Mathematical Physics 3rd Year Lab Report

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1 Learning Outcomes

The Learning Objective of Part 1 of this lab is to understand the connection between Planck's Law and the Stefan-Boltzman Law by deriving the relationship between them and to become familiar with analysing experimental data from an excel spreadsheet and deriving formulae and results from user-generated plots and tables.

In part 2, the Learning Objective is to use Planck's Radiation Law to generate plots of excel from which Wien's Displacement Law can be derived and the power emitted from a blackbody can be analysed.

2 Part I: The Stefan-Boltzman Relationship

In Part I, the Experimental Data in the lab manual will be used to plot log(P) vs log(T) which can be used to find the values of σ and in the Equation for the Stefan Boltzman Law

$$P = \sigma T^4 \tag{1}$$

First the Values of Voltage V and Current I must be entered into a table in Excel. From this one can find the Power ($P = VI$) and Resistance R(T) ($R(T) = V/I$)

Using Data in the lab manual, a plot of $\rho(T)/\rho(300)$ vs T can be created.

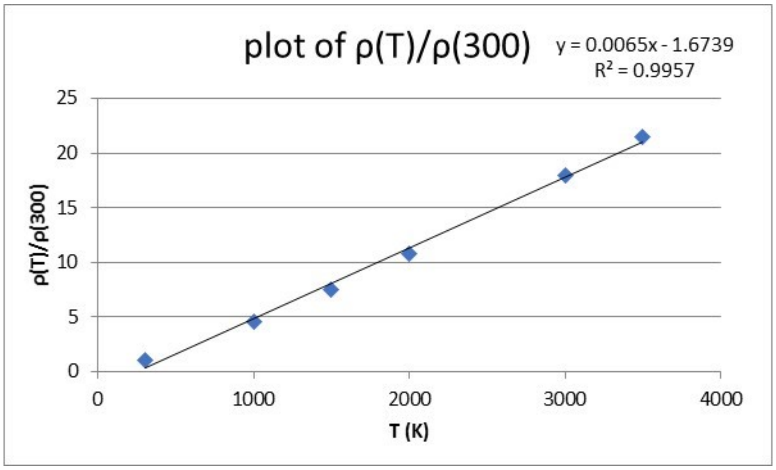

Figure 1: $\rho(T)/\rho(300)$ VS T

Since $\rho(T)/\rho(300) = R(T)/R(300)$ The temperatures for each measurement can be evaluated using the equation of the trendline which corresponds to

$$\frac{R(T)}{R(300)} = m(T) + c$$

$$\frac{R(T)}{R(300)} = 0.0065T - 1.6739$$

so then we obtain an equation for the values of T

$$T = \frac{1}{0.0065}\left(\frac{R(T)}{R(300)} + 1.6739\right)$$

We need to derive an equation so we can plot log(P) against log(T)
we know Radiated Power P depends on filament temperature as follows

$$P = \sigma T^4$$

we want to show that

$$log(P) = n \cdot log(T) + log(\sigma)$$

first take logs on both sides

$$log(P) = log(\sigma T^4)$$

using the rules of logarithms we can rewrite in the required format

$$log(P) = log(T^n) + log(\sigma)$$

$$log(P) = n \cdot log(T) + log(\sigma)$$

as required

Now, log(P) vs log(T) can be graphed

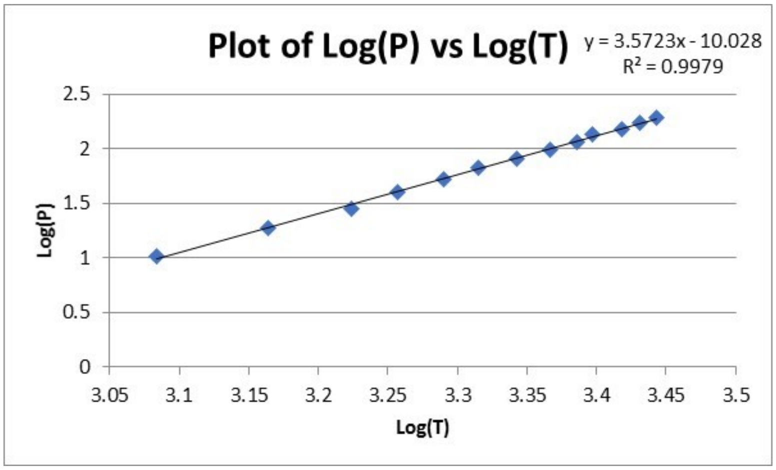

Figure 2: Plot of $log(P)$ vs $log(T)$

Using the equation of the trendline from this graph we can find approximate values for n and σ since

$$y = 3.5723 \cdot x - 10.028$$

corresponds to

$$log(P) = n \cdot log(T) + log(\sigma)$$

.
From the Excel Linest() function, the uncertainties in slope and y-intercept can be known

