① Non-Linear Systems Assignment 1 Dara Corr

Q1 $\frac{dx}{dt} = -2x + ax + \frac{x^3}{3}$ ID = 18483836

$a = 6 + 0.5 = 6.5$

$\frac{dx}{dt} = 4.5x + \frac{x^3}{3}$

equilibrium points $\dot{x} = 0$

$x(4.5 + \frac{x^2}{3}) = 0$

$x = 0$ or $\frac{x^2}{3} + 4.5 = 0$

$x^2 = -13.5 = -\frac{27}{2}$

$x = \pm\sqrt{-\frac{27}{2}}$ → imaginary number

not possible since we are dealing with the real 2D-plane here.

therefore for $a = 6.5$, $x = 0$ is the only equilibrium point.

$\dot{x} = 0 = x(a-2) + \frac{x^3}{3}$

$x\left((a-2) + \frac{x^2}{3}\right) = 0$

$x = 0$ or $(a-2) + \frac{x^2}{3} = 0$

$x^2 = -3(a-2)$

2 equilibrium points → $x = 0$ or $x = \sqrt{-3(a-2)}$

$= \sqrt{6-3a}$ ↖ a must be less than/equal to 2 $(a \leq 2)$

Stabilities: $(a-2)x + \frac{x^3}{3} > 0$

$\left(\frac{x^3}{3}\right) > -(a-2)x$

$\left(\frac{x^3}{3}\right) > (2-a)x$

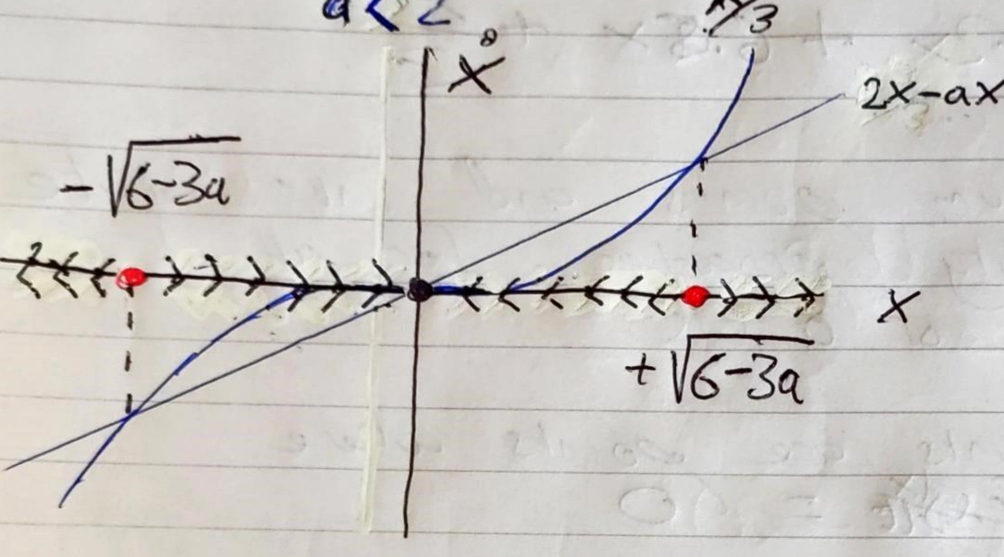
②

$$(a-2)x + \frac{x^3}{3} < 0 \Rightarrow \text{trajectory moves to left}$$

$$\frac{x^3}{3} < -(a-2)x$$

$$\frac{x^3}{3} < (2-a)x$$

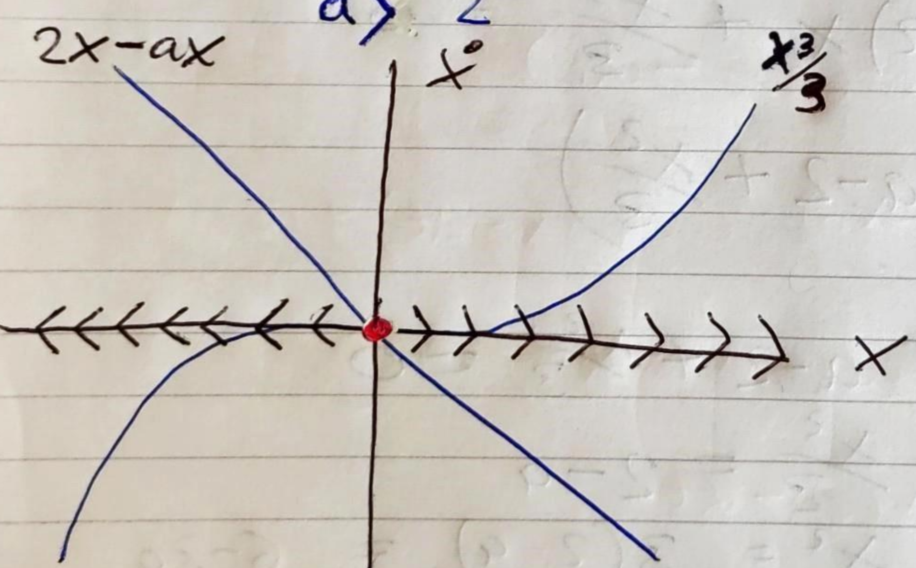
$a < 2$



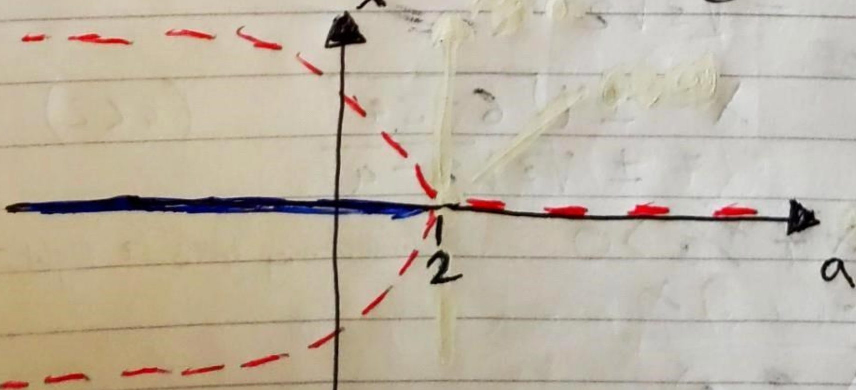
● - unstable

● - stable

$a > 2$



Bifurcation Diagram



Subcritical pitchfork bifurcation point

- - - - unstable

—— stable

Q2. Explicit Solutions

Identify the nature and stability of the equilibrium point at (0,0) for the following system of ODEs

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = -2x + 7y$$

& by analysing the coefficient matrix A, find the general solutions $x(t)$ & $y(t)$

$$\begin{aligned} \dot{x} &= 3x + 2y \\ \dot{y} &= -2x + 7y \end{aligned} \Rightarrow A = \begin{bmatrix} 3 & 2 \\ -2 & 7 \end{bmatrix}$$

$$\tau = tr(A) = 3 + 7 = 10$$

$$\delta = \det(A) = 3 \cdot 7 - (-2) \cdot (2) = 25$$

$$\Delta = \tau^2 - 4\delta = (10)^2 - 4 \cdot (25) = 0$$

e-values:

$$\lambda_{1,2} = \frac{\tau}{2} \pm \frac{\sqrt{\Delta}}{2} \Rightarrow \frac{10}{2} \pm 0 = 5$$

eigenvector $\lambda = 5 \Rightarrow \begin{bmatrix} 3-5 & 2 \\ -2 & 7-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{V_1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

④

find second linearly independent solution

$$(A-\lambda I)^2 \underline{v} = 0 \qquad \text{where} \quad (A-\lambda I)\underline{v} \neq 0$$

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}^2 \underline{V_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \underline{V_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A-\lambda I) V_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A-\lambda I) V_2 = V_1$$ ← V_1 & V_2 need to satisfy this

