Quantum Mechanics II Homework II

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1 (a) Consider the states of a system with spin angular momentum $s = 3/2$ with standard orthonormal basis $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$ and $|\frac{3}{2}, -\frac{3}{2}\rangle$

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |\tfrac{3}{2}, -\tfrac{3}{2}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\begin{cases} S_+ |sm\rangle = \hbar\sqrt{(s-m)(s+m+1)}\,|s\,m+1\rangle \\ S_- |sm\rangle = \hbar\sqrt{(s+m)(s-m+1)}\,|s\,m-1\rangle \\ S_z = \langle s m_1 | S_z | s m_2\rangle = \hbar m_2 \delta_{m_1 m_2} = \hbar m_2 \end{cases}$$

$s = 3/2$

$$S_+ : \quad S_+|\tfrac{3}{2}\tfrac{3}{2}\rangle = \hbar\sqrt{(s-m)(s+m+1)}\,|s\,m+1\rangle = \hbar\sqrt{0(4)}\,|s\,m+1\rangle = 0 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$S_+|\tfrac{3}{2}\tfrac{1}{2}\rangle = \hbar\sqrt{1\cdot 3}\,|\tfrac{3}{2}\tfrac{3}{2}\rangle = \begin{pmatrix}\sqrt{3}\\0\\0\\0\end{pmatrix}\hbar$$

$$S_+|\tfrac{3}{2}-\tfrac{1}{2}\rangle = \hbar\sqrt{(2)\cdot(2)}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle = \begin{pmatrix}0\\2\\0\\0\end{pmatrix}\hbar$$

$$S_+|\tfrac{3}{2}-\tfrac{3}{2}\rangle = \hbar\sqrt{3(1)}\,|\tfrac{3}{2}-\tfrac{1}{2}\rangle = \begin{pmatrix}0\\0\\\sqrt{3}\\0\end{pmatrix}\hbar$$

$$S_+ = \hbar\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$S_-|sm\rangle = \hbar\sqrt{(s+m)(s-m+1)}\,|s\,m-1\rangle$$

$$S_-|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = \hbar\sqrt{3(1)}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\begin{pmatrix}0\\ \sqrt{3}\\ 0\\ 0\end{pmatrix}$$

$$S_-|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{2(2)}\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \hbar\begin{pmatrix}0\\ 0\\ 2\\ 0\end{pmatrix}$$

$$S_-|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \hbar\sqrt{1(3)}\,|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = \hbar\begin{pmatrix}0\\ 0\\ 0\\ \sqrt{3}\end{pmatrix}$$

$$S_-|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = \hbar\sqrt{0(s-m+1)}\,|s\,m-1\rangle = \begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}$$

$$S_- = \hbar\begin{pmatrix}0 & 0 & 0 & 0\\ \sqrt{3} & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}$$

$$S_z = \hbar m_z = \hbar\begin{pmatrix}s & 0 & 0 & 0\\ 0 & s-1 & 0 & 0\\ 0 & 0 & -s+1 & 0\\ 0 & 0 & 0 & -s\end{pmatrix} = \hbar\begin{pmatrix}3/2 & 0 & 0 & 0\\ 0 & 1/2 & 0 & 0\\ 0 & 0 & -1/2 & 0\\ 0 & 0 & 0 & -3/2\end{pmatrix}$$

(b) $S_x|sm\rangle = \tfrac{1}{2}(S_+ + S_-)|sm\rangle$

$$S_y|sm\rangle = \tfrac{1}{2i}(S_+ - S_-)|sm\rangle$$

$$S_x = \tfrac{1}{2}\hbar\left(\begin{bmatrix}0 & \sqrt{3} & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & \sqrt{3}\\ 0 & 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 & 0\\ \sqrt{3} & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & \sqrt{3} & 0\end{bmatrix}\right)$$