3.57226	-10.0282022
0.04892	0.162346752

Figure 3: clockwise: Slope, Intercept, Uncertainty in Slope, Uncertainty in Intercept

Hence

$$n = 3.57226 \pm 0.04892$$

and $log(\sigma) = -10.028 \pm 0.16235$ so

$$\sigma = 9.3756 \cdot 10^{-11} \pm 1.5178 \cdot 10^{-12} W m^{-2} K^{-n}$$

Now, the derivation of the real, exact value for the constant n. Planck's Radiation Law for Blackbodies is used here. We take the integral of $U(\lambda)$ from 0 to ∞ to find the total emitted Power accross all wavelengths and hence we can find The Stefan-Boltzmann constant σ.

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1}}$$

$$P = \int_0^\infty u(\lambda) d\lambda = \int_0^\infty \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1} d\lambda}$$

Make the Substitution $x = \frac{hc}{kT\lambda} \to \lambda = \frac{hc}{kTx}$

$$d\lambda = -\frac{kT}{hc}\lambda^2 dx$$

$$\int_0^\infty -8\pi kT\lambda^{-3}\frac{1}{e^x - 1}dx$$

$$-8\pi kT \int_0^\infty \frac{k^3T^3x^3}{h^3c^3} \cdot \frac{1}{e^x - 1}dx$$

$$\frac{-8\pi k^4T^4}{h^3c^3} \int_0^\infty \frac{x^3}{e^x - 1}dx$$

Note $\int_0^\infty \frac{x^{n-1}}{e^x-1} = \zeta(n)(n-1)!$
Notice n = 4 here $\to \zeta(4) = \frac{\pi^4}{90}$

$$\frac{-8\pi k^4T^4}{h^3c^3} \cdot \frac{\pi^4}{90} \cdot 6$$

$$P = \sigma T^n = \frac{-8}{15} \cdot \frac{\pi^5k^4T^4}{h^3c^3}$$

Since σ is a constant we see that n must equal 4 as T^4 is on both LHS and RHS
Thus n =4

From this we obtain

$$\sigma = \frac{-8}{15} \cdot \frac{\pi^5k^4T^4}{h^3c^3} = -7.5646 \cdot 10^{-16}$$

$$\sigma = -7.5646 \cdot 10^{-16}\ WK^{-4}$$

This value is verified as the modulus of $-7.5646 \cdot 10^{-16} = \frac{4\sigma}{c} = 7.56 \cdot 10^{-16}$

3 Part I: Discussion

In Part 1 the relationship $P = \sigma T^n$ was derived using Planck's Radiation Law and thus found the exact values of n and σ. I verified that this value of σ was correct by showing its modulus was equal to $\frac{4\sigma}{c}$. There is a difference in these two values of σ since they are defined in terms of different units (the value calculated here is for U in terms of wavelength whereas the actual value is found with U in terms of frequency).

The relationship $log(P) = n \cdot log(T) + log(\sigma)$ was verified and then this relationship was used with experimental data in the lab manual to find an approximate value for n and σ. The value obtained for n was $n = 3.5723 \pm 0.04892$ and the value obtained for σ was $9.3756 \times 10^{-11} \pm 1.5178 \times 10^{-12} Wm^{-2}K^{-n}$. There is a very large discrepency here between the approximate result of σ and the exact result of σ that was derived and this discrepency is not reflected in the uncertainties that were calculated using the linest() function in excel. This approximation gives a result that is somewhat precise but it is not very accurate.

The most likely source of error in the approximation is due to a large fraction of input power being dissipated by the supports holding the filament in the light bulb in the experiment. There may be other sources of random errors in the experiment the data is from that are not mentioned in the lab manual that may be other causes of this large discrepancy in the approximate value for σ.

From these results, it can be concluded that the relationships $log(P) = n \cdot log(T) + log(\sigma)$ and $P = \sigma T^n$ hold for blackbodies. We also now know that the exact values for σ and n in this experiment are $-7.5646 \cdot 10^{-16}$ Wk^-4 and 4 respectively.

