MS6032 Networks and Complex Systems

A Review of the Voter Model on Networks

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Abstract

In the last two decades, there has been rapid development in areas in Economics, Biology and the Social Sciences through the applications of theories from statistical physics. The voter model is a rather simple stochastic process that is used to describe the formation and transfer of opinions between individuals on a network or lattice. This simple model provides an interesting insight into how the seemingly random behaviour of opinion formation between individuals can create interesting patterns and structures in networks. This review aims to give a background behind some of the voter model algorithms, how they work and study how they perform on different network configurations.

1 Introduction

In the mid-19th century physicists like Boltzmann discovered that the underlying properties and rules governing thermodynamics were stochastic by nature. Instead of looking at systems from a macroscopic perspective, there was a shift in physics to begin looking at systems from a microscopic perspective. Statistical physics sought to explain macroscopic processes through statistics and mechanics on a microscopic level. This approach to physics proved very successful in the fields of thermodynamics and other branches of physics.

This statistical framework of physics was a great success and scientists began to try to apply ideas from statistical physics to other fields, such as Economics, Sociology and Biology [1]. In this review paper I will primarily focus on the statistical approaches in modelling social dynamics on networks and in particular the voter model.

I aim to give an overview of the algorithms behind the standard voter model, the majority voter model and the noisy majority voter model. I will also be looking at variations of the voter model with the inclusion of strong opinionated individuals and stubborn 'zealots' who are reluctant to change their opinions. I will also be investigating the mean times for voter model convergence for a selection of different network configurations and looking at general scaling rules for different connected networks in terms of number of nodes N.

A simple voter model may be considered on a Complete Graph or on a regular lattice consisting of voters (nodes) connected to one another by some kind of social contact (edges), Each node has a binary state which can be considered the voter's opinion. the simplest voter model consists of voters having 2 different opinions, e.g. 'red' and 'blue' or 'republican' and 'democrat'. These opinions can be numerically represented as [-1,+1].

In the standard voter model, a node is chosen at random, and it adopts the opinion of one randomly chosen neighbouring node [2]. This type of voter model is sometimes known as 'classic voter model'. It is sometimes called the basic voter model in the case where there are only two opinions, but models can be generalised to include as many opinions as there are nodes in the network.

Other variations of the voter model exist such as the 'edge-update' model where a random edge between two neighbouring nodes is chosen, if the two nodes share the same state, nothing happens. If the two nodes are of different states, then one of the two nodes is chosen at random and takes the state of the node at the opposing end of the edge. Another variation is a majority-vote model where a node is chosen at random and this node adopts the opinion of the majority opinion of its nearest neighbours [3].

These voting algorithms try to capture a sense of peer-pressure in social networks. The standard voter model gives a sense of pressure from peers but they do not factor in pressure from the majority into the model. The majority opinion model better models the pressure on an individual to align with the majority of people in society, but it is more deterministic in nature and loses an element of randomness in the decision-making process. The Majority vote model also means that opinions held by a few individuals are likely going to be wiped out very quickly. Another iteration of the majority voter model is the noisy majority voter model, which introduces a probability q that a minority opinion is chosen.

Probabilists were able to solve the voter model in any dimension d by using the fact that the model can be mapped to random walkers that coalesce upon encounter. It has been proven that the standard voter model converges to one network-wide consensus for all finite networks with dimension $d \leq 2$, and for $d > 2$, no consensus is reached [4] [5]. This means that given enough time, these models will always eventually reach a consensus in the 1 and 2 dimensional cases. Some properties of voter models that are of interest are how fast do the models converge for different network structures, and the differences between the voting model algorithms as well as the differences in opinion propagation for different network structures.

Another topic of interest is how do voter models behave for more than 2 dimensions? And how do voter models on infinite networks behave? There are many interesting variations on the voter model which look at opinions between communities and 'zealot' nodes who do not change their opinions for example, which we will investigate in this paper also.

