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Electromag. HW 1

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Q1 $\underline{A} = A_1(x,y,z)\hat{\underline{i}} + A_2(x,y,z)\hat{\underline{j}} + A_3(x,y,z)\hat{\underline{k}}$

$\underline{B} = B_1(x,y,z)\hat{\underline{i}} + B_2(x,y,z)\hat{\underline{j}} + B_3(x,y,z)\hat{\underline{k}}$

$f = f(x,y,z) \qquad g = g(x,y,z)$

(i) Show $\nabla(fg) = f\nabla g + g\nabla f$

$\nabla = \partial/\partial x\,\hat{\underline{i}} + \partial/\partial y\,\hat{\underline{j}} + \partial/\partial z\,\hat{\underline{k}}$

$\nabla(fg) = \partial/\partial x\, f.g + \partial/\partial y\, fg + \partial/\partial z\, fg$

product rule $\rightarrow$ $= f_x g + g_x f + f_y g + g_y f + f_z g + g_z f$

$= f(g_x + g_y + g_z) + g(f_x + f_y + f_z)$

$= f\nabla g + g\nabla f$

$\therefore \nabla(fg) = f\nabla g + g\nabla f$

(ii) $\nabla\left(\frac{f}{g}\right) = \partial/\partial x\left(f/g\right) + \frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \frac{\partial}{\partial z}\left(\frac{f}{g}\right)$

quotient rule $\rightarrow$ $d/dx\left(\frac{f(x)}{g(x)}\right) = \frac{d/dx\, f \cdot g - f \cdot d/dx\, g}{(g(x))^2}$

$\nabla(f/g) = \frac{f_x g - f g_x}{g^2} + \frac{f_y g - g_y f}{g^2} + \frac{f_z g - f g_z}{g^2}$

$= \frac{g(f_x + f_y + f_z) - f(g_x + g_y + g_z)}{g^2}$

$\therefore \nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}$

(iii) Show $\underline{\nabla}\cdot(fA) = f(\underline{\nabla}\cdot\underline{A}) + \underline{A}\cdot(\underline{\nabla}f)$

$$\nabla\cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla\cdot(fA) = \partial/\partial x\,(f\cdot A_1(x,y,z)) + \partial/\partial y\,(f\cdot A_2) + \partial/\partial z\,(f\cdot A_3)$$

$$= f_x A_1 + f\cdot\frac{\partial A_1}{\partial x} + f_y A_2 + f\frac{\partial A_2}{\partial y} + f_z A_3 + f\frac{\partial A_3}{\partial z}$$

$$= \underline{A}\,(f_x + f_y + f_z) + f\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= \underline{A}\cdot(\nabla f) + f\,(\underline{\nabla}\cdot\underline{A})$$

$$\therefore \underline{\nabla}\cdot(fA) = f(\underline{\nabla}\cdot\underline{A}) + \underline{A}\cdot(\underline{\nabla}f)$$

(iv) show $\underline{\nabla}\times(f\underline{A}) = f(\underline{\nabla}\times\underline{A}) + (\nabla f)\times\underline{A}$

$$\underline{\nabla}\times(f\underline{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ fA_1 & fA_2 & fA_3 \end{vmatrix}$$

$$= \left[\partial/\partial y\,(fA_3) - \partial/\partial z\,(fA_2)\right]\hat{i} - \left[\partial/\partial x\,(fA_3) - \partial/\partial z\,(fA_1)\right]\hat{j} + \left[\partial/\partial x\,(fA_2) - \partial/\partial y\,(fA_1)\right]\hat{k}$$

$$\underline{\nabla}\times\underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\,(\partial/\partial y\,A_3 - \partial/\partial z\,A_2) - \hat{j}\,(\partial/\partial x\,A_3 - \partial/\partial z\,A_1) + \hat{k}\,(\partial/\partial x\,A_2 - \partial/\partial y\,A_1)$$

$$\nabla f\times\underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\left(\frac{\partial f}{\partial y}A_3 - \frac{\partial f}{\partial z}A_2\right) - \hat{j}\left(\frac{\partial f}{\partial x}A_3 - \frac{\partial f}{\partial z}A_1\right) + \hat{k}\left(\frac{\partial f}{\partial x}A_2 - \frac{\partial f}{\partial y}A_1\right)$$

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$$\underline{\nabla} \times (fA) = \left(\partial/\partial y(fA_3) - \partial/\partial z(fA_2)\right)\hat{\underline{i}}$$
$$- \left(\partial/\partial x(fA_3) - \partial/\partial z(fA_1)\right)\hat{\underline{j}}$$
$$+ \left(\partial/\partial x(fA_2) - \partial/\partial y(fA_1)\right)\hat{\underline{k}}$$

$$(*)\begin{cases} = \left(\frac{\partial f}{\partial y}\cdot A_3 + f\frac{\partial A_3}{\partial y} - \frac{\partial f}{\partial z}\cdot A_2 - \frac{\partial A_2}{\partial z}f\right)\hat{i} \\ + \left(-\frac{\partial f}{\partial x}A_3 - \frac{\partial A_3}{\partial x}f + \frac{\partial f}{\partial z}A_1 + \frac{\partial A_1}{\partial z}f\right)\hat{\underline{j}} \\ + \left(\frac{\partial f}{\partial x}A_2 + \frac{\partial A_2}{\partial x}f - \frac{\partial f}{\partial y}A_1 - \frac{\partial A_1}{\partial y}f\right)\hat{\underline{k}} \end{cases}$$

