MP345 Maths Methods I Assignment 3
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Q1 $bxy'' + 5y' + y = 0$ $b=8$

$\boxed{8xy'' + 5y' + y = 0}$

(a) Check if analytical for $x=0$

normal form $y'' + \frac{5}{8x}y' + \frac{1}{8x}y$

$p = \frac{5}{8x}$ $q = \frac{1}{8x}$

$\frac{5}{8(0)} \to \infty$ $q\ \frac{1}{8(0)} \to \infty$

$\therefore$ $p\&q$ not analytic for $x=0$

Check if $xp(x)$ and $x^2q(x)$ are analytic for $x=0$

$xp = \frac{5}{8}$ $x^2q = \frac{x}{8}$

$xp = \frac{5}{8}$ $x\to0$ $x^2q = \frac{0}{8} = 0$ for $x=0$

$xp(x)$ and $x^2q(x)$ are analytic for $x=0$

Since this 2nd order homogeneous ODE has $p(x)$ and $q(x)$ not analytic for $x=0$ but $xp(x)$ & $x^2q(x)$ are analytic for $x=0$, We can use the Frobenius Method to find solutions.

(b) $y = x^r\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$

$y' = \sum_{n=0}^{\infty}(r+n)a_n x^{n+r-1}$ want in form $\sum_{n=0} \_\_\ x^{n+r}$

$n=p+1$ $p=n-1$ $y' = \sum_{p=-1}^{\infty}(r+p+1)\,a_{p+1}\,x^{p+r}$

$y' = \sum_{n=-1}(r+n+1)\,a_{n+1}\,x^{n+r}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = r a_0 x^{r-1} + \sum_{n=0}^{\infty} (n+r+1) a_{n+1} x^{n+r}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$xy'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$p = n-1 \quad n = p+1$

$$xy'' = \sum_{p=-1}^{\infty} (p+1+r)(p+r) a_{p+1} x^{p+r}$$

$$xy'' = \sum_{n=-1}^{\infty} (n+1+r)(n+r) a_{n+1} x^{n+r}$$

$$xy'' = \cancel{xy''} \; r(r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} (n+r+1)(n+r) a_{n+1} x^{n+r}$$

$\boxed{8xy'' + 5y' + y}$ $\cancel{ra_0 x^{r-1}}$ $5ra_0 x^{r-1} + 8r(r-1)a_0 x^{r-1} +$

$$\sum_{n=0}^{\infty} \left[8(n+r+1)(n+r)a_{n+1} + 5(n+r+1)a_{n+1} + a_n \right] x^{n+r} = 0$$

Coefficients of $x^{r-1} = 0$

$x^{n+r} = 0$

$$x^{r-1}(5ra_0 + 8r(r-1)a_0) = x^{r-1} a_0 (5r + 8r(r-1)) = 0$$

$$a_0(5r + 8r^2 - 8r) = a_0(8r^2 - 3r) = a_0 r(8r-3) = 0$$

indicial equation

$r = 0, \; r = 3/8$

Find recursion relation

$$8(n+r+1)(n+r)a_{n+1} + 5(n+r+1)a_{n+1} + a_n = 0$$

$$a_{n+1}(8(n+r+1)(n+r) + 5(n+r+1)) = -a_n$$

$\cancel{a_{n+1}}$

$$a_{n+1} = \frac{-a_n}{8(n+r+1)(n+r) + 5(n+r+1)}$$

$$\boxed{a_{n+1} = \frac{-a_n}{(n+r+1)(8(n+r)+5)}}$$ Recursion Relation

$r=0$, Choose $a_0 = 1$ ($a_0 = 0 \to$ trivial soln)

(n=0) $$a_1 = a_{0+1} = \frac{-1}{(0+0+1)(8(0+0)+5)} = \frac{-1}{5}$$

(n=1) $$a_2 = \frac{-(-1/5)}{(1+1)(8(1)+5)} = \frac{1}{130}$$

(n=2) $$a_3 = \frac{-(1/130)}{(2+1)(8(2)+5)} = \frac{-1}{8190}$$

$$y_1 = x^0(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

$$y_1 = 1 - \tfrac{1}{5}x + \tfrac{1}{130}x^2 - \tfrac{1}{8190}x^3$$

$r = 3/8$ $a_0 = 1$

(n=0) $$a_1 = \frac{-1}{(0+3/8+1)(8(0+3/8)+5)} = \frac{-1}{11}$$

(n=1) $$a_2 = \frac{-(-1/11)}{(1+3/8+1)(8(1+3/8)+5)} = \frac{1}{418}$$

(n=2) $$a_3 = \frac{-1/418}{(2+3/8+1)(8(2+3/8)+5)} = \frac{-1}{33858}$$

$$y_2 = x^{3/8}\left(1 - \tfrac{1}{11}x + \tfrac{1}{418}x^2 - \tfrac{1}{33858}x^3\right)$$