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & \sqrt{3} & 0 & 0\\ \sqrt{3} & 0 & 2 & 0\\ 0 & 2 & 0 & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 0\end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-)$$

$$S_y = \frac{\hbar}{2i}\left(\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\right)$$

$$S_y = \frac{\hbar}{2i}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3}i & 0 & 0 \\ +\sqrt{3}i & 0 & -2i & 0 \\ 0 & 2i & 0 & -\sqrt{3}i \\ 0 & 0 & \sqrt{3}i & 0 \end{pmatrix} = \frac{i\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

(c) Show that $S_x^2 + S_y^2 + S_z^2 = \hbar^2 \frac{15}{4} \underline{I}_4$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad S_y = \frac{i\hbar}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$S_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & 2\sqrt{3} & 0 \\ 0 & 7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 7 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{pmatrix}$$

$$S_y^2 = -\frac{\hbar^2}{4}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = -\frac{\hbar^2}{4}\begin{pmatrix} -3 & 0 & 2\sqrt{3} & 0 \\ 0 & -7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & -7 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 7 & 0 & -2\sqrt{3} \\ -2\sqrt{3} & 0 & 7 & 0 \\ 0 & -2\sqrt{3} & 0 & 3 \end{pmatrix}$$

$$S_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

$$S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}\left[\begin{pmatrix} 3 & 0 & 2\sqrt{3} & 0 \\ 0 & 7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 7 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 7 & 0 & -2\sqrt{3} \\ -2\sqrt{3} & 0 & 7 & 0 \\ 0 & -2\sqrt{3} & 0 & 3 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\left[\begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 14 & 0 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}\right] = \frac{\hbar^2}{4}\begin{pmatrix} 15 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 15 \end{pmatrix}$$

$$= \frac{15}{4}\hbar^2 \underline{\underline{I}}_4$$

Q2 have spin $1/2$ particle with known Spin component $S_z = 1/2 \hbar$ along Z axis. First make measurement along a direction $\hat{n}$ at angle θ to Z axis. Then spin is measured along Z axis again. Show after the two measurements that the particle has spin $-1/2 \hbar$ along Z axis with probability $1/2 \sin^2\theta$

Spin wave function $|\psi\rangle$ is measured along Z axis initially with $S_z = 1/2 \hbar$ so wavefunction collapses to $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Choose axes so that $\hat{n}$ lies on XZ plane.

$\hat{n} = \cos\theta \, \hat{k} + \sin\theta \, \hat{i}$

$\underline{S} \cdot \hat{n} = \sin\theta S_x + \cos\theta S_z$

$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\underline{S} \cdot \hat{n} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$\underline{S} \cdot \hat{n}$ has eigenvalues of $\pm\hbar/2$ with eigenvectors $\begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix}$ for $\frac{\hbar}{2}$

& $\begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{bmatrix}$ for $-\frac{\hbar}{2}$

We can write the probabilities of obtaining the different spins in a probability tree with probabilities a^2 and b^2 on each branch.

$$
\begin{tikzcd}
 & \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix} \arrow[r, "a_2^2"] \arrow[dr, "b_2^2"'] & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 |\uparrow\rangle \arrow[ur, "a_1^2"] \arrow[dr, "b_1^2"'] & & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
 & \begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{bmatrix} \arrow[r, "a_3^2"] \arrow[dr, "b_3^2"'] & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 & & \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{tikzcd}
$$

where $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Each wavefunction can be written as a linear combination of the two possible spin states that can be measured

i.e. $$|\uparrow\rangle = a_1 \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix} + b_1 \begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{bmatrix}$$

where a_1^2 is the probability of the spin being in the first spin state & b_1^2 is probability of the spin being the second state when measured.

here we find $$|\uparrow\rangle = \cos\theta/2 \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix} + \sin\theta/2 \begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{bmatrix}$$

$$a_1^2 = \cos^2\theta/2 \qquad b_1^2 = \sin^2\theta/2$$

Then for the 2nd measurement, we measure S_z after measuring $\underline{S}\cdot\hat{n}$.