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} V_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X_1(t) = e^{\lambda_1 t} \underline{V_1}$$

$$X_2(t) = e^{\lambda_1 t}(t\underline{V_1} + \underline{V_2})$$

$$x_1(t) = e^{5t}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$x_2(t) = e^{5t}\left(t\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \underline{x}_1(t) + C_2 \underline{x}_2(t)$$

$$= \left(C_1\begin{bmatrix} 2 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} 2t+0 \\ 2t+1 \end{bmatrix}\right) e^{5t}$$

$$\begin{cases} x(t) = (C_1 \cdot 2 + C_2 \cdot 2t)\, e^{5t} \\ y(t) = (C_1 \cdot 2 + C_2 \cdot (2t+1)) \cdot e^{5t} \end{cases}$$

general solutions of $x(t)$ & $y(t)$

$\tau = 10$
$\delta = 25$
$\Delta = 0$

Nature of Equilibrium Points:

eigenvalues have the same real, positive value

$$\lambda_1 = \lambda_2 = \lambda = 5$$

2 independent eigenvectors → Star node

$\lambda > 0$ → gives repulsive improper node

$x^* = 0$ is <u>unstable</u> node

Q3 Phase Plane Portraits

identify nature & stability of the equilibrium point at (0,0)

$$\frac{dx}{dt} = -2x + 4y$$

$$\frac{dy}{dt} = -3x + 2y$$

$$A = \begin{bmatrix} -2 & 4 \\ -3 & 2 \end{bmatrix}$$

$\tau = tr(A) = 0$
$\delta = det(A) = -4 + 12 = 8$
$\Delta = \tau^2 - 4\delta = -32$

$$\lambda_{1,2} = \tau/2 \pm \frac{\sqrt{\Delta}}{2}$$

$$= \pm \frac{\sqrt{32}}{2} = \pm 16i$$

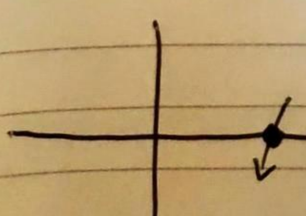
$\tau^2 - 4\Delta = -128 < 0$

$\lambda = \pm 16i$

→ 2 imaginary eigenvalues
→ equilibrium point is a stable centre.

neutrally stable → neither attracting nor repelling

find trajectories:
let $x = 2$, $y = 0$

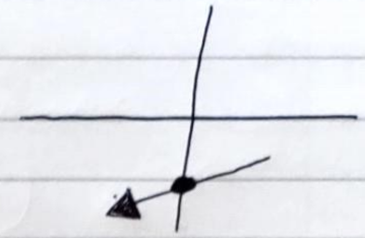


$\dot{x} = -4$, $\dot{y} = -3(2) = -6$

Let $y=-2$, $x=0$

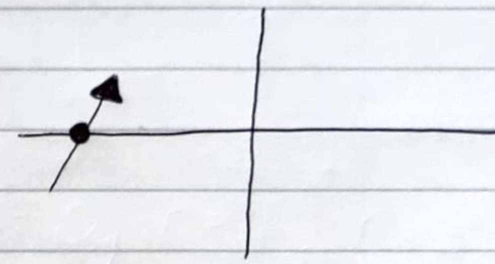
$\dot{x}=-8$

$\dot{y}=-4$



Let $x=-2$, $y=0$

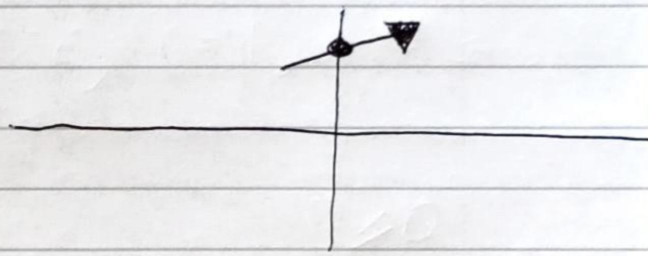
$\dot{x}=4$, $\dot{y}=6$



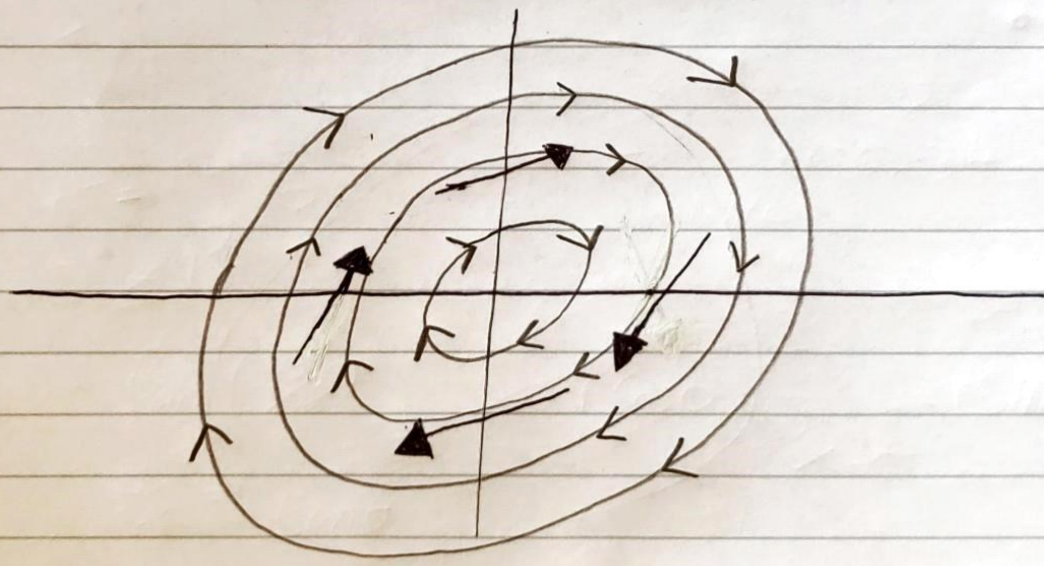
Let $x=0$, $y=2$

$\dot{x}=8$

$\dot{y}=4$



Phase Plane Portrait



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Q4

$a \in \mathbb{R}$

$$\frac{dx}{dt} = (2-a)x + y$$
$$\frac{dy}{dt} = -x + y$$

$$A = \begin{bmatrix} 2-a & 1 \\ -1 & 1 \end{bmatrix}$$

$\tau = tr(A) = 2-a+1 = 3-a$
$\delta = det(A) = 2-a+1 = 3-a$
$\Delta = \tau^2 - 4\delta = (3-a)^2 - 4 \cdot (3-a)$
$= 9-6a+a^2 - 12 + 4a$
$= a^2 - 2a - 3$
$= (a-3)(a+1)$

$\tau = 0$ for $a=3$
$\delta = 0$ for $a=3$
$\Delta = 0$ for $a=3$ or $a=-1$

Critical points → if $\delta = 0$ or $\tau = 0$, $a=3$
if $\Delta = 0$, $a=3$ or $a=-1$

$a < 3 \rightarrow \tau > 0 \quad \delta > 0$
$a = 3 \rightarrow \tau = 0 \quad \delta = 0$
$a > 3 \rightarrow \tau < 0 \quad \delta < 0$

$a < -1 \rightarrow \Delta > 0$
$a = -1 \rightarrow \Delta = 0$
$-1 < a < 3 \rightarrow \Delta < 0$
$a = 3 \rightarrow \Delta = 0$
$a > 3 \rightarrow \Delta > 0$