2 First Voter Models

The first voter model was proposed by Clifford and Sudbury in 1973 where they considered a mathematical model for the competition of species of species [6]. It would not be until Holley and Liggett's 1975 paper that it would be known as the voter model [7].

Clifford and Sudbury's paper describes a Markov process for the competition of two fairly matched species competing for territory. These species are denoted as 'black' or 'white' cells. At each time step either a swapping of two cells' position on the lattice occurs through a swapping process or else an invasion process occurs where within a pair of adjacent black and white cells, a new black or white cell is created and replaces the cell of the other colour.

Random walks determine the probabilities of positions being occupied in both the swapping and invasion processes.

Holley and Liggett's paper [7] introduced the standard voter model where sites flip at a rate equal to the fraction of neighbours that have the other opinion. In their paper they also stated that for the standard voter model, consensus is reached for one or two dimensional lattice networks as $t \to \infty$, but in higher dimensions disagreements in opinion continue to exist.

3 Introducing the Ising Model

Physicists who had an interest in solving problems outside of physics introduced ideas from statistical physics to problems in the other sciences and the humanities. One such idea is the Ising model which models spins in ferromagnets. This model treats each particle like an electron in a ferromagnet. The property that gives ferromagnets their magnetism is the alignment of the spins of its particles.

We consider a 2-d lattice where each site i has 4 neighbours - up, left, down and right. Each site has a spin of $\pm S_i$. Neighbouring particles want to have spins aligned in the same direction - each particle wants to be aligned parallel to neighbouring particles' spins. particles have an energy $-J$ if they are in the same spin state and energy J if they are in the same state. A magnetic field (denoted by H or sometimes also B) tries to orientate particles' spin in the direction of the magnetic field also [8]
. This gives way to the following equation for the energy of the system:

$$E = -J \sum_{<ij>} S_i S_j - H \sum S_i \, . \tag{1}$$

Another term of interest is the "magnetisation" which is given by the number of up spins minus the number of down spins. It can be written as:

$$M = \sum_i S_i \, . \tag{2}$$

This framework is quite useful for simulating computer simulations of voter models and can also be generalised into d dimensions. Terminology from physics models such as the Ising and Glauber dynamics like "spin" and "magnetisation" are used quite frequently in papers on voter models and social science viewed through a mathematical lens.

The Ising model can be applied to a situation where everyone tries to convince their own opinion to others (J). H may represent the opinion a government or group may be trying to convince other people of. Then temperature (energy) is a measure of conformity. A temperature of 0 means that there is complete conformity - everyone agrees on an opinion - and a temperature of $+\infty$ means that nobody agrees on an opinion, everyone has differing opinions.

As long as the network is connected and finite, the probability that the system converges to one single state is always greater than zero (for $d \leq 2$). This applies for latices and complete graphs like in early models but also to any voter model which is connected and finite [9] [10]. However, it may take a very long time for the model to reach a saturated state. The convergence time for voter networks will be examined in the next section.

4 Standard Voter Model

The most common iteration of the voter model used is called the standard voter model. In this algorithm a node in the network is chosen at random at each time step and it randomly chooses a neighbouring node it is connected to and takes the opinion of that neighbouring node. We consider a set of opinions S in a network of total size N. The maximum possible size of the set S is the number of nodes in the network where $S - 1, 2, ..., N$. Usually we are concerned with cases with a much smaller opinion set, say for example a 2-state voter model with $S = 1, 2$.