two linearly independent solns
$$\begin{cases} y_1 = 1 - \tfrac{1}{5}x + \tfrac{1}{130}x^2 - \tfrac{1}{8190}x^3 \\ y_2 = x^{3/8}\left(1 - \tfrac{1}{11}x + \tfrac{1}{418}x^2 - \tfrac{1}{33858}x^3\right) \end{cases}$$

Q2 (a) $6xy'' + 26y' + ty = 0$

$\underline{8xy''} + 16y' + ty = 0$

get into normal form

$$y'' + \frac{2}{x}y' + \frac{1}{8x}y = 0$$

$p(x) = \frac{2}{x}$ $\quad q(x) = \frac{1}{8x}$ $\quad$ see if analytic for $x=0$

$p(0) = \frac{2}{0} \to \infty$ $\quad q(0) = \frac{1}{8(0)} \to \infty$ $\to p \,\&\, q$ not analytic for $x=0$

see if p & q are analytic for $xp(x)$, $x^2q(x)$ $x=0$

$xp(x) = 2 \quad x^2q(x) = \frac{x}{8}$

$xp(0) = 2 \quad x^2q(0) = \frac{0}{8} = 0$

$xp(x)$ & $x^2q(x)$ analytic for $x=0$

Since this ODE is not analytic for $p(x)$ & $q(x)$ with $x=0$ in normal form but $xp(x)$ and $x^2q(x)$ are analytic for $x=0$. This means we can use the Frobenius Method to find at least one Solution to this ODE!

(b) $8xy'' + 16y' + ty = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \to \text{shift } p = n-1 \quad n = p+1$$

$$y' = \sum_{n=-1}^{\infty} a_{n+1}(n+r+1)x^{n+r}$$

$$y' = a_0 r x^{r-1} + \sum_{n=0}^{\infty} a_{n+1}(n+r+1)x^{n+r}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

$$xy'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1}$$

$$xy'' = \sum_{n=-1}^{\infty} a_{n+1}(n+r+1)(n+r)X^{n+r}$$

$$xy'' = a_0(\quad r)(r-1)X^{r-1} + \sum_{n=0}^{\infty} a_{n+1}(n+r+1)(n+r)X^{n+r}$$

$$\begin{cases} 16a_0 r X^{r-1} + 8a_0 r(r-1)X^{r-1} \\ + \sum_{n=0}^{\infty} \left[8a_{n+1}(n+r+1)(n+r) + 16a_{n+1}(n+r+1) + a_n \right] X^{n+r} = 0 \end{cases}$$

Coefficients of $X^{r-1} = 0$

$X^{n+r} = 0$

$$16a_0 r X^{r-1} + 8a_0 r(r-1)X^{r-1} = 0$$ indicial eqn

$$X^{r-1}(16a_0 r + 8a_0 r(r-1)) = 0$$

$$a_0(16r + 8r(r-1)) = 0$$

$$a_0(16r + 8r^2 - 8r) = 0$$

$$a_0(8r^2 + 8r) = 0$$

$$r=0, \quad r=-1$$

$$8a_{n+1}(n+r+1)(n+r) + 16a_{n+1}(n+r+1) + a_n = 0$$

$$-a_n = 8a_{n+1}(n+r+1)((n+r)+2)$$

$$\boxed{a_{n+1} = \frac{-a_n}{8(n+r+1)((n+r)+2)}}$$ Recurrence Relation

$r=0$, set $a_0 = 1$

$n=0$ $\quad a_1 = \dfrac{-1}{8(1)((0)+2)} = \dfrac{-1}{16}$

$n=1$ $\quad a_2 = \dfrac{-(-1/16)}{8(1+0+1)((1+0)+2)} = \dfrac{1}{768}$

$n=2$ $\quad a_3 = \dfrac{-(1/768)}{8(2+0+1)((2+0)+2)} = -\dfrac{1}{73728}$

$$y_1 = 1 - \tfrac{1}{16}X + \tfrac{1}{768}X^2 - \tfrac{1}{73728}X^3$$

(c) $r_1 = 0 \quad r_2 = -1 \quad r_1 - r_2 = 1 = \text{integer}$

try get y_2 try $r = -1$ $a_0 = 1$

$$a_{n+1} = \frac{-a_n}{8(n+r+1)(n+r+2)}$$

$a_0 = 1 \quad a_1 = \frac{-1}{8(0-1+1)(0-1+2)} = \frac{-1}{8(0)(1)}$

a_1 is undefined.

