Quantum Mechanics HW 1 Dara Carr

Q1(a) 1D Schrödinger Equation

$$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad \hat{p} = -i\hbar\partial/\partial x$$

$$\hat{H}\Psi = \hat{E}\Psi \qquad \hat{H} = 1/2m\,\hat{p}^2 + V(x)$$

$$\langle[H,Q]\rangle = \langle\hat{H}\hat{Q} - \hat{Q}\hat{H}\rangle$$

$$\langle[H,Q]\rangle = \int_{-\infty}^{\infty}\Psi^*[H,Q]\Psi\,dx$$

$$= \int_{-\infty}^{\infty}\Psi^*\hat{H}\hat{Q} - \hat{Q}\hat{H}\Psi$$

$$d/dt\langle\hat{Q}\rangle = \int_{-\infty}^{\infty}\partial/\partial t(\Psi^*\hat{Q}\Psi)dx$$

1 $$= \int_{-\infty}^{\infty}\left(\frac{\partial\Psi^*}{\partial t}\hat{Q}\Psi\right) + \left(\Psi^*\hat{Q}\frac{\partial\Psi}{\partial t}\right)dx$$

We want to show that

$$d/dt\langle Q\rangle = \frac{i}{\hbar}\langle[H,Q]\rangle$$

$$\frac{i}{\hbar}\int_{-\infty}^{\infty}\Psi^*(\hat{H}\hat{Q} - \hat{Q}\hat{H})\Psi\,dx \qquad 2$$

$$\frac{\partial\Psi}{\partial t} = \frac{-\hbar}{2mi}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{\hbar i}$$

$$\boxed{\begin{aligned}\frac{\partial\Psi}{\partial t} &= \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V(x)\Psi \\ \frac{\partial\Psi^*}{\partial t} &= -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V(x)\Psi^*\end{aligned}}$$

sub into 1

$$\frac{d}{dt}\langle \hat{Q} \rangle = \int_{-\infty}^{\infty} \Big[\Big(-\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V(x) \psi^* \Big) \hat{Q} \psi$$

$$+ \psi^* \hat{Q} \Big(\frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V(x) \psi \Big) \Big] dx$$

$$[\hat{H}\hat{Q} - \hat{Q}\hat{H}) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \hat{Q} + V(x) \hat{Q}$$
$$+ \hat{Q} \frac{\hbar^2}{2m} \frac{\partial}{\partial x} - \hat{Q} V(x)$$

$$\frac{i}{\hbar}[\hat{H}\hat{Q} - \hat{Q}\hat{H}) = -\frac{i\hbar}{2m} \frac{\partial^2}{\partial x^2} \hat{Q} + \frac{i}{\hbar} V(x) \hat{Q}$$
$$+ \hat{Q} \frac{i\hbar}{2m} \frac{\partial}{\partial x} - \hat{Q} \frac{i}{\hbar} \hat{Q} V(x)$$

$$\frac{i}{\hbar}(\hat{H}\hat{Q} - \hat{Q}\hat{H}) =$$

$$\frac{i}{\hbar}\langle [H, Q] \rangle = \int_{-\infty}^{\infty} \Big[\psi^* - \frac{i\hbar}{2m} \frac{\partial^2}{\partial x^2} \hat{Q} \psi + \psi^* \frac{i}{\hbar} V(x) \hat{Q} \psi$$
$$+ \psi^* \hat{Q} \frac{i\hbar}{2m} \frac{\partial^2}{\partial x^2} \psi - \psi \hat{Q} \frac{i}{\hbar} V(x) \psi \Big] dx$$

note $\hat{H}$ is Hermitian, $H = H^*$

$$\int \psi^* H \psi \, dx = \int (H\psi)^* \psi \, dx = \int \psi (H\psi)^* dx = \int [\psi^* (H\psi)]^*$$

~~rearrange~~ write 1
$$\int_{-\infty}^{\infty} \Big[-\frac{i\hbar}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \hat{Q} \psi + \frac{i}{\hbar} V(x) \psi^* \hat{Q} \psi$$
$$+ \psi^* \hat{Q} \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \psi^* \hat{Q} \frac{i}{\hbar} V(x) \psi \Big] dx$$

rearrange 2
$$\int_{-\infty}^{\infty} -\frac{i\hbar^*}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \hat{Q} \psi + \frac{i^*}{\hbar} V(x)^* \psi^* \hat{Q} \psi$$
$$+ \psi^* \hat{Q} \frac{i\hbar}{2m} \frac{\partial^2 \psi}{\partial x^2} - \psi^* \hat{Q} \frac{i}{\hbar} V(x) \psi \, dx$$

$-\frac{i\hbar^*}{2m}$ term $= \frac{i\hbar}{2m}$ & $\frac{i^*}{\hbar} V(x)^*$ term $= \frac{i}{\hbar} V(x)$

because H is Hermitian $H = H^*$

thus I have shown $d/dt\langle \hat{Q} \rangle = \frac{i}{\hbar}\langle \hat{H}, \hat{Q} \rangle$