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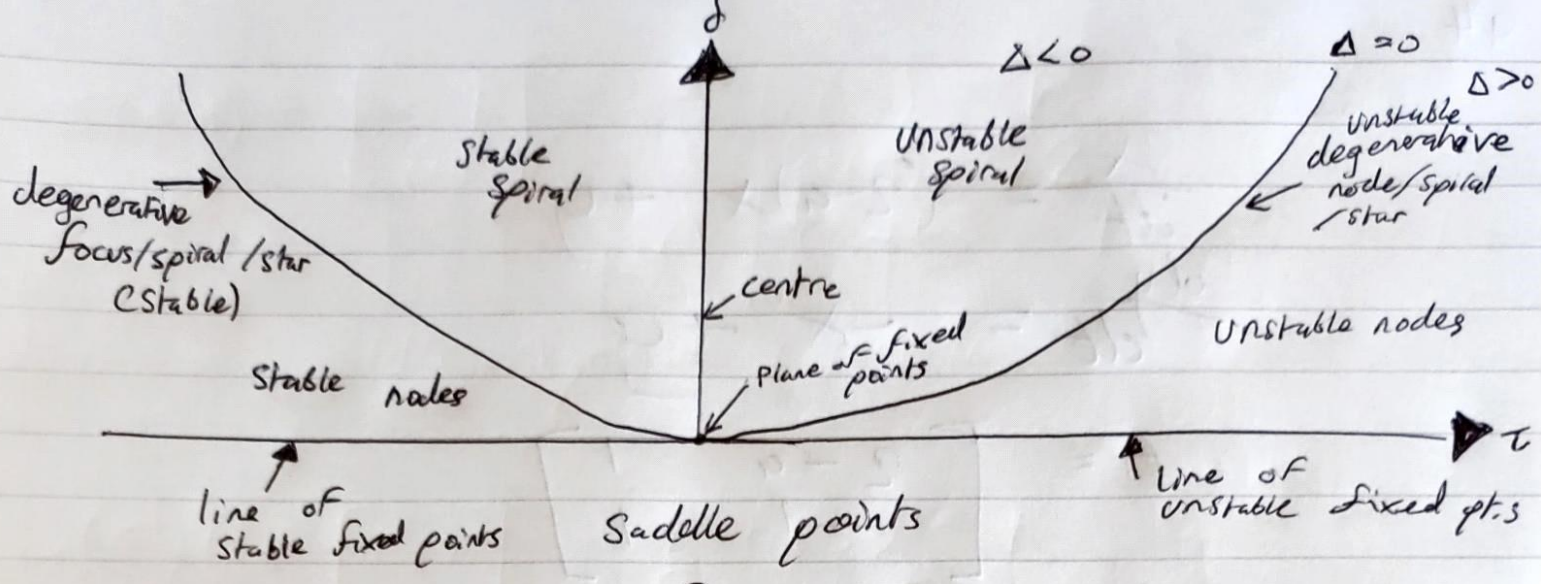
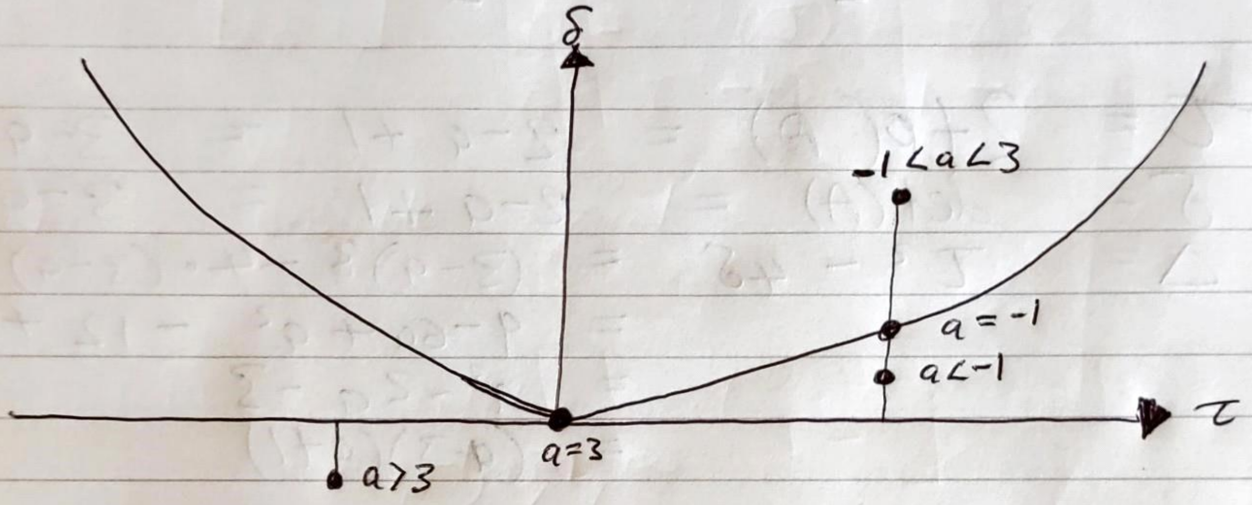




Nature of Equilibrium points

a < -1 : repulsive node (unstable)
a = -1 : repulsive degenerative node (unstable)
-1 < a < 3 : repulsive spiral (unstable)
a = 3 : plane of fixed points
a > 3 : saddle point

For particular case a = 6.5, a > 3
the equilibrium point is a saddle point

Q5 Consider system of nonlinear ODEs

$$\frac{dx}{dt} = y-1$$
$$\frac{dy}{dt} = x^2+y-2$$

(a) locate and classify all equilibrium points in the associated phase plane

$$\dot{x} = y-1 = X(x,y)$$
$$\dot{y} = x^2+y-2 = Y(x,y)$$

Find equilibria: $\dot{x}=0$ and $\dot{y}=0$

$\dot{x}=0 \Rightarrow y-1=0$ so $\boxed{y=1}$

$\dot{y}=0 \Rightarrow x^2+y-2=0$
sub in $y=1$

$$x^2+1-2=0 \rightarrow x^2-1=0$$
$$\rightarrow x^2=1$$
$$\boxed{x=\pm 1}$$

equilibrium points are: $(1,1)$ $(-1,1)$

Calculate Jacobian $J(x,y) = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} \end{bmatrix}$

$\frac{\partial X}{\partial x} = 0, \quad \frac{\partial X}{\partial y} = 1$

$\frac{\partial Y}{\partial x} = 2x \quad \frac{\partial Y}{\partial y} = 1 \qquad J(x,y) = \begin{bmatrix} 0 & 1 \\ 2x & 1 \end{bmatrix}$

(11)

$$J(1,1) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= tr(J) = 1 > 0 \\ \delta &= det(J) = -2 < 0 \\ \Delta &= \tau^2 - 4\delta = 1 - 4(-2) = 9 > 0 \end{aligned}$$

(1,1) is a saddle point

$$J(-1,1) = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \qquad \begin{aligned} \tau &= tr(J) = 1 > 0 \\ \delta &= det(J) = +2 > 0 \\ \Delta &= \tau^2 - 4\delta = 1 - 8 = -7 < 0 \end{aligned}$$

(-1,1) is unstable spiral

(b) Identify & sketch isoclines corresponding to $k = -1, 0, 1, \infty$

where $\dfrac{dy}{dx} = k = \dfrac{Y(x,y)}{X(x,y)} = \dfrac{dy/\cancel{dt}}{dx/\cancel{dt}} = \dfrac{\dot{y}}{\dot{x}}$

Direction of linearised field near (1,1):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= y &= X(x,y) \\ \dot{y} &= 2x + y &= Y(x,y) \end{aligned}$$

$\leftarrow$ or $\rightarrow$ $\boxed{k=0} \rightarrow \dot{y} = 0 \rightarrow 2x + y = 0$

$y = -2x$

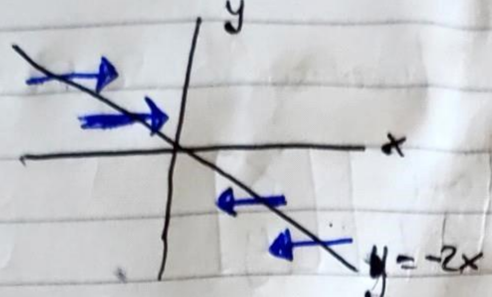
for $k=0$

$(\dot{x}, \dot{y})|_{(1,-2)} = (-2, 0) \leftarrow$

$(\dot{x}, \dot{y})|_{(-1,2)} = (2, 0) \rightarrow$

(12)

$k=0$

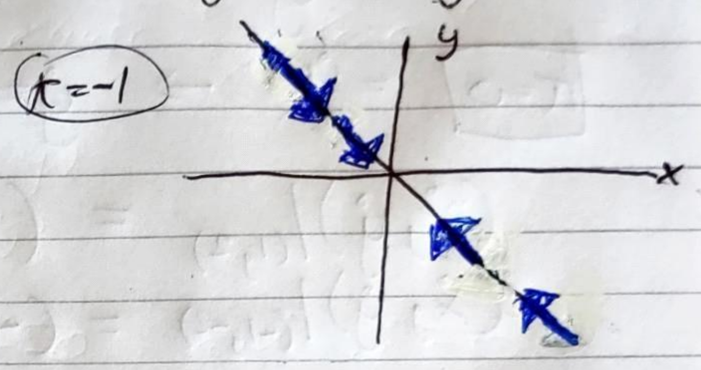


$\boxed{k=-1} = \dot{y}/\dot{x}$ $\quad \dot{y} = -\dot{x}$

$2x+y = -y \rightarrow 2y=-2x \rightarrow y=-x$

$(\dot{x},\dot{y})|_{(-1,1)} = (+1,-1)$ ↘

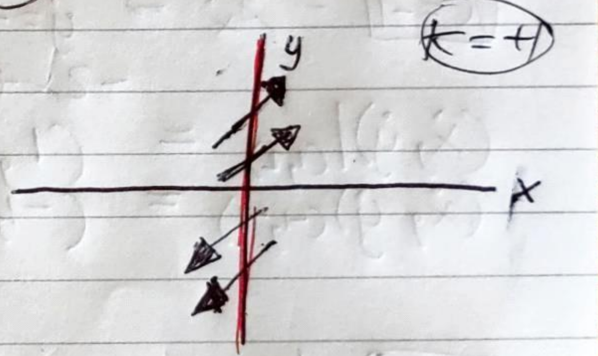
$(\dot{x},\dot{y})|_{(1,-1)} = (-1,+1)$ ↖



$\boxed{k=+1} = \dot{y}/\dot{x}$ $\quad \dot{y}=\dot{x} \rightarrow y = 2x+y \rightarrow x=0$