Using k to an iteration of the node update procedure at a given timestep, at the iteration $k + 1$, the node $\mathcal{I}_{k+1}$ is updated as follows:

$$c_{\mathcal{I}_{[k+1]}}[k+1] = f(c_{i,1}[k], c_{i,2}[k], ..., c_{i,d_i}[k]) \,. \tag{3}$$

This means that at each timestep, a randomly chosen node updates its state to the state of a randomly chosen neighbour. $c_{i,j}[k]$ is the colour of the j^{th} neighbour of node i at iteration k. Only one node can change state during an iteration, so $c_j[k+1] = c_j[k] \forall j \neq \mathcal{I}[k+1]$

We focus on undirected graphs with no self-loops here. The graphs of interest to us here are finite random networks which are fully connected. A property of particular interest with random networks is the small world property. The mean distance between nodes in a random network is $< d > \approx logN$, which is much smaller than that seen in a 2D or 3D lattice which each have approximate mean distances of $N^{1/2}$ and $N^{1/3}$ respectively.

The mean time T_N to reach consensus in a finite network of N voters can be estimated. For regular lattices in 1 dimension, time scales with N^2, T_N scales with $Nln(N)$ in a 2-dimensional lattice and T_N scales with N in a lattice with dimension $d > 2$ [11] [12]. In heterogeneous networks with broad degree distributions T_N tends to scale sublinearly with N. T_N scales as $N\frac{\mu_1^2}{\mu_2}$ where μ_k is the kth moment of the degree distribution where the degree distribution is arbitrary and uncorrelated. [13]. For power law degree distributions $n_k\ k^{-\gamma}$, T_N scales as N for $\gamma > 3$ and scales as $N/ln(N)$ for $\gamma = 3$, as $N^{(2\gamma-4/\gamma-1)}$ for $2 < \gamma < 3$ and as $(ln(N))^2$ for $\gamma = 2$ [13].

In the paper by M. Yildiz et al. they showed that there are upper bounds on the time it takes for a voter model on a network to converge to a saturated steady state [9]. They discovered the maximum bound on a voter network's expected convergence time is

$$E[T] \leq \frac{4\,e\,log(N+2)\,|\epsilon|}{1-\lambda_2(\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2})} \max_j D_{jj}^{-1}\,. \tag{4}$$

Where A is the adjacency matrix of the network, D is the degree matrix of the network, $\lambda_2\,(\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2})\lambda_2)$ is the second largest eigenvalue. There are N number of nodes and $|\epsilon|$ number of edges in the network. The eigenvalues v_k and eigenvectors $\lambda_k(.)$ are well defined since $\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ is a real symmetric matrix.

Equation 4 allows us to put bounds on the expected convergence times for some network structures, namely the complete graph, star graph, cycle graph and name node. To do this computation for the complete graph case, we take the number of edges $|\epsilon| = N(N-1)/2$, the degree for each node equal to $N-1$ and $1-\lambda_2(\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}) = N/(N-1)$. From equation 4 for the complete graph, we obtain:

$$E[T] \leq e(N-1)log(N+2).$$

This process can be done in a similar way to obtain the results for the other network structures mentioned above. The results can be found in table 1 [9].

Table 1: Convergence time for different Network structures

Network Structure	Bounds on expected time for convergence
Complete Graph	$e(N-1)log(N+2)$
Star Graph	$e(2N-2)log(N+2)$
Cycle Graph	$eN^2log(N+2)$
Line Graph	$e(N-1)^2log(N+2)$

The authors [9] also showed the upper bound on the convergence of Erdos-Renyi graphs with $p >> \frac{log^2(N)}{N}$. They that the expected time for an Erdos-Renyi graph to reach a consensus scales as:

$$E[T] \approx O\left(\frac{4e\,log(N+2)N\,(1+2\sqrt{log(N)/Np})}{1-\frac{8}{Np}-\frac{g(n)log^2(N)}{Np}}\right).$$

where $g(n)$ is a function that tends to infinity at a very slow rate. This shows that the convergence time for the voter model in a network is dependent on the network's size and the degrees of the nodes in the network. Cox showed in his 1989 paper that the time to reach a consensus depends on the size N of the system and its dimension d [14]. Here we have focused on 1-d and 2-d voter models in particular. For finite cases, the 1-d and 2-d voter model will always reach a consensus.

5 Majority rule voter model

Another variation of the voter model is instead of using the standard voter model algorithm, instead a random node is chosen from the network and it takes on the opinion of the majority opinion of its neighbours. This is known as the majority If we define the number of node i's neighbours who share the same opinion c at iteration k to be $Q_{i,c}[k]$, then this update rule can be written as

$$c_i[k+1] = arg \max_c Q_{i,c}[k] \,. \tag{5}$$

If more than one opinion is tied for the majority opinion, then one of the opinions is chosen randomly from the group of tied opinions. Once the initial node i is chosen in an iteration, the way the opinion is adopted by node i is in a deterministic fashion, which means that the model behaves rather differently to the stochastic fashion of adopting the opinion of a randomly selected neighbouring node.

The majority vote model has some interesting properties. As the number of iterations increases, the number of opinions decreases. Unlike the previous voter model where a node's state is taken from a random nearest neighbour, the majority vote model is not guaranteed to reach convergence to a consensus. Over time communities - subgraphs which have a greater number of internal edges than external edges - end up adopting their own opinions [9]. There is disagreement between the two update rules: from the node update rule seen in equation 3 we saw that a consensus is always reached as $t \to \infty$ and using the update rule from equation 5 we see that there is no guarantee of reaching a consensus.

This differs from the behaviour of the standard voter model we saw previously which behaves much more randomly, the standard voter model manages to model the random peer-to-peer infuence of opinions but fails to incorporate a sense of pressure on individuals from an overwhelming majority opinion. The majority voter model also tends to assign one opinion to entire communities which is something we observe in real life where people tend to make connections with people similar to them in some way [15] [16].

6 Voter model on network partitioned into two cliques

In the previous two sections we have touched on how the standard voter model is insensitive to changes in community structure i.e. the presence of communities or cliques in the network does not affect the outcome of the model. This was believed to be true based on networks with two large cliques of similar size. It was believed that the mean consensus time was proportional to N except for cases where the connections between cliques are extremely sparse [10]. An investigation into the standard voter model behaviour with two cliques of unequal size, each clique with a different opinion showcasing a polarising society. The inves-

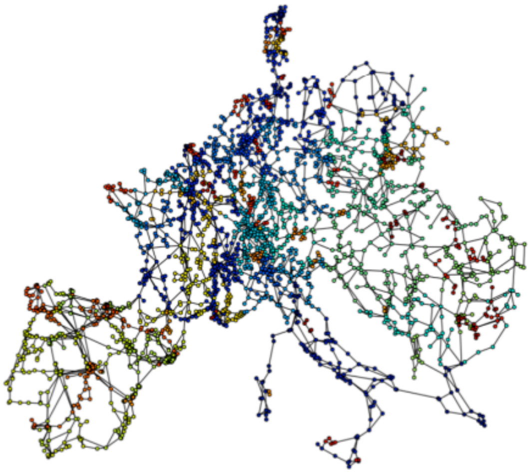

Figure 1: Majority voter algorithm applied to European power grid network [9]

tigation showed that to minimize the mean consensus time, the number of edges joining the two cliques should scale with $N^{2/3}$ [10]. This was shown using a combination of methods including monte carlo simulations and an equation-based analysis.

7 Noisy majority voter system on scale free

In section 5 we looked at the majority voter model and established that communities tend to share the same opinions in these networks and that there is no guarantee a consensus is reached. Now we will introduce the noisy majority vote system and investigate how it behaves and look at the special case of how it behaves on a power-law network.

Consider a 2-state system with spins (opinions) of ± 1. At a time t (the kth instance) a node takes the opinion of the majority of nodes in its neighbourhood with probability $p = 1 - q$ and the minority opinion has probability $p = q$. The probability q is called the noise parameter and quantifies social temperature. Large values of q indicates that the system has higher temperature in regions with this q value - meaning that in these regions there is dispute over the opinions in question. Low q values mean that the minority opinion is improbable in these regions and the network moves towards a consensus in these low temperature regions of the network.