Q1(b) Want to show that $\langle [\hat{H}, \hat{Q}] \rangle = 0$
for a stationary state

For stationary state $\Psi(x,t) = \psi(x) e^{-\frac{iEt}{\hbar}}$
Note also E is constant for stationary state
$\hat{E}\Psi = i\hbar \partial/\partial t \Psi = E\Psi$
$\langle \hat{H} \rangle = E = \text{constant}$

$$\langle [H, Q] \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{H}\hat{Q}\Psi - \Psi^* \hat{Q}\hat{H}\Psi \, dx$$

$\hat{H}$ is Hermitian and time independent $(\hat{H}\Psi = E\Psi)$
$H^* = H$

$$\int_{-\infty}^{\infty} \hat{H}\Psi^* \hat{Q}\Psi - \Psi^* \hat{Q}\hat{H}\Psi \, dx$$

$$= \int_{-\infty}^{\infty} E\Psi^* \hat{Q}\Psi - \Psi^* \hat{Q}E\Psi \, dx$$

E is constant for stationary state

$$E\int_{-\infty}^{\infty} \Psi^* \hat{Q}\Psi - \Psi^* \hat{Q}\Psi \, dx$$

$$= E\int_{-\infty}^{\infty} dx = [0]_{-\infty}^{\infty} = 0$$

thus for a stationary state,
$\langle [\hat{H}, \hat{Q}] \rangle = 0$

Q1 (c) $Q = xp$ for a stationary state

Show that

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle$$

$$\langle \hat{T}\rangle = \frac{1}{2m}\langle \hat{p}^2\rangle = \boxed{\frac{-\hbar^2}{2m}\int_{-\infty}^{\infty}\psi^*\frac{\partial^2\psi}{\partial x^2}\,dx}$$

$$= \left[\frac{-\hbar^2}{2m}\psi^*\psi_x\right]_{-\infty}^{\infty} + \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left|\frac{\partial\psi}{\partial x}\right|^2 dx$$

$$\langle \hat{T}\rangle = 0 + \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left|\frac{\partial^2\psi}{\partial x^2}\right| dx$$

for stationary state: $\langle [H,Q]\rangle = \int_{-\infty}^{\infty}\psi^*\hat{H}\hat{Q}\psi - \psi^*\hat{Q}\hat{H}\psi\,dx = 0$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \qquad \hat{Q} = xp = x\hat{p}$$

$$\int_{-\infty}^{\infty}\Big[\psi^* -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}x\hat{p}\psi + \psi^* V(x)\,x\hat{p}\psi - \psi^* x\hat{p} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$$
$$- \psi^* x\hat{p}V(x)\psi\,dx\Big]$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \qquad \hat{p}\psi = -i\hbar\frac{\partial\psi}{\partial x}$$

$$\int_{-\infty}^{\infty}\Big[\psi^* -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(-i\hbar x\frac{\partial\psi}{\partial x}\right) + \psi^* V(x)\left(-i\hbar x\frac{\partial\psi}{\partial x}\right) - \psi^*\cdot x\cdot -i\hbar\frac{\partial}{\partial x}\left(-\frac{\hbar^2}{2m}\right)\left(\frac{\partial^2\psi}{\partial x^2}\right)$$
$$-\psi^*\cdot -i\hbar x\frac{\partial}{\partial x}\left(V(x)\psi(x)\right)\Big]dx$$

$$\int_{-\infty}^{\infty}\Big[\psi^*\frac{i\hbar^3}{2m}\frac{\partial^2}{\partial x^2}\left(x\frac{\partial\psi}{\partial x}\right) - i\hbar x\psi^* V(x)\left(\frac{\partial\psi}{\partial x}\right)$$
$$-\psi^*\frac{i\hbar^3}{2m}x\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right) + \psi^* i\hbar x\frac{\partial}{\partial x}\left(V(x)\psi(x)\right)\Big]dx$$

Simplify integrand

$$\psi^*\frac{i\hbar^3}{2m}\left(\frac{\partial^2}{\partial x^2}\left(x\frac{\partial\psi}{\partial x}\right)\right) = \psi^*\frac{i\hbar^3}{2m}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x} + x\frac{\partial^2\psi}{\partial x}\right)$$
$$= \psi^*\frac{i\hbar^3}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial x} + x\frac{\partial^3\psi}{\partial x^3}\right)$$
$$= \psi^*\frac{i\hbar^3}{2m}\left(2\frac{\partial^2\psi}{\partial x^2} + x\frac{\partial^3\psi}{\partial x^3}\right)$$

$$-\underline{\psi^* \frac{i\hbar^3}{2m} x \frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right)} = -\psi^* \frac{i\hbar^3}{2m} x \left(\frac{\partial^3\psi}{\partial x^3}\right)$$

