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Maths Methods II Assignment 1

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IBVP:

$\alpha = 1$ $\quad \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$ ①

$u(0,t) = 0$ $\quad \frac{\partial u}{\partial x}(1,t) = 0$, $t > 0$ ②

$u(x,0) = 6$, $0 < x < 1$ ③

1. Find Separable Solutions to ① for $u(x,t) = X(x)T(t)$

given ① $\quad \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

sub in $u(x,t) = X(x)T(t)$

$$\frac{\partial}{\partial t}(X(x)T(t)) = \frac{\partial^2}{\partial x^2}\alpha^2(X(x)T(t))$$

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

divide by $X(x)$ and $T(t)\alpha^2$

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

equation is now of the form $g(t) = h(x)$
differentiating w.r.t. x gives us $h'(x) = 0$
So both equations are equal to a constant.
Define this constant as $-\lambda$

$$\frac{1}{\alpha^2}\cdot\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

then we have two equations

$$T'(t) = -\lambda T(t)\alpha^2$$

$$X''(x) = -\lambda X(x)$$

i $\boxed{T'(t) + \lambda\alpha^2 T(t) = 0}$

ii $\boxed{X''(x) + \lambda X(x) = 0}$

② $u(0,t) = 0 \quad \frac{\partial u}{\partial x}(1,t) = 0$

Can solve (i) using the Integration factor method

(i) $T'(t) + \lambda\alpha^2 T(t) = 0$

$I = e^{\int \lambda\alpha^2 dt} = e^{\lambda\alpha^2 t}$

$$e^{\lambda\alpha^2 t}T'(t) + \lambda\alpha^2 e^{\lambda\alpha^2 t}T(t) = 0$$

$$\frac{d}{dt}\left(e^{\lambda\alpha^2 t}\cdot T(t)\right) = 0$$

$$e^{\lambda\alpha^2 t}T(t) = C = \text{constant}$$

$$\boxed{T(t) = Ce^{-\lambda\alpha^2 t}}$$

$$\boxed{X''(x) + \lambda X(x) = 0}$$

2 $\boxed{\lambda < 0}$

$$T(t) \to \infty \quad \text{as} \quad t \to \infty$$

This does not make sense physically as we expect $T(t)$ to tend to 0 as t goes to +infinity.
This solution is discarded.

$\boxed{\lambda = 0}$ $T(t) = C$

$$X''(x) + 0(X(x)) = 0$$

$$X''(x) = 0$$

$$X(x) = Ax + B$$

apply BC 2

$u(0,t) = 0 \to C(A(0) + B) = 0 \quad CB = 0$

$\frac{\partial u}{\partial x}(1,t) = 0 \to C(A) = 0 \to CA = 0$

either $C = 0$ or A & $B = 0$

only trivial solution is possible for $\lambda = 0$ so this solution is ignored.

③ $\boxed{\lambda > 0}$ $T(t) = Ce^{-\lambda\alpha^2 t}$

$$X''(x) + \lambda X(x) = 0$$

$t \to \infty$, $T \to 0$ ✓

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

$$u(x,t) = X(x)T(t)$$

$$u(x,t) = Ce^{-\lambda\alpha^2 t}\cdot\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right)$$

$$= e^{-\lambda\alpha^2 t}\cdot\left(D\cos(\sqrt{\lambda}x) + E\sin(\sqrt{\lambda}x)\right)$$

② $u(0,t) = 0 \to e^{-\lambda\alpha^2 t}\cdot(D + 0) = 0$

$\to D = 0$

$\frac{\partial u}{\partial x}(1,t) = 0 \to e^{-\lambda\alpha^2 t}\left(D\sin(\sqrt{\lambda}(1)) + E\cos(\sqrt{\lambda}(1))\right)$

$D=0 \to e^{-\lambda\alpha^2 t}\left(E\cos(\sqrt{\lambda})\right) = 0$

Since $\lambda > 0$, $\alpha = 1$, $E \neq 0$

then $\cos(\sqrt{\lambda}) = 0$ must be true.