Studies of noisy majority voter models show that the network experiences a phase transition when a critical value of q_c is reached [17] [18]. In the case of complex networks, opinions are represented by spins of $\sigma = \pm 1$ in the two state voter model and then in the three state voter model opinions are represented by spin states $\sigma = 1, 2, 3$. In a regular square lattice for the two-state voter model, there is an order-disorder phase transition at $q_c \approx 0.075$ and for the three-state voter model in a regular square lattice this phase transition occurs for

$q_c \approx 0.118$ - the point at which there is no longer consensus in the system [17].

We move away from the regular lattice and look at Barabási-Albert models with preferential attachment. We examine a Barabáshi-Albert model that is distributed with scale-free degree distribution k^{λ} where $\lambda = 3$. In this voter system there are z neighbours selected by each added site in the network, this number z is called the growth parameter. For instance, if $z = 2$, then a new node added into the model will have links to two neighbouring nodes.

In the 3-opinion noisy voter model, the node σ_i adopts the majority opinion with probability $p = 1 - q$ and adopts the a minority opinion (any other opinion from nearest neighbours which is not the majority opinion) with probability $p = q$. In the case of a tie between three opinions there is an equal probability that either of the opinion states will be chosen - $p = 1/3$. A tie 2 between majorities means that σ_i can adopt the opinion of either majority state with probability $p = \frac{(1-q)}{2}$ and the minority opinion state with $p = q$. Then for the case where there is a tie between 2 minority opinions, the majority opinion is adopted by σ_i with $p = 1 - q$ and the minority opinion state is chosen by σ_i with $p = \frac{q}{2}$. These update rules can be summarised as follows:

$$
\begin{aligned}
P(1|n_1 > n_2, n_3) &= 1 - q, \\
P(1|n_1 = n_2 > n_3) &= \frac{(1-q)}{2}, \\
P(1|n_1 < n_2 = n_3) &= q, \\
P(1|n_1, n_2 < n_3) &= \frac{q}{2}, \\
P(1|n_1 = n_2 = n_3) &= \frac{1}{3}\,.
\end{aligned}
\tag{6}
$$

Monte Carlo simulations were carried out by the A. L. M. Vilela et al. to estimate $q_c(z)$ the critical noise parameter as a function of the growth parameter z. After some time, the system reaches a state of complete order, semi-ordered or complete disorder. Complete order is achieved where $q = 0$ and the majority opinion takes over. An upper limit can be placed on q where the system has infinite social temperature - this occurs when the probability of agreeing with the majority is equal to the probability of agreeing with the minority: $1 - q = \frac{q}{2} \rightarrow q = 2/3$. In this case where $q = \frac{2}{3}$, the magnetisation of the system tends to zero as $N \rightarrow \infty$ [17].

Since q_c is a function of z and because of the preferential attachment nature of Barabási-Albert networks, z also determines the value of the mean degree $< k >$. The results of the investigation showed that second order phase transition occurs for $z > 1$ and that $q_c(Z)$ converges to $2/3$ as $z \rightarrow \infty$. The interpretation behind this is that even with a high degree of disorder (q) that a society with three opinions remains ordered. This is a significant result as 'scale-free' networks with power-law distributions are seen frequently in real world networks. This result agrees with similar results in the literature which show that small-world networks are robust to noise in voter models [19]

8 Effect of strong opinions on the majority vote model

Another interesting modification to the voter model is to consider agents who have stronger views on their opinions than other agents in the model. This kind of model includes regular actors σ, which act like regular voters with states ± 1 in the two state model, and what are called strong actors μ which have spins of ± 1.5 [18]. The strong actors are included in the model to represent individuals with strong stances on issues who may be more influential on neighbouring nodes. The model is considered on a 2-dimensional lattice of size $N = N_\sigma + N_\mu$, where N_σ and N_μ represent the number of regular and strong actors in the network. A parameter $r = \frac{N_\mu}{N}$ is defined to measure the fraction of strong agents relative to the total number of agents in the network. Using $r = 0$ or $r = 1$ results in the regular noisy voter model, so values of $0 \leq r \leq 1/2$ are considered in this model, as we are interested in the effect of strong opinions when the strong opinions do not form an overall majority in the network.