4 Part II: The Planck Radiation Formula

In part II we are asked to analyse Planck's Radiation Formula

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1} Jm^{-3}}$$

U gives the radiation energy per unit volume as a function of wavelength λ in terms of planck's constant h, Boltzmann constant k and the speed of light c. Here the spectrum of radiation emitted from a body is investigated by plotting the emission spectrum accross a large distribution of wavelengths for given temperatures. Then Wien's Displacement Law is derived and the corresponding Wien's Displacement Constant calculated.

Q2.1 Given Planck's Radiation Formula in SI

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1} Jm^{-3}}$$

we want to rewrite it in terms of λ in nm so that:

$$u(\lambda) = \frac{5 \times 10^{21}}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{14.4 \times 10^6}{kT\lambda}\right)-1} Jm^{-3}}$$

evaluating constants

$$u(\lambda) = \frac{5 \times 10^{-24}}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{0.0144}{kT\lambda}\right)-1} Jm^{-3}}$$

λ in nm $= \lambda \times 10^{-9}$

$$u(\lambda) = \frac{5 \times 10^{-24}}{\lambda^5 \cdot (10^{-9})^5} \cdot \frac{1}{e^{\left(\frac{0.0144}{kT\lambda \cdot 10^{-9}}\right)-1} Jm^{-3}}$$

$$u(\lambda) = \frac{5 \times 10^{21}}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{14.4 \times 10^6}{kT\lambda}\right)-1} Jm^{-3}}$$

Now that a Planck's Radiation Formula is rewritten in terms of λ in nm, excel can be used to plot 2 graphs of the U vs λ, one for T = 6000 K and another for T = 2000 K. Values of λ increase in steps of 10 nm on the x axis to generate the radiation spectrum. λ starts at 10 nm instead of at 0 nm because Planck's Formula is undefined at $\lambda = 0$ since $e^{\left(\frac{hc}{0}\right)}$ is undefined.

lambda	u (T=6000)		lambda	u (T=2000)
10	7.99077E-88		10	0
20	3.25669E-37		20	1.9E-141
30	1.00948E-20		30	3.29E-90
40	1.16224E-12		40	8.91E-65
50	6.1984E-08		50	1.26E-49
60	7.42556E-05		60	1.34E-39
70	0.010415443		70	1.73E-32

Figure 4: Extract from table used to generate U vs λ plots in Excel

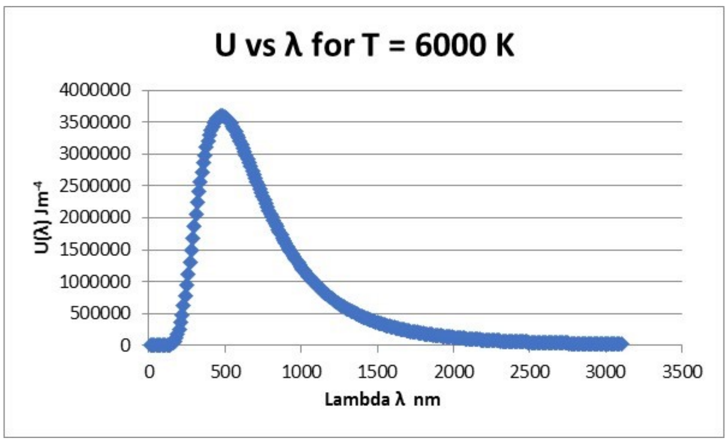

Figure 5: Plot of U(λ) vs λ for T = 6000

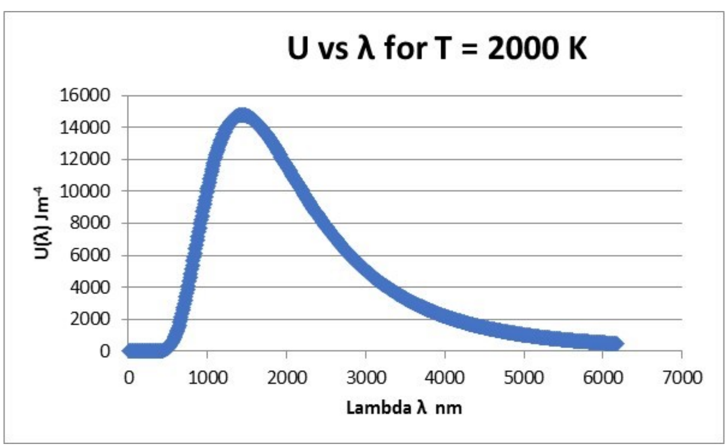

Figure 6: Plot of U(λ) vs λ for T = 2000

Once the plots have been obtained, it is now possible do some analysis on the radiations emitted from a blackbody using Planck's Radiation Law.

