+a)^2 - \alpha(y-a)^2 + z^2 - \alpha z^2 = 0$$

$$(y+a)^2 = y^2 + 2ay + a^2$$

$$(y-a)^2 = y^2 - 2ay + a^2$$

$$y^2 + 2ay + a^2 - \alpha(y^2 - 2ay + a^2) + z^2 - \alpha z^2 = 0$$

$$y^2(1-\alpha) + 2a(1+\alpha)y + z^2(1-\alpha) + (1-\alpha)a^2 = 0$$

divide by $(1-\alpha)$

$$y^2 + 2a\frac{(1+\alpha)}{(1-\alpha)}y + z^2 + a^2 = 0$$

① $$y^2 + z^2 + 2a\frac{(1+\alpha)}{(1-\alpha)}y + a^2 = 0$$

eqn of circle on y, z plane

$$y^2 + z^2 + 2gy + 2fz + c = 0$$

$$2g = 2a\frac{1+\alpha}{1-\alpha}, \quad f=0, \quad c=a^2$$

circle centred on $(-g,-f)$

centre is $\left(-a\left(\frac{1+\alpha}{1-\alpha}\right), 0\right) = (y, z)$

$$(y-g)^2 + (z-f)^2 = r^2$$

$$\left(y + a\left(\frac{1+\alpha}{1-\alpha}\right)\right)^2 + z^2 = r^2$$

$$r = \sqrt{\left(y + a\left(\frac{1+\alpha}{1-\alpha}\right)\right)^2 + z^2}$$

$$r^2 = y^2 + 2ay\left(\frac{1+\alpha}{1-\alpha}\right) + a^2\left(\frac{1+\alpha}{1-\alpha}\right)^2 + z^2$$

① $y^2+z^2 = -2a\frac{(1+\alpha)}{(1-\alpha)}y - a^2$ sub on

$$r^2 = \cancel{2ay\left(\frac{1+\alpha}{1-\alpha}\right)} + a^2\left(\frac{1+\alpha}{1-\alpha}\right)^2 - \cancel{2a\left(\frac{1+\alpha}{1-\alpha}\right)y} - a^2$$

$$r^2 = a^2\left(\left(\frac{1+\alpha}{1-\alpha}\right)^2 - 1\right) \quad r^2 = a^2\left(\frac{1+2\alpha+\alpha^2}{1-2\alpha+\alpha^2} - 1\right)$$

$$r^2 = a^2\left(\frac{\cancel{1}+2\alpha+\cancel{\alpha^2}-\cancel{1}+2\alpha-\cancel{\alpha^2}}{1-2\alpha+\alpha^2}\right) \rightarrow r^2 = a^2\left(\frac{4\alpha}{(1-\alpha)^2}\right)$$

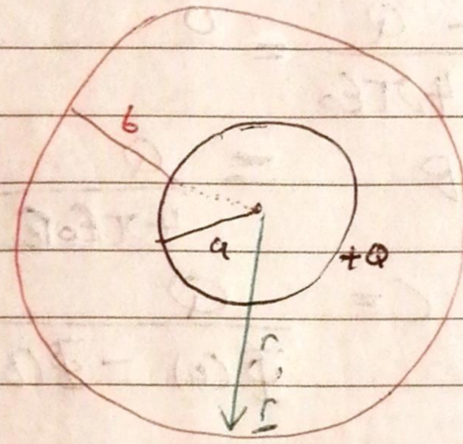
radius of circle y^2+z^2 $\boxed{r = a\left(\frac{2\sqrt{\alpha}}{1-\alpha}\right)}$ circle continues in x direction from $-\infty$ to $+\infty$

so therefore the equipotential surfaces are circular cylinders.

Q2 Two conducting spherical shells have radii a and b with $a < b$. inner shell has charge $+Q$ & outter shell has charge $-Q$. What is the capacitance of this system? What is the energy stored in the capacitor?