$\underline{S}\cdot\hat{n}$ can then be expressed as a linear combination of $|\uparrow\rangle$ and $|\downarrow\rangle$ states as follows:

$$a_2^2 = \cos^2\frac{\theta}{2} \qquad b_2^2 = \sin^2\theta/2$$

$$\begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = \cos\frac{\theta}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin\frac{\theta}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{bmatrix} = \sin\theta/2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + -\cos\theta/2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$a_3^2 = \sin^2\frac{\theta}{2} \qquad b_3^2 = \cos^2\theta/2$$

Probability tree now looks like this:

$$\begin{tikzcd}[column sep=large]
 & \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix} \arrow[r, "\cos^2\theta/2"] \arrow[dr, "\sin^2\theta/2"'] & {|\uparrow\rangle} \\
{|\uparrow\rangle} \arrow[ur, "\cos^2\theta/2"] \arrow[dr, "\sin^2\theta/2"'] & & \boxed{|\downarrow\rangle} \\
 & \begin{bmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{bmatrix} \arrow[r, "\sin^2\theta/2"] \arrow[dr, "\cos^2\theta/2"'] & {|\uparrow\rangle} \\
 & & \boxed{|\downarrow\rangle}
\end{tikzcd}$$

1st measurement ($\underline{S}\cdot\underline{\hat{n}}$) 2nd measurement S_z

The probability of finding the particle has spin $S_z = -\hbar/2$ after 2 measurements is given by:

$$P(S_z = -\hbar/2) = \cos^2(\theta/2)\sin^2(\theta/2) + \sin^2(\theta/2)\cos^2(\theta/2)$$
$$= 2(\cos^2(\theta/2)\sin^2(\theta/2))$$

$$\sin^2\theta/2 = 1/2(1-\cos\theta), \quad \cos^2\theta/2 = 1/2(1+\cos\theta)$$

$$= 2[1/2(1+\cos\theta)\cdot 1/2(1-\cos\theta)$$
$$= 2[(1/2 + 1/2\cos\theta)\cdot(1/2 - 1/2\cos\theta)]$$
$$= 2[1/4 - 1/4\cos\theta + 1/4\cos\theta - 1/4\cos^2\theta]$$
$$= 2[1/4 - 1/4\cos^2\theta]$$
$$= 2[1/4(1-\cos^2\theta)$$
$$= 2/4[\sin^2\theta]$$

$\therefore$ after the two measurements Probability of Measuring $S_z = -\hbar/2$

$$= 1/2\sin^2\theta$$

Q3(a) Two particle system consisting of a spin $\frac{3}{2}$ particle together with a spin $\frac{1}{2}$ particle.
Describe the total spin eigenstates $|sm\rangle$ of the system in terms of the basis $\{|\frac{3}{2} m_1; \frac{1}{2} m_2\rangle\}$ for:
$m_1 = \frac{3}{2}, \ldots, -\frac{3}{2}$ $\quad m_2 = \frac{1}{2}, -\frac{1}{2}$

$S_1 = \frac{3}{2}, \quad S_2 = \frac{1}{2}.$
possible values of $s = s_1 + s_2, s_1 + s_2 - 1, \ldots |s_1 - s_2|$
$S = 2$ or 1
$m_1 = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$
$m_2 = \frac{1}{2}, -\frac{1}{2}$

basis $\{|\frac{3}{2} m_1; \frac{1}{2} m_2\rangle\}$

$$\# |s_1 m_1; s_2 m_2\rangle \text{ states} = (2s_1 + 1)(2s_2 + 1)$$
$$= (2 \cdot \tfrac{3}{2} + 1)(2 \cdot \tfrac{1}{2} + 1)$$
$$= 8 \text{ states}$$

Describe Total spin eigenstates in terms of basis $|\frac{3}{2} m_1; \frac{1}{2} m_2\rangle$:
$|2, m\rangle$ for $m = 2, 1, 0, -1, -2$
$|1, m\rangle$ for $m = 1, 0, -1$