$(\dot{x},\dot{y})|_{(0,1)} = (1,1)$ ↗

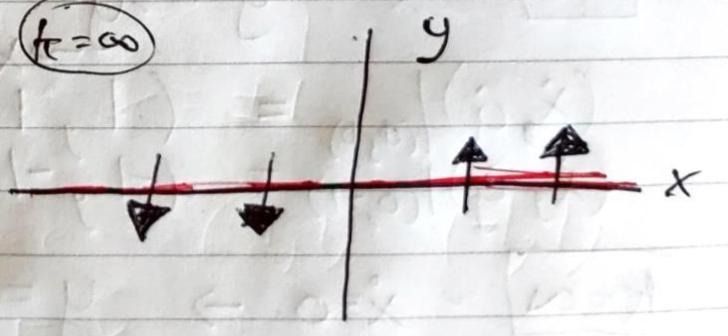
$(\dot{x},\dot{y})|_{(0,-1)} = (-1,-1)$ ↙



$\boxed{k=\infty}$ $\quad \dot{x}=0 \rightarrow y=0$

$(\dot{x},\dot{y})|_{(1,0)} = (0,2)$ ↑

$(\dot{x},\dot{y})|_{(-1,0)} = (0,-2)$ ↓



(13)

Direction field of linearised system near (-1, 1)

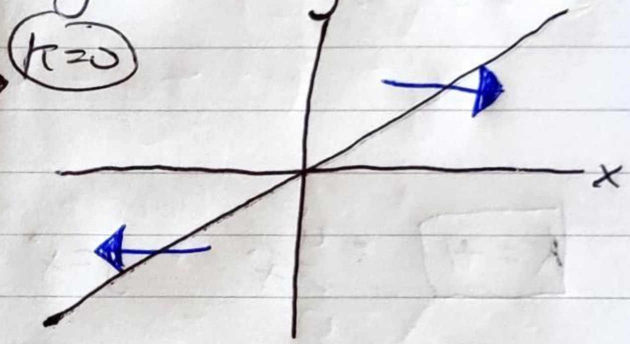
$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$$\dot x = y$$
$$\dot y = -2x + y$$

$\boxed{k=0} = \dot y/\dot x \to \dot y = 0 \to y = 2x$

$(\dot x, \dot y)|_{(1,2)} = (2, 0) \rightarrow$

$(\dot x, \dot y)|_{(-1,-2)} = (-2, 0) \leftarrow$

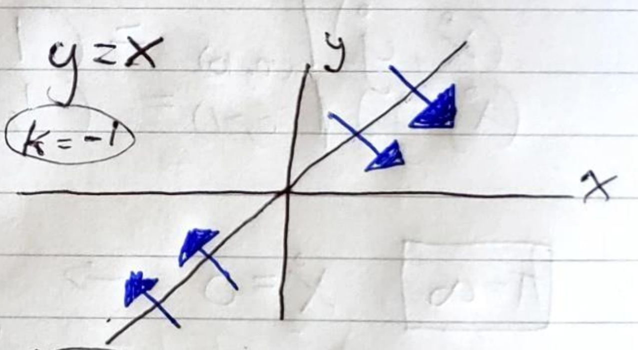


$\boxed{k=-1} = \dot y/\dot x \to \dot y = -\dot x$

$-2x + y = -y \to y = x$

$(\dot x, \dot y)|_{(1,1)} = (1, -1) \searrow$

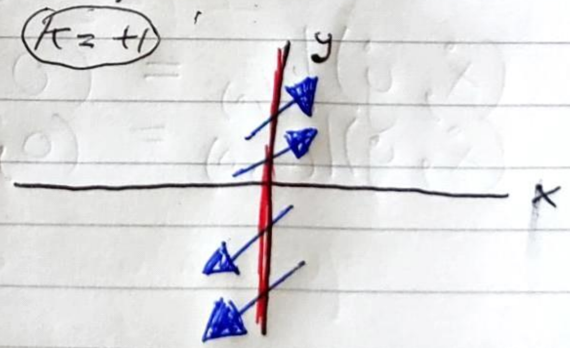
$(\dot x, \dot y)|_{(-1,-1)} = (-1, 1) \nwarrow$



$\boxed{k=+1} = \dot y/\dot x \to \dot y = \dot x \to y = -2x + y \to x = 0$

$(\dot x, \dot y)|_{(0,1)} = (1, 1) \nearrow$

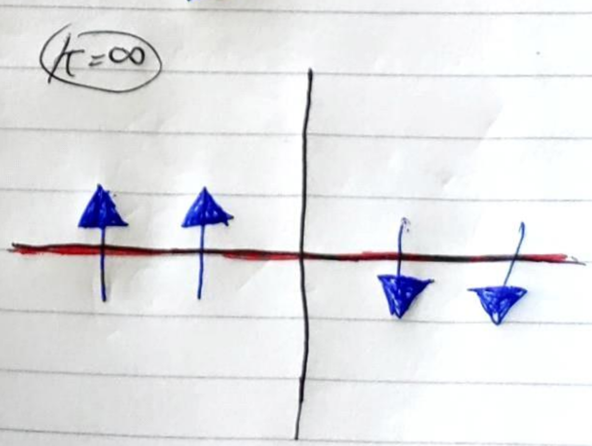
$(\dot x, \dot y)|_{(0,-1)} = (-1, -1) \swarrow$



$\boxed{k=\infty} \to \dot x = 0 \to y = 0$

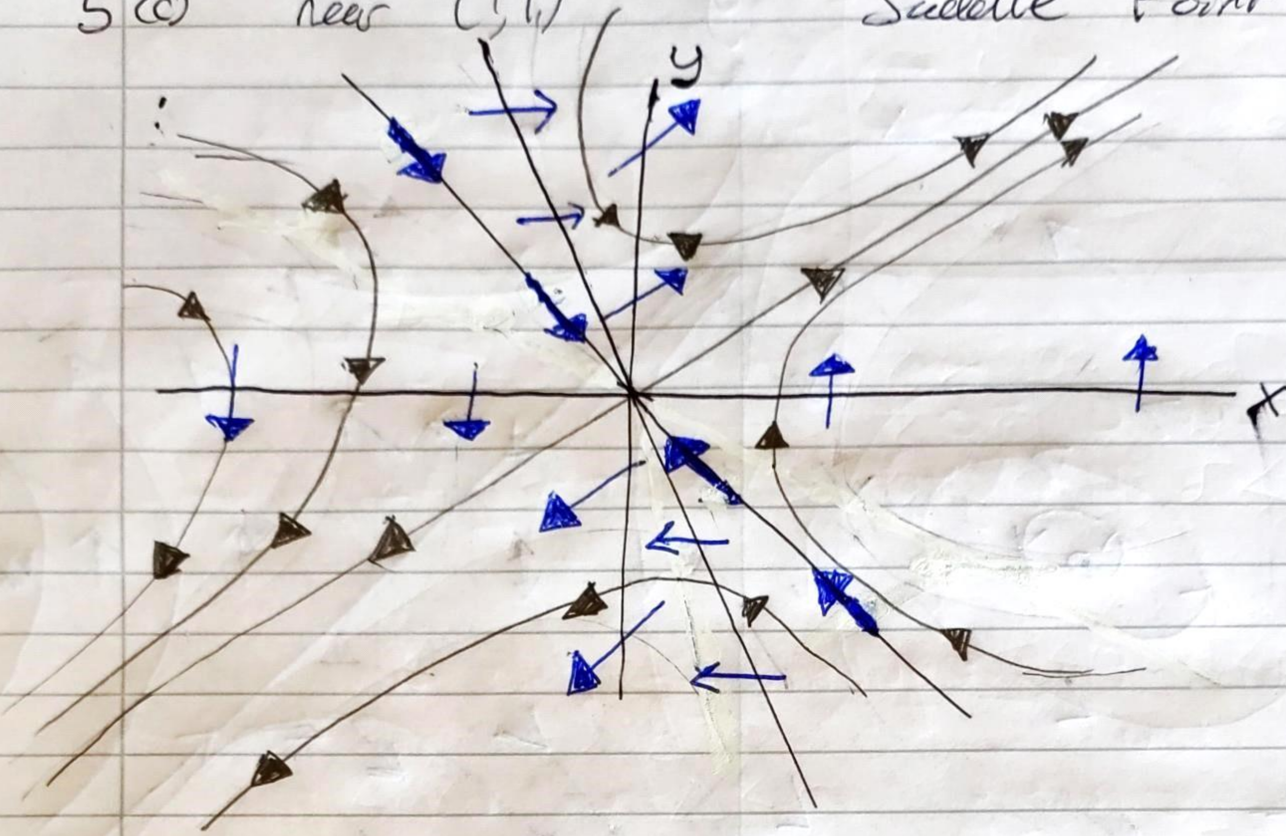
$(\dot x, \dot y)|_{(1,0)} = (0, -2) \downarrow$