Q2.2 The Wavelengths for maximum power emission are found to be:

$$\lambda_{max} = 480nm for T = 6000K$$

$$\lambda_{max} = 1440nm for T = 2000K$$

Q2.3 Wien's Displacement Law states that $\lambda_{max} \times T$ = constant Derive the formula from $U(\lambda)$ and thus evaluate the constant

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1}}$$

find λ_{max}:

$$\frac{\partial}{\partial\lambda}\left(u(\lambda) = \frac{\partial}{\partial\lambda}\left(\frac{8\pi hc}{\lambda^5}\right) \cdot \frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1}}\right) = 0$$

$$\frac{\partial}{\partial\lambda}\left(\frac{8\pi hc}{\lambda^5}\right) \cdot \left(\frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1}}\right) + \frac{\partial}{\partial\lambda}\left(\frac{1}{e^{\left(\frac{hc}{kT\lambda}\right)-1}}\right) \cdot \frac{8\pi hc}{\lambda^5} = 0$$

using chain rule and product rule, the following is obtained:

$$\frac{\partial u(\lambda)}{\partial\lambda} = \frac{hc}{kT}\lambda^{-2} \cdot \frac{\exp\left(\frac{hc}{\lambda kT}\right)}{\left(ex\left(\frac{hc}{\lambda kT}\right)-1\right)^2} \cdot \frac{8\pi hc}{\lambda^6} - 5 \cdot \frac{8\pi hc}{\lambda^6} \cdot \frac{1}{\exp\left(\frac{hc}{t-\lambda T}\right)-1}$$

$$= \frac{hc}{kT\lambda} \cdot \frac{\exp\left(\frac{hc}{kT\lambda}\right)}{\left(exp\left(\frac{hc}{k\tau\lambda}\right)-1\right)^2} = 5 \cdot \frac{1}{\exp\left(\frac{hc}{KT\lambda}\right)-1}$$

$$\frac{hc}{kt\lambda} \cdot \exp\left(\frac{hc}{kt\lambda}\right) = 5 \cdot \left(\exp\left(\frac{hc}{kt\lambda}\right) - 1\right)$$

$$\frac{hc}{kt\lambda} = 5 \cdot \frac{\left(\exp\left(\frac{hc}{kt\lambda}\right)-1\right)}{\exp\left(\frac{hc}{kt\lambda}\right)} = 5 \cdot \left(1 - \frac{1}{\exp\frac{hc}{kT\lambda}}\right)$$

$$\lambda T = \frac{hc}{5k} \cdot \left(1 - \frac{1}{\exp\frac{hc}{kT\lambda}}\right)^{-1}$$

Therefore, it is shown that $\lambda_{max}T$ is a constant
Thus it is possible to find the value of that constant
Substitute $x = \frac{hc}{\lambda kT}$ into $\frac{hc}{kt\lambda} = 5 \cdot \left(1 - \frac{1}{\exp\frac{hc}{kT\lambda}}\right)$

$$x = 5(1 - e^{-x})$$

which yields x = 4.965
Substitute this x value into $\lambda_{max}T = \frac{hc}{kx}$ which gives us Wien's Displacement Constant = $\lambda T = 2.9 \times 10^{-3}$ mK

Q2.4

$$T = 6000 \rightarrow \lambda_{max} \times T = 480 \times 10^{-9} \cdot 6000 = 2.9 \times 10^{-3}$$

$$T = 2000 \rightarrow \lambda_{max} \times T = 1440 \times 10^{-9} \cdot 2000 = 2.9 \times 10^{-3}$$

Thus Wien's Displacement Law is verified from the graphs, as these two values of Wien's Displacement constant agree with the theoretical value that was calculated in Q2.3.

Q2.5 Radiated Power at λ_{max} for T = 6000 K = 5.1755×10^9 W Radiated Power at λ_{max} for T = 2000 K = 14790.39 W

Ratio between the two radiated powers at λ_{max} is 349923.16 times. This means that radiated power at λ_{max} for T = 6000 K is 349923.16 times the radiated power at λ_{max} for T = 2000 K.

Q2.6 Find the % Power total output that lies in the visible spectrum for T = 2000 and T = 6000. Use both an approximate method and an exact method.

For the Approximate method I used the Trapezoidal Rule which is done in the following way:

$$A = \sum_{i=0}^{n-1} \frac{h}{2}(f_i + f_{i+1})$$

To do this in Excel, it is possible to create a collumn equal to the nth $U(\lambda)$ term + n+1th $U(\lambda)$ term all multiplied by $\frac{h}{2}$ where h is the step size in λ, which h = 10 in this case.

