Maths Methods Assignment 5
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Q1 (a)

$$y_{i+1} \simeq y(x_i+h) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \frac{h^3}{6}y'''(x_i)$$

$$y_{i+2} \simeq y(x_i+2h) = y(x_i) + 2hy'(x_i) + \frac{(2h)^2}{2}y''(x_i) + \frac{(2h)^3}{6}y'''(x_i)$$
$$= y(x_i) + 2hy'(x_i) + 2h^2y''(x_i) + \tfrac{4}{3}h^3y'''(x_i)$$

$$y_{i+3} \simeq y(x_i+3h) = y(x_i) + 3hy'(x_i) + \frac{(3h)^2}{2}y''(x_i) + \frac{(3h)^3}{6}y'''(x_i)$$

$$= y(x_i) + 3hy'(x_i) + \tfrac{9}{2}h^2y''(x_i) + \tfrac{9}{2}h^3y'''(x_i)$$

b) Compute $ay_{i+1} + by_{i+2} + cy_{i+3}$
adjust constants so that $y''(x_i)$ & $y'''(x_i)$ vanish.

$a y_{i+1} + b y_{i+2} + c y_{i+3} =$
$$(a+b+c)y(x_i) + (a+2b+3c)hy'(x_i)$$
$$+\left(\tfrac{a}{2} + 2b + \tfrac{9}{2}c\right)h^2y''(x_i) + \left(\tfrac{1}{6}a + \tfrac{4}{3}b + \tfrac{9}{2}c\right)h^3y'''(x_i)$$

we want $\left(\frac{a}{2} + 2b + \frac{9}{2}c\right) = 0$
$\left(\frac{1}{6}a + \frac{4}{3}b + \frac{9}{2}c\right) = 0$

a, b, c = 0 is trivial $\longrightarrow$ $a, b, c \neq 0$, $(a+b+c) \neq 0$, $(a+2b+3c) \neq 0$

① $\frac{a}{2} + 2b + \frac{9}{2}c = 0$
② $\frac{1}{6}a + \frac{4}{3}b + \frac{9}{2}c = 0$

① $\frac{9}{2}c = -\frac{a}{2} - 2b \longrightarrow c = \frac{2}{9}\left[-\frac{a}{2} - 2b\right]$
② $\frac{9}{2}c = -\frac{1}{6}a - \frac{4}{3}b \longrightarrow c = \frac{2}{9}\left[-\frac{1}{6}a - \frac{4}{3}b\right]$

① $c = \frac{-a}{9} - \frac{4}{9}b = \frac{-3a}{27} - \frac{12b}{27}$
② $c = -\frac{1}{27}a - \frac{8}{27}b$

① = ② $\frac{2a}{27} = \frac{-4b}{27}$

$\boxed{a = -2b} \rightarrow \boxed{b = -a/2}$

$c = -a/9 - 4/9\, b \rightarrow$ $c = \frac{2b}{9} - \frac{4b}{9} = \frac{-2b}{9}$

$c = \frac{-2}{9} \cdot \frac{-a}{2} = a/9$

$$\boxed{\begin{aligned} a &= a \\ b &= -a/2 \\ c &= a/9 \end{aligned}}$$

in terms of a, $ay_{i+1} + by_{i+2} + cy_{i+3}$ becomes:

$$\left(\frac{11}{18}a\right)y(x_i) + \left(\frac{a}{3}\right)h\,y'(x_i) + (0)h^2y''(x_i) + (0)h^3y'''(x_i)$$

$$= \left(\frac{11}{18}a\right)y(x_i) + \left(\frac{a}{3}\right)h\,y'(x_i)$$

take $(a + 2b + 3c) = 1$

$\cancel{a} - \cancel{a} + 3a/9 = 1$ $\quad a = 3$

$b = -a/2 = -3/2$

$c = a/9 = 1/3$

now $ay_{i+1} + by_{i+2} + cy_{i+3} =$

$3y_{i+1} - 3/2\, y_{i+2} + 1/3\, y_{i+3} = \left(\frac{11}{6}\right)y(x_i) + h\,y'(x_i)$

(c) $h\,y'(x_i) = -11/6\, y(x_i) + 3y_{i+1} - 3/2\, y_{i+2} + \frac{1}{3}y_{i+3}$

$$y'(x_i) = \frac{1}{h}\left[-\frac{11}{6}y_i + 3y_{i+1} - \frac{3}{2}y_{i+2} + 1/3\, y_{i+3}\right] + O(h^3$$

(d) want $y'''(x_i)$ to vanish $\rightarrow (1/6\, a + 4/3\, b + 9/2\, c) = 0$

want $y'(x_i)$ to vanish $\rightarrow (a + 2b + 3c) = 0$

① $1/6\, a + 4/3\, b + 9/2\, c = 0$

② $a + 2b + 3c = 0$

② $a = \cancel{-2b} - 3c$

① ×6 $\rightarrow a + 8b + 27c = 0$

② $a = -2b - 3c$ ↳ sub in

① $-2b - 3c + 8b + 27c = 0$

$2b + 24c = 0$

$\boxed{b = -4c}$

$a = -2b - 3c \qquad a = -2(-4)c - 3c = 8c - 3c$

$\boxed{a = 5c}$

Set $(9/2 + 2b + 9/2\, c) = 1$

$5/2\, c - 8 + 9/2\, c = 1$

$-1c = 1 \rightarrow \boxed{\begin{array}{l} c = -1 \\ b = -4c = 4 \\ a = 5c = -5 \end{array}}$

$$a y_{i+1} + b y_{i+2} + c y_{i+3} = (-5+4-1)\, y(x_i) + (-5 + 2(4) + 3(-1))\, h y'(x_i) + (-1/2 + 2(4) + 9/2(-5))\, h^2 y''(x_i) + (1/6(-5) + 2(4) + 9/2(-1))\, h^3 y'''(x_i)$$

$$-5y_{i+1} + 4y_{i+2} - 1\, y_{i+3} = -2\, y(x_i) + (0)\, h y'(x_i) + h^2 y''(x_i) + (0)\, h^3 y'''(x_i)$$

$$-5y_{i+1} + 4y_{i+2} - y_{i+3} = -2y(x_i) + h^2 y''(x_i)$$

$$h^2 y''(x_i) = 2y(x_i) - 5y_{i+1} + 4y_{i+2} - y_{i+3}$$

$$y''(x_i) = \frac{1}{h^2}\left[2y_i - 5y_{i+1} + 4y_{i+2} - y_{i+3}\right] + O(h^2)$$

Maths Methods Assignment 5

Q2 $y'' + \alpha y = \sin x$ $\quad y(0)=1 \quad y'(0)=0$

$\alpha=4$ $\quad \boxed{y'' + 4y = \sin x}$

$$\underline{Y} = \begin{bmatrix} y \\ y' \end{bmatrix} \quad \underline{Y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ \sin x - 4y \end{bmatrix}$$

Approximate $y(0.1)$ using Taylor Series method

Truncated Taylor Series

$$\underline{Y}(h) = \underline{Y}(0) + h\underline{Y}'(0) + \frac{h^2}{2}Y''(0) + \frac{h^3}{6}Y'''(0) + \frac{h^4}{24}Y''''(0) + \dots$$

$$\underline{Y} = \begin{bmatrix} y \\ y' \end{bmatrix} \Rightarrow \underline{Y}(0) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\underline{Y}' = F(x, \underline{Y})$ $\quad \underline{Y}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ \sin x - 4y \end{bmatrix} \rightarrow Y'(0) = \begin{bmatrix} y'(0) \\ \sin x - 4y(0) \end{bmatrix}$

$$Y'(0) = \begin{bmatrix} 0 \\ \sin x - 4 \end{bmatrix} = \begin{bmatrix} y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$ $\sin(0)=0$