$(\dot x, \dot y)|_{(-1,0)} = (0, 2) \uparrow$



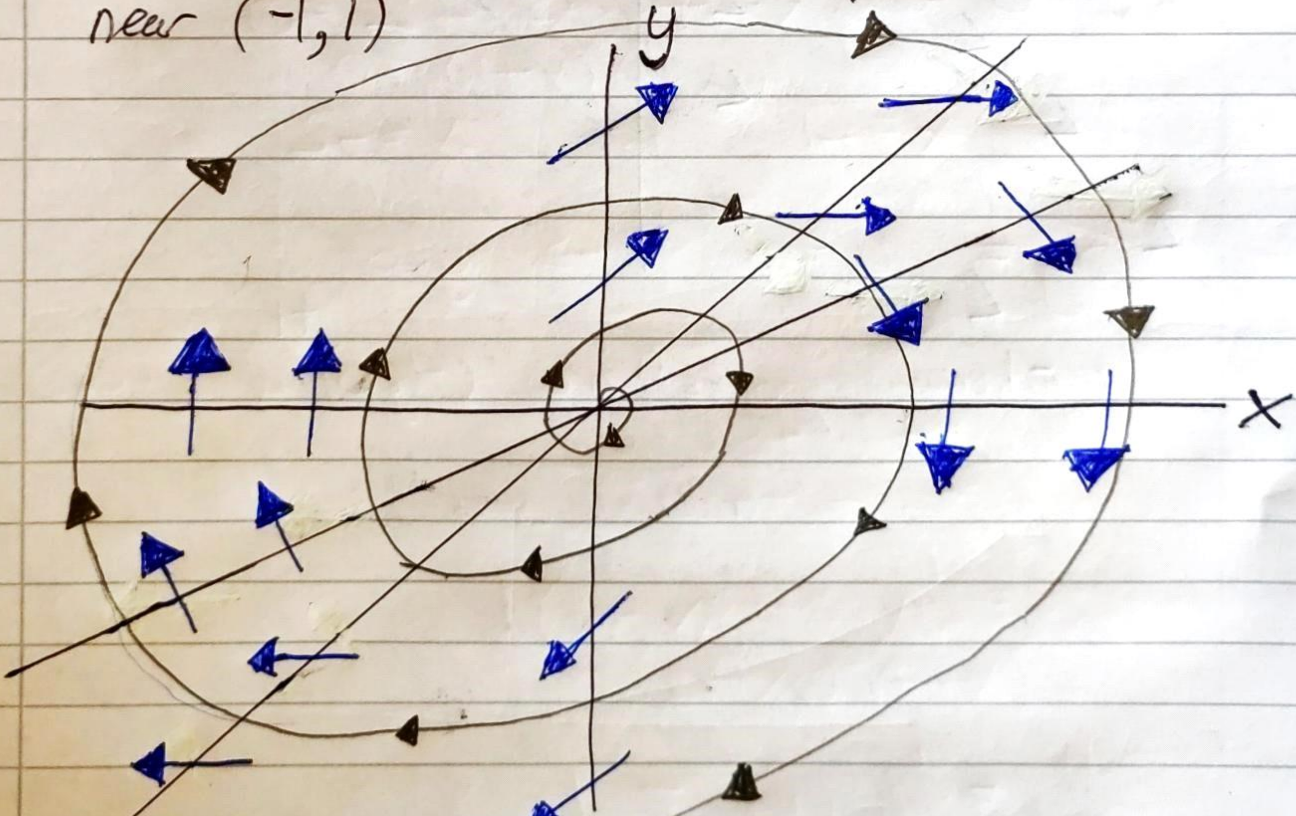
(14)

5 (c) near (1,1) Saddle Point



Unstable spiral

near (-1,1)



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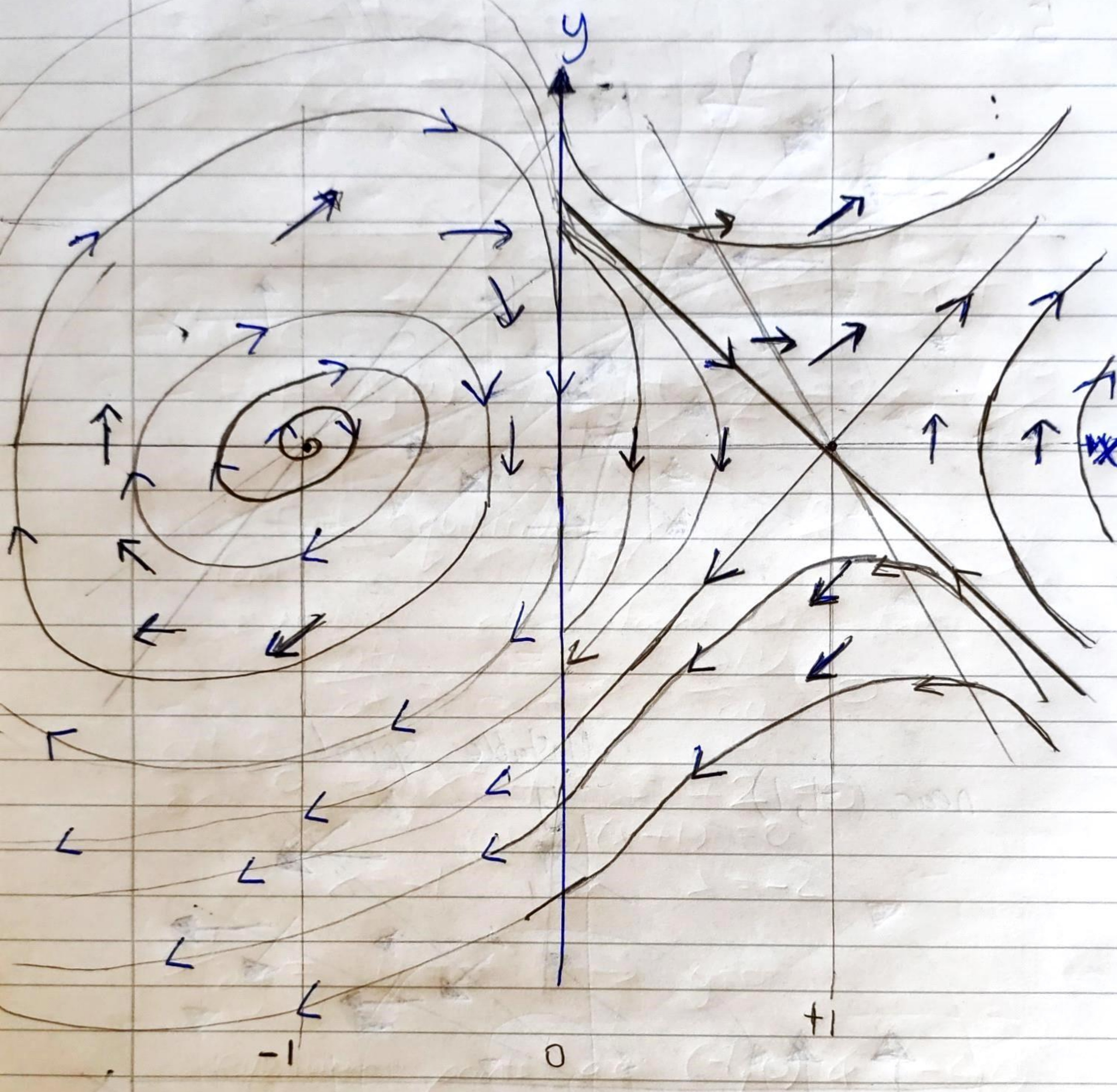
Phase Portrait



Q6 $\frac{dx}{dt} = y \quad = X(x,y)$

$$\frac{dy}{dt} = -x^2 - 3x - 2 \quad = Y(x,y)$$

(a) Show system is Hamiltonian & find the Hamiltonian Function

System is Hamiltonian if

$$\frac{\partial X}{\partial x} = 0 \quad \text{and} \quad \frac{\partial Y}{\partial y} = 0$$

and $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$

$$\partial/\partial x\, X = \partial/\partial x\, y = 0 \quad \checkmark$$

$$\partial/\partial y\, Y = \partial/\partial y\, (-x^2 - 3x - 2) = 0 \quad \checkmark$$