$$f\cdot(\underline{\nabla}\times\underline{A}) = \hat{i}\left(\frac{\partial A_3}{\partial y}f - \frac{\partial A_2}{\partial z}f\right)$$
$$-\hat{j}\left(\frac{\partial A_3}{\partial x}f - \frac{\partial A_1}{\partial z}f\right)$$
$$+\hat{\underline{k}}\left(\frac{\partial A_2}{\partial x}f - \frac{\partial A_1}{\partial y}f\right)$$

$$\nabla f \times \underline{A} = \hat{\underline{i}}\left(\frac{\partial f}{\partial y}A_3 - \frac{\partial f}{\partial z}A_2\right) - \hat{j}\left(\frac{\partial f}{\partial x}A_3 - \frac{\partial f}{\partial z}A_1\right)$$
$$+\hat{\underline{k}}\left(\frac{\partial f}{\partial x}A_2 - \frac{\partial f}{\partial y}A_1\right)$$

$$(+)\begin{cases} f\cdot(\underline{\nabla}\times\underline{A}) + (\nabla f)\times\underline{A} = \left(\frac{\partial f}{\partial y}A_3 + f\frac{\partial A_3}{\partial y} - A_2\frac{\partial f}{\partial z} - \frac{\partial A_2}{\partial z}f\right)\hat{\underline{i}} \\ + \left(-\frac{\partial f}{\partial x}A_3 - \frac{\partial A_3}{\partial x}f + \frac{\partial f}{\partial z}A_1 + \frac{\partial A_1}{\partial z}f\right)\hat{\underline{j}} \\ + \left(\frac{\partial f}{\partial x}A_2 + \frac{\partial A_2}{\partial x}f - \frac{\partial f}{\partial y}A_1 - \frac{\partial A_1}{\partial y}f\right)\hat{k} \end{cases}$$

Since $(+) = (*)$ → shown that

$$\underline{\nabla}\times(f\underline{A}) = f(\underline{\nabla}\times\underline{A}) + (\nabla f)\times\underline{A}$$

(v) Show $\underline{\nabla} \times (\nabla f) = 0$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\underline{i}} + \frac{\partial f}{\partial y}\hat{\underline{j}} + \frac{\partial f}{\partial z}\hat{\underline{k}}$$

$$\underline{\nabla} \times (\nabla f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\partial/\partial y \cdot \frac{\partial f}{\partial z} - \partial/\partial z \frac{\partial f}{\partial y}\right)\hat{i}$$
$$- \left(\partial/\partial x \frac{\partial f}{\partial z} - \partial/\partial z \frac{\partial f}{\partial x}\right)\hat{\underline{j}}$$
$$+ \left(\partial/\partial x \frac{\partial f}{\partial y} - \partial/\partial y \frac{\partial f}{\partial x}\right)\hat{\underline{k}}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right)\hat{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{j}$$
$$+ \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right)\hat{\underline{k}}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$
$$= \underline{0}$$

(vi) show $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0$

$$\underline{\nabla} \times \underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{\underline{i}}(\partial/\partial y\, A_3 - \partial/\partial z\, A_2) - \hat{\underline{j}}(\partial/\partial x\, A_3 - \partial/\partial z\, A_1)$$
$$+ \hat{\underline{k}}(\partial/\partial x\, A_2 - \partial/\partial y\, A_1)$$

$$\underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = \partial/\partial x(\partial/\partial y\, A_3 - \partial/\partial z\, A_2)$$
$$+ \partial/\partial y(-\partial/\partial x\, A_3 + \partial/\partial z\, A_1)$$
$$+ \partial/\partial z(\partial/\partial x\, A_2 - \partial/\partial y\, A_1)$$

$$= \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z}\right) + \left(-\frac{\partial^2 A_3}{\partial x \partial y} + \frac{\partial^2 A_1}{\partial y \partial z}\right) + \left(\frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_1}{\partial y \partial z}\right)$$

$$= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_3}{\partial x \partial y} + \frac{\partial^2 A_2}{\partial x \partial z} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_1}{\partial y \partial z} - \frac{\partial^2 A_1}{\partial y \partial z}$$

$$= 0 + 0 + 0$$
$$= \underline{0}$$

(vii) Show $\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \nabla(\underline{\nabla} \cdot \underline{A}) - \underline{\nabla}^2 \underline{A}$

$$\underline{\nabla} \cdot \underline{A} = \partial/\partial x\, A_1 + \partial/\partial y\, A_2 + \partial/\partial z\, A_3$$

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{A}) = \partial/\partial x \left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right] \hat{\underline{i}}$$
$$+ \frac{\partial}{\partial y}\left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right] \hat{\underline{j}}$$
$$+ \frac{\partial}{\partial z}\left[\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right] \hat{\underline{k}}$$

$$\underline{\nabla} \times \underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \left[\partial/\partial y\, A_3 - \partial/\partial z\, A_2\right] \hat{\underline{i}}$$
$$- \left[\partial/\partial x\, A_3 - \partial/\partial z\, A_1\right] \hat{\underline{j}}$$
$$+ \left[\partial/\partial x\, A_2 - \partial/\partial y\, A_1\right] \hat{k}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & -\frac{\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left[\partial/\partial y \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right] - \partial/\partial z \left[-\frac{\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z}\right]\right] \hat{\underline{i}}$$
$$- \left[\partial/\partial x \left[\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right] - \partial/\partial z \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right]\right] \hat{\underline{j}}$$
$$+ \left[\partial/\partial x \left[-\frac{\partial A_3}{\partial x} + \frac{\partial A_1}{\partial z}\right] - \partial/\partial y \left[\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right]\right] \hat{\underline{k}}$$