As with the noisy majority voter model we looked at in the previous section, a random node i takes the spin (± 1) of the majority of its neighbours with probability $p = 1 - q$ and takes the spin of the minority opinion of its neighbours with probability $p = q$.

A random spin α_i is chosen in the network. α can be equal to σ_i or μ_i. α_i adopts the spin of the majority of its neighbours with probability

$$w(\alpha_i) = \frac{1}{2}\left[1 - (1-2q)\ \mathrm{sgn}(\alpha_i)\ \mathrm{sgn}\left(\sum_{\delta=1}^{k_i} \alpha_{i+\delta}\right)\right], \tag{7}$$

where $\mathrm{sgn}(x)$ denotes the sign of a spin x and $\mathrm{sgn}(x) = -1, 0, +1$ where $x < 0, x = 0, x > 0$ respectively. The sum runs over all k_i nodes attached to node α_i, in the case of a 2-dimensional square lattice the value of $k_i = 4$ for all $i \in N$.At each step the selected node has probability q of choosing the opinion opposite to the majority of its neighbours and probability $1 - q$ of adopting the majority opinion of its neighbours. During each Monte Carlo step the procedure is repeated N times, on average every site in the lattice can flip its sign once per Monte Carlo step.

In this system, increasing the concentration r, weakens the consensus of the network. Consider figure 2, In fig. 2(a) the central node α_i is surrounded by 2 neighbours with spin +1 and 2 neighbours with spin -1. There is a tie here since there are two positive opinions and two negative opinions. There is no strongly opinionated node present so $r = 0$ and the probability of spin flip is independent of the noise parameter q since there is a tie for the sign of the spins of node α_i's neighbours. The probability of spin flip = 1/2 here, $w(\sigma_i) = 1/2$.

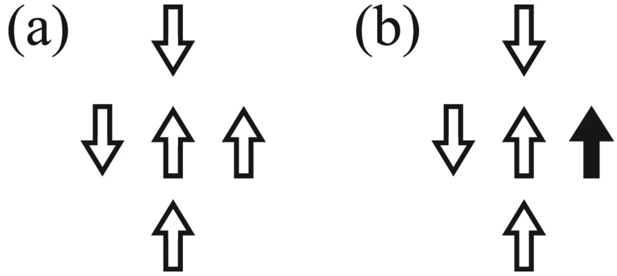

Figure 2: Strong opinion model - In fig.2 **(a)** the central spin α_i is surrounded by an equal number of up-spins and down-spins where $r = 0$. There is a tie for majority opinion. Spin flip probability is independent of q and equal to 1/2. fig.2 **(b)** there is a tie for the signs of the spins amongst the central spin's neighbours again but in this case one of the nodes with spin up is a node who is strongly opinionated. There is a majority amongst node α_i's neighbours here because of the presence of strong opinionated μ. [18]

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In figure 2 (b), there is a tie for the sign of the spins of central node α_i's neighbours like in fig.2(a). However, $r \neq 0$ here since one of neighbours with positive spin is a strongly opinionated node μ. The signs of the neighbours' spins are tied here but since the node μ has spin value of +1.5, the sum of all the neighbours' spins values is +0.5, which means that positive spin is the majority spin in this case. Since the central node α_i is already in the positive state, probability of node α_i's spin flipping is determined by the noise parameter q $\rightarrow w(\sigma_i) = q$. The opinion of spin σ_i is influenced by the social temperature of the network q now. Monte Carlo simulations show that there is a critical temperature q_c that when reached the network undergoes a phase transition from an ordered state to a disordered one [18].

