# MP346 Assignment 2

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$a=6$

IBVP: $y'' + 12y' + (36+\lambda)y = 0$ ①

$y(0)=0 \quad , \quad y(1)=0$ ②

1) Show $\lambda=0$ does not yield a solution

try $y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad , \quad y'' = r^2 e^{rx}$

$$y'' + 12y' + 36y = 0$$

$$(r^2+12r+36)(e^{rx}) = 0$$

$$(r+6)(r+6)(e^{rx}) = 0$$

roots are $r=-6$

$$y(x) = Ae^{-6x} + Be^{-6x}.x$$

BCs: $y(0)=0$

$$A + B(0) = 0 \rightarrow A = 0$$

$y(1)=0 \quad Be^{-6(1)}(1) = Be^{-6} = 0$

$$e^{-6} \neq 0$$

so $B=0$

then $y(x) = 0$

thus for $\lambda=0$ we get the trivial solution

$y(x) = 0$

2) $\underline{\lambda > 0}$ $\quad y'' + 12y' + (36+\lambda)y = 0$

try $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2e^{rx}$

$(r^2 + 12r + (36+\lambda))e^{rx} = 0$

$r^2 + 12r + (36+\lambda) = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{144 - 4(1)(36+\lambda)}}{2}$$

$$= \frac{-12 \pm \sqrt{-4\lambda}}{2}$$

$$r = \frac{-12 \pm 2i\sqrt{\lambda}}{2} = -6 \pm i\sqrt{\lambda}$$

$y(x) = Ae^{-6+i\sqrt{\lambda}x} + Be^{-6-i\sqrt{\lambda}x}$

$y(x) = Ae^{-6x} \cdot e^{i\sqrt{\lambda}x} + Be^{-6x} \cdot e^{-i\sqrt{\lambda}x}$

apply BCs $\quad -y(0) = 0$

$A + B = 0 \quad \rightarrow -A = +B$

$-y(1) = 0$

$Ae^{-6x} \cdot e^{i\sqrt{\lambda}x} - Ae^{-6x} \cdot e^{-i\sqrt{\lambda}x} = 0$

$Ae^{-6(1)} \cdot e^{i\sqrt{\lambda}} - Ae^{-6} \cdot e^{-i\sqrt{\lambda}} = 0$

$e^{-6}A(e^{i\sqrt{\lambda}} - e^{-i\sqrt{\lambda}}) = 0$

$A(e^{i\sqrt{\lambda}} - e^{-i\sqrt{\lambda}}) = 0$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$e^{i\sqrt{\lambda}} = \cos(\sqrt{\lambda}) + i\sin(\sqrt{\lambda})$

$e^{-i\sqrt{\lambda}} = \cos(\sqrt{\lambda}) - i\sin(\sqrt{\lambda})$

$A(\cos(\sqrt{\lambda}) + i\sin(\sqrt{\lambda}) - \cos(\sqrt{\lambda}) + i\sin(\sqrt{\lambda})) = 0$

$A(2i\sin(\sqrt{\lambda})) = 0$

$\Rightarrow \sin(\sqrt{\lambda}) = 0$ gives eigenvalues

$\sin(n\pi) = 0 \rightarrow \sqrt{\lambda} = n\pi$

$\boxed{\lambda = n^2\pi^2}$ eigenvalues

$\boxed{y_n(x) = A_n e^{-6x}(2i\sin(n\pi x))}$ eigensolutions

3) Find $p(x)$, $q(x)$, $r(x)$ so that Equation (1) is equivalent to the Sturm-Liouville form

$$(py')' + (q+\lambda r)y = 0$$

$$y'' + 12y' + (36+\lambda)y = 0$$

~~can be written in the form~~

take $p = e^{12x}$

first term of Strum-Liouville is:

$$(e^{12x}y')' \xrightarrow[\text{product rule}]{} e^{12x}y'' + 12e^{12x}y'$$

then take $q = 32e^{12x}$ & $r = e^{12x}$

second term $\to$ $(32e^{12x} + \lambda e^{12x})y$

factorise accross by $e^{12x}$:

$$e^{12x}(y'' + 12y' + (32+\lambda)y) = 0$$

and dividing both sides by $e^{12x}$ gives our original equation (1)