$$= \left[-\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial}{\partial y}\frac{\partial A_2}{\partial x} + \partial/\partial z \frac{\partial A_3}{\partial x} - \frac{\partial^2 A_1}{\partial z^2}\right] \hat{\underline{i}}$$
$$+ \left[-\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial A_1}{\partial y} + \frac{\partial}{\partial y}\frac{\partial A_3}{\partial z} - \frac{\partial^2 A_2}{\partial z^2}\right] \hat{\underline{j}}$$
$$+ \left[-\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial A_1}{\partial z} + \partial/\partial y \frac{\partial A_2}{\partial z} - \frac{\partial^2 A_3}{\partial y^2}\right] \hat{\underline{k}}$$

$$= \left(\partial/\partial x \left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right) \hat{i} + \hat{\underline{j}}\left(\partial/\partial y \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_3}{\partial z}\right) - \frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial z^2}\right)$$
$$+ \hat{\underline{k}}\left(\partial/\partial z \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y}\right) - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)$$

$$= \left(\frac{\partial}{\partial x}\left(\cancel{\frac{\partial A_1}{\partial x}} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) - \cancel{\frac{\partial^2 A_1}{\partial x^2}} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) \underline{\hat{i}}$$

$$+ \left(\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \cancel{\frac{\partial A_2}{\partial y}} + \frac{\partial A_3}{\partial z} \right) - \frac{\partial^2 A_2}{\partial x^2} - \cancel{\frac{\partial^2 A_2}{\partial y^2}} - \frac{\partial^2 A_2}{\partial z^2} \right) \underline{\hat{j}}$$

$$+ \left(\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \cancel{\frac{\partial A_3}{\partial z}} \right) - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \cancel{\frac{\partial^2 A_3}{\partial z^2}} \right) \underline{\hat{k}}$$

$$= \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \underline{\nabla}^2 \underline{A}$$

so have shown

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \underline{\nabla}^2 \underline{A}$$

(viii)

show $\underline{\nabla} \cdot (\underline{A} \times \underline{B}) = (\underline{\nabla} \times \underline{A}) \cdot \underline{B} - \underline{A} \cdot (\underline{\nabla} \times \underline{B})$

$$\underline{A} \times \underline{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \underline{\hat{i}}(A_2 B_3 - A_3 B_2) - \underline{\hat{j}}(A_1 B_3 - A_3 B_1) + \underline{\hat{k}}(A_1 B_2 - A_2 B_1)$$

$$\underline{\nabla} \times \underline{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\left(\tfrac{\partial}{\partial y} A_3 - \tfrac{\partial}{\partial z} A_2\right) - \hat{j}\left(\tfrac{\partial}{\partial x} A_3 - \tfrac{\partial}{\partial z} A_1\right) + \hat{k}\left(\tfrac{\partial}{\partial x} A_2 - \tfrac{\partial}{\partial y} A_1\right)$$

similarly

$$\underline{\nabla} \times \underline{B} = \hat{i}\left(\tfrac{\partial}{\partial y} B_3 - \tfrac{\partial}{\partial z} B_2\right) - \hat{j}\left(\tfrac{\partial}{\partial x} B_3 - \tfrac{\partial}{\partial z} B_1\right) + \hat{k}\left(\tfrac{\partial}{\partial x} B_2 - \tfrac{\partial}{\partial y} B_1\right)$$

$$(\underline{\nabla} \times \underline{A}) \cdot B = B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) - B_2\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$A \cdot (\underline{\nabla} \times \underline{B}) = A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right) - A_2\left(\frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z} \right) + A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right)$$

$$\underline{\nabla} \cdot (\underline{A} \times \underline{B}) = \partial/\partial x \left[A_2 B_3 - A_3 B_2\right]$$
$$- \frac{\partial}{\partial y}\left[A_1 B_3 - A_3 B_1\right]$$
$$+ \partial/\partial z \left[A_1 B_2 - A_2 B_1\right]$$

$$= \frac{\partial A_2}{\partial x} B_3 + \frac{\partial B_3}{\partial x} A_2 - \frac{\partial A_3}{\partial x} B_2 - \frac{\partial B_2}{\partial x} A_3$$
$$- \frac{\partial A_1}{\partial y} B_3 - \frac{\partial B_3}{\partial y} A_1 + \frac{\partial A_3}{\partial y} B_1 + \frac{\partial B_1}{\partial y} A_3$$
$$+ \frac{\partial A_1}{\partial z} B_2 + \frac{\partial B_2}{\partial z} A_1 - \frac{\partial A_2}{\partial z} B_1 - \frac{\partial B_1}{\partial z} A_2$$

$$= B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - B_2\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right)$$
$$+ B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$- A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) - A_2\left(\frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z}\right)$$
$$- A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

$$= (\underline{\nabla} \times \underline{A}) \cdot \underline{B} - \underline{A} \cdot (\underline{\nabla} \times \underline{B}) \quad \blacksquare$$

Q1 (ix) Show $\underline{\nabla} \times (\underline{A} \times \underline{B}) = A(\underline{\nabla} \cdot \underline{B}) - B(\underline{\nabla} \cdot \underline{A})$
$+ (B \cdot \nabla)A - (A \cdot \nabla)B$

$$\underline{A} \times \underline{B} = \hat{i}(A_2B_3 - A_3B_2) - \hat{j}(A_1B_3 - A_3B_1) + \hat{k}(A_1B_2 - A_2B_1)$$

$$\underline{\nabla} \times (\underline{A} \times \underline{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (A_2B_3 - A_3B_2) & (-A_1B_3 + A_3B_1) & (A_1B_2 - A_2B_1) \end{vmatrix}$$