$$\underline{\psi^* i\hbar x \frac{\partial}{\partial x}(V(x)\psi(x))} = \psi^* i\hbar x \left(\frac{\partial V(x)}{\partial x}\psi(x) + V(x)\frac{\partial \psi}{\partial x}\right)$$

$$\rightarrow \int_{-\infty}^{\infty}\Big[\psi^* \frac{i\hbar^3}{2m}\left(2\frac{\partial^2\psi}{\partial x^2} + \cancel{x\frac{\partial^3\psi}{\partial x^3}}\right) - i\hbar \cancel{x\psi^* V(x)\left(\frac{\partial\psi}{\partial x}\right)}$$

$$\cancel{-\psi^* \frac{i\hbar^3}{2m} x\frac{\partial^3\psi}{\partial x^3}} + \psi^* i\hbar x\left(\frac{\partial V(x)}{\partial x}\psi(x) + \cancel{V(x)\frac{\partial\psi}{\partial x}}\right)\Big] dx$$

$$= \int_{-\infty}^{\infty} \psi^* \frac{i\hbar^3}{2m}\left(2\frac{\partial^2\psi}{\partial x^2}\right) + \psi^* i\hbar x\left(\frac{\partial V(x)}{\partial x}\psi(x)\right) dx$$

$i\hbar$ constants

$$\Rightarrow i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\hbar^2}{2m}\left(2\frac{\partial^2\psi}{\partial x^2}\right) + \psi^* x \left(\frac{\partial V(x)}{\partial x}\psi(x)\right) dx$$

$\boxed{\langle[H,\hat{Q}]\rangle = 0 \text{ Stationary State}}$

$$i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\hbar^2}{2m}\left(2\frac{\partial^2\psi}{\partial x^2}\right) + i\hbar\int_{-\infty}^{\infty}\psi^* x\left(\frac{\partial V(x)}{\partial x}\psi(x)\right)dx$$

$$= 0$$

$$\frac{\cancel{2} i\hbar^3}{\cancel{2}m}\int_{-\infty}^{\infty}\psi^*\left(\frac{\partial^2\psi}{\partial x^2}\right) = -i\hbar\int_{-\infty}^{\infty}\psi^* x\frac{dV}{dx}\psi(x)\, dx$$

$$\boxed{\frac{-\hbar^2}{m}\int_{-\infty}^{\infty}\psi^*\left(\frac{\partial^2\psi}{\partial x^2}\right) = \int_{-\infty}^{\infty}\psi^* x\frac{dV}{dx}\psi(x)\,dx} \quad (*)$$

recall: $\langle\hat{T}\rangle = \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty}\psi^*\frac{\partial^2\psi}{\partial x^2}dx$

& $\langle x\frac{dV}{dx}\rangle = \int_{-\infty}^{\infty}\psi^* x\frac{dV}{dx}\psi(x)\,dx$

Thus from $(*)$ we see that

$$2\langle\hat{T}\rangle = \langle x\frac{dV}{dx}\rangle$$

From that result, we can show $\langle T \rangle = \langle V \rangle$ for a Harmonic Oscillator System.

$V(x) = \frac{1}{2} m \omega^2 x^2$ for Harmonic oscillator

$\frac{\partial V}{\partial x} = m \omega^2 x \qquad x \frac{\partial V}{\partial x} = m \omega^2 x^2$

from above we know $2\langle T \rangle = \langle x \frac{dV}{dx} \rangle$

$$\langle T \rangle = \frac{1}{2} \langle x \frac{dV}{dx} \rangle = \frac{1}{2} m \omega^2 x^2 = \langle V \rangle$$

$\therefore \langle T \rangle = \langle V \rangle$ for Harmonic Oscillator System ∎

Q2 (i) $a = \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x) \quad a^{\dagger} = \frac{1}{\sqrt{2m}}(-i\hat{p} + m\omega x)$

$$\hat{H} = \frac{p^2}{2m} + V(x,t)$$

$$aa^{\dagger} = \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x) \cdot \frac{1}{\sqrt{2m}}(-i\hat{p} + m\omega x)$$

$$= \frac{1}{\sqrt{2m}} \cdot \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x)(-i\hat{p} + m\omega x)$$

$$= \frac{1}{2m}\left(\hat{p}^2 \cancel{+} - im\omega x\hat{p} + im\omega\hat{p}x + (m\omega x)^2\right)$$