$|\frac{3}{2} m_1; \frac{1}{2} m_2\rangle$ for $m_1 = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$
$m_2 = \frac{1}{2}, -\frac{1}{2}$

recall: $S_- |sm\rangle = \hbar\sqrt{(s+m)(s-m+1)}\, |s\, m-1\rangle$
$S_- = S_-^{(1)} + S_-^{(2)}$

First consider $|2, m\rangle$ states:

$$\boxed{|2,2\rangle = |\tfrac{3}{2}\,\tfrac{3}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle \text{ is the max } m \text{ state}}$$

Act with $S_- = S_-^{(1)} + S_-^{(2)}$ to find other states.

$$S_-|sm\rangle = \hbar\sqrt{(s+m)(s-m+1)}\,|s\,m-1\rangle$$
$$S_-|22\rangle = \hbar\sqrt{4(1)}\,|21\rangle = \hbar\sqrt{4}\,|21\rangle$$

$$(S_-^{(1)} + S_-^{(2)})|\tfrac{3}{2}\,\tfrac{3}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|\tfrac{3}{2}\,\tfrac{3}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\hbar\sqrt{4}\,|21\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\,|\tfrac{3}{2}\,\tfrac{3}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\boxed{|21\rangle = \sqrt{\tfrac{3}{2}}\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \tfrac{1}{2}\,|\tfrac{3}{2}\,\tfrac{3}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

$\sqrt{\tfrac{3}{4}}^2 + \tfrac{1}{\sqrt{4}}^2 = 1$ ✓ $\quad C^{s_1\,s_2\,s}_{m_1\,m_2\,m}$

$$C^{\frac{3}{2}\,\frac{1}{2}\,2}_{\frac{1}{2}\,\frac{1}{2}\,1} = \sqrt{\tfrac{3}{2}}\,, \quad C^{\frac{3}{2}\,\frac{1}{2}\,2}_{\frac{3}{2}\,-\frac{1}{2}\,1} = \tfrac{1}{2}$$

Consider $S_-|21\rangle$: $\quad S_-|21\rangle = \hbar\sqrt{3\cdot(2-1+1)}\,|20\rangle$
$$= \hbar\sqrt{6}\,|20\rangle$$

$$\hbar\sqrt{6}\,|20\rangle = \sqrt{\tfrac{3}{4}}\left[\hbar\sqrt{2\cdot 2}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \hbar\cdot 1\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\right]$$
$$+ \tfrac{1}{\sqrt{4}}\left[\hbar\sqrt{3\cdot 1}\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + 0\right]$$

$$\hbar\sqrt{6}\,|20\rangle = \hbar\sqrt{3}\cdot\tfrac{\sqrt{4}}{\sqrt{4}}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + 2\sqrt{\tfrac{3}{4}}\,\hbar\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$\sqrt{\tfrac{3}{6}} = \tfrac{1}{\sqrt{2}}$

$$\boxed{|20\rangle = \tfrac{1}{\sqrt{2}}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{2}}\,|\tfrac{3}{2}\,\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

$$C^{\frac{3}{2}\,\frac{1}{2}\,2}_{-\frac{1}{2}\,\frac{1}{2}\,0} = \tfrac{1}{\sqrt{2}}\,, \quad C^{\frac{3}{2}\,\frac{1}{2}\,2}_{\frac{1}{2}\,-\frac{1}{2}\,0} = \tfrac{1}{\sqrt{2}} \qquad \sqrt{\tfrac{3}{6}}^2 = \tfrac{1}{2}$$
$$\tfrac{1}{2} + \tfrac{1}{2} = 1 \checkmark$$

consider $S_- |20\rangle$:

$$S_-|20\rangle = \hbar\sqrt{2\cdot 3}\ |2\,-1\rangle = \hbar\sqrt{6}\,|2\,-1\rangle$$