$\cos(x) = 0$ when $x = \frac{(2(n)+1)\pi}{2}$

Then there exists an infinity of solutions for

$$\sqrt{\lambda_n} = \frac{(2n+1)\pi}{2} \to \lambda_n = \frac{(2n+1)^2\pi^2}{4}$$

$$\lambda_n = (n^2 + n + 1/4)\pi^2$$

(4) Superpose all corresponding eigenfunctions & show that soln satisfying ~~opp~~ condition (3) can be of the form

$$u(x,t) = \sum_{n=0}^{\infty} C_n \left[\sin(n+\tfrac{1}{2})\pi x\right] e^{-(n+\frac{1}{2})^2\pi^2 t}$$

found $u_n(x,t) = e^{-\lambda\alpha^2 t}\left(E \sin\left(\frac{(2n+1)\pi}{2}x\right)\right)$

Superpose all u_n's

$E = C_n$ $\quad u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{2n+1}{2}\pi x\right) \cdot e^{-\frac{(2n+1)^2\pi^2}{2^2}t}$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin((n+\tfrac{1}{2})\pi x) \cdot e^{-(n+\frac{1}{2})^2\pi^2 t}$$

as required.

need to satisfy (3) $\quad u(x,0) = 6, \quad 0 < x < 1$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin((n+\tfrac{1}{2})\pi x) = 6$$

Since C_n's are fourier sine coefficients, can multiply ~~this~~ by $\sin((m+\frac{1}{2})\pi x)$ and integrate from 0 to 1

$$\sum_{n=1}^{\infty} C_n \int_0^1 \underbrace{\sin((n+\tfrac{1}{2})\pi x)\sin((m+\tfrac{1}{2})\pi x)}_{=0 \text{ unless } m=n}\,dx = \int_0^1 \sin((m+\tfrac{1}{2})\pi x)\,6\,dx$$

$$\sum_{n=1}^{\infty} C_n \int_0^1 \sin^2((n+\tfrac{1}{2})\pi x)$$

$\sin^2(A) = \frac{1}{2}(1-\cos(2A))$

$$= \sum_{n=1}^{\infty} C_n \int_0^1 \tfrac{1}{2}\left(1-\cos(2(n+\tfrac{1}{2})\pi x)\right)dx$$

$$= \sum_{n=1}^{\infty} C_n \int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2}\cos((2n+1)\pi x)\right)dx$$

$$= \sum_{n=1}^{\infty} C_n \left[\frac{x}{2} - \frac{1}{(2n+1)\pi}\underbrace{\sin((2n+1)\pi x)}_{0}\right]_0^1$$

$$= \sum_{n=1}^{\infty} C_n \cdot \tfrac{1}{2} = 6\int_0^1 \sin((n+\tfrac{1}{2})\pi x)\,dx$$

$$\sum_{n=1}^{\infty} C_n \cdot \tfrac{1}{2} = -6 \cdot \frac{1}{(n+\frac{1}{2})\pi} \cos\left((n+\tfrac{1}{2})\pi x\right)\Big|_0^1$$

$$\sum_{n=1}^{\infty} C_n \cdot \tfrac{1}{2} = \frac{-6}{(n+\frac{1}{2})\pi}\left(\underbrace{\cos\left((n+\tfrac{1}{2})\pi\right)}_{=0} - 1\right)$$

$$\sum_{n=1}^{\infty} C_n = \sum_{n=1}^{\infty} \frac{-12}{(n+\frac{1}{2})\pi}\left(\cancel{\cos\left((n+\tfrac{1}{2})\pi\right)} - 1\right)$$

$$\sum_{n=1}^{\infty} C_n = \sum_{n=1}^{\infty} \frac{12}{(n+\frac{1}{2})\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{12}{(n+\frac{1}{2})\pi} \cdot \sin\left((n+\tfrac{1}{2})\pi x\right) e^{-(n+\frac{1}{2})^2\pi^2 t}$$

> $assume(n, integer);$

> $solve\left(c\_n \cdot int\left(\left(\sin\left(\left(n+\frac{1}{2}\right)\mathrm{Pi}\cdot x\right)\right)^2, x=0..1\right) = 6\cdot int\left(\left(\sin\left(\left(n+\frac{1}{2}\right)\mathrm{Pi}\cdot x\right)\right), x=0..1\right), c\_n\right)$ *#solve for c_ns*

Warning, solving for expressions other than names or functions is not recommended.