$$LHS = \hat{i}\left(\partial/\partial y\,(A_1B_2 - A_2B_1) - \partial/\partial z\,(A_3B_1 - A_1B_3)\right) - \hat{j}\left(\partial/\partial x\,(A_1B_2 - A_2B_1) - \partial/\partial z\,(A_2B_3 - A_3B_2)\right) + \hat{k}\left(\partial/\partial x\,(A_3B_1 - A_1B_3) - \partial/\partial y\,(A_2B_3 - A_3B_2)\right)$$

$$RHS = \underline{A}(\underline{\nabla} \cdot \underline{B}) - \underline{B}(\underline{\nabla} \cdot \underline{A}) + (\underline{B} \cdot \underline{\nabla})\underline{A} - (\underline{A} \cdot \underline{\nabla})\underline{B}$$

$$\underline{A}(\underline{\nabla} \cdot \underline{B}) = \underline{A}\left(\partial/\partial x\,B_1 + \partial/\partial y\,B_2 + \partial/\partial z\,B_3\right)$$

$$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\left(\partial/\partial x\,B_1 + \partial/\partial y\,B_2 + \partial/\partial z\,B_3\right)$$

$$= A_1\left(\partial B_1/\partial x + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\hat{i} + A_2\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\hat{j} + A_3\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\hat{k}$$

$$\underline{B}(\underline{\nabla} \cdot \underline{A}) = \underline{B}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= B_1\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\hat{i} + B_2\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\hat{j} + B_3\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\hat{k}$$

$$(\underline{B}\cdot\underline{\nabla})\underline{A} = \left(B_1 \partial/\partial x + B_2 \partial/\partial y + B_3 \partial/\partial z\right)\underline{A}$$

$$= (\underline{B}\cdot\underline{\nabla})(A_1\hat{\underline{i}} + A_2\hat{\underline{j}} + A_3\hat{\underline{k}})$$

$$= (\underline{B}\cdot\underline{\nabla}(\underline{A}_1))\hat{\underline{i}} + (\underline{B}\cdot\underline{\nabla}\underline{A}_2)\hat{\underline{j}} + (\underline{B}\cdot\underline{\nabla}\underline{A}_3)\hat{\underline{k}}$$

$$= \left(B_1\frac{\partial A_1}{\partial x} + B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z}\right)\hat{\underline{i}} + \left(B_1\frac{\partial A_2}{\partial x} + B_2\frac{\partial A_2}{\partial y} + B_3\frac{\partial A_2}{\partial z}\right)\hat{\underline{j}} + \left(B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} + B_3\frac{\partial A_3}{\partial z}\right)\hat{\underline{k}}$$

$$(\underline{A}\cdot\nabla)\underline{B} = \left(A_1 \partial/\partial x + A_2\partial/\partial y + A_3\partial/\partial z\right)\underline{B}$$

$$= \left(A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z}\right)\hat{\underline{i}} + \left(A_1\frac{\partial B_2}{\partial x} + A_2\frac{\partial B_2}{\partial y} + A_3\frac{\partial B_2}{\partial z}\right)\hat{\underline{j}} + \left(A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} + A_3\frac{\partial B_3}{\partial z}\right)\hat{\underline{k}}$$

$$\text{RHS} = A_1\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\hat{\underline{i}} + A_2\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\hat{\underline{j}} + A_3\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right)\hat{\underline{k}}$$

$$- B_1\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\hat{i} - B_2\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)\hat{\underline{j}} - B_3\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$+ \left(B_1\frac{\partial A_1}{\partial x} + B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z}\right)\hat{\underline{i}} + \left(B_1\frac{\partial A_2}{\partial x} + B_2\frac{\partial A_2}{\partial y} + B_3\frac{\partial A_3}{\partial z}\right)\hat{\underline{j}} + \left(B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} + B_3\frac{\partial A_3}{\partial z}\right)\hat{\underline{k}}$$

$$- \left(A_1\frac{\partial B_1}{\partial x} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z}\right)\hat{\underline{i}} - \left(A_1\frac{\partial B_2}{\partial x} + A_2\frac{\partial B_2}{\partial y} + A_3\frac{\partial B_3}{\partial z}\right)\hat{\underline{j}} - \left(A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} + A_3\frac{\partial B_3}{\partial z}\right)\hat{\underline{k}}$$

$$RHS = \Big[\cancel{A_1}\Big(\cancel{\frac{\partial B_1}{\partial x}} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\Big) - B_1\Big(\cancel{\frac{\partial A_1}{\partial x}} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\Big)$$
$$+ \Big(\cancel{B_1\frac{\partial A_1}{\partial x}} + B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z}\Big)$$
$$- \Big(\cancel{A_1\frac{\partial B_1}{\partial x}} + A_2\frac{\partial B_1}{\partial y} + A_3\frac{\partial B_1}{\partial z}\Big)\Big]\underline{\hat{i}}$$

$$+ \Big[A_2\Big(\frac{\partial B_1}{\partial x} + \cancel{\frac{\partial B_2}{\partial y}} + \frac{\partial B_3}{\partial z}\Big) - B_2\Big(\frac{\partial A_1}{\partial x} + \cancel{\frac{\partial A_2}{\partial y}} + \frac{\partial A_3}{\partial z}\Big)$$
$$+ \Big(B_1\frac{\partial A_2}{\partial x} + \cancel{B_2\frac{\partial A_2}{\partial y}} + B_3\frac{\partial A_2}{\partial z}\Big)$$
$$- \Big(A_1\frac{\partial B_2}{\partial x} + \cancel{A_2\frac{\partial B_2}{\partial y}} + A_3\frac{\partial B_2}{\partial z}\Big)\Big]\underline{\hat{j}}$$

