Fluid Mechanics Homework 1

Dara Carr
ID = 18483836

Q1

given velocity field $\underline{v} = (u, v)$ in 2 dimensional space $\underline{x} = (x, y)$

$u = u_0$ $\qquad$ $v = V_0 \cos(kx - \alpha t)$

u_0, V_0, α, k = constants $\qquad$ $x_0 = (0,0)$

(a) Derive equation of trajectories.

$\underline{x}(x, y) = x(t)\,\underline{\hat{i}} + y(t)\,\underline{\hat{j}}$

$\underline{v} = \frac{d\underline{x}}{dt} = \frac{dx(t)}{dt}\,\underline{\hat{i}} + \frac{dy(t)}{dt}\,\underline{\hat{j}}$

$\underline{v} = u_0\,\hat{i} + V_0 \cos(kx - \alpha t)\,\underline{\hat{j}}$

$x_0 = 0$ at $t = 0$

integrate $\frac{dx(t)}{dt}$ wrt t.

$x(t) = (u_0 t + c)\,\hat{i}$

$x(0) = 0 \rightarrow x(0) = u_0 \cdot 0 + c$

$\rightarrow c = 0$

$\boxed{x(t) = u_0 t}$

$\frac{dy}{dt} = V_0 \cos(kx - \alpha t)$

$= V_0 \cos(k \cdot u_0 t - \alpha t)$

$= V_0 \cos((ku_0 - \alpha)t)$

$$y(t) = V_0 \int \cos((ku_0 - \alpha)t)\, dt$$

$$y(t) = \frac{V_0}{(ku_0 - \alpha)} \cdot \sin((ku_0 - \alpha)t) + y_0$$

but $x(t) = u_0 t \rightarrow t = x/u_0$

$$y(t) = \frac{V_0}{ku_0 - \alpha} \cdot \sin\left((ku_0 - \alpha) \cdot \frac{x}{u_0}\right) + y_0$$

$$y(t) = \frac{V_0}{ku_0 - \alpha} \cdot \sin\left(\left(k - \frac{\alpha}{u_0}\right)x\right) + y_0$$

y_0 is arbitrary const. of integration.

$$y(0) = \frac{V_0}{ku_0 - \alpha} \cdot \sin\left(\left(k - \frac{\alpha}{u_0}\right) \cdot 0\right) + y_0$$

$$= 0 + y_0 = 0$$

$$\rightarrow y_0 = 0$$

$$\rightarrow \boxed{y(t) = \frac{V_0}{ku_0 - \alpha} \cdot \sin\left(\left(k - \frac{\alpha}{u_0}\right)x\right)}$$

equation of trajectories

(b) Find the streamlines (keep t constant)

$$\frac{d\underline{x}}{ds} = \frac{dx}{ds}\,\underline{\hat{i}} + \frac{dy}{ds}\,\underline{\hat{j}}$$

$$\boxed{\begin{aligned}\frac{dx}{ds} &= U_0 \\ \frac{dy}{ds} &= V_0 \cos(kx - \alpha t)\end{aligned}}$$

integrate $\frac{dx}{ds}$ wrt s

$$x(s) = U_0 s + x_0$$

@ $t=0$ $\quad \underline{x} = (0,0)$; $x_0 = 0$

$$\boxed{x(s) = U_0 s}$$

$$s = \frac{x - x_0}{U_0} = \frac{x}{U_0}$$

$$\frac{dy}{ds} = V_0 \cos(kx - \alpha t) \quad \Rightarrow \text{integrate}$$

$$y = V_0 \cdot s \cdot \cos(kx - \alpha t)$$

$$y = \frac{V_0}{U_0} \cdot (x - x_0) \cdot \cos(kx - \alpha t) + y_0 \quad x_0 = 0$$

$$y = \frac{V_0}{U_0} \cdot x \cdot \cos(kx - \alpha t) + y_0$$

we can let $u = kx(s) - \alpha t$

$$\frac{dy}{ds} = \frac{du}{dx} \cdot \frac{dx}{ds} = k\dot{x}$$

$$\boxed{x(s) = U_0 \cdot s}$$

$$\dot{y}(s) = V_0 \cos(kU_0 s - \alpha t)$$

$$\Rightarrow y(s) = \frac{-V_0}{kU_0} \sin(\alpha t - kU_0 s) + y_0$$

Streamline equation.