$$\hbar\sqrt{6}\,|2\,-1\rangle = \tfrac{\sqrt{3}}{\sqrt{6}}\Big[\hbar\sqrt{(\tfrac{3}{2}-\tfrac{1}{2})(\tfrac{3}{2}+\tfrac{1}{2}+1)}\,|\tfrac{3}{2}\,-\tfrac{3}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$
$$+\ \hbar\sqrt{(1)(1)}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\Big]$$
$$+\ \tfrac{\sqrt{3}}{\sqrt{6}}\Big[\hbar\sqrt{2\cdot 2}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + 0\Big]$$

$$\hbar\sqrt{6}\,|2\,-1\rangle = \tfrac{\sqrt{3}}{\sqrt{6}}\cdot\hbar\sqrt{3}\,|\tfrac{3}{2}\,-\tfrac{3}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$
$$+\ \tfrac{\sqrt{3}}{\sqrt{6}}\cdot 3\hbar\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle\ ]$$

$$|2\,-1\rangle = \tfrac{1}{2}\,|\tfrac{3}{2}\,-\tfrac{3}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \tfrac{3\sqrt{3}}{6}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$\boxed{|2\,-1\rangle = \tfrac{1}{2}\,|\tfrac{3}{2}\,-\tfrac{3}{2};\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \tfrac{\sqrt{3}}{2}\,|\tfrac{3}{2}\,-\tfrac{1}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

$$C^{\frac{3}{2}\ \frac{1}{2}\ 2}_{-\frac{3}{2}\ \frac{1}{2}\ -1} = \tfrac{1}{2}\quad,\quad C^{\frac{3}{2}\ \frac{1}{2}\ 2}_{-\frac{1}{2}\ -\frac{1}{2}\ -1} = \tfrac{\sqrt{3}}{2}$$

$$(\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{2})^2 = \tfrac{1}{4}+\tfrac{3}{4} = 1\quad ✓$$

and the unique minimum eigenstate $m=-2$

$$\boxed{|2\,-2\rangle = |\tfrac{3}{2}\,-\tfrac{3}{2};\tfrac{1}{2}\,-\tfrac{1}{2}\rangle}$$

Now consider $|1,m\rangle$ states $m = 1, 0, -1$

$$|1,1\rangle = \alpha\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + \beta\,|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$|1,1\rangle$ is orthogonal to $|2,1\rangle$

$$|2,1\rangle = \sqrt{3/4}\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + 1/\sqrt{4}\,|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

since they are eigenstates with different $\underline{S}^2$ & $\underline{S_z}$ values (both are dependent on s)

so, $\langle 11|21\rangle = 0 = \alpha\sqrt{3/4} + \beta\frac{1}{\sqrt{4}}$

choose α +:

$$\alpha = -\beta\cdot\sqrt{\tfrac{4}{3}}\cdot\tfrac{1}{\sqrt{4}} \rightarrow \alpha = -\beta\cdot\tfrac{1}{\sqrt{3}}$$

$$\alpha = 1/\sqrt{4} = 1/2,\quad \beta = -\sqrt{\tfrac{3}{4}} = -\sqrt{3}/2$$

so,

$$\boxed{|1,1\rangle = \tfrac{1}{2}\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + -\tfrac{\sqrt{3}}{2}\,|\tfrac{3}{2}\tfrac{3}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle}$$

$$C^{\frac{3}{2}\,\frac{1}{2}\,1}_{\frac{1}{2}\,\frac{1}{2}\,1} = \tfrac{1}{2},\qquad C^{\frac{3}{2}\,\frac{1}{2}\,1}_{\frac{3}{2}\,-\frac{1}{2}\,1} = -\tfrac{\sqrt{3}}{2}$$

$$(\tfrac{1}{2})^2 + (-\tfrac{\sqrt{3}}{2})^2 = \tfrac{1}{4} + \tfrac{3}{4} = 1 \checkmark$$

act with S_- to find next state:

$$S_-|1,1\rangle = \hbar\sqrt{1\cdot(2)}\,|1\,0\rangle = \hbar\sqrt{2}\,|10\rangle$$