$$+ \Big[A_3\Big(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \cancel{\frac{\partial B_3}{\partial z}}\Big) - B_3\Big(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \cancel{\frac{\partial A_3}{\partial z}}\Big)$$
$$+ \Big(B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} + \cancel{B_3\frac{\partial A_3}{\partial z}}\Big)$$
$$- \Big(A_1\frac{\partial B_3}{\partial x} + A_2\frac{\partial B_3}{\partial y} + \cancel{A_3\frac{\partial B_3}{\partial z}}\Big)\Big]\underline{\hat{k}}$$

$$LHS = \underline{\hat{i}}\Big(\frac{\partial}{\partial y}(A_1B_2 - A_2B_1) - \frac{\partial}{\partial z}(A_3B_1 - A_1B_3)\Big)$$
$$- \underline{\hat{j}}\Big(\frac{\partial}{\partial x}(A_1B_2 - A_2B_1) - \frac{\partial}{\partial z}(A_2B_3 - A_3B_2)\Big)$$
$$+ \underline{\hat{k}}\Big(\frac{\partial}{\partial x}(A_3B_1 - A_1B_3) - \frac{\partial}{\partial y}(A_2B_3 - A_3B_2)\Big)$$

$$= \Big[A_1\frac{\partial B_2}{\partial y} + B_2\frac{\partial A_1}{\partial y} - A_2\frac{\partial B_1}{\partial y} - B_1\frac{\partial A_2}{\partial y} - \frac{\partial A_3}{\partial z}B_1 - \frac{\partial B_1}{\partial z}A_3$$
$$+ \frac{\partial A_1}{\partial z}B_3 + \frac{\partial B_3}{\partial z}A_1\Big]\hat{i}$$

$$+ \Big[-\frac{\partial A_1}{\partial x}B_2 - \frac{\partial B_2}{\partial x}A_1 + \frac{\partial A_2}{\partial x}B_1 + \frac{\partial B_1}{\partial x}A_2 + \frac{\partial A_2}{\partial z}B_3 + \frac{\partial B_3}{\partial z}A_2$$
$$- \frac{\partial A_3}{\partial z}B_2 - \frac{\partial B_2}{\partial z}A_3\Big]\underline{\hat{j}}$$

$$+ \Big[\frac{\partial A_3}{\partial x}B_1 + \frac{\partial B_1}{\partial x}A_3 - \frac{\partial A_1}{\partial x}B_3 - \frac{\partial B_3}{\partial x}A_1 - \frac{\partial A_2}{\partial y}B_3 - \frac{\partial B_3}{\partial y}A_2$$
$$+ \frac{\partial A_3}{\partial y}B_2 + \frac{\partial B_2}{\partial y}A_3\Big]\underline{\hat{k}}$$

$$LHS = \hat{i}\Big[A_1\Big(\frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\Big) - B_1\Big(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\Big) + B_2\frac{\partial A_1}{\partial y} + B_3\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial y} - A_3\frac{\partial B_1}{\partial z}\Big]$$
$$+ \hat{j}\Big[A_2\Big(\frac{\partial B_1}{\partial x} + \frac{\partial B_3}{\partial z}\Big) - B_2\Big(\frac{\partial A_1}{\partial x} + \frac{\partial A_3}{\partial z}\Big) + B_1\frac{\partial A_2}{\partial x} + B_3\frac{\partial A_2}{\partial z} - A_1\frac{\partial B_2}{\partial x} - A_3\frac{\partial B_2}{\partial z}\Big]$$
$$+ \hat{k}\Big[A_3\Big(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y}\Big) - B_3\Big(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y}\Big) + B_1\frac{\partial A_3}{\partial x} + B_2\frac{\partial A_3}{\partial y} - A_1\frac{\partial B_3}{\partial x} - A_2\frac{\partial B_3}{\partial y}$$

$= RHS$ ∴ shown that $\underline{\nabla}\times(\underline{A}\times\underline{B}) = \underline{A}(\underline{\nabla}\cdot\underline{B}) - \underline{B}(\underline{\nabla}\cdot\underline{A}) + (\underline{B}\cdot\underline{\nabla})\underline{A} - (\underline{A}\cdot\underline{\nabla})\underline{B}$

Q2 EM

Divergence Theorem:

$$\iiint_V \underline{\nabla} \cdot \underline{F} \, dV$$

$$= \iint_S \underline{F} \cdot \underline{\hat{n}} \, dS$$

$$F = 3x \, \hat{i} + xy \, \hat{j} + 2xz \, \hat{k}$$

$$\underline{\nabla} \cdot \underline{F} = \partial/\partial x \, 3x + \partial/\partial y \, xy + \partial/\partial z \, 2xz$$

$$= 3 + x + 2x$$

$$= 3 + 3x$$

$$= 3(1+x)$$

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=1} 3+3x \; dz\,dy\,dx$$

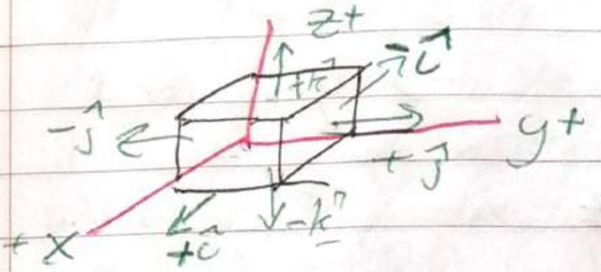
$$\int_{y=0}^{1} \int_{z=0}^{1} 3x + \frac{3x^2}{2} \Big|_0^1 \; dz\,dy$$

$$\int_{y=0}^{1} \int_{z=0}^{1} 3 + 3/2 \; dz\,dy \qquad : 4 + 1/2 = 9/2$$

$$\int_0^1 9/2 \, z \Big|_0^1 \, dy = 9/2 \, y \Big|_0^1 = 9/2$$

EM

Q2 (a) $\iint_S \underline{F} \cdot \underline{\hat{n}} \, dS$ $\qquad \underline{F} = 3x\underline{\hat{i}} + xy\underline{\hat{j}} + 2xz\underline{\hat{k}}$