no soln for $r = -1$, $a_0 = 1$

try $r = -1$ $a_0 = 0$, $a_1 = 1$

$a_0 = 0$

$a_1 = 1$

$$a_2 = \frac{-1}{8(1+-1+1)(1-1+2)} = -\frac{1}{16}$$

$$a_3 = \frac{-(-1/16)}{8(2-1+1)(2-1+2)} = +\frac{1}{768}$$

$$a_4 = \frac{-1/768}{8(3-1+1)(3-1+2)} = \frac{-1}{73728}$$

$$\boxed{y_2 = x^{-1}\left(0 + x - \tfrac{1}{16}x^2 + \tfrac{1}{768}x^3 - \tfrac{1}{73728}x^4\right)}$$

$$y_2 = 1 - \tfrac{1}{16}x + \tfrac{1}{768}x^2 - \tfrac{1}{73728}x^3$$

$$y_1 = 1 - \tfrac{1}{16}x + \tfrac{1}{768}x^2 - \tfrac{1}{73728}x^3$$

Since $r_1 - r_2 = 1$, which is an integer & it can be seen here that $y_1 = y_2$, it is shown that the Frobenius method generates only one solution here.

Q2 (d) From maple we obtain soln.

$$y_2 = \frac{1}{x}\left(1 - \frac{3}{256}x^2 + \frac{7}{18432}x^3 - \frac{35}{7077888}x^4 \dots\right)$$

$$+ \ln(x)\left(-\frac{1}{8} + \frac{1}{128}x - \frac{1}{6144}x^2 + \frac{1}{589824}x^3 + \dots\right)$$

which matches the format for y_2 given in the question (From reduction of order method)

$$y_2 = x^{-1}\sum_{n=0}^{\infty} b_n x^n + \ln x \sum_{n=0}^{\infty} c_n x^n$$

thus $c_0 = -\frac{1}{8}$ and $c_1 = \frac{1}{128}$

> $y_2 = x^{-1} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n + \ln(x) \cdot \sum_{n=0}^{\infty} c_n \cdot x^n$

$$y = \frac{\sum_{n=0}^{\infty} b_n x^n}{x} + \ln(x) \left(\sum_{n=0}^{\infty} c_n x^n \right) \qquad (1)$$

> $with(DEtools):$

> $deq := 8 \cdot x \cdot diff(y(x), x, x) + 16 \cdot diff(y(x), x) + y(x) = 0$

$$deq := 8\,x \left(\frac{d^2}{dx^2}\, y(x) \right) + 16\, \frac{d}{dx}\, y(x) + y(x) = 0 \qquad (2)$$

> $dsolve(deq, y(x), series);$

$$y(x) = \_C1 \left(1 - \frac{1}{16}\, x + \frac{1}{768}\, x^2 - \frac{1}{73728}\, x^3 + \frac{1}{11796480}\, x^4 - \frac{1}{2831155200}\, x^5 + \mathrm{O}(x^6) \right) + \_C2 \left(\frac{\ln(x) \left(-\frac{1}{8}\, x + \frac{1}{128}\, x^2 - \frac{1}{6144}\, x^3 + \frac{1}{589824}\, x^4 - \frac{1}{94371840}\, x^5 + \mathrm{O}(x^6) \right)}{x} + \frac{1 - \frac{3}{256}\, x^2 + \frac{7}{18432}\, x^3 - \frac{35}{7077888}\, x^4 + \frac{101}{2831155200}\, x^5 + \mathrm{O}(x^6)}{x} \right) \qquad (3)$$

> $dsolve(deq, y(x));$

$$y(x) = \frac{\_C1\, \mathrm{BesselJ}\left(1, \frac{\sqrt{2}\,\sqrt{x}}{2}\right)}{\sqrt{x}} + \frac{\_C2\, \mathrm{BesselY}\left(1, \frac{\sqrt{2}\,\sqrt{x}}{2}\right)}{\sqrt{x}} \qquad (4)$$

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