We see that the inclusion of strong-opinionated agents into the voter model on a network weakens the consensus of the network and leads to disorder in the network when a critical value of q is reached.

9 Constrained voter models

There have been many attempts to incorporate stubborn individuals who are resistant to changing their opinions into dynamical voter models. One example of this is the constrained three-state voter model (3CVM) in which incompatible 'left-wing' and 'right-wing' nodes cannot interact with each other, they can only interact with 'centrist' nodes [20]. The outcome is either a consensus with one of the three parties or a polarised state with a mixture of leftists and rightists. In the case of this polarised state,

Another interesting constraint placed on the voter model is the inclusion of 'zealot' voters who never change their opinion. This model consists of a population of N voters, with a fixed number of zealots who never change their opinion, other voters in the model have the ability to change their opinion in the model. Each voter can be in 2 states +1/-1 or 'democrat'/'republican'. We can denote the population as having N_+ democrat and N_- republican susceptible voters and Z_+ democrat zealots and Z_- republican zealots . members of Z_+ and Z_- sets cannot change set membership, whereas voters who are members of N_+ can become members of N_- and individuals in N_- cohort can move into N_+ group. Each agent in the model - whether susceptible or zealot - is equally persuasive. the total number of voters N is the sum of these 4 groups [21].

The update rule for this model is

(1) pick a random voter, if they are a zealot, nothing happens;

(2) if the voter is a susceptible node, they take the opinion of a random neighbouring node that it shares an edge with;

(3) repeat steps 1 and 2 until consensus is reached.

This model is quite effective at modelling a population of voters that maintain a steady state and never reach consensus when a small number of zealots is used. When there is an equal number of democrat zealots and republican zealots, there is a steady-state 50-50 split between republicans and democrats, with a magnetisation of zero. An unequal number of zealots between democrats and republicans results in a steady state magnetisation $m^* = (Z_+ - Z_-)/(Z_+ + Z_-)$. The magnetisation distribution is also generally Gaussian with $P(m) \propto e^{-(m-m^*)^2/2\sigma^2}$ with $\sigma \propto 1/\sqrt{Z_- + Z_+}$. This Gaussian magnetisation distribution can be seen for the 1 and 2 dimensional lattices as well as a complete graph voter network [21].

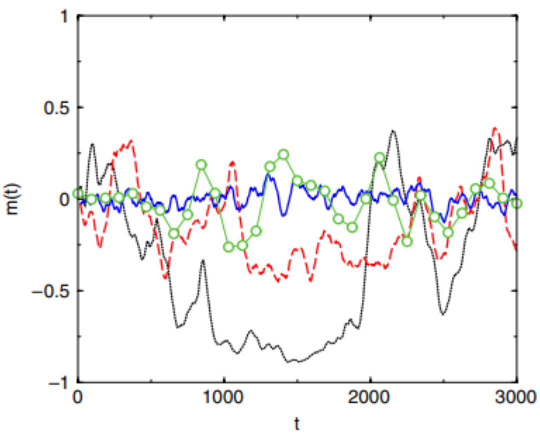

Figure 3: Magnetisation over time for simulations of 1000 voters on the complete graph for different number of zealots Z. $Z = 2$ (black), $Z = 16$ (red) and $Z = 128$ (blue). US presidential election results from 1876 to 2004 are shown in this plot also (green circles) [21]

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Conclusions

While the voter model is a simplification of the complex nature of opinion formation, it does give some insight into how different random interactions may be responsible for large shifts in changes in opinion across demographics in society. In this review we covered the algorithms behind the standard voter model and the majority voter model and investigated whether they reach a consensus and how long it takes for these networks to reach a consensus.