$$S = \underset{x=0}{S_1} \cup \underset{x=1}{S_2} \cup \underset{y=0}{S_3} \cup \underset{y=1}{S_4} \cup \underset{z=0}{S_5} \cup \underset{z=1}{S_6}$$

$$\iint_{S_1} \underline{F} \cdot \underline{\hat{n}} \, dS = \iint_{S_1} (3x\underline{\hat{i}} + xy\underline{\hat{j}} + 2xz\underline{\hat{k}}) \cdot -\hat{i} \, dS_1$$

$$= \iint_{S_1} -3x \, dS_1 \qquad S_1 \Rightarrow x = 0$$

$$= \int_{y=0}^{y=1}\int_{z=0}^{z=1} -3(0) \, dy \, dz = 0 |_0^1 |_0^1 = 0$$

$$\iint_{S_2} (3x\underline{\hat{i}} + xy\underline{\hat{j}} + 2xz\underline{\hat{k}}) \cdot \hat{i} \, dS_2$$

$$\iint_{S_2} 3x \quad dS_2 \qquad x = 1 \text{ for } S_2$$

$$\int_{y=0}^{y=1}\int_{z=0}^{z=1} 3x \, dz \, dy = 3$$

$$\iint_{S_3} (3x\underline{\hat{i}} + xy\underline{\hat{j}} + 2xz\underline{\hat{k}}) \cdot -\hat{j} \, dS_3$$

$$\iint_{S_3} -xy \quad dS_3$$

$$\int_{x=0}^{x=1}\int_{z=0}^{z=1} -xy \, dz \, dx \qquad y = 0 \text{ for } S_3$$

$$= \int_0^1\int_0^1 0 \, dz \, dx = 0$$

$$\iint_{S_4} (3x\hat{i} + xy\hat{j} + 2xz\hat{k}) \cdot \hat{j} \, dS_4$$

$$= \iint_{S_4} xy \, dS_4 \quad \rightarrow y=1 \text{ for } S_4$$

$$\int_{x=0}^{x=1} \int_{z=0}^{z=1} x \, dx \, dz$$

$$= \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\iint_{S_5} (3x\hat{i} + xy\hat{j} + 2xz\hat{k}) \cdot -\hat{k} \, dS_5$$

$$\iint_{S_5} -2xz \, dS_5 = \int_{x=0}^{x=1}\int_{y=0}^{y=1} -2xz \, dy \, dx$$

$$z=0$$

$$\int_0^1\int_0^1 0 \, dy \, dx = 0$$

$$\iint_{S_6} (3x\hat{i} + xy\hat{j} + 2xz\hat{k}) \cdot \hat{k} \, dS_6$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1} 2xz \, dy \, dx \qquad z=1$$

$$= \int_0^1 2x \, dx = \cancel{2}\frac{x^2}{\cancel{2}}\Big|_0^1 = 1$$

$$\iint_S \underline{F} \cdot \hat{n} \, dS = 0 + 3 + 0 + \frac{1}{2} + 0 + 1$$

$$\iint_S \underline{F} \cdot \hat{\underline{n}} \, dS = \frac{9}{2}$$

$$\iiint_V \underline{\nabla} \cdot \underline{F} \, dV = \frac{9}{2}$$

hence Divergence Theorem is verified here.

EM

Q2 (b) $\iiint_V \underline{\nabla} \cdot \underline{F} \, dV = \iint_S \underline{F} \cdot \underline{\hat{n}} \, dS$

V is unit ball $x^2+y^2+z^2 \leq 1$
vector field $F = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}}$

ball of radius r, $r=1$ (unit sphere)
$x^2+y^2+z^2 = r^2 = 1 \rightarrow r=1$

$\underline{F} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}} \Rightarrow \underline{F} \cdot \underline{F} = r^2$
$\rightarrow \underline{F} = \underline{r} = r\hat{\underline{r}}$
$\hat{\underline{n}} = \hat{\underline{r}}$

$\iiint_V \underline{\nabla} \cdot \underline{F} \, dV \rightarrow$ use spherical coords

integral becomes
$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \underline{\nabla} \cdot \underline{F} \; r^2 \sin\theta \, dr \, d\theta \, d\phi$$

in spherical coords: $\underline{\nabla} \cdot \underline{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta} \frac{\partial A_\phi}{\partial \phi}$

$\underline{F} = \underline{r}$
$$\underline{\nabla} \cdot \underline{F} = \frac{1}{r^2} \frac{\partial (r^2 \cdot r)}{\partial r} = \frac{1}{r^2} 3r^2 = 3$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 3r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{\cancel{3} r^3}{\cancel{3}} \Big|_0^1 \sin\theta \, d\theta \, d\phi$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{2\pi} -\cos\theta \Big|_0^{\pi} = \int_0^{2\pi} -[-1-1]$$

$$= \int_0^{2\pi} 2 \, d\phi = 4\pi$$

$$\oiint_S \underline{F} \cdot \underline{\hat{n}} \, ds \qquad \underline{F} = \underline{r} = \underline{\hat{r}} r$$

$$\underline{\hat{n}} = \underline{\hat{r}}$$

$$\oiint_S r\underline{\hat{r}} \cdot \underline{\hat{r}} \, ds = \oiint_S r \, ds$$

