Sean Tobin - 18483232
& Dara Corr - 18483836

Maths Methods : Assignment 4

1.(a) $t = s^2$, $\frac{dt}{ds} = 2s \therefore dt = 2s\,ds$

$t^{-1/2} = (s^2)^{-1/2} = s^{-1} \Rightarrow \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty 2\cancel{s} \cdot \frac{1}{\cancel{s}} e^{-s^2}\,ds = 2\int_0^\infty e^{-s^2}\,ds$

(b) $x = r\cos\theta$, $y = r\sin\theta$, θ ranges from $0 - \frac{\pi}{2}$ (not 2π, since that's from $\int_{-\infty}^{\infty}$ [$\int_0^\infty$ only positive answers])

$4\int_0^\infty \int_0^{\pi/2} e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\,d\theta\,dr = 4\int_0^\infty \left[e^{-r^2} r\,\theta\right]_0^{\pi/2} dr = 2\pi \int_0^\infty r e^{-r^2}\,dr$

Substitution: $u = r^2$, $du = 2r\,dr$, $dr = \frac{du}{2r} \Rightarrow 2\pi \int_0^\infty \frac{1}{2} e^{-u}\,du = 2\pi \cdot \frac{1}{2} = \pi = \Gamma(\tfrac{1}{2})^2$

$\therefore \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

2. $z^2 \frac{d^2\omega}{dz^2} + z\frac{d\omega}{dz} + (z^2 - \nu^2)\omega = 0$, where $z = ax^b$, $\omega = yx^c$, $\frac{dz}{dx} = bax^{b-1}$

$\frac{d\omega}{dx} = \frac{dy}{dx}x^c + cyx^{c-1}$, $\frac{d^2\omega}{dx^2} = \frac{d^2y}{dx^2}x^c + \frac{dy}{dx}cx^{c-1} + \frac{dy}{dx}cx^{c-1} + c(c-1)yx^{c-2}$

$\frac{d\omega}{dx} = \frac{d\omega}{dz}\cdot\frac{dz}{dx} = \frac{d\omega}{dz}\cdot bax^{b-1} \Rightarrow \frac{d}{dz} = \frac{1}{bax^{b-1}}\frac{d}{dx} \Rightarrow \frac{d\omega}{dz} = \frac{1}{bax^{b-1}}\cdot\frac{d\omega}{dx}$

$\therefore \frac{d^2\omega}{dz^2} = \frac{d}{dz}\left(\frac{d\omega}{dz}\right) = \frac{1}{bax^{b-1}}\cdot\frac{d}{dx}\left(\frac{1}{bax^{b-1}}\cdot\frac{d\omega}{dx}\right) = \frac{1}{bax^{b-1}}\left(\frac{1}{ba}(1-b)x^{-b}\frac{d\omega}{dx} + \frac{1}{ba}x^{1-b}\cdot\frac{d^2\omega}{dx^2}\right)$

Can factor out $\frac{1}{ba}$ and x^{-b} (and recall $z^2 = a^2x^{2b}$ [and $\frac{1}{x^{-1}} = x$])

$z^2\frac{d^2\omega}{dz^2} = \cancel{a^2}\cancel{x^{2b}}\cdot\frac{1}{b^2\cancel{a^2}\cancel{x^{2b}}}x\left((1-b)\frac{d\omega}{dx} + x\frac{d^2\omega}{dx^2}\right)$

$z\frac{d\omega}{dz} = ax^b\cdot\frac{1}{bax^{b-1}}\cdot\frac{d\omega}{dx} = \frac{x}{b}\cdot\frac{d\omega}{dx}$

$(z^2 - \nu^2)\omega = (a^2x^{2b} - \nu^2)yx^c$

sum of these = original equation = 0

$\Rightarrow \frac{x}{b^2}\left((1-b)\frac{d\omega}{dx} + x\frac{d^2\omega}{dx^2}\right) + \frac{x}{b}\cdot\frac{d\omega}{dx} + (a^2x^{2b} - \nu^2)yx^c = 0$

Separate eqn. into parts for convenience:

$\frac{x}{b^2}\left(x\frac{d^2\omega}{dx^2}\right) = \frac{1}{b^2}x^2\left(\frac{d^2y}{dx^2}x^c + 2\frac{dy}{dx}cx^{c-1} + c(c-1)yx^{c-2}\right) = \frac{1}{b^2}x^c\left(x^2\frac{d^2y}{dx^2} + 2\frac{dy}{dx}cx + c(c-1)y\right)$

$\frac{x}{b^2}\left((1-b)\frac{d\omega}{dx}\right) = \frac{1}{b^2}\left((x-bx)\left(\frac{dy}{dx}x^c + cyx^{c-1}\right)\right) = \frac{1}{b^2}x^c\left(x\frac{dy}{dx} + cy - bx\frac{dy}{dx} - bcy\right)$

$\frac{x}{b}\cdot\frac{d\omega}{dx} = \frac{1}{b}\left(x\frac{dy}{dx}x^c + \cancel{x}cyx^{c-1}\right) = \frac{1}{b}x^c\left(x\frac{dy}{dx} + cy\right)$

Multiply across by b^2 and divide across by x^c:

$x^2\frac{d^2y}{dx^2} + \frac{dy}{dx}(2cx + x - \cancel{bx} + \cancel{bx}) + (c^2y - \cancel{cy} + \cancel{cy} - \cancel{bcy} + \cancel{bcy}) + (a^2b^2x^{2b} - \nu^2b^2)y = 0$

$\Rightarrow x^2\frac{d^2y}{dx^2} + (2c+1)x\frac{dy}{dx} + (a^2b^2x^{2b} + c^2 - \nu^2b^2)y = 0$

↳ the form we're looking for

For $\nu \neq 0$ and not an integer

$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu}$ $\qquad J_{-\nu}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n-\nu+1)}\left(\frac{z}{2}\right)^{2n-\nu}$

$y = \frac{\omega}{x^c}$, $z = ax^b \therefore x = \left(\frac{z}{a}\right)^{1/b}$

$y = \omega / \left(\frac{z}{a}\right)^{c/b}$ $\qquad \omega(z) = c_1J_\nu(z) + c_2J_{-\nu}(z)$

$\therefore y(z) = \frac{c_1}{x^c}J_\nu(z) + \frac{c_2}{x^c}J_{-\nu}(z)$ $\qquad y(x) = y(z(x))$

$\therefore y(x) = c_1x^{-c}J_\nu(ax^b) + c_2x^{-c}J_{-\nu}(ax^b)$ = general solution

(where $J_\nu(ax^b) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{ax^b}{2}\right)^{2n+\nu}$

Note: $J_\nu(x) = \sqrt[b]{\left(\frac{a}{2}\right)^{2n+\nu}}\sum_{n=0}^{\infty}\sqrt[b]{\frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}}\cdot x^{2n+\nu}$ ($J_{-\nu}(x)$ just swaps signs of ν)

$\therefore J_\nu(ax^b) = \left(J_\nu(x)\cdot\frac{1}{a^{2n+\nu}}\right)^b$, $\left(J_{-\nu}(x)\cdot\frac{1}{a^{2n-\nu}}\right)^b$

$\alpha = 4 \therefore \alpha^2 = 16$

$y(x) = \sum_{n=0}^{\infty} a_n x^n \rightarrow xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \Rightarrow \overset{\text{as}}{\underset{\downarrow}{n}} = n+1,\ \overset{\text{as}}{\underset{\downarrow}{n}} = n-1 \Rightarrow \sum_{n=1}^{\infty} a_{n-1} x^n$

$y''(x) = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} \Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n = 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} + 16a_{n-1}]x^n = 0 \quad \Rightarrow 2a_2 = 0 \therefore a_2 = 0$

$(n+2)(n+1)a_{n+2} + 16a_{n-1} = 0 \therefore a_{n+2} = \frac{-16a_{n-1}}{(n+2)(n+1)}$

Take $a_0 = 1,\ a_1 = 0;\ a_3 = \frac{-16a_0}{(1+2)(1+1)} = -\frac{8}{3} = \frac{-16}{3!},\ a_4 = 0,\ a_5 = 0,$

$a_6 = \frac{-16a_3}{(6)(5)} = \frac{-16(-16a_0)(4)}{(6)(5)(3)(2)} = \frac{-16(-16a_0)(4)}{6!},\ a_7 = 0,\ a_8 = 0,\ a_9 = \frac{-16(-16)(-16)a_0(4)(7)}{9!}, \ldots$

Take $a_0 = 0,\ a_1 = 1,\ a_2 = a_3 = 0,\ a_4 = \frac{-16a_1}{(4)(3)} = \frac{-16a_1(2)}{4!},\ a_5 = a_6 = 0$

$a_7 = \frac{-16(-16a_1)(2)(5)}{7!},\ a_8 = a_9 = 0,\ a_{10} = \frac{-16(-16)(-16a_1)}{10(9)} = \frac{(-16)^3 a_1 (2)(5)(8)}{10!}, \ldots$

$y_1 = 1 + -\frac{16}{3!}x^3 + \frac{(16^2)4}{6!}x^6 - \frac{(16)^3(4)(7)}{9!}x^9 + \ldots = 16\left(\frac{(-1)^n}{n!\Gamma}\right)x^n$

$y_2 = x - \frac{16(2)}{4!}x^4 + \frac{(16^2)(2)(5)}{7!}x^7 - \frac{(16)^3(2)(5)(8)}{10!}x^{10} + \ldots$

Regular general solution:

$$y(x) = c_1\left(1 - \frac{16}{3!}x^3 + \frac{(16)^2(4)}{6!}x^6 - \frac{(16)^3(4)(7)}{9!}x^9 + \ldots\right)$$
$$+ c_2\left(x - \frac{16(2)}{4!}x^4 + \frac{(16)^2(2)(5)}{7!} - \frac{(16)^3(2)(5)(8)}{10!}x^{10} + \ldots\right)$$

Bessel functions: $J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu}$

$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-\nu+1)}\left(\frac{x}{2}\right)^{2n-\nu}$

Try convert $n \rightarrow n+1$ in recursion relation

$a_{n+3} = \frac{-16a_n}{(n+3)(n+2)} \Rightarrow -16a_{3n} = a_{3n+3}(3n+3)(3n+2)$

From rearranging, we find $y(x) = c_1\sqrt{x}\cdot 16 J_{1/3}\left(\frac{2}{3}ix^{3/2}\right) + c_2\sqrt{x}J_{-1/3}\left(\frac{2}{3}ix^{3/2}\right)$