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0 \quad \checkmark$$

$\rightarrow$ System is Hamiltonian

Find Hamiltonian Function

$$\frac{\partial x}{\partial t} = \frac{\partial H}{\partial y} \quad , \quad \frac{\partial y}{\partial t} = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial x} = -\frac{\partial y}{\partial t}$$

$$\cdot\ \frac{\partial H}{\partial y} = y \qquad \cdot\ \frac{\partial H}{\partial x} = x^2 + 3x + 2$$

$$H = \int y\, dy$$

$$H = y^2/2 + \cancel{f}(x)$$

$$\frac{\partial H}{\partial x} = f'(x)$$

(17)

$$\frac{\partial H}{\partial x} = x^2+3x+2$$

$$\frac{\partial H}{\partial x} = f'(x) \rightarrow \quad f(x) = \int \frac{\partial H}{\partial x}\,dx$$

$$f(x) = \int x^2+3x+2\,dx$$

$$f(x) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c$$

Let $c = 0$

$$H(x,y) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + \frac{y^2}{2}$$

b) Find equilibrium points & classify them

Equilibria are found by taking

$$\frac{\partial H}{\partial x} = 0 \quad \& \quad \frac{\partial H}{\partial y} = 0$$

$$\frac{\partial H}{\partial x} = 0 \rightarrow x^2 + 3x + 2 = 0$$

$$(x+2)(x+1) = 0$$

$$x = -1 \quad \text{or} \quad x = -2$$

$$\partial H/\partial y = y = 0 \quad \rightarrow \quad y = 0$$

equilibrium points are $(-1, 0)$ & $(-2, 0)$

Classify equilibrium points

$$\frac{\partial^2 H}{\partial x^2} = d/dx\,(x^2+3x+2)$$

$$= 2x+3$$

$$\frac{\partial^2 H}{\partial y^2} = d/dy\,(y) = 1$$

$$\frac{\partial^2 H}{\partial x \partial y} = 0$$

Stationary Points test:

$$\delta = \left(\frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2}\right) - \frac{\partial^2 H}{\partial x \partial y}$$

$$\delta = ((2x+3) \cdot 1) - 0 \quad = \quad 2x+3$$

for $(-1, 0) \quad \rightarrow \delta = 3-2 = 1 > 0$
$\rightarrow$ centre

for $(-2, 0) \quad \rightarrow \delta = 3-4 = -1 < 0$
$\rightarrow$ saddle point

$$\left.\frac{\partial^2 H}{\partial x^2}\right|_{(-1,0)} = 2(-1)+3 = 1 > 0 \Rightarrow \text{centre (local min)}$$

$$H(-1, 0) = \frac{(-1)^3}{3} + \frac{3(-1)^2}{2} + 2(-1)$$

$$= -\frac{1}{3} + \frac{3}{2} - 2 = -\frac{5}{6}$$

$$H(-2, 0) = \frac{(-2)^3}{3} + \frac{3(-2)^2}{2} + 2(-2)$$

$$= \frac{-8}{3} + \frac{12}{2} - 4 = -\frac{2}{3}$$

(c) Want to draw phase portrait by drawing level curves for $H(x, y) = c$

$C = -5/6$ at centre
$C = -2/3$ at saddle point

(19)

ranges:

$C < -5/6$	H less than centre
$C = -5/6$	H is same as centre
$-5/6 < C < -2/3$	H between saddle pt. & centre
$C = -2/3$	H is same as saddle point
$C > -2/3$	H is greater than saddle pt.

note for $x \to \infty$, $H \to \infty$ for fixed y

& $x \to -\infty$, $H \to -\infty$ for fixed y

$\boxed{C = -5/6}$ $\quad \frac{x^3}{3} + \frac{3x^2}{2} + 2x + \frac{y^2}{2} = \frac{-5}{6}$

curve cuts x-axis at $y=0$

$$\frac{x^3}{3} + \frac{3x^2}{2} + 2x = \frac{-5}{6}$$

$\times 6$ $\quad \frac{x^3}{3} + \frac{3x^2}{2} + 2x + \frac{5}{6} = 0$ $\qquad$ we know $x=-1$ is a factor

$\hookrightarrow \quad 2x^3 + 9x^2 + 12x + 5 = 0$

$$\begin{array}{r} 2x^2 + 7x + 5 \\ (x+1) \overline{) \, \cancel{2}x^3 + 9x^2 + 12x + 5} \\ -2\cancel{x}^3 - 2x^2 + 12x \\ \hline \cancel{7}x^2 + 12x \\ -\cancel{7}x^2 - 7x \\ \hline \cancel{5}x + \cancel{5} \\ -5\cancel{x} - \cancel{1} \\ \hline 0 \end{array}$$

$(x+1)(2x^2 + 7x + 5) = 0$

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(5)}}{2(2)} \rightarrow x = \frac{-7 \pm \sqrt{9}}{4} = \frac{-7 \pm 3}{4}$$

$x = -1$ or $x = -5/2$

$= (x+1)^2(x + 5/2)$

curve cuts y-axis at $x = 0$, $\frac{y^2}{2} = -5/6$

$y = \pm\sqrt{-5/3} \notin \mathbb{R}$

$\rightarrow$ doesn't cross y-axis.

(20)

Calculate points near $x = -1$ and $x = -\frac{5}{2}$ $(x = -2.5)$

$x = -3 \quad y = \pm\sqrt{-\frac{2x^3}{3} - 3x^2 - 4x - \frac{5}{3}} = \pm\frac{2\sqrt{3}}{3} = \pm 1.155$

$x = -2 \quad y =$ complex

$x = -2.6 \quad y = \pm 0.413$

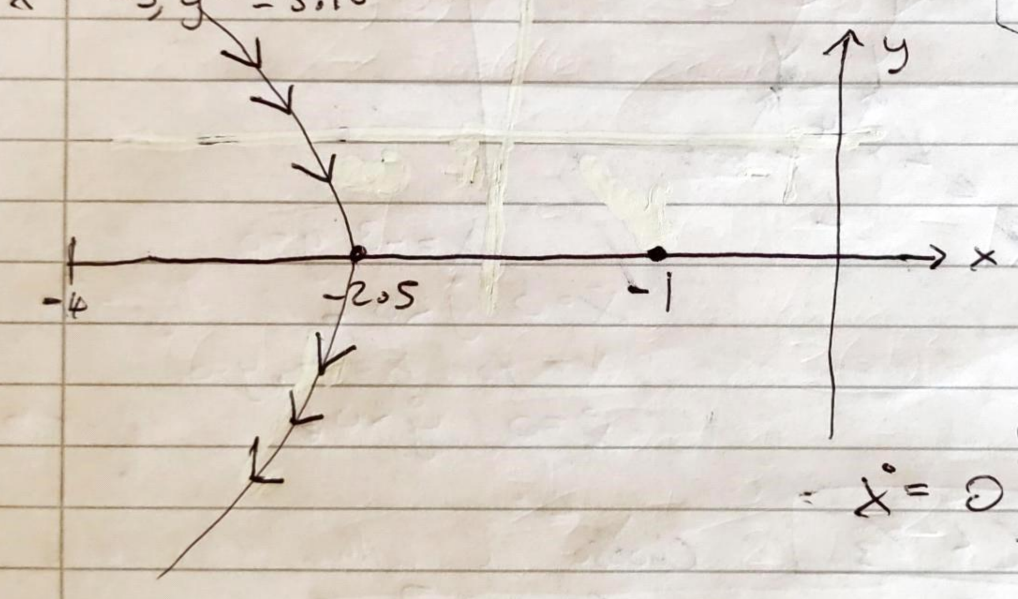
$x = -2.9 \quad y = \pm 0.984$

$x = -3.6 \quad y = \pm 2.227$

$x = -4 \quad y = \pm 3$

$x = -5, \ y = \pm 5.16$

above $x = -2.5$, y has no real values



$x = -2.5$
$y = 0$
$\dot{x} = 0, \ \dot{y} = -\frac{3}{4}$

$\boxed{C < -\frac{5}{6}}$

$$\frac{x^3}{3} + \frac{3x^2}{2} + 2x + \frac{y^2}{2} < -\frac{5}{6}$$

y has no real values above $x = -2.5$

choose points for $x < -2.5$ & calculate H

$\underline{x = -3}, \ y = 0, \ H = \frac{(-3)^3}{3} + \frac{3(-3)^2}{2} + 2(-3) + 0$

$H = -1.5 < -\frac{5}{6}$

Find curves $x = -3$ $\quad y = \sqrt{-\frac{2x^3}{3} - 3x^2 - 4x - 1.5}$

$y = \pm 1.225$

$x = -4 \quad y = \pm 3.03 \qquad x = -5, \ y = \pm 5.18$

(21)

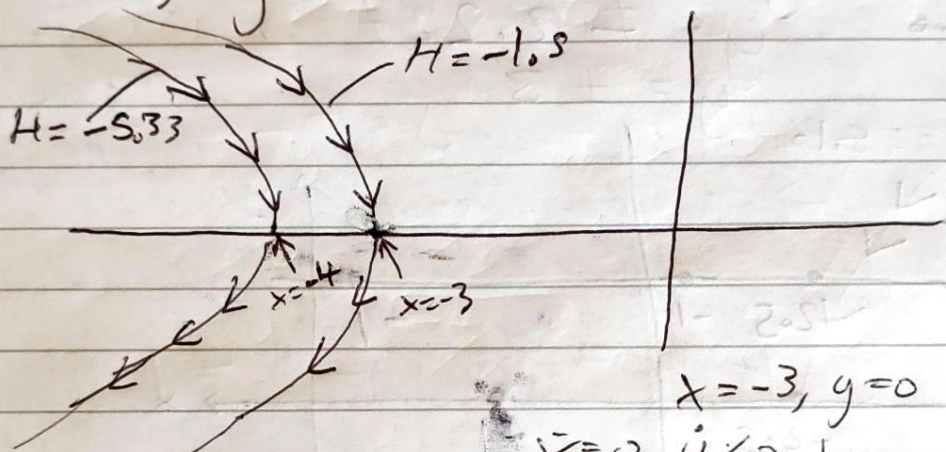
$\underline{x=-4}$ $y=0$ $H=\frac{(-4)^3}{3}+\frac{3(-4)^2}{2}+2(-4)$

$H=-5.33$

$x=-5,\quad y=\sqrt{\frac{2(-5)^3}{3}-3(-5)^2-4(-5)-5.33}$

$y=\pm 4.80$

$x=-6\quad ,\ y=\pm 7.39$



$x=-3,\ y=0$

$\dot{x}=0\quad \dot{y}<0 \downarrow \qquad \dot{y}=-2$

for $x=-4,\ y=0$

$\dot{x}=0\quad \dot{y}<0 \downarrow \qquad \dot{y}=-6$

$\boxed{-5/6 < C < -2/3}$ Choose C within this range

Choose $C=-0.7$.