$r=1$ so $= \oiint_S 1 \, ds$

integrate over θ and ϕ in polar coords for surface integral

$$\int_0^{2\pi}\int_0^{\pi} 1 \, r^2 \sin\theta \, d\theta \, d\phi \qquad r=1$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} -\cos\theta \Big|_0^{\pi} d\phi$$

$$= \int_0^{2\pi} 2 \, d\phi$$

$$= 4\pi$$

$\therefore$ have verified Divergence Theorem for this case

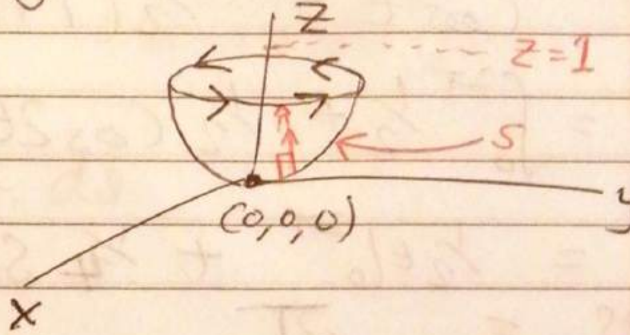
Q3 (a) $\int_C \underline{F} \cdot dr = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \hat{\underline{n}} \, dS$

$S =$ part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ oriented upwards

vector Field $F = y^2 \hat{\underline{i}} + x \hat{\underline{j}} + z^2 \hat{\underline{k}}$

$z = x^2 + y^2$

$0 \le z \le 1$



$\boxed{z=0}$: $x = y = 0 \quad = \quad (0,0,0)$

$\boxed{z=1}$ $z = x^2 + y^2 = 1$

on C, $\left.\begin{array}{l} x(t) = \cos(t) \\ y(t) = \sin(t) \end{array}\right\} 0 \le z \le 1$ ①

$0 \le t \le 2\pi$

$\underline{r} = \cos(t)\hat{\underline{i}} + \sin(t)\hat{\underline{j}} + 1\hat{\underline{k}} \qquad 0 \le t \le 2\pi$

$\frac{dr}{dt} = -\sin(t)\hat{\underline{i}} + \cos(t)\hat{\underline{j}} + 0\hat{\underline{k}}$

$\underline{dr} = (-\sin(t)\hat{\underline{i}} + \cos(t)\hat{\underline{j}})\,dt$

from ①

$\underline{F} = y^2\hat{\underline{i}} + x\hat{\underline{j}} + z^2\hat{\underline{k}} = \sin^2(t)\hat{\underline{i}} + \cos(t)\hat{\underline{j}} + 1\hat{\underline{k}}$

$\underline{F} \cdot \underline{dr} = (-\sin^3(t) + \cos^2(t))\,dt$

$\oint_C \underline{F} \cdot \underline{dr} = \int_{t=0}^{t=2\pi} (-\sin^3(t) + \cos^2(t))\,dt$

$$\int_0^{2\pi} -\sin^3(t) = 0$$ since $\sin^3(t)$ is an odd function

$$\oint_c \underline{F} \cdot dr = \int_{t=0}^{t=2\pi} \cos^2(t)\,dt$$

$$\cos^2 t = \tfrac{1}{2}(1+\cos 2t)$$

$$= \int_0^{2\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos 2t \; dt$$

$$= \tfrac{1}{2}t\Big|_0^{2\pi} + \underbrace{\tfrac{1}{4}\sin 2t\Big|_0^{2\pi}}_{=0}$$

LHS $= 2\pi$

RHS $= \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{n}} \; ds$

$$F = y^2 \underline{\hat{i}} + x\underline{\hat{j}} + z^2 \underline{\hat{k}}$$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & z^2 \end{vmatrix}$$

$$= \hat{i}\left(\partial/\partial y\, z^2 - \partial/\partial z\, x\right)$$
$$- \hat{j}\left(\partial/\partial x\, z^2 - \partial/\partial z\, y^2\right)$$
$$+ \hat{k}\left(\partial/\partial x\, x - \partial/\partial y\, y^2\right)$$

$$= 0\underline{\hat{i}} + 0\underline{\hat{j}} + (1-2y)\underline{\hat{k}}$$

surface $S = x^2+y^2 = z$

$$z - x^2 - y^2 = 0 = \phi(x,y,z)$$

$$\underline{\hat{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2x\hat{i} - 2y\hat{j} + \underline{\hat{k}}}{\sqrt{4x^2+4y^2+1}}$$

$x^2+y^2=z$

$$\underline{\hat{n}} = \frac{-2x\underline{\hat{i}} - 2y\underline{\hat{j}} + \underline{\hat{k}}}{\sqrt{4z+1}}$$

$$\iint_S (\underline{\nabla}\times\underline{F})\cdot\hat{n}\,dS$$

$$= \iint_S \frac{(1-2y)\underline{\hat{k}}}{1} \cdot \frac{-2x\hat{i} - 2y\hat{j} + \hat{k}}{\sqrt{4z+1}}\,dS$$

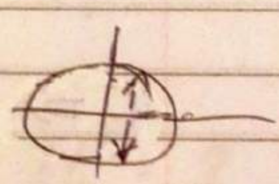
$$= \iint_S \frac{(1-2y)}{\sqrt{4z+1}}\,ds$$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_S \frac{1-2y}{\sqrt{4z+1}}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\underline{\hat{n}}\cdot\hat{k} = \frac{1}{\sqrt{4z+1}} \rightarrow ds = \sqrt{4z+1}\,dx\,dy$$

$$\iint_S \frac{1-2y}{\sqrt{4z+1}}\sqrt{4z+1}\,dx\,dy$$



$$= \iint_S 1-2y\,dx\,dy$$

S = unit circle

$$\iint_{\text{unit circle}} y\,dx\,dy = 0$$

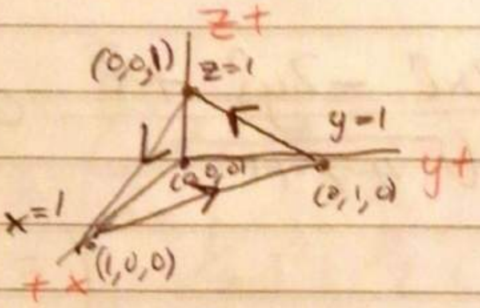
$$\iint_{\text{unit circle}} 1\,dx\,dy = \text{area of circle}$$

$$1^2\pi = \pi$$

LHS = RHS ∴ Stokes's Theorem verified.