Spherical Capacitor

$a < r < b$ by radial symmetry



$\underline{E} = E(r)\,\hat{\underline{r}}$

$E = -\dfrac{d\Phi}{dr}\hat{\underline{r}}$ from $\underline{E} = -\nabla\Phi$ in spherical polar coords. between plates

also $\nabla \cdot \underline{E} = 0$ between plates. $\rho = 0 \rightarrow$ no charge between plates

$$\nabla^2 \Phi = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 0 \quad \leftarrow \text{Laplacian in Spherical Polars no } \theta, \phi \text{ dependence.}$$

$$\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 0$$

$$r^2 \frac{d\Phi}{dr} = A = \text{const.}$$

$$\frac{d\Phi}{dr} = \frac{A}{r^2}$$

$$\Phi = -\frac{A}{r} + B$$

Where A & B are constants of integration.

$$\underline{E} = -\frac{d\Phi}{dr}\hat{\underline{r}} \rightarrow \underline{E} = -\frac{d}{dr}\left(-\frac{A}{r} + B\right)\hat{\underline{r}}$$

$$= +\left(-r^{-2}A\right)\hat{\underline{r}}$$

$$\boxed{\underline{E} = \frac{-A}{r^2}\hat{\underline{r}}}$$

immediately above a, $\underline{E} = \underline{E}^{+} = \dfrac{\sigma}{\varepsilon_0}\hat{\underline{r}}$

with $\sigma = \dfrac{Q}{4\pi\varepsilon_0 a^2}$.

$$\sigma = \frac{Q}{4\pi r^2} \qquad \underline{E}_+ = \frac{\sigma}{\varepsilon_0}\hat{\underline{r}}, \boxed{r=a}$$

$$\underline{E} = \frac{-A}{r^2}\hat{\underline{r}} \rightarrow \underline{E}_+ = \frac{Q}{4\pi\varepsilon_0 a^2}\hat{\underline{r}}$$ r=a here

$$\underline{E}_+ = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{1}{a^2}\hat{\underline{r}}$$

$$A = \frac{-Q}{4\pi\varepsilon_0}$$

$$\Phi = -\frac{A}{r} + B = \frac{Q}{4\pi r\varepsilon_0} + B$$

Capacitance $C = \dfrac{Q}{\Phi(a) - \Phi(b)}$

$$C = \frac{Q}{-A/a + \cancel{B} + A/b - \cancel{B}}$$

→ don't need to know constant B here since it cancels

$$C = \frac{Q}{+\frac{Q}{4\pi a\varepsilon_0} - \frac{Q}{4\pi b\varepsilon_0}}$$

$$C = Q \cdot 4\pi\varepsilon_0 \cdot \frac{1}{\frac{Q}{a} - Q/b} = \frac{4\pi\varepsilon_0}{(1/a - 1/b)}$$

$$1/a - 1/b = \frac{b}{ab} - \frac{a}{ab} \qquad \frac{1}{\left(\frac{b-a}{ab}\right)} = \frac{ab}{b-a}$$

$$C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$

$$W = 1/2\, C\,(\Phi(a) - \Phi(b))^2$$

$$= 1/2\, 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)\left[\frac{Q}{4\pi\varepsilon_0 a} - \frac{Q}{4\pi\varepsilon_0 b}\right]^2$$

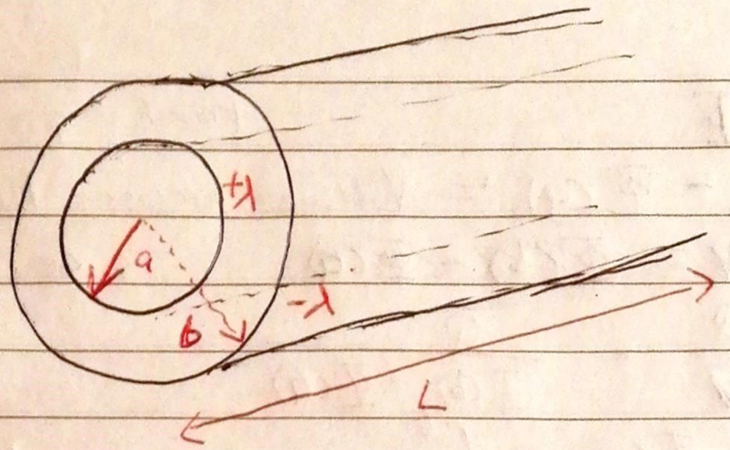
$$= 2\pi\varepsilon_0\left(\frac{ab}{b-a}\right)\left[\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]\right]^2$$

$$= \frac{2\pi\varepsilon_0 Q^2}{16\pi^2\varepsilon_0^2}\left[\frac{b-a}{ab}\right]^2\left(\frac{ab}{b-a}\right)$$

$$W = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{b-a}{ab}\right]\cancel{\left[\frac{b-a}{ab}\right]}\cancel{\left[\frac{ab}{b-a}\right]}$$

$$W = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$ energy stored in the capacitor