$\hat{p} = i\hbar\,\partial/\partial x$

$$= \frac{1}{2m}\left(\hat{p}^2 - im\omega[x\hat{p} - \hat{p}x] + m^2\omega^2x^2\right)$$

$$= \frac{1}{2m}\left(\hat{p}^2 + - im\omega\left[x\hat{p}(0) + i\hbar\frac{\partial x}{\partial x}\right] + m^2\omega^2x^3\right)$$

$$= \frac{1}{2m}\left(\hat{p}^2 + \hbar m\omega + m^2\omega^2x^2\right)$$

$$aa^{\dagger} = \frac{1}{2m}\underbrace{(\hat{p}^2 + m\omega^2x^2)}_{\boxed{=\hat{H} + \frac{\hbar\omega}{2}}} + \frac{1}{2}\hbar\omega$$

$$a^{\dagger}a = \frac{1}{\sqrt{2m}}(-i\hat{p} + m\omega x) \cdot \frac{1}{\sqrt{2m}}(+i\hat{p} + m\omega x)$$

$$= \frac{1}{2m}\left(\hat{p}^2 - im\omega\hat{p}x + im\omega x\hat{p} + (m\omega x)^2\right)$$

$$= \frac{1}{2m}\left(\hat{p}^2 - im\omega[\hat{p}x - x\hat{p}] + (m\omega x)^2\right)$$

$$= \frac{1}{2m}\left(\hat{p}^2 - im\omega\left[-i\hbar\frac{\partial x}{\partial x}\right] + (m\omega x)^2\right)$$

$$= \frac{1}{2m}\left(\hat{p}^2 + m\omega^2x^2\right) - \frac{1}{2}\hbar\omega$$

$$a^{\dagger}a = \boxed{\hat{H} - \frac{\hbar\omega}{2}}$$

$$aa^{\dagger} = \boxed{\hat{H} + \frac{\hbar\omega}{2}}$$

thus $[a, a^{\dagger}] = \hbar\omega$

(ii) $\xi = x\sqrt{\frac{m\omega}{\hbar}}$

Show $a = \sqrt{\frac{\hbar\omega}{2}}\left(\frac{d}{d\xi} + \xi\right)$ $a^{+} = \sqrt{\frac{\hbar\omega}{2}}\left(-\frac{d}{d\xi} + \xi\right)$

$\xi = x\sqrt{\frac{m\omega}{\hbar}}$ $x = \xi\sqrt{\frac{\hbar}{m\omega}}$

$$a = \frac{1}{\sqrt{2m}}(i\hat{p} + m\omega x)$$

$$\hat{p} = i\hbar\,\partial/\partial x = i\hbar\,\partial/\partial\xi\,\frac{\partial\xi}{\partial x}$$ sub in x, p in terms of ξ

$$a = \frac{1}{\sqrt{2m}}\left(i \cdot i\hbar\,\partial/\partial\xi\,\frac{\partial\xi}{\partial x} + m\omega\xi\sqrt{\frac{\hbar}{m\omega}}\right)$$

$$a = \frac{1}{\sqrt{2m}}\left(\hbar\,\partial/\partial\xi\,\partial/\partial x\, x\sqrt{\frac{m\omega}{\hbar}} + m\omega\xi\sqrt{\frac{\hbar}{m\omega}}\right)$$

$$a = \frac{1}{\sqrt{2m}}\left(\hbar\sqrt{\frac{m\omega}{\hbar}}\,\partial/\partial\xi + m\omega\xi\sqrt{\frac{\hbar}{m\omega}}\right)$$

$$a = \left(\frac{1}{\sqrt{2m}}\right)\left(\sqrt{\frac{m\omega}{\hbar}}\right)\left(\hbar\,\partial/\partial\xi + m\omega\xi\frac{\hbar}{m\omega}\right)$$

$$a = \left(\frac{\hbar}{\sqrt{2m}}\right)\left(\sqrt{\frac{m\omega}{\hbar}}\right)\left(\partial/\partial\xi + \frac{m\omega}{m\omega}\xi\right)$$

$$a = \left(\frac{\hbar}{\sqrt{\hbar}}\right)\left(\frac{\sqrt{m}\sqrt{\omega}}{\sqrt{2m}}\right)\left(\partial/\partial\xi + \xi\right)$$

$$a = \sqrt{\frac{\hbar\omega}{2}}\left(\frac{d}{d\xi} + \xi\right)$$ as required ■

Similarly for $a^{+} = \frac{1}{\sqrt{2m}}(-i\hat{p} + m\omega x)$

$$a^{+} = \frac{1}{\sqrt{2m}}\left(-i \cdot -i\hbar\,\partial/\partial\xi\,\partial/\partial x\, x\sqrt{\frac{m\omega}{\hbar}} + m\omega\xi\sqrt{\frac{\hbar}{m\omega}}\right)$$

$$a^{+} = \frac{1}{\sqrt{2m}}\left(-\hbar\,\partial/\partial\xi\sqrt{\frac{m\omega}{\hbar}} + m\omega\xi\sqrt{\frac{\hbar}{m\omega}}\right)$$

$$a^{+} = \frac{\hbar}{\sqrt{2m}}\sqrt{\frac{m\omega}{\hbar}}\left(-\partial/\partial\xi + \frac{m\omega}{m\omega}\xi\right)$$

$$a^{+} = \frac{\hbar}{\sqrt{2m}}\sqrt{\frac{m\omega}{\hbar}}\left(-\partial/\partial\xi + \xi\right)$$

$$a^{+} = \left(\frac{\hbar}{\sqrt{\hbar}}\right)\left(\frac{\sqrt{m}\sqrt{\omega}}{\sqrt{2m}}\right)\left(-\partial/\partial\xi + \xi\right)$$

we see $a^{+} = \sqrt{\frac{\hbar\omega}{2}}\left(-\frac{d}{d\xi} + \xi\right)$ as required.