$$\boxed{y = \frac{-V_0}{kU_0} \cdot \sin(\alpha t - kx) + y_0}$$

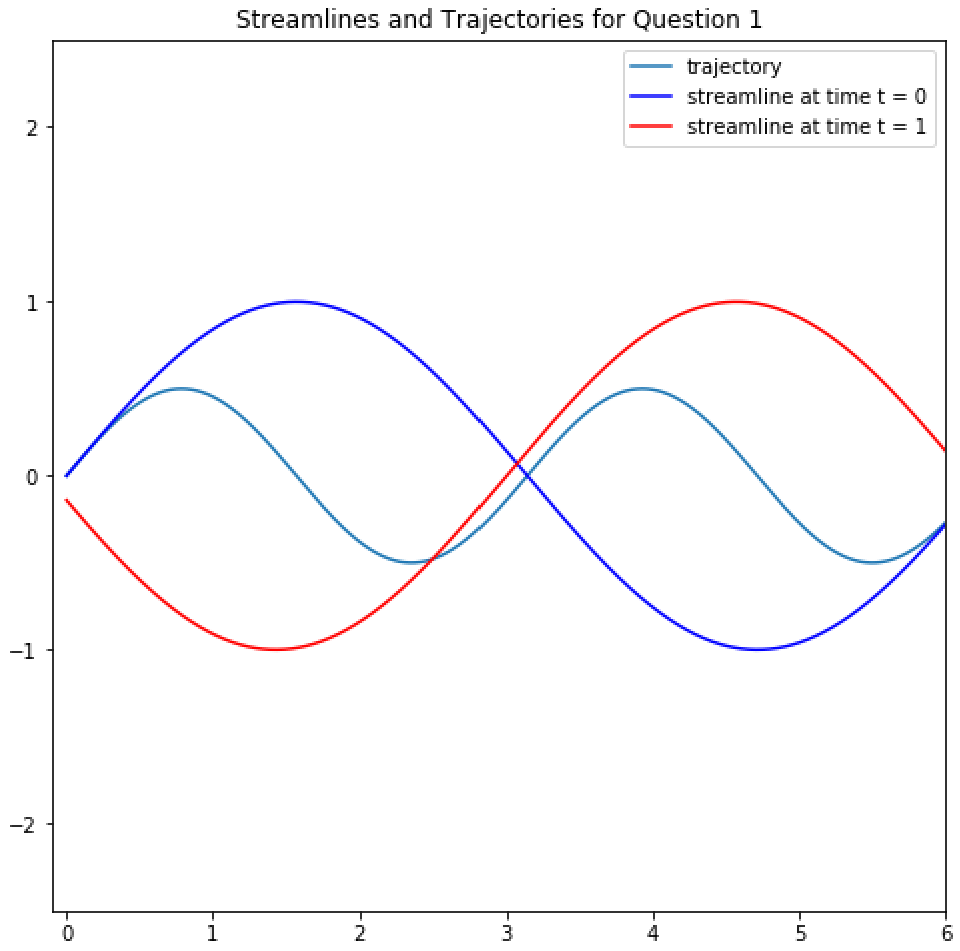
$y_0 = 0$

In [7]:

```
import matplotlib.pyplot as plt
import scipy
import numpy as np

N = 2000 # number of points on plot
x_start, x_end = 0 , 10.0
y_start, y_end = 0, 10.0
x = np.linspace(x_start,y_end, N) #create x and y arrays for plotting
y = np.linspace(y_start, y_end, N)

#########question 1: Streamlines and Trajectory plots

alpha = 3 #define constants
k = 1
u_0 = 1
v_0 = 1

traj = (np.sin((alpha/u_0 - k)*x))/alpha #trajectory function

for i in range(N):
    y[i] = ( (v_0)/(k*u_0 - alpha) * np.sin((k - alpha/u_0)*x[i])) # y = trajectory

plt.figure(figsize = [8,8] ) #plot trajectory
plt.title("Streamlines and Trajectories for Question 1")

plt.xlim([-0.1,6])
plt.ylim([-2.5,2.5])
plt.plot(x,y, label = "trajectory")




# plotting streamlines for t = 0 and t = 1

t0 = 0 #set times
t1 = 1

y0 = np.linspace(y_start, y_end, N) #streamlines for t0 and t1
y1 = np.linspace(y_start, y_end, N)

for i in range(N):
    y0[i] = -(v_0/(u_0*k)) * np.sin(alpha*t0 - k*x[i]) #fill arrays
    y1[i] = -(v_0/(u_0*k)) * np.sin(alpha*t1 - k*x[i])



# plot streamlines
plt.plot(x,y0, label = "streamline at time t = 0", color = "blue")
plt.plot(x,y1, label ="streamline at time t = 1", color = "red")
plt.legend(loc = 'upper right')