$H=\frac{x^3}{3}+\frac{3x^2}{2}+2x+\frac{y^2}{2}$

$y=\pm\sqrt{\frac{-2x^3}{3}-3x^2-4x+2H}$

$H=C=-0.7 \rightarrow y=\pm\sqrt{\frac{-2x^3}{3}-3x^2-4x-1.4}$

Curve cuts x-axis when $y=0$

$0=\sqrt{\frac{-2x^3}{3}-3x^2-4x-1.4}$

$+\frac{2x^3}{3}+3x^2+4x+1.4=0$

(22)

$H = c = -0.7$

$y=0$ where it cuts x-axis, $0 = \frac{x^3}{3} + \frac{3x^2}{2} + 2x - H$

$$\frac{x^3}{3} + \frac{3x^2}{2} + 2x + 0.7 = 0$$

$\rightarrow$ wolfram alpha $\rightarrow$ Solns are
$x \simeq -2.24$
$x \simeq -1.71$
$x \simeq -0.547$

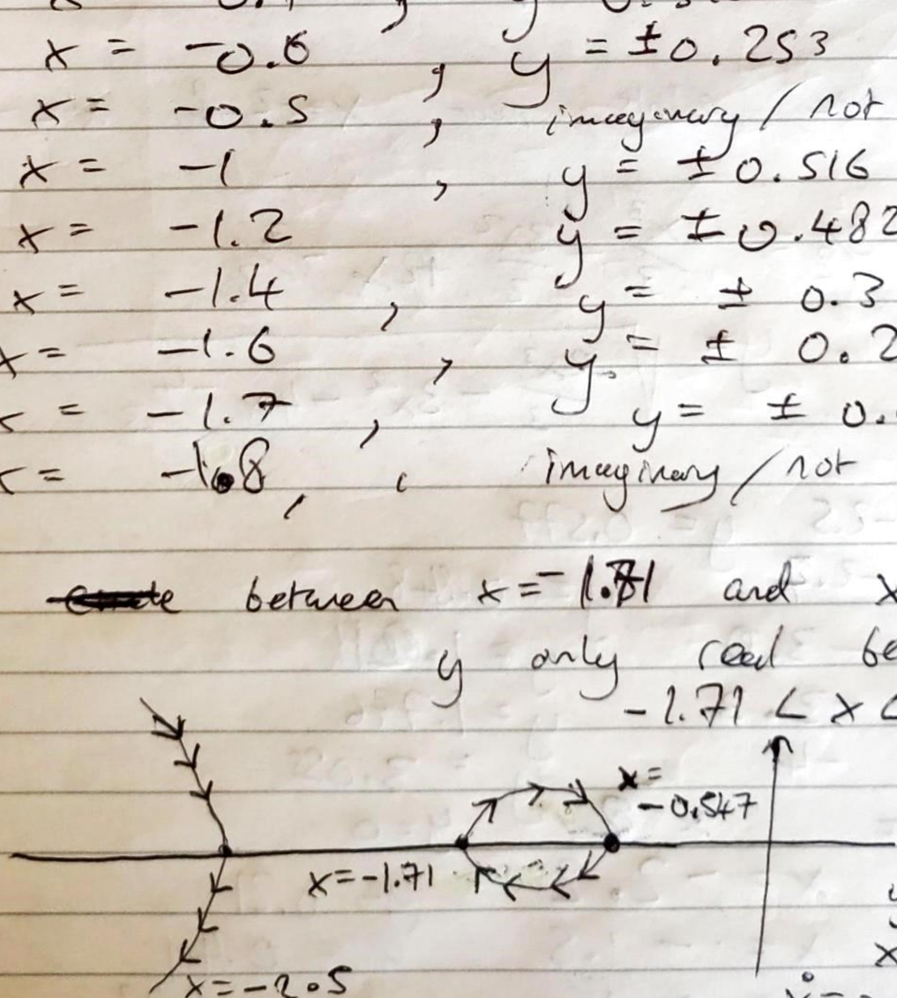
Find values of y between $x=-2.24$ & $x=-0.547$

$$y = \sqrt{-\frac{2x^3}{3} - 3x^2 - 4x + 2H}$$

$2H = -1.4$ here.

$x = -2.1$ imaginary / not real
$x = -2$ imaginary / not real
$x = -0.7$, $y = \pm 0.398$
$x = -0.6$, $y = \pm 0.253$
$x = -0.5$, imaginary / not real
$x = -1$, $y = \pm 0.516$
$x = -1.2$ $y = \pm 0.482$
$x = -1.4$, $y = \pm 0.386$
$x = -1.6$, $y = \pm 0.225$
$x = -1.7$, $y = \pm 0.073$
$x = -1.8$, imaginary / not real

~~curve~~ between $x = -1.71$ and $x = -0.547$
y only real between
$-1.71 < x < -0.547$



$\dot{x} = 0$ $\dot{y} = -0.65 < 0$

$x = -1.71$ $\dot{x} = 0$ $\dot{y} = 0.206 > 0$
$y = 0$

(23)

$\boxed{C=-2/3}$ $\frac{x^3}{3}+\frac{3x^2}{2}+2x+\frac{y^2}{2}=-\frac{2}{3}$

cuts x-axis at $y=0$

$\frac{x^3}{3}+\frac{3x^2}{2}+2x+\frac{2}{3}=0$

$(x+2)^2$ is a factor since $(-2,0)$ is a saddle point

$(x+2)^2(x+1/2)=0$ $x=-1/2$

curve cuts y-axis at $x=0$

$\frac{y^2}{2}=-\frac{2}{3}$

$y=\sqrt{-4/3}$ not real $\rightarrow$ doesn't cut y axis.