Q2 (c) Show LHS $\psi_n = \frac{1}{\sqrt{n!(\hbar\omega)^n}} (a^+)^n \psi_0$ for $n=0,1,2\ldots$

implies RHS $\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) \exp(-\frac{1}{2}\xi^2)$

Where $H_n(\xi) = (-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2)$

& $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{1}{2}\xi^2)$

$a^+ = \sqrt{\frac{\hbar\omega}{2}} \left(-\frac{d}{d\xi} + \xi\right)$

Proof by induction

Part 1 • Prove true for $n=0$:

LHS $\psi_0 = \frac{1}{\sqrt{1(\hbar\omega)^0}} \sqrt{\frac{\hbar\omega}{2}}^0 \left(-\frac{d}{d\xi} + \xi\right)^0 \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{1}{2}\xi^2)$

$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{1}{2}\xi^2)$

RHS $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{1}{\sqrt{2^0 0!}}\right) (-1)^0 \exp(\xi^2) \exp(-\xi^2) \exp(-\frac{1}{2}\xi^2)$

$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{1}{2}\xi^2) \cdot (1)(1)$

$\therefore \psi_0 = \psi_0$ for $n=0$ LHS = RHS proved.

Part 2 • Now assume true for $n=k$:

LHS $\psi_k = \frac{1}{\sqrt{k!(\hbar\omega)^k}} \left(\sqrt{\frac{\hbar\omega}{2}} \left(-\frac{d}{d\xi} + \xi\right)\right)^k \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(-\frac{1}{2}\xi^2)$

doesn't depend on ξ

$= \left(\frac{m\omega}{\pi\hbar}\right) \frac{\sqrt{\hbar\omega}^k}{\sqrt{\hbar\omega}^k} \sqrt{\frac{1}{k!2^k}} \left(-\frac{d}{d\xi} + \xi\right)^k \exp(-\frac{1}{2}\xi^2)$

LHS $= \sqrt{\frac{1}{k!2^k}} \left(-\frac{d}{d\xi} + \xi\right)^k \exp(-\frac{1}{2}\xi^2)$

$$RHS = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{k!\,2^k}} H_k(\xi) \exp(-\tfrac{1}{2}\xi^2)$$

Part 3 • Prove true for $n=k+1$

LHS $\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{k+1!\,2^{k+1}}} \left(-d/d\xi + \xi\right)^{k+1} \exp(-\tfrac{1}{2}\xi^2)$

$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{k+1!\,2^{k+1}}} \left(-d/d\xi + \xi\right)\left(-d/d\xi + \xi\right)^{k} \exp(-\tfrac{1}{2}\xi^2)$

Equating LHS and RHS in Part 2 gives us:

$$(-d/d\xi + \xi)^k = H_k(\xi)$$

$$H_k(\xi) = (-1)^k \exp(\xi^2) \frac{d^k}{d\xi^k} \exp(-\xi^2)$$

now LHS $= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{k+1!\,2^{k+1}}} \left[\left(-d/d\xi + \xi\right)(-1)^k \exp(\xi^2) d^k/d\xi^k \exp(-\xi^2)\right] \cdot \exp(-\tfrac{1}{2}\xi^2)$

inside square brackets $\left[(-d/d\xi + \xi) \cdot \underbrace{(-1)^k \exp(\xi^2) \frac{d^k}{d\xi^k} \exp(-\xi^2)}_{A}\right]$

$-d/d\xi\, A + \xi A$

• $-d/d\xi\, A = \left[-(-1)^k d/d\xi \exp(\xi^2) d^k/d\xi^k \exp(-\xi^2) - (-1)^k \exp(\xi^2) d^k/d\xi^k\, d/d\xi \exp(-\xi^2)\right]$

$= \left[(-1)^{k+1} \cdot 2\xi \exp(\xi^2) d^k/d\xi^k \exp(-\xi^2) - (-1)^{k+1} \cdot 2\xi \exp(\xi^2) d^k/d\xi^k \exp(-\xi^2)\right] = 0$

• $\xi A = (-1)^k \exp(\xi^2) d^k/d\xi^k \cdot \xi \cdot \exp(-\tfrac{1}{2}\xi) \exp(-\tfrac{1}{2}\xi)$

note: $-d/d\xi \exp(-\tfrac{1}{2}\xi^2) = \xi \exp(-\tfrac{1}{2}\xi)$

$\xi A = -(-1)^k \exp(\xi^2) d^k/d\xi^k\, d/d\xi \exp(-\tfrac{1}{2}\xi) \exp(-\tfrac{1}{2}\xi)$