$$\underline{Y}'' = \begin{bmatrix} y'' \\ y''' \end{bmatrix} = \begin{bmatrix} y'' \\ \cos x - 4y' \end{bmatrix}$$

$$Y''(0) = \begin{bmatrix} -4 \\ 1 - 4y'(0) \end{bmatrix} = \begin{bmatrix} y''(0) \\ y'''(0) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$ $\boxed{y'(0) = 0}$

don't need for 2nd order Taylor Series

$$Y''' = \begin{bmatrix} y''' \\ y'''' \end{bmatrix} = \begin{bmatrix} y''' \\ -\sin x - 4y'' \end{bmatrix}$$

$$Y'''(0) = \begin{bmatrix} y'''(0) \\ y''''(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 - 4(-4) \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \end{bmatrix}$$

$$Y'''' = \begin{bmatrix} y'''' \\ y''''' \end{bmatrix} = \begin{bmatrix} y'''' \\ -\cos x - 4y''' \end{bmatrix}$$

$$Y''''(0) = \begin{bmatrix} y''''(0) \\ y'''''(0) \end{bmatrix} = \begin{bmatrix} 16 \\ -1 - 1 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \end{bmatrix}$$

2nd order Truncated Taylor Series

$$\underline{Y}(h) = \underline{Y}(0) + h\,\underline{Y}'(0) + \frac{h^2}{2}\,\underline{Y}''(0)$$

$$\underline{Y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \underline{Y}'(0) = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \quad \underline{Y}''(0) = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\underline{Y}(0.1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \frac{(0.1)^2}{2} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\underline{Y}(0.1) \simeq \begin{bmatrix} 1 + 0 - 2(0.01) \\ 0 - 0.4 + \frac{1}{2}(0.01) \end{bmatrix}$$

$$\underline{Y}(0.1) \simeq \begin{bmatrix} 0.98 \\ -0.395 \end{bmatrix} \simeq \begin{bmatrix} y(0.1) \\ y'(0.1) \end{bmatrix}$$

$$y(0.1) \simeq 0.98$$

From Wolfram Alpha find exact solution:

$$y(x) = C_2 \sin(2x) + C_1 \cos(2x) + \frac{\sin(x)}{3}$$

$$y(0) = 1 \rightarrow \quad C_1 = 1$$

$$y'(x) = 2C_2 \cos(2x) - 2C_1 \sin(2x) + \frac{\cos(x)}{3}$$

$$y'(0) = 0 \quad 0 = 2C_2 + \frac{1}{3} - 0 \rightarrow \quad C_2 = -\frac{1}{6}$$

$$y(x) = -\frac{1}{6} \sin(2x) + \cos(2x) + \frac{\sin(x)}{3}$$

exact value of $y(0.1) = 0.9802328282$

error of approximate value from Taylor series $= 0.00023752$

error $= 0.0237\%$

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> x := 0;
  Y := ⟨1, 0⟩;
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$$x := 0$$

$$Y := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1}$$

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> Yp := ⟨Y(2),  sin(x) - 4·Y(1)⟩;
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$$Yp := \begin{bmatrix} 0 \\ -4 \end{bmatrix} \tag{2}$$

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> Ypp := ⟨Yp(2), cos(x) - 4·Yp(1)⟩;
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$$Ypp := \begin{bmatrix} -4 \\ 1 \end{bmatrix} \tag{3}$$

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> h := 0.1 :
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> $Y\_01 := Y + h \cdot Yp + \frac{h^2}{2} \cdot Ypp;$

$$Y\_01 := \begin{bmatrix} 0.980000000000000 \\ -0.395000000000000 \end{bmatrix} \tag{4}$$

> $x := 0.1; y\_exact := -\frac{1}{6} \cdot \sin(2\,x) + \cos(2\,x) + \frac{\sin(x)}{3};$

$$x := 0.1$$

$$y\_exact := 0.9802328282 \tag{5}$$

> $err := \frac{(y\_exact - Y\_01(1))}{y\_exact};$

$$err := 0.000237523365165781 \tag{6}$$

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