```

Out[7]:

```
<matplotlib.legend.Legend at 0x79d8d60fd0>
```


Streamlines and Trajectories for Question 1
trajectory
streamline at time t = 0
streamline at time t = 1
2
1
0
−1
−2
0
1
2
3
4
5
6

Question 2

Q2 (a) Calculate the stream and potential functions of the line source's flow at $z_0 = a + ib$

Complex potential is given by

$$w(z) = m\ln(z)$$
$$= m\ln(z - z_0) \quad \text{for a line source}$$

$$z = x + iy$$
$$z_0 = -6 + 3i$$

$$w(z) = m\ln((x+6) + i(y-3))$$

$$w(z) = m\ln(re^{i\theta})$$
$$= m\ln r + im\theta$$

$$\phi = m\ln r \qquad \psi = m\theta$$

$$r = \sqrt{(x+6)^2 + (y-3)^2}$$

$$\theta = \arg(x+6 + i(y-3)) = \tan^{-1}\left(\frac{y-3}{x+6}\right)$$

Stream function $\Rightarrow \psi = m\theta$

$\boxed{\text{Unit strength } m=1}$ $\psi = m \cdot \tan^{-1}\left(\frac{y-3}{x+6}\right) = \tan^{-1}\left(\frac{y-3}{x+6}\right)$

Streamlines are found by keeping ψ constant

$\psi =$ const. c

$$\tan\left(\frac{c}{m}\right) = \frac{y-3}{x+6}$$

$$\boxed{y = \tan(c)\cdot(x+6) + 3}$$ streamline equation for line source of unit strength.

Velocity potential $\phi = m\ln(r)$

$$\phi = m\ln\left(\sqrt{(x+6)^2+(y-3)^2}\right)$$

potential function for line source of unit strength:

$$\phi = \ln\left(\sqrt{(x+6)^2+(y-3)^2}\right)$$

Equipotential lines can be found by taking $\phi =$ constant (i.e. d)