$= (-1)^{k+1} \exp(\xi^2) d^{k+1}/d\xi^{k+1} \exp(-\xi^2)$

$= H_{n+1}(\xi)$

$$\text{LHS} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{(k+1)!}\sqrt{2^{k+1}}} H_{k+1}(\xi) \exp\left(-\tfrac{1}{2}\xi^2\right)$$

$$\text{RHS} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{(k+1)!\,2^{k+1}}} H_{k+1}(\xi) \exp\left(-\tfrac{1}{2}\xi^2\right)$$

LHS = RHS for $n = k+1$ thus proven by induction in n ■

$H_n(\xi) = (-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2)$

Q2(d) Show $a\psi_n = \sqrt{n\hbar\omega}\,\psi_{n-1}$

Implies that $\frac{d}{d\xi} H_n(\xi) = 2_n H_{n-1}(\xi)$

LHS $a = \sqrt{\frac{\hbar\omega}{2}}\left(\frac{d}{d\xi} + \xi\right)$ $\quad \psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) \exp(-\frac{1}{2}\xi^2)$

RHS $\sqrt{n\hbar\omega}\,\psi_{n-1} = \sqrt{n\hbar\omega}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^{n-1} n-1!}} H_{n-1}(\xi) \exp(-\frac{1}{2}\xi^2)$

LHS $\frac{\sqrt{\hbar\omega}}{\sqrt{2}} \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n(\xi) \exp(-\frac{1}{2}\xi^2)$

$= \frac{\sqrt{\hbar\omega}}{\sqrt{2}\sqrt{2^n}\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n(\xi) = \frac{\sqrt{\hbar\omega}}{2^{\frac{n+1}{2}}\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n(\xi)$

$2^{1/2} 2^{n/2} = 2^{\frac{n+1}{2}}$

RHS $= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{\sqrt{n\hbar\omega}}{\sqrt{2^{n-1} n-1!}} H_{n-1}(\xi) \exp(-\frac{1}{2}\xi^2)$

Forgot to add $(\partial/d\xi + \xi)$ on LHS ~~RHS~~

LHS $= \frac{\sqrt{\hbar\omega}}{2^{\frac{n+1}{2}}\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{\partial}{d\xi} + \xi\right) H_n(\xi) \exp(-\frac{1}{2}\xi^2)$

RHS $= \frac{\sqrt{n\hbar\omega}}{\sqrt{2^{n-1}(n-1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(H_{n-1}(\xi)\right) \exp(-\frac{1}{2}\xi^2)$

LHS $= \frac{\sqrt{\hbar\omega}}{2^{\frac{n+1}{2}}\sqrt{n!}} * \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{\partial}{\partial\xi} + \xi\right)\left[(-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2)\right] \exp(-\frac{1}{2}\xi^2)$

$= \frac{\sqrt{\hbar\omega}}{2^{n+1/2}\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[\quad \exp(-\frac{1}{2}\xi^2)\right]$

$[\,] = \left(\partial/\partial\xi + \xi\right)\left[\cancel{(-1)^n \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2)}\right]$ $H_n(\xi)\exp(-\frac{1}{2}\xi^2)$

$= \left(\partial/\partial\xi + \xi\right) H_n \exp(-\frac{1}{2}\xi^2)$

$$= \frac{\partial}{\partial \xi}\left(H_n \exp(-\tfrac{1}{2}\xi^2)\right) + \xi H_n \exp(-\tfrac{1}{2}\xi^2)$$

Product Rule

$$= \frac{\partial}{\partial \xi} H_n \cdot \exp(-\tfrac{1}{2}\xi^2) + H_n \cdot -\xi \cancel{\exp(-\tfrac{1}{2}\xi^2)} + \cancel{\xi H_n \exp(-\tfrac{1}{2}\xi^2)}$$

$$= \frac{\partial}{\partial \xi} H_n \cdot \exp(-\tfrac{1}{2}\xi^2)$$

$$\cancel{+ (-1)^n \cdot -\frac{\partial}{\partial \xi} \exp(\xi^2) \frac{d}{d\xi} \exp(-\xi^2)}$$

$$\cancel{+ (-1)^n \xi \exp(\xi^2) \frac{d^n}{d\xi^n} \exp(-\xi^2)}$$

$$\cancel{(-1)^n \frac{\partial}{\partial \xi} \exp(\xi^2) \frac{d^{n+1}}{d\xi^{n+1}} \exp(-\xi^2)}$$

$$LHS = \frac{\sqrt{\hbar\omega}}{2^{\frac{n+1}{2}} \sqrt{n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{\partial}{\partial \xi} H_n \exp(-\tfrac{1}{2}\xi^2)$$