We saw how strongly opinionated voters and zealots disrupt the ability for the network to reach a consensus and raise the social temperature of the system. We saw that for power-law networks, it is possible to predict the mean time to reach a consensus and also to put a theoretical limit on that time for a number of different network configurations. We also saw that for Barabási-Albert networks, when there is a large network with 3 differing opinions, the network is robust against noise in the system and tends towards an ordered state with a consensus.

There are many different iterations and modifications to the voter model algorithm that I did not discuss in this review - such as the threshold q-voter model. It will be interesting to see if future developments on voter models applied to 'real' networks such as Scale-free and small world networks will be able to reveal more detailed insights into human behaviour and opinion dynamics in the future.

References

[1] Claudio Castellano, Santo Fortunato, and Vittorio Loreto. Statistical physics of social dynamics. *Reviews of Modern Physics*, 81(2):591–646, may 2009.

[2] Mason A. Porter and James P. Gleeson. Dynamical systems on networks: A tutorial, 2015.

[3] M. J. de Oliveira. Isotropic majority-vote model on a square lattice. *Journal of Statistical Physics*, 66:273–281, 1992.

[4] Thomas M. Liggett. *Interacting Particle Systems.* Springer Berlin Heidelberg, 1985.

[5] Inés Caridi, Sergio Manterola, Viktoriya Semeshenko, and Pablo Balenzuela. Topological study of the convergence in the voter model. *Applied Network Science*, 4(1), dec 2019.

[6] Peter Clifford and Aidan Sudbury. A model for spatial conflict. *Biometrika*, 60(3):581–588, 1973.

[7] Richard A. Holley and Thomas M. Liggett. Ergodic theorems for weakly interacting infinite systems and the voter model. *The Annals of Probability*, 3(4):643–663, 1975.

[8] Dietrich Stauffer. Statistical physics for humanities: A tutorial, 2011.

[9] Mehmet Yildiz, Roberto Pagliari, Asuman Ozdaglar, and Anna Scaglione. Voting models in random networks. pages 1 – 7, 03 2010.

[10] Michael T Gastner and Kota Ishida. Voter model on networks partitioned into two cliques of arbitrary sizes. *Journal of Physics A: Mathematical and Theoretical*, 52(50):505701, nov 2019.

[11] T.M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013.

[12] P. L. Krapivsky. Kinetics of monomer-monomer surface catalytic reactions. *Phys. Rev. A*, 45:1067–1072, Jan 1992.

[13] V. Sood and S. Redner. Voter model on heterogeneous graphs. *Phys. Rev. Lett.*, 94:178701, May 2005.

[14] J. T. Cox. Coalescing Random Walks and Voter Model Consensus Times on the Torus in $\mathbb{Z}^d$. *The Annals of Probability*, 17(4):1333 – 1366, 1989.

[15] Vincent Blondel, Adeline Decuyper, and Gautier Krings. A survey of results on mobile phone datasets analysis. *EPJ Data Science*, 4, 02 2015.

[16] Miller McPherson, Lynn Smith-Lovin, and James M. Cook. Birds of a feather: Homophily in social networks. *Annual Review of Sociology*, 27:415–444, 2001.

[17] André L. M. Vilela, Bernardo J. Zubillaga, Chao Wang, Minggang Wang, Ruijin Du, and H. Eugene Stanley. Three-state majority-vote model on barabási-albert and cubic networks and the unitary relation for critical exponents, 2019.

[18] André L. M. Vilela and H. Eugene Stanley. Effect of strong opinions on the dynamics of the majority-vote model. *Scientific Reports*, 8(1), 6 2018.

[19] Paulo R. A. Campos, Viviane M. de Oliveira, and F. G. Brady Moreira. Small-world effects in the majority-vote model. *Phys. Rev. E*, 67:026104, Feb 2003.

[20] Mauro Mobilia. Polarization and consensus in a voter model under time-fluctuating influences. *Physics*, 5(2):517–536, 2023.

[21] M Mobilia, A Petersen, and S Redner. On the role of zealotry in the voter model. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08029–P08029, aug 2007.