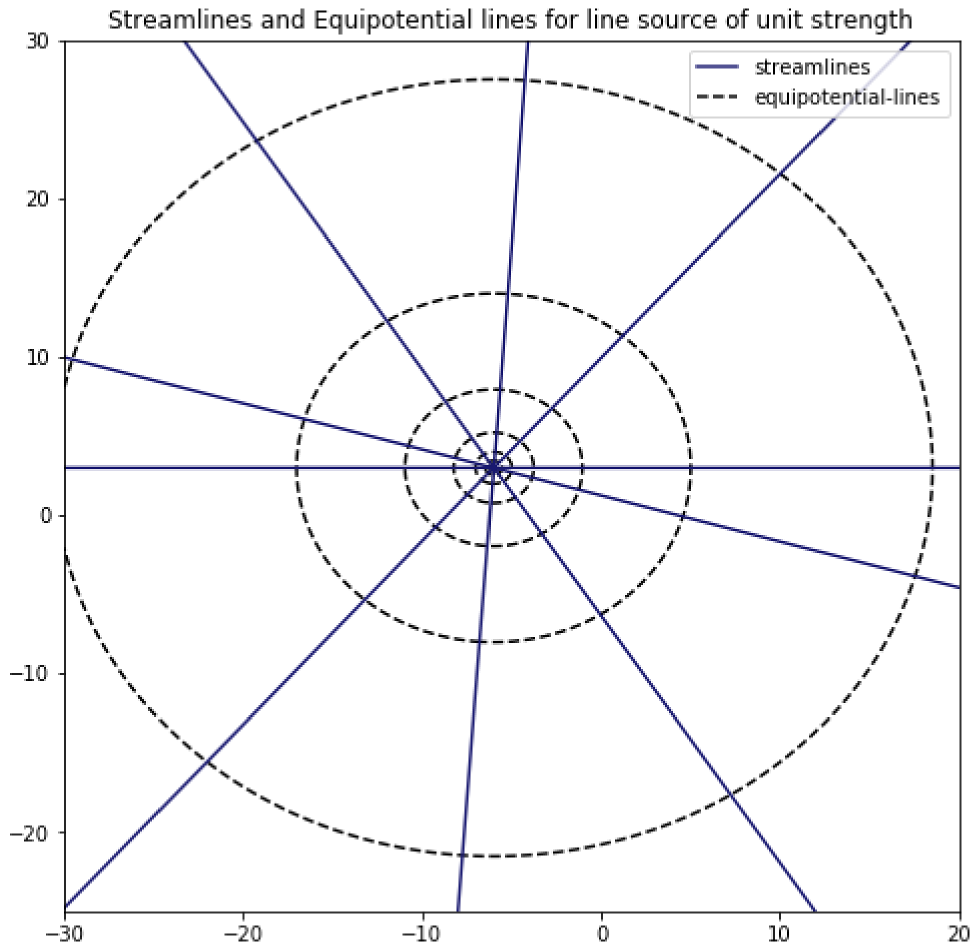
In [8]:

```
##### question 2


#plotting streamlines for m = 1 and 5 different choices of c

N = 2000 # number of points on plot
x_start, x_end = -30.0 , 20.0
y_start, y_end = -25.0, 30.0
x = np.linspace(x_start,y_end, N)
y= np.linspace(x_start,y_end, N)
y1 = np.linspace(y_start, y_end, N)
y2 = np.linspace(y_start, y_end, N)
y3 = np.linspace(y_start, y_end, N)
y4 = np.linspace(y_start, y_end, N)
y5 = np.linspace(y_start, y_end, N)

m = 1 # set m = 1 for unit source strength

c1 = 6 #take different values of constant c to plot streamlines
c2 = -1
c3 = 0
c4 = 3/2
c5 = 4



for i in range(N):
    y1[i] = np.tan(c1/m) * (x[i]+6) + 3 #fill arrays
    y2[i] = np.tan(c2/m) * (x[i]+6) + 3
    y3[i] = np.tan(c3/m) * (x[i]+6) + 3
    y4[i] = np.tan(c4/m) * (x[i]+6) + 3
    y5[i] = np.tan(c5/m) * (x[i]+6) + 3



plt.figure(figsize = [8,8] ) #plotting streamlines
plt.xlim([-30,20])
plt.ylim([-25,30])
plt.title("Streamlines and Equipotential lines for  Question 2")
plt.plot(x,y1, label = "streamlines", color = "midnightblue")
plt.plot(x,y2, color = "midnightblue")
plt.plot(x,y3, color = "midnightblue")
plt.plot(x,y4, color = "midnightblue")
plt.plot(x,y5, color = "midnightblue")


## plotting equipotentials

#plot for 5 different values of phi = d = constant


x = np.linspace(-30,30, N)
y= np.linspace(-30,30, N)


X, Y = np.meshgrid(x,y)

phi = m*np.log(((X+6)**2 + (Y-3)**2)**(1/2)) #potential function
```

```
plt.title("Streamlines and Equipotential lines for line source of unit strength")

phiplot2 = plt.contour(X,Y, phi, 10 ,linestyles = 'dashed', colors= "black") #plot equipote

labels = ["streamlines", "equipotential-lines"]

phiplot2.collections[1].set_label(labels[1])

plt.legend(loc = "upper right") #display legend
```

Out[8]:

```
<matplotlib.legend.Legend at 0x79dee0dfd0>
```



Question 3

Q3 (a) Calculate potential function and stream function of line doublet with unit moment parallel to the y-axis.

Line doublet has source of strength m at $z_0 + he^{i\alpha}$ and a sink of strength m at $z_0 - he^{i\alpha}$

Let moment $M = 2mh$, unit moment $\to M = 1$

Complet Potential :

$$w(z) = m\ln\left(z - (z_0 + he^{i\alpha})\right) - m\ln\left(z - (z_0 - he^{i\alpha})\right)$$

$$\to \quad w(z) = \frac{1}{2h}\left[\ln\left(x+iy+6-3i-he^{i\alpha}\right) - \ln\left(x+iy+6-3i+he^{i\alpha}\right)\right]$$

line is parallel to the y-axis

$\to \alpha = \pi/2$

$$e^{i\alpha} \to e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i + 0$$

$$w(z) = \frac{1}{2h}\left[\ln\left(x+6+i(y-3-h)\right) - \ln\left(x+6+i(y-3+h)\right)\right]$$

$$w(z) = \frac{1}{2h}\left[\ln(r_1e^{i\theta_1}) - \ln(r_2e^{i\theta_2})\right]$$

$$= \frac{1}{2h}\left[\ln\left(\sqrt{(x+6)^2+(y-3-h)^2}\cdot e^{i\tan^{-1}\left(\frac{y-3-h}{x+6}\right)}\right)\right.$$

$$\left. - \ln\left(\sqrt{(x+6)^2+(y-3+h)^2}\cdot e^{i\tan^{-1}\left(\frac{y-3+h}{x+6}\right)}\right)\right]$$

velocity potential fn

$$\phi = \frac{1}{2h}\left[\ln\left(\sqrt{(x+6)^2+(y-3-h)^2}\right) - \ln\left(\sqrt{(x+6)^2+(y-3+h)^2}\right)\right]$$

$$\phi = \frac{1}{2h}\ln\left[\sqrt{\frac{(x+6)^2+(y-3-h)^2}{(x+6)^2+(y-3+h)^2}}\right]$$

stream function

$$\psi = \frac{1}{2h}\left[\tan^{-1}\left(\frac{y-3-h}{x+6}\right) - \tan^{-1}\left(\frac{y-3+h}{x+6}\right)\right]$$