$$RHS = \frac{\sqrt{n\hbar\omega}}{\sqrt{2^{n-1}(n-1)!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_{n-1} \exp(-\tfrac{1}{2}\xi^2)$$

$$\boxed{\frac{1}{(n-1)!} = \frac{n}{n!}}$$

$$= \frac{\sqrt{n\hbar\omega}\sqrt{n}}{\sqrt{2^{n-1} n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_{n-1} \exp(-\tfrac{1}{2}\xi^2)$$

$$= \frac{\sqrt{2}\cdot\sqrt{2}\sqrt{n\hbar\omega}\sqrt{n}}{\sqrt{2}\sqrt{2}\sqrt{2^{n-1}}\sqrt{n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_{n-1} \exp(-\tfrac{1}{2}\xi^2)$$

$$= \frac{2n\sqrt{\hbar\omega}}{\sqrt{2^{n+1}}\sqrt{n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_{n-1} \exp(-\tfrac{1}{2}\xi^2)$$

Now LHS = RHS $\quad \frac{\sqrt{\hbar\omega}}{\sqrt{2^{n+1} n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{\partial}{\partial \xi} H_n(\xi) \exp(-\tfrac{1}{2}\xi^2)$

$$= \frac{2n\sqrt{\hbar\omega}}{\sqrt{2^{n+1}}\sqrt{n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_{n-1} \exp(-\tfrac{1}{2}\xi^2)$$

Cancel similar terms

$$\frac{\partial}{\partial \xi} H_n(\xi) = 2n H_{n-1}(\xi)$$

as we wanted to show ■

Q3 Let $\{\psi_n, n=1,2,\ldots\}$ be an orthogonal set of energy eigenfunctions with eigenvalues E_n

general wavefunction $\Psi = \sum_{n\geq 1} c_n \psi_n$

(a) recall $\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}$

$$\Psi = \sum_{m=1}^{\infty} c_m \psi_m$$

$$\Psi^* = \sum_{n=1}^{\infty} c_n \psi_n^*$$

sub in

$\boxed{\text{Similarly } \int_{-\infty}^{\infty} \psi_n^* \psi_m \, dx = \delta_{nm}}$

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = \int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1$$

$$\int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} c_n \psi_n^*\right)\left(\sum_{m=1}^{\infty} c_m \psi_m\right) dx = 1$$

$$\int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty}\left(c_n \psi_n^* \sum_{m=1}^{\infty} c_m \psi_m\right)\right) dx = 1$$

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\underbrace{c_n c_m}_{\text{constants}} \psi_n^* \psi_m\right) dx = 1$$

→ $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(c_n c_m \int_{-\infty}^{\infty} \psi_n^* \psi_m \, dx\right) = 1$$ $\boxed{\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}}$

→ $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(c_n c_m \delta_{nm}\right) = 1$$

$\boxed{\delta_{mn} = 1 \text{ for } m=n \text{ thus } c_m = c_n}$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_n c_m (1) = \sum_{n\geq 1} c_n c_n = 1$$

$$= \sum_{n\geq 1}^{\infty} |c_n|^2 = 1$$

as required ∎

(b) Show $\langle H \rangle = \sum_{n \geq 1} E_n |C_n|^2$

$$\hat{H} = \hat{T} + \hat{V} = E \rightarrow \hat{H}\Psi_n = E_n \Psi_n$$

$$\langle \hat{H} \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{H} \Psi \, dx$$

$$\langle \hat{H} \rangle = \int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} C_n \Psi_n^* \right) \hat{H} \left(\sum_{m=1}^{\infty} C_m \Psi_m \right) dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} C_n \Psi_n^* \right) \left(\sum_{m=1}^{\infty} C_m \hat{H} \Psi_m \right) dx$$

$\hat{H}\Psi_m = E_m \Psi_m$

$$\int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} C_n \Psi_n^* \right) \left(\sum_{m=1}^{\infty} C_m E_m \Psi_m \right) dx$$

$E = \text{const.}$

$$\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_n \Psi_n^* C_m E_m \Psi_m \, dx$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_n C_m E_m \int_{-\infty}^{\infty} \Psi_n^* \Psi_m \, dx$$

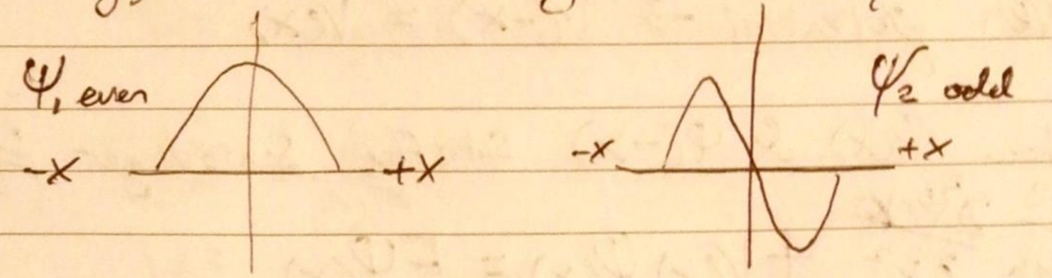
$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_n C_m E_m \delta_{nm}$$