Warning, solve may be ignoring assumptions on the input variables.

$$\frac{24}{(2\,n\sim+1)\,\pi} \tag{1}$$

> $12\left(\int_0^1 \sin\left(\left(n+\frac{1}{2}\right)\pi\,x\right)\mathrm{d}x\right)$

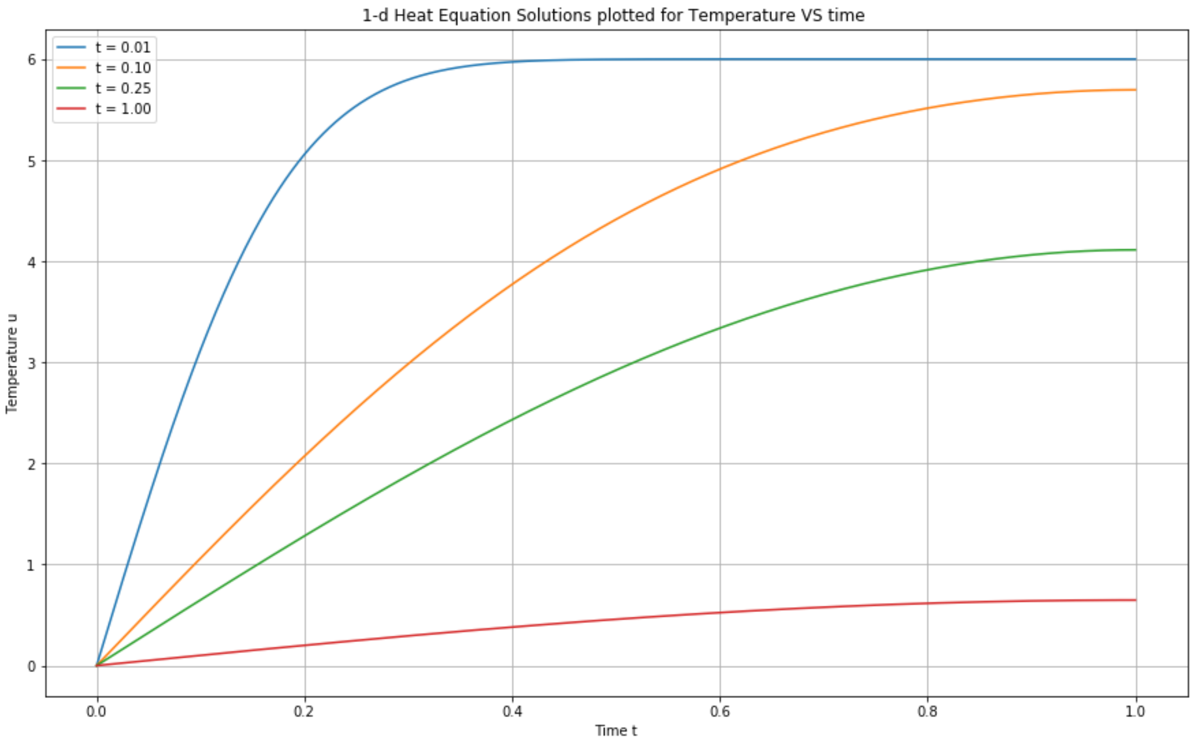
$$\frac{24}{(2\,n\sim+1)\,\pi} \tag{2}$$

> $c\_n := \frac{24}{(2\,n+1)\mathrm{Pi}} = \frac{12}{\left(n+\frac{1}{2}\right)\mathrm{Pi}}$

$$c\_n := \frac{24}{(2\,n\sim+1)\,\pi} = \frac{12}{\left(n\sim+\frac{1}{2}\right)\pi} \tag{3}$$

>

In [21]:
```
import numpy as np
import matplotlib.pyplot as plt

#set N = 1000
N = 1000

#create range of x values
x = np.linspace(0,1,N)

def u(x,t): # defiine function for solution
    total = 0
    for n in range(N):
        total += (12/((n+0.5)*(np.pi)) * np.sin((n+0.5)*np.pi*x) * np.exp(-(n+
0.5)**2 *(np.pi)**2 * t))
    return total

#create 4 arrays to plot solutions at 4 times
u_1 = []
u_2 = []
u_3 = []
u_4 = []

#populate arrays
for i in range(N):
    u_1.append(u(x[i],0.01))
    u_2.append(u(x[i],0.10))
    u_3.append(u(x[i],0.25))
    u_4.append(u(x[i],1.00))

#plot
plt.figure(figsize=(15,9))
plt.plot(x,u_1, label = "t = 0.01")
plt.plot(x,u_2, label = "t = 0.10")
plt.plot(x,u_3, label = "t = 0.25")
plt.plot(x,u_4, label = "t = 1.00")
plt.title("1-d Heat Equation Solutions plotted for Temperature VS time ")
plt.ylabel("Temperature u")
plt.xlabel("Time t")
plt.legend()
plt.grid()
plt.show()
```


1-d Heat Equation Solutions plotted for Temperature VS time
t = 0.01
t = 0.10
t = 0.25
t = 1.00
Temperature u
Time t
0
1
2
3
4
5
6
0.0
0.2
0.4
0.6
0.8
1.0