Thus the Sturm-Liouville form of (1) is:

$$(e^{12x}y')' + (32e^{12x} + e^{12x}\lambda)y = 0$$

with

$$p = e^{12x}$$
$$q = 32e^{12x}$$
$$r = e^{12x}$$

Show orthogonality of $y_n$'s

given $\begin{cases} -(q+\lambda_m r)y_m = (p y_m')' & (a) \\ -(q+\lambda_n r)y_n = (p y_n')' & (b) \end{cases}$

integrate $y_n \times (a) - y_m \times (b)$:

$$(\lambda_n - \lambda_m)\int_0^1 y_m y_n r\,dx = \int_0^1 [(p y_m')' y_n - (p y_n')' y_m]\,dx$$

$$= \int_0^1 [(p y_m')' y_n + p y_m' y_n' - p y_m' y_n' - (p y_n')' y_m]\,dx$$

$$= \int_0^1 [p(y_m' y_n - y_n' y_m)]'\,dx$$

$$= [p(y_m' y_n - y_n' y_m)]_0^1$$

applying Boundary conditions:

- $y(0) = 0$
- $y(1) = 0$

$$RHS = [p(0-0)] - [p[0-0]] = 0$$

so $(\lambda_n - \lambda_m)\int_0^1 y_m y_n r\,dx = 0$

therefore $\lambda_n \neq \lambda_m$

$$\int_0^1 r(x)\, y_m(x)\, y_n(x)\,dx = 0 \quad , \; m \neq n$$

$\therefore$ $y_n$'s are orthogonal

4) The $5^{th}$ Property of regular Sturm-Liouville Systems states: Any piecewise continuous function $f(x)$ on $(0,1)$ can be expanded as a generalised Fourier Series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n y_n(x)}$$

where $c_n$'s are orthogonal and are given by

$$c_n = \frac{\int_0^1 r(x) f(x) y_n(x)\, dx}{\int_0^1 r(x) y_n^2(x)\, dx}$$

5) $$f(x) = 10 = \sum_{n=0}^{\infty} c_n y_n(x)$$

$$c_n = \frac{\int_0^1 r(x) f(x) y_n(x)\, dx}{\int_0^1 r(x) y_n^2(x)\, dx}$$

$f(x) = 10$ $\quad r(x) = e^{12x}$ $\quad y_n(x) = A_n e^{-6x}(2i\sin(n\pi x))$

$A_n$ & $c_n$s are both arbitrary constants.

drop $A_n$ from $y_n(x)$ since it can now be part of the $c_n$'s to be calculated.

$f(x) = 10$

$r(x) = e^{12x}$

$y_n = e^{-6x} \cdot 2i \sin(n\pi x)$

$$c_n = \frac{\int_0^1 e^{12x} \cdot 10 \cdot e^{-6x} \cdot 2i \sin(n\pi x)\,dx}{\int_0^1 e^{12x} \left(e^{-6x}(2i \sin(n\pi x))\right)^2 dx}$$

$$c_n = \frac{20 \int_0^1 e^{6x} \cdot i \sin(n\pi x)\,dx}{\int_0^1 e^{12x} \left(e^{-6x}(2i \sin(n\pi x))\right)^2 dx}$$

From Maple ....

$$c_n = \frac{-10i \cdot n\pi(1-(-1)^n \cdot e^6)}{n^2\pi^2 + 36}$$

$$y_n \cancel{f}(x) = e^{-6x}(2i \sin(n\pi x))$$

6) Plot $\displaystyle\sum_{n=0}^{200} c_n y_n = \sum_{n=0}^{200} \frac{20 \cdot e^{-6x}(\sin(n\pi x))(n\pi)(1-(-1)^n e^6)}{n^2\pi^2 + 36}$

> $I\_1 := int(\exp(6\,x)\cdot 20\cdot I\cdot \sin(n\cdot \mathrm{Pi}\cdot x), x = 0\,..1);$

$$I\_1 := \frac{20\,\mathrm{I}\,n{\sim}\,\pi\,\left(1-(-1)^{n\sim}\,\mathrm{e}^{6}\right)}{n{\sim}^{2}\,\pi^{2}+36} \quad (1)$$