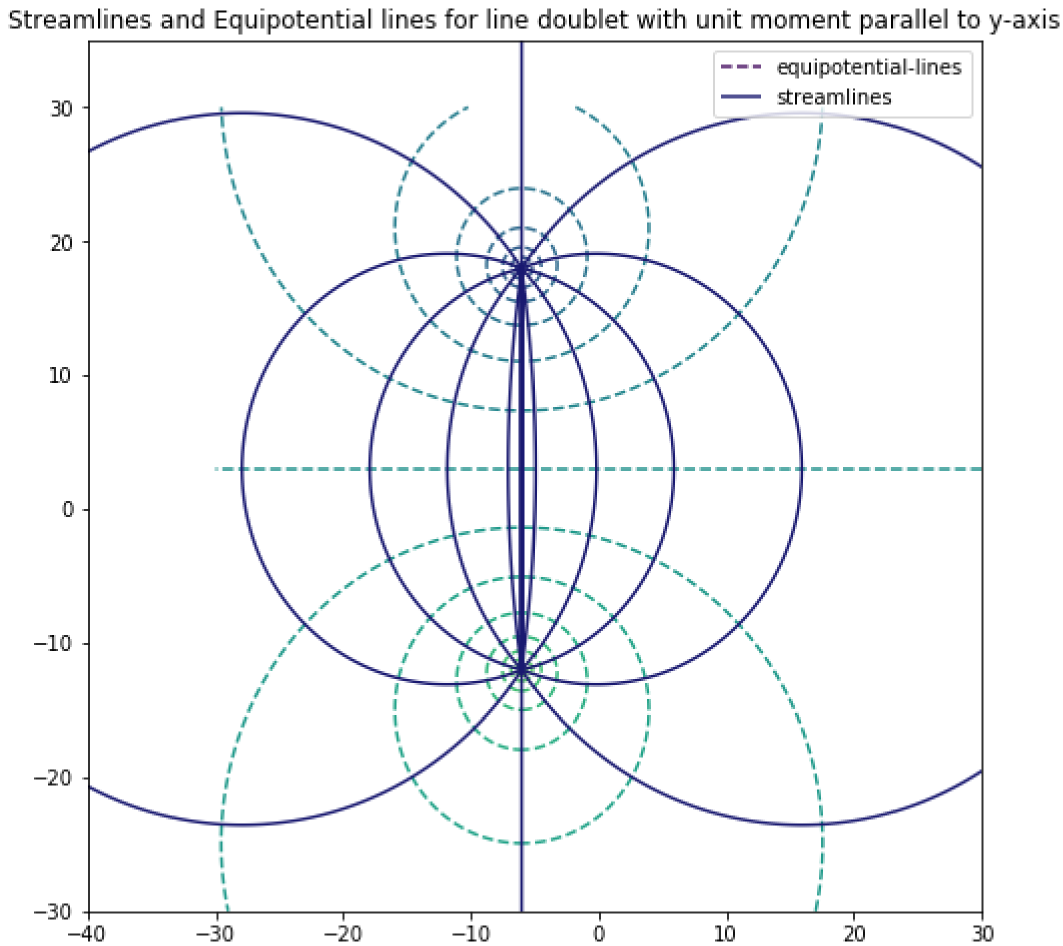
take ϕ, ψ = constants & plot to get streamlines & equipotential lines

In [9]:

```
##question 3


##plot potentials

N = 2000 # number of points on plot
x_start, x_end = -55.0 , 50.0
y_start, y_end = -50.0, 55.0
x = np.linspace(x_start,y_end, N) #create x and y arrays
y= np.linspace(x_start,y_end, N)


h = 15

phi1 = 1/(2*h) * np.log(np.sqrt(( (X+6)**2 + (Y-3-h)**2 )/( (X+6)**2 + (Y-3+h)**2 ) ) )




plt.figure(figsize = [8,8] )
plt.xlim([-40,30])
plt.ylim([-30,35])


phiplot3 = plt.contour(X,Y, phi1, 25 ,linestyles = 'dashed')



#plot streamlines



x = np.linspace(x_start,y_end, N)
y= np.linspace(x_start,y_end, N)

X, Y = np.meshgrid(x,y)



psi1 = 1/(2*h) * (np.arctan((Y-3-h)/(X+6)) - np.arctan((Y-3+h)/(X+6)))


plt.title("Streamlines and Equipotential lines for line doublet with unit moment parallel t
psiplot3 = plt.contour(X,Y, psi1, 10 , colors = 'midnightblue', linestyles = 'solid')


labels = ["streamlines", "equipotential-lines"]

psiplot3.collections[0].set_label(labels[0])
phiplot3.collections[1].set_label(labels[1])

plt.legend(loc = "upper right")
```