Q3 (b) S is part of the plane $x+y+z=1$ that lies on the first octant oriented upwards

Vector Field $F = -y\hat{\underline{i}} + x\hat{\underline{j}}$



$LHS = \int_C \underline{F} \cdot \underline{dr}$ $\quad F = -y\hat{\underline{i}} + x\hat{\underline{j}}$

$z = 1-x-y$

$C_1 : (1,0,0) \to (0,1,0)$
$C_2 : (0,1,0) \to (0,0,1)$
$C_3 : (0,0,1) \to (1,0,0)$

$r_1 \Rightarrow \langle 1,0,0 \rangle + t[\langle 0,1,0 \rangle - \langle 1,0,0 \rangle]$
$= \langle \underset{x}{1-t}, \underset{y}{+t}, \underset{z}{0} \rangle$

$r_2 \Rightarrow \langle 0,1,0 \rangle + t[\langle 0,0,1 \rangle - \langle 0,1,0 \rangle]$
$= \langle \underset{x}{0}, \underset{y}{1-t}, \underset{z}{+t} \rangle$

$r_3 \Rightarrow \langle 0,0,1 \rangle + t[\langle 1,0,0 \rangle - \langle 0,0,1 \rangle]$
$= \langle \underset{x}{t}, \underset{y}{0}, \underset{z}{1-t} \rangle$

$\frac{dr_1}{dt} = \langle -1, 1, 0 \rangle$
$\frac{dr_2}{dt} = \langle 0, -1, 1 \rangle$
$\frac{dr_3}{dt} = \langle 1, 0, -1 \rangle$

$$\int_C \underline{F} \cdot \underline{dr} =$$

$$\int_{C_1} \underline{F} \cdot dr_1 + \int_{C_2} \underline{F} \cdot dr_2 + \int_{C_3} \underline{F} \cdot dr_3$$

$$= \int_0^1 \langle -y, x, 0 \rangle \cdot \langle -1, 1, 0 \rangle \, dt$$

$$+ \int_0^1 \langle -y, x, 0 \rangle \cdot \langle 0, -1, 1 \rangle \, dt$$

$$+ \int_0^1 \langle -y, x, 0 \rangle \cdot \langle 1, 0, -1 \rangle \, dt$$

$$= \int_0^1 \langle -t, 1-t, 0 \rangle \cdot \langle -1, 1, 0 \rangle \, dt$$

$$+ \int_0^1 \langle t-1, 0, 0 \rangle \cdot \langle 0, -1, 1 \rangle \, dt$$

$$+ \int_0^1 \langle 0, t, 0 \rangle \cdot \langle 1, 0, -1 \rangle \, dt$$

$$= \int_0^1 t + 1 - t \, dt = \int_0^1 1 \, dt = t|_0^1 = 1$$

$$+ \int_0^1 0 \, dt = 0$$

$$+ \int_0^1 0 \, dt = 0$$

$$LHS = 1$$

RHS: $\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{n}} \; dS$

Surface $x + y + z = 1$

$z = 1 - x - y$

$r = \langle x, y, z \rangle$

$r = \langle x, y, 1-x-y \rangle$

$\underline{\hat{n}} = (\underline{r_x} \times \underline{r_y}) \, dx \, dy$

$\underline{\hat{n}} = (\langle 1, 0, -1 \rangle \times \langle 0, 1, -1 \rangle) \, dx \, dy$

$$\underline{r_x} \times \underline{r_y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{i}(+1) - \hat{j}(-1) + \underline{\hat{k}}$$

$$= +\underline{\hat{i}} + \underline{\hat{j}} + \underline{\hat{k}}$$

$F = -y\underline{\hat{i}} + x\underline{\hat{j}}$

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} =$$

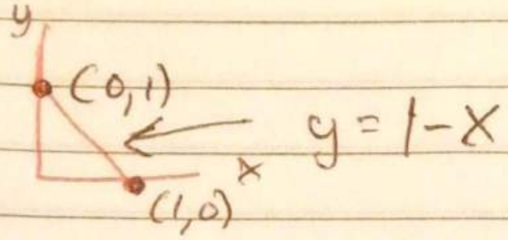
$$\hat{i}\left(\partial/\partial y(0) - \partial/\partial z\, x\right) - \hat{j}\left(\partial/\partial x(0) + \partial/\partial z\, y\right) + \hat{k}\left(\partial/\partial x\, x - \partial/\partial y - y\right)$$

$$= 0\hat{i} + 0\hat{j} + \hat{k}(1+1)$$

$$= 2\hat{k}$$

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{\hat{n}} = \iint_S (2\hat{k}) \cdot (+\hat{i} + \hat{j} + \hat{k}) \, dx\,dy$$

$$= \iint_S 2 \, dx\,dy$$



$$\int_0^1\int_0^{1-x} 2\,dy\,dx$$

$$= \int_0^1 2(1-x)\,dx$$

=

$$= 2x - \frac{\cancel{2}x^2}{\cancel{2}}\Big|_0^1$$

$$= 2-1-0 = 1$$

RHS = 1
LHS = 1

LHS = RHS
∴ Stoke's Theorem verified here.