$\delta_{nm} = 1$ for $m = n$
$\rightarrow C_m = C_n$
$\rightarrow E_m = E_n$

$$\sum_{n=1}^{\infty} C_n C_n E_n \delta_{nm} = \sum_{n=1}^{\infty} C_n C_n E_n (1)$$

Thus $\langle \hat{H} \rangle = \sum_{n \geq 1}^{\infty} |C_n|^2 E_n$ as we wanted to show ■

Q4 Show that if $\Psi(x)$ is an energy eigenstate with energy E for a 1-d system with Potential $V(x)$, then $\Psi(-x)$ is an energy eigenstate with Energy E for the system with potential $V(-x)$.



$\hat{H} = \hat{T} + \hat{V} = E$ 1-D Schrödinger Equation

$$-\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \Psi(x)}{\partial x^2} + V(x)\Psi(x) = i\hbar\frac{\partial \Psi}{\partial t} = E\Psi$$

$\Psi(x)$, E, $V(x)$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x)}{\partial x^2} + V(x)\Psi(x) = E\Psi(x) \quad \checkmark$$

then $\Psi(-x)$ eigenstate for Energy E & potential $V(-x)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(-x) + V(-x)\Psi(-x) = E\Psi(-x)$$

$\Psi(x)$ odd $\rightarrow$ $\Psi(-x) = -\Psi(x)$

$$-\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) + (-V(-x)\Psi(x)) = -E\Psi(x)$$

$$= -\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x)}{\partial x^2} + V(x)\Psi(x)\right] = -E\Psi(x)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x)}{\partial x^2} + V(-x)\Psi(x) = E\Psi(x)$$

$\Psi(x)$ even $\Psi(-x) = \Psi(x)$ Eqn becomes

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x)}{\partial x^2} + V(-x)\Psi(x) = E\Psi(x)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x)}{\partial x^2} + V(-x)\Psi(x) = E\Psi(x)$$

Thus $\Psi(-x)$ is an energy eigenstate with Energy E for the system with potential $V(-x)$.

Show that the energy eigenstates with an even potential function can be taken to be either even or odd functions of x.

even potential $\rightarrow V(-x) = V(x)$

Show $\Psi(x)$ & $\Psi(-x)$ satisfy Schrödinger Eqn.

1 $$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi(x)}{\partial x^2} + V(x)\Psi(x) = E\Psi(x)$$

2 $$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi(-x)}{\partial x^2} + V(-x)\Psi(-x) = E\Psi(-x) \quad ✓.$$

2 becomes $$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi(-x)}{\partial x^2} + V(x)\Psi(-x) = E\Psi(-x)$$

Since $V(x) = V(-x)$,
If $\Psi(x)$ & $V(x)$ are even, this implies $\Psi(-x)$ satisfies Schrödinger Equation.

Can construct a 3rd Solution as a linear Combination of two solutions

$$\Psi_3 = C_1\Psi_1(x) + C_2\Psi_2(x)$$

take $\Psi_+ = \Psi(+x) + \Psi(-x)$

$$\Psi_+(-x) = \Psi(-x) + \Psi(-(-x)) = \Psi_+(x)$$

$\rightarrow \Psi_+$ is an even function, linear combination of two solns to time independent 1D Schrödinger Eqn.

Ψ_+ is a solution to Schrödinger equation, following from 1

define $\Psi_-(x) = \Psi(x) - \Psi(-x)$

$$\Psi_-(-x) = \Psi(-x) - \Psi(x) = -\Psi_-(x)$$

thus $\Psi_-(x)$ is an odd wave function

$\psi_-(x)$ is also a solution to the time-independent Schrödinger Eqn

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi_-(-x)}{\partial x^2} + V(-x)\,\psi_-(-x) = E\,\psi(-x)$$

$$\frac{+\hbar^2}{2m}\frac{\partial^2\psi_-(x)}{\partial x^2} - V(x)\,\psi_-(x) = -E\,\psi_-(-x)$$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2\psi_-(x)}{\partial x^2} + V(x)\,\psi_-(x) = E\,\psi_-(x) \quad \checkmark$$

Show for $\psi_+ \rightarrow$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi_+(-x)}{\partial x^2} + V(-x)\,\psi_+(-x) = E\,\psi_+(-x)$$

$V(x)$ & ψ_+ even

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_+(x)}{\partial x^2} + V(x)\,\psi_+(x) = E\,\psi_+(x) \quad \checkmark$$

$\blacksquare$

Thus energy eigenstates with even potentials can be taken to have even or odd functions of x.