2(b)



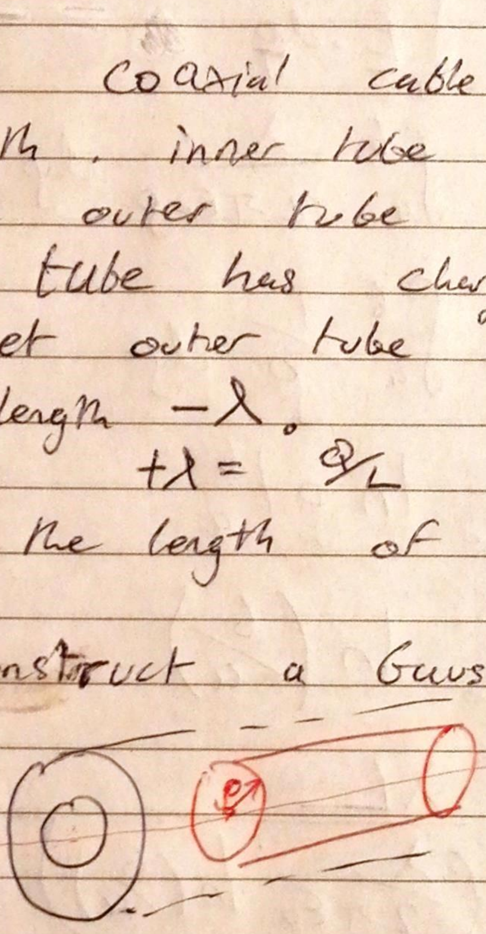
Coaxial cable

given coaxial cable of infinite length. inner tube of radius a and outer tube of radius b.
Let Inner tube has charge per unit length $+\lambda$ and let outer tube have charge per unit length $-\lambda$.

$$+\lambda = Q/L \quad , \quad -\lambda = -Q/L$$

L is the length of the coaxial cable

Construct a Gaussian cylinder of length h & radius ρ



Gauss' Law $\int \underline{E} \cdot d\underline{a} = \frac{q_{encl}}{\varepsilon_0}$

$$\int_{cylinder} E \cdot da = E(\rho) \cdot 2\pi\rho L = \frac{\lambda L}{\varepsilon_0}$$

$$E(\rho) = \frac{\lambda L}{\varepsilon_0 2\pi \rho L}$$

$$E(\rho) = \frac{\lambda}{2\pi\varepsilon_0 \rho}$$

$E = -\nabla \Phi$ Potential goes from $-Q$ to Q

$+V = \Phi(b) - \Phi(a)$ = Voltage accross the capacitor

~~$V = \Phi(b) - \Phi(a)$~~ since $\Phi = 0$ for $\rho > b$

$$C = \frac{Q}{V} = \frac{Q}{\Phi(b) - \Phi(a)}$$

$$\Phi(b) - \Phi(a) = \int_a^b \underline{E} \cdot d\underline{\rho}$$

$$V = \int_a^b E \cdot d\rho = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 \rho}\, d\rho = \Phi(b) - \Phi(a)$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \int_a^b \frac{1}{\rho}\, d\rho$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \rho \Big|_a^b$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{\lambda L}{\left(\frac{\lambda}{2\pi\varepsilon_0} \ln\left(b/a\right)\right)}$$

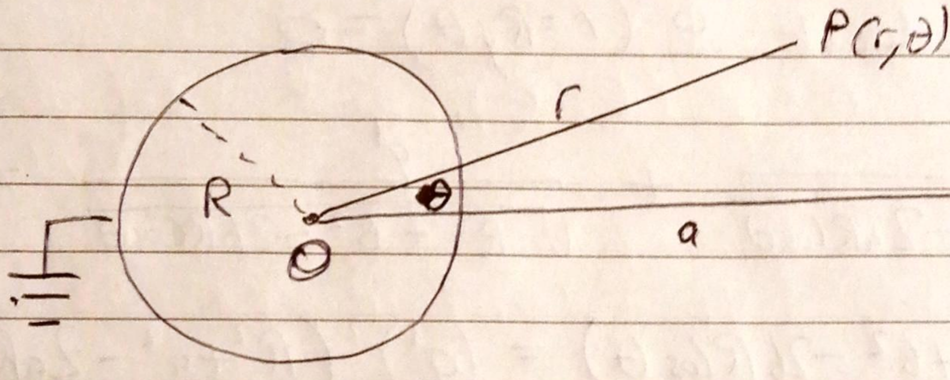
$$C = \frac{\cancel{\lambda} L}{\cancel{\lambda} \ln(b/a)} \cdot 2\pi\varepsilon_0$$

$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

$$C/L = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

= Capacitance per unit length for the coaxial cable.

Q3 (a) A point charge q is placed outside a grounded (zero potential) conducting sphere of radius R. The charge q is at a distance $a>R$ from centre O of sphere. Electrostatic potential at point P with coords (r,θ), $r>R$ is denoted by $\Phi(r,\theta)$



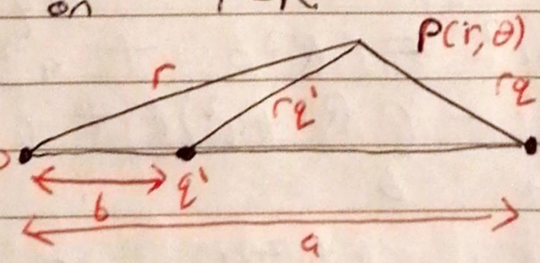
(a) Find $\Phi(r,\theta)$ in $r>R$ by placing an image charge somewhere along Oq and inside the sphere

$\Phi = 0$ on $r=R$

$\nabla^2\Phi = 0$ except at q

remove conducting sphere & place image charge inside where the sphere was so that

$\Phi = 0$ on $r=R$



q' = image charge

call Oq, 'a' and Oq', 'b'

pick q' and b so that $\Phi = 0$ on $r=R$

$$\Phi(r,\theta) = \left\{ \frac{q}{r_q} + \frac{q'}{r_{q'}} \right\} \frac{1}{4\pi\varepsilon_0}$$

cosine rule: $r_q^2 = r^2 + a^2 - 2ar\cos\theta$

$r_{q'}^2 = r^2 + b^2 - 2br\cos\theta$

$$\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2br\cos\theta}} \right\}$$

We want $\Phi(r=R,\theta) = 0$

$$\frac{q}{\sqrt{R^2+a^2-2aR\cos\theta}} + \frac{q'}{\sqrt{R^2+b^2-2bR\cos\theta}} = 0$$

$$q^2(R^2+b^2-2bR\cos\theta) = (q')^2(R^2+a^2-2aR\cos\theta)$$

has to be true for all θ such that $0 \le \theta \le \pi$

~~Set $\theta = \frac{\pi}{2}$~~ ~~$q^2(R^2+b^2) =$~~ $\quad +q^2\,\cancel{2}b\cancel{R} = q'^2\,\cancel{2}a\cancel{R}$

$bq^2 = q'^2 a$ (1)

AND

$q^2(R^2+b^2) = (q')^2(R^2+a^2)$ (2)

(2) ÷ (1) $\quad \dfrac{q^2(R^2+b^2)}{bq^2} = \dfrac{q'^2(R^2+a^2)}{aq'^2}$

$\rightarrow \quad aR^2 + ab^2 = R^2 b + a^2 b$

$ab^2 - (R^2+a^2)b + aR^2 = 0$

quadratic in b

2 solns: $b = a$ or $b = R^2/a$

choose soln inside sphere, $b = R^2/a$

$b = R^2/a < R$

From (1) $\quad q^2 b = (q')^2 a \Rightarrow \dfrac{q^2R^2}{a} = (q')^2 a$

$\Rightarrow (q')^2 = \dfrac{R^2}{a^2}(q)^2$

$\Rightarrow q' = \pm \dfrac{R}{a} q$

Choose $\boxed{q' = -\dfrac{R}{a} q}$

$$\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{\frac{R}{a}q}{\sqrt{R^2+\frac{R^4}{a^2}-\frac{2R^2}{a}r\cos\theta}}\right\}$$

(b) Calculate the force on q

$$q' \bullet \!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\circ\, q \longrightarrow \underline{F}$$

Force on q $\quad \underline{F}_q = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q'q}{(a-b)^2}$

$$q' = -\frac{R}{a}q \qquad b = \frac{R^2}{a}$$

$$\underline{F}_q = \frac{1}{4\pi\varepsilon_0}\frac{-\frac{R}{a}q\cdot q}{\left(a-\frac{R^2}{a}\right)^2}$$

$$\left(a-\frac{R^2}{a}\right)^2 = a^2 - 2\cancel{a}\frac{R^2}{\cancel{a}} + \frac{R^4}{a^2}$$

$$a^2 - 2R^2 + \frac{R^4}{a^2}$$

$$= \frac{a^4 - 2R^2a^2 + R^4}{a^2}$$

$$= \frac{(a^2-R^2)^2}{a^2}$$

$$= \frac{((a-R)(a+R))^2}{a^2}$$

$$= \frac{(a-R)^2(a+R)^2}{a^2} = \left(\frac{(a-R)(a+R)}{a}\right)^2$$

$$\underline{F}_q = \frac{1}{4\pi\varepsilon_0}\cdot\frac{-R}{a}\frac{q^2}{\left(\frac{(a-R)(a+R)}{a}\right)^2}$$

$$= \frac{-q^2}{4\pi\varepsilon_0}\frac{R}{a}\cdot\left(\frac{a}{(a-R)(a+R)}\right)^2$$

$$\underline{F}_q = \frac{-q^2}{4\pi\varepsilon_0} \frac{aR}{(a-R)^2(a+R)^2}$$

(c) electric field in $r>R$ is $\qquad (\underline{E} = -\nabla\Phi)$

$$\underline{E} = -\frac{\partial \Phi}{\partial r}\hat{\underline{r}} - \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{\underline{\theta}}$$

charge density on sphere is:

$$\sigma = \varepsilon_0 (\underline{E} \cdot \hat{\underline{r}})_{r=R^+} = -\varepsilon_0 \left(\frac{\partial \Phi}{\partial r}\right)_{r=R^+}$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{\frac{R}{a}q}{\sqrt{R^2+\frac{R^4}{a^2}-\frac{2R^2}{a}r\cos\theta}}\right\}$$

$$\frac{\partial \Phi}{\partial r} = \frac{1}{4\pi\varepsilon_0}\left[\frac{(2r-2a\cos(\theta))\ q}{(r^2+a^2-2ar\cos\theta)^{3/2}}\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\frac{q'(2r-b\cos\theta)}{(r^2+b^2-2br\cos\theta)^{3/2}}\right]$$

by chain rule & quotient rule

$$\sigma = \frac{1}{4\pi}\left\{q\frac{(R-a\cos\theta)}{(R^2+a^2-2aR\cos\theta)^{3/2}} + \frac{q'(R-b\cos(\theta))}{(R^2+b^2-2bR\cos\theta)^{3/2}}\right\}$$

$$= \frac{q}{4\pi}\left\{\frac{R-a\cos\theta}{(R^2+a^2-2aR\cos\theta)^{3/2}} + \frac{\frac{R}{a}\left(R-\frac{R^2\cos\theta}{a}\right)}{\left(R^2+\frac{R^4}{a^2}-\frac{2R^3}{a}\cos\theta\right)^{3/2}}\right\}$$

$$\sigma = \frac{q}{4\pi}\left[\frac{R-a\cos\theta - \frac{R}{a}\cdot\frac{1}{R^3}a^3\cdot\frac{R}{a}(a-R\cos\theta)}{(R^2+a^2-2aR\cos\theta)^{3/2}}\right]$$

$$\sigma = \frac{q}{4\pi}\left[\frac{(R-\cancel{a\cos\theta}-\frac{a^2}{R}+\cancel{a\cos\theta})}{(R^2+a^2-2aR\cos\theta)^{3/2}}\right]$$

$$\sigma = \frac{q}{4\pi R}\left(\frac{R^2-a^2}{(R^2+a^2-2aR\cos\theta)^{3/2}}\right) < 0$$

is the ~~energy~~ charge density induced on surface of sphere due to q.