Out[9]:

```
<matplotlib.legend.Legend at 0x79d8e5d1d0>
```


Streamlines and Equipotential lines for line doublet with unit moment parallel to y-axis
equipotential-lines
streamlines
30
20
10
0
−10
−20
−30
−40
−30
−20
−10
0
10
20
30

Question 4

Q4 Calculate stream function and the potential function of the flow at $z_0 = a + ib$ for a line vortex of unit strength.

for a line vortex, the complex potential is given by

$$w(z) = -ik \ln z \quad = -ik \ln(z - z_0)$$

$$= -ik \ln(re^{i\theta})$$
$$= -ik \ln(r) \quad + \quad k\theta$$

$$\phi = k\theta \quad , \quad \psi = -k \ln(r)$$

$z - z_0$ is the same as in Q2
$= (x+6) + i(y-3)$

$$r = \sqrt{(x+6)^2 + (y-3)^2}$$

$$\theta = \tan^{-1}\left(\frac{y-3}{x+6}\right)$$

Potential fn. $$\phi = k \cdot \tan^{-1}\left(\frac{y-3}{x+6}\right)$$

$$\tan\left(\frac{\phi}{k}\right) = \frac{y-3}{x+6}$$

$$y = (x+6) \cdot \tan\left(\frac{\phi}{k}\right) + 3$$

equation for equipotential lines for ϕ const.
→ unit strength : $k = 1$

stream function

$$\psi = -k \ln\left(\sqrt{(x+6)^2+(y-3)^2}\right)$$

$k=1 \rightarrow$ unit strength

$$\psi = -\ln\left(\sqrt{(x+6)^2+(y-3)^2}\right)$$

take $\psi=$ const. to find streamlines.

In [11]:

```
## question 4 plots


#streamlines


k=1

N = 2000 # number of points on plot
x_start, x_end = -30.0 , 20.0
y_start, y_end = -25.0, 30.0
x = np.linspace(x_start,y_end, N)
y= np.linspace(x_start,y_end, N)



X, Y = np.meshgrid(x,y)

psi = -k*np.log(np.sqrt((X+6)**2 + (Y-3)**2)) #stream function

plt.figure(figsize = [8,8] )

plt.title("Streamlines and Equipotential-lines for line vortex of unit strength") # plot eq

plt.xlim([-20,10])
plt.ylim([-15,20])

psiplot4 = plt.contour(X,Y, phi, 12) #plotting streamlines as contour plot




#equipotential lines

l1 = 1 #define different values for constant l to plot different equipotential lines
l2 =2
l3 = 3/2
l4 = 23/7
l5 = 9

for i in range(N):
    y1[i] = (x[i]+6)*np.tan(l1/k) + 3 #fill arrays
    y2[i] = (x[i]+6)*np.tan(l2/k) + 3
    y3[i] = (x[i]+6)*np.tan(l3/k) + 3
    y4[i] = (x[i]+6)*np.tan(l4/k) + 3
    y5[i] = (x[i]+6)*np.tan(l5/k) + 3

plt.plot(x,y1, 'k--', label = "equipotential lines") #plotting equipotential lines
plt.plot(x,y2, 'k--')
plt.plot(x,y3, 'k--')
plt.plot(x,y4, 'k--')
plt.plot(x,y5, 'k--')

labels = ["streamlines", "equipotential-lines"]

psiplot4.collections[0].set_label(labels[0])

plt.legend(loc = "upper right")
```

Out[11]:

```
<matplotlib.legend.Legend at 0x79da91a048>
```