This results in the following table where the results for total power emitted and power emitted over the visible spectrum (380 nm - 700 nm) can be found.

trapezoidal rule T = 6000					trapezoidal rule T = 2000			
1.62835E-36	2437965750	total emitted power			9.5735E-141	29405059.3	total emitted power	
5.04742E-20					1.64419E-89			
5.8112E-12	1075215441	emmitted power over			4.45582E-64	282790.381	emitted power over	
3.09926E-07		visible spectrum			6.29476E-49		visible spectrum	
0.000371588					6.70101E-39			
0.052448495					8.63883E-32			
1.992740095	0.44102976	44.10%			1.69935E-26	0.00961707	0.96%	
32.12876144					2.0773E-22			
286.7350894					3.65829E-19			

Figure 7: Table used to find total power emission and total power emission over the visible spectrum

The approximate values for total emitted power and emitted power over the visible spectrum are displayed here in Figure 7.
The % Power output to the visible spectrum for a blackbody of T = 6000 is 44.10%.
The % Power output to the visible spectrum for a blackbody of T = 2000 is 0.96%

To obtain the exact % of power emitted to the visible light spectrum, we must integrate $U(\lambda)$ over the range of λ values to find the total power output over that range.

First find definite integral of $U(\lambda)$

$$\int \frac{5\times 10^{21}}{\lambda^5}\cdot\frac{1}{e^{\left(\frac{14.4\times 10^6}{kT\lambda}\right)-1}}$$

which turns out to be:
$\lambda^{-3}\cdot 5\times 10^{21} T e^{\frac{-14.4\times 10^6}{T\lambda}}\cdot(3.79311\times 10^{-28}T^3\lambda^3$
$+5.46208\times 10^{-21}T^2\lambda^2 + 3.9327\times 10^{-14}T\lambda + 1.8877\times 10^{-7}) + C$

setting limits as 0 to ∞ gives the total power emitted:
Total power emitted for T = 6000 K = 2.45794×10^9 W
Total power emitted for T = 2000 K = 30344900 W

setting limits as 380 nm to 700 nm gives power emitted over visible wavelengths:
Power output to visible spectrum for T = 6000 K = 1.04948×10^9 W
Power output to visible spectrum for T = 2000 K = 253972 W

% Power output to visible spectrum for T = 6000 k = 42.70 %
% Power output to visible spectrum for T = 2000 k = 0.84%

5 Part II: Discussion

In Part 2, Wien's Displacement Law was derived using Planck's Radiation formula. It was shown that $\lambda_{max} T$ = a constant. That constant is calleed Wien's Displacement constant and it was evaluated to be 2.9×10^{-3}mK.

The plots created in Excel further verify Wien's Displacement law as for both T = 2000 K and T = 6000 K, $\lambda_{max} T$ was found to equal 2.9×10^{-3} mK.

In Q2.5 we see that there is a large change in emitted power at peak wavelengths when Temperature T is tripled. The ratio between the two values of $U(\lambda)_{max}$ for T = 2000 and T = 6000 is $\simeq$ 350000. This is not a surprising result as the Stefan-Boltzman Law shows that Power output from a blackbody is mostly dependant on Temperature since T is to the highest power in $P = \sigma T^4$.

in Q2.6 it is shown that very little power is output to the visible spectrum of light for low temperatures but a considerable fraction of power output from a blackbody can be seen for higher temperatures. This is shown using both an approximation and an exact calculation. The approximation finds that for a blackbody with temperature T = 2000 K, that 0.96% of output power is seen in the visible light spectrum (λ = 380nm - 700nm). The approximate method also finds that for T = 6000 K 44.10% of a blackbody's power output is seen over the visible light spectrum. The exact method finds similar percentages, 0.84% of power output for a blackbody at 2000 K is visible accross the visible spectrum of light. 42.70% of power output from a blackbody at T = 6000 K is found in the visible wavelengths for light. The errors between the approximate method and the exact percentages are relatively low but they could be reduced by using a better fit model than the Trapezoidal rule, by making a smaller step size and by making a larger range of λ values for the plot in Excel.

From these results we can conclude that $\lambda_{max} T = 2.9 \times 10^{-3}$ and that a greater % of light radiated from a blackbody is visible for bodies with higher temperatures. We also see the Ratio between values of U at peak λ values is not linear and increases exponentially with T.