$$\hbar\sqrt{2}\,|10\rangle = \tfrac{1}{2}\left[\hbar\sqrt{2\cdot 2}\,|\tfrac{3}{2}-\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + \hbar\sqrt{1}\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle\right.$$
$$\left.-\tfrac{\sqrt{3}}{2}\left[\hbar\sqrt{3\cdot 1}\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle + 0\right]\right.$$

$$\hbar\sqrt{2}\,|10\rangle = \hbar\,|\tfrac{3}{2}-\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle + \tfrac{\hbar}{2}\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle - \tfrac{3}{2}\hbar\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$\boxed{|10\rangle = 1/\sqrt{2}\,|\tfrac{3}{2}-\tfrac{1}{2};\tfrac{1}{2}\tfrac{1}{2}\rangle - 1/\sqrt{2}\,|\tfrac{3}{2}\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\rangle}$$

$$C^{3/2\ 1/2\ 1}_{-1/2\ 1/2\ 0} = 1/\sqrt{2} \quad , \quad C^{3/2\ 1/2\ 1}_{1/2\ -1/2\ 0} = -\frac{1}{\sqrt{2}}$$

$$(1/\sqrt{2})^2 + (-1/\sqrt{2})^2 = 1/2 + 1/2 = 1 \checkmark$$

$$S_-|10\rangle = \hbar\sqrt{1\cdot 2}\,|1\,-1\rangle = \hbar\sqrt{2}\,|1\,-1\rangle$$

$$\hbar\sqrt{2}\,|1\,-1\rangle = 1/\sqrt{2}\Big[\hbar\sqrt{1\cdot 3}\,|3/2\ -3/2;\ 1/2\ 1/2\rangle + \hbar\sqrt{1\cdot 1}\,|3/2\ -1/2;\ 1/2\ -1/2\rangle\Big]$$
$$+ -1/\sqrt{2}\Big[\hbar\sqrt{2\cdot 2}\,|3/2\ -1/2;\ 1/2\ -1/2\rangle + 0\Big]$$

$$\hbar\sqrt{2}\,|1\,-1\rangle = \frac{\hbar\sqrt{3}}{\sqrt{2}}|3/2\ -3/2;\ 1/2\ 1/2\rangle + \frac{\hbar}{\sqrt{2}}|3/2\ -1/2;\ 1/2\ -1/2\rangle - \frac{2\hbar}{\sqrt{2}}|3/2\ -1/2;\ 1/2\ -1/2\rangle$$

$$\hbar\sqrt{2}\,|1\,-1\rangle = \frac{\hbar\sqrt{3}}{\sqrt{2}}|3/2\ -3/2;\ 1/2\ 1/2\rangle - \frac{\hbar}{\sqrt{2}}|3/2\ -1/2;\ 1/2\ -1/2\rangle$$

$$\boxed{|1\,-1\rangle = \frac{\sqrt{3}}{2}|3/2\ -3/2;\ 1/2\ 1/2\rangle - \frac{1}{2}|3/2\ -1/2;\ 1/2\ -1/2\rangle}$$

$$C^{3/2\ 1/2\ 1}_{-3/2\ 1/2\ -1} = \sqrt{3}/2 \qquad C^{3/2\ 1/2\ 1}_{-1/2\ -1/2\ -1} = -1/2$$

$$(\sqrt{3}/2)^2 + (-1/2)^2 = 3/4 + 1/4 = 1 \checkmark$$

(b) Determine tables of Clebsch-Gordon coefficients $C^{\frac{3}{2}\,\frac{1}{2}\,J}_{m_1 m_2 m}$

<u>m = 2</u>

		2
		2
$\frac{3}{2}$	$\frac{1}{2}$	1

<u>m = 1</u>

		2	1
		1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

<u>m = 0</u>

		2	1
		0	0
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

<u>m = -1</u>

		2	1
		-1	-1
$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

<u>m = -2</u>

		2
		-2
$-\frac{3}{2}$	$-\frac{1}{2}$	1