Calculate points near $x=-2$ & $x=-1/2$

$\frac{y^2}{2}=-\frac{x^3}{3}-\frac{3x^2}{2}-2x-\frac{2}{3}$

$y=\sqrt{-\frac{2x^3}{3}-3x^2-4x-4/3}$

$x=-2.5$ $y=0.577$

$x=-2.7$ $y=0.848$

$x=-3.0$ $y=1.24$

$x=-3.3$ $y=1.776$

$x=-4$ $y=3.055$

$x=-2$ $y=0$

$x=-2.2$ $y=0.21$

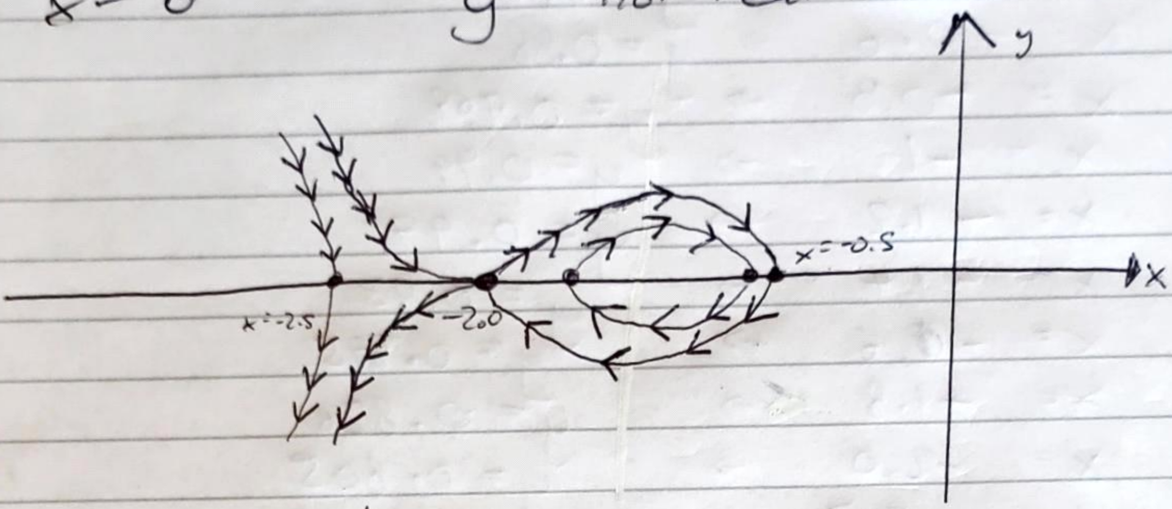
$x=-1.5$ $y=0.41$

$x=-1$ $y=0.577$

$x = -0.6 \quad y = 0.36$

$x = -0.5 \quad y = 0$

$x = 0 \quad y =$ not real For $x > -0.5$



$x = -2.1 \quad y = \pm 0.103$

$\dot{y} = -0.11 \quad \dot{x} = \pm 0.103$ ↘ or ↙

$x = -1.9 \quad y = \pm 0.0966$

$\dot{y} = 0.09 \uparrow \quad \dot{x} = \pm 0.0966$ ↗ or ↖

$\boxed{C > -\frac{2}{3}}$

find a point with $H > -\frac{2}{3}$

take $x = -0.2, y = 0 \longrightarrow H = 0.343 > -\frac{2}{3}$

$$\frac{x^3}{3} + \frac{3x^2}{2} + 2x + \frac{y^2}{2} = -0.343$$

$$y = \sqrt{-\frac{2x^3}{3} - 3x^2 - 4x - 0.685}$$

curve cuts x-axis when $y = 0$

$$\frac{x^3}{3} + \frac{3x^2}{2} + 2x + 0.343 = 0$$

Soln for $x = -0.20023$

cuts y axis when $x = 0 \rightarrow y = \sqrt{-0.685} \rightarrow$ not real

↓

doesn't cut y-axis

25

Calculate points $\boxed{\pm}$

$x = -0.20023 \rightarrow y = 0$

$x = -0.4 \rightarrow y = 0.6911$

$x = -0.6 \rightarrow y = 0.88$

$x = -0.8 \rightarrow y = 0.968$

$x = -1 \rightarrow y = 0.99$

$x = -1.2 \rightarrow y = 0.97$

$x = -1.4 \rightarrow y = 0.93$

$x = -1.6 \rightarrow y = 0.875$

$x = -1.8 \rightarrow y = 0.826$

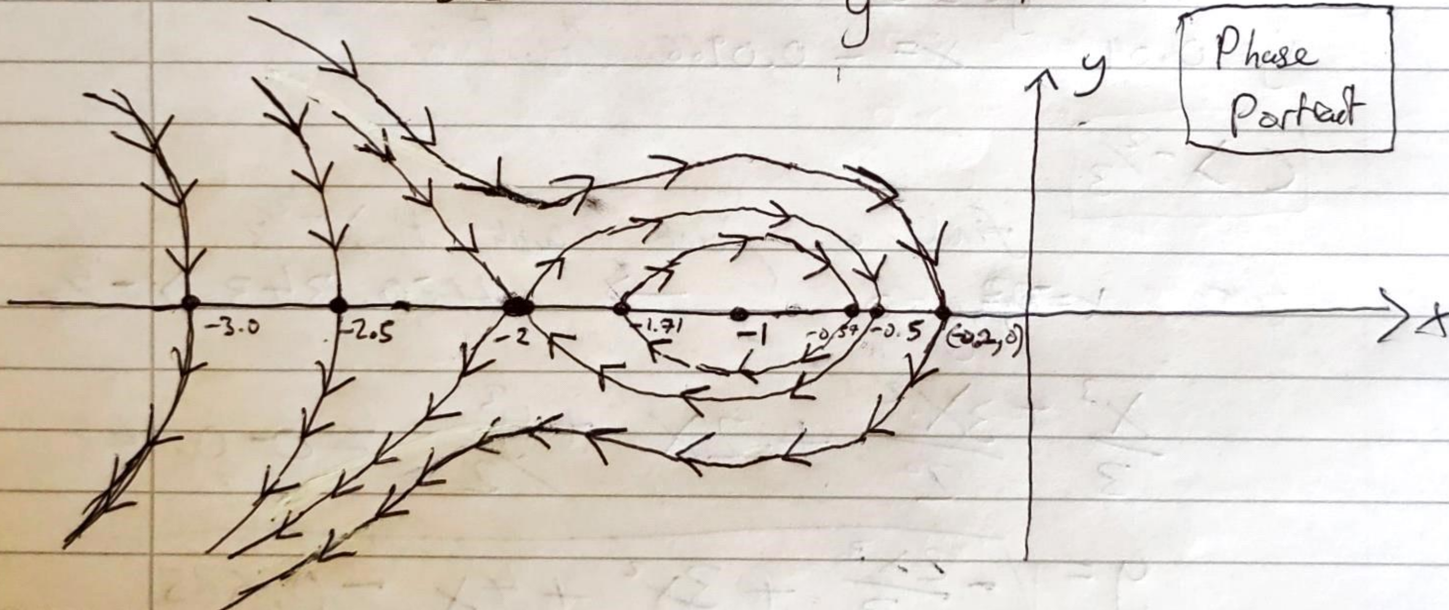
$x = -2.0 \rightarrow y = 0.805$

$x = -2.3 \quad y = 0.87$

$x = -2.7 \quad y = 1.17$

$x = -3 \quad y = 1.52$

$x = -3.5 \quad y = 2.27$



$\boxed{\text{Phase Portrait}}$

$x = -0.4, \quad y = \pm 0.6911$

$\dot{y} = -0.96 \downarrow \quad \dot{x} = \pm 0.6911$

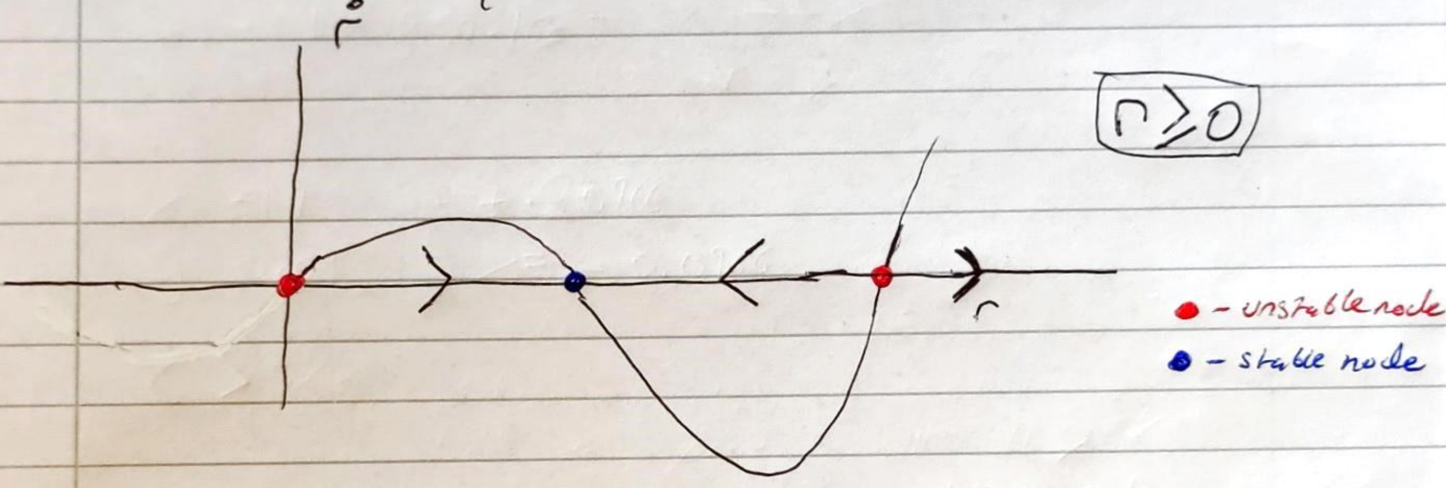
Q7 Consider system of nonlinear ordinary differential equations in polar coordinates:

$$\frac{dr}{dt} = r(1-r^2)(9-r^2)$$

$$\frac{d\theta}{dt} = 1$$

(a) $\frac{dr}{dt} = 0$ for $r=0,\ r=1,\ r=3$

fixed points $r=0,\ r=1,\ r=3$



$r^* = 0, 3$ are unstable
$r^* = 1$ is stable

$\theta = \tan^{-1}(y/x)$, $x = r\cos\theta$, $y = r\sin\theta$

$y/x = \tan\theta$

$\dot{\theta} = 1 > 0$ → rotation is anti-clockwise

$\theta = t$, $r = 1$ is the explicit limit cycle

(b)