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> I_2:= int((exp(12*x)*(exp(-6*x) * (2*I*sin(n*Pi*x)))**2), x = 0 .
. 1)
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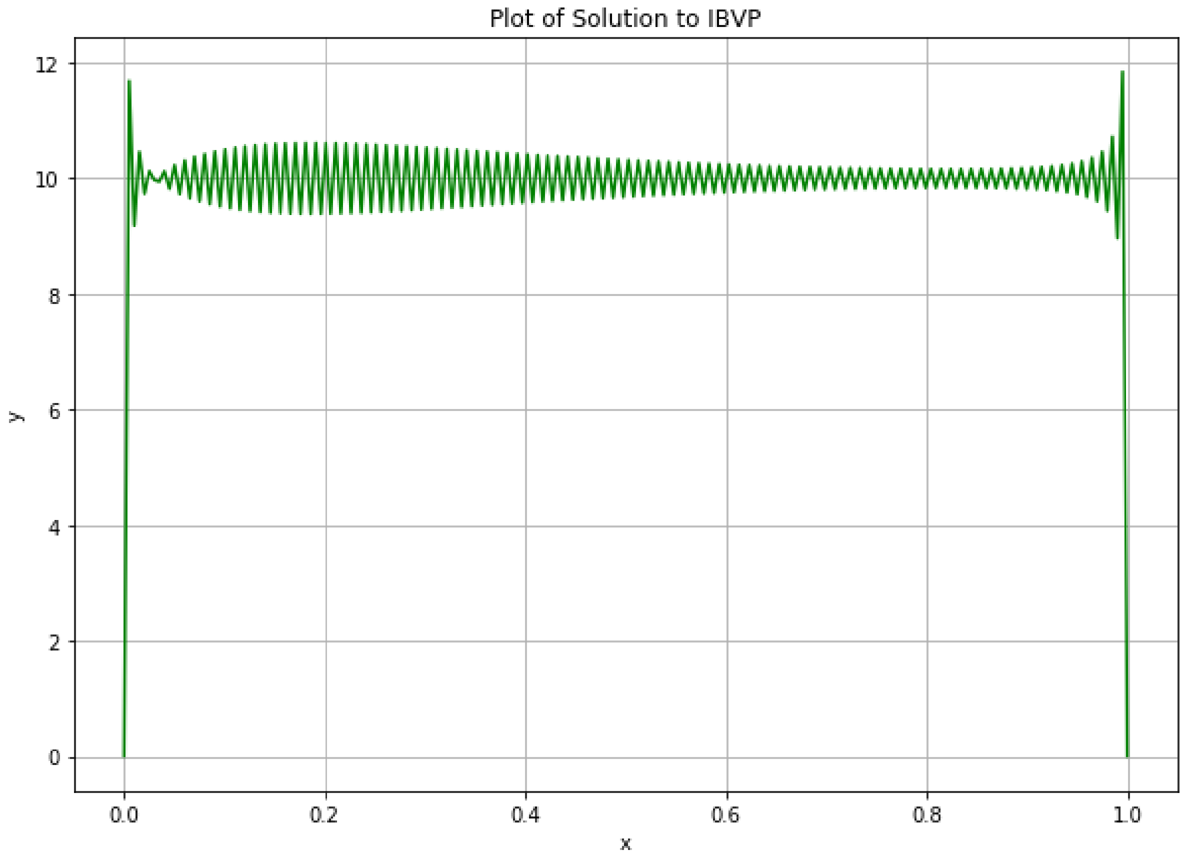
$$I\_2 := -2 \quad (2)$$

> $eval\left(\frac{I\_1}{I\_2}\right)$

$$\frac{-10\,\mathrm{I}\,n{\sim}\,\pi\,\left(1-(-1)^{n\sim}\,\mathrm{e}^{6}\right)}{n{\sim}^{2}\,\pi^{2}+36} \quad (3)$$

>

In [20]:
```
import numpy as np
import matplotlib.pyplot as plt

#set N = 200
N = 200

#create range of x values
x = np.linspace(0,1,N)

def y(x): # defiine function for solution
    total = 0
    for n in range(N):
        total += ( (20* n*np.pi * np.exp(-6*x) * np.sin(n*np.pi*x) ) * (1 - (-
1)**n * np.exp(6) ) ) / (n**2 * np.pi**2  + 36)
    return total

#create arrays to plot solution
y_vals = []

#populate array
for i in range(N):
    y_vals.append(y(x[i]))

#plot
plt.figure(figsize=(10,7))
plt.plot(x,y_vals, 'g')
plt.title("Plot of Solution to IBVP")
plt.ylabel("y")
plt.xlabel("x")
plt.grid()
plt.show()
```




In [ ]: