18483836
Dara Carr

PDEs Assignment 2

Q1 $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + [1 - q(y)]\frac{\partial^2 u}{\partial y^2} = 0$

where $q(y) = \begin{cases} -1 & \text{for } y < -1 \\ 0 & \text{for } |y| \leq 1 \\ 1 & \text{for } y \geq 1 \end{cases}$

Classify this equation:

$a = 1 \quad b = 2/2 = 1, \quad c = 1 - q(y)$

$b^2 - ac = 1 - 1\cdot(1 - q(y))$
$= 1 - 1 + q(y)$

$b^2 - ac = q(y)$

$b^2 - ac = \begin{cases} < 0 & \text{for } y < -1 \\ 0 & \text{for } |y| \leq 1 \rightarrow -1 \leq y \leq +1 \\ > 0 & \text{for } y > 1 \end{cases}$

equation is elliptic for $y < -1$
equation is parabolic for $-1 \leq y \leq +1$
equation is hyperbolic for $y > 1$

(b) Find canonical form in each region

elliptic: $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$

$a = 1, b = 1, c = 2$

$b^2 - ac = -1$

to find canonical form for elliptic case:

$dy/dx = \frac{b + i\sqrt{ac - b^2}}{a}$

$dy/dx = \frac{1 + \sqrt{2 - 1}\, i}{1} = 1 + i$

$y = (1 + i)x + \text{constant}$

$$\phi = x - y + ix = \text{constant}$$

$$\xi = Re(\phi) \qquad \eta = Im(\phi)$$

$$\xi = x - y \qquad \eta = x$$

$$u(x,y) = u(x(\xi,\eta), y(\xi,\eta)) = \omega(\xi,\eta)$$

$$\frac{\partial u}{\partial x} = \frac{\partial \omega}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \omega}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$= \frac{\partial \omega}{\partial \xi}(1) + \frac{\partial \omega}{\partial \eta}(1)$$

$$\frac{\partial u}{\partial x} = \left(\partial/\partial\xi + \partial/\partial\eta\right)\omega$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \omega}{\partial \xi^2} + 2\frac{\partial^2 \omega}{\partial \xi \partial \eta} + \frac{\partial^2 \omega}{\partial \eta^2}$$

$$\partial u/\partial y = \frac{\partial \omega}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \omega}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$= -\frac{\partial \omega}{\partial \xi} + \frac{\partial \omega}{\partial \eta} \cdot (0) = -\frac{\partial \omega}{\partial \xi} = -\partial/\partial\xi\, \omega$$

$$\frac{\partial^2 u}{\partial y^2} = +\frac{\partial^2 \omega}{\partial \xi^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 \omega}{\partial \xi^2} - \frac{\partial^2 \omega}{\partial \xi \partial \eta}$$

Sub into eqn $\rightarrow$ $$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$$

$$\frac{\partial^2 \omega}{\partial \xi^2} + \cancel{2\frac{\partial^2 \omega}{\partial \xi \partial \eta}} + \frac{\partial^2 \omega}{\partial \eta^2} - \cancel{2\frac{\partial^2 \omega}{\partial \xi^2}} - \cancel{2\frac{\partial^2 \omega}{\partial \xi \partial \eta}} + \cancel{2\frac{\partial^2 \omega}{\partial \xi^2}} = 0$$

$$\frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \eta^2} = 0 \rightarrow \nabla^2_{(\xi,\eta)}\omega = 0$$

Canonical form for elliptic case

$$\boxed{\nabla^2_{(\xi,\eta)} u = 0}$$

parabolic case: $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

$a = 1 \ , \ b = \frac{2}{2} = 1 \ , \ c = 1$

to find canonical form for parabolic case:

$$\frac{dy}{dx} = \frac{b}{a} = 1$$

$(x, y) \to (\xi, \eta)$

$$y = x + \text{constant}$$
$$y - x = \text{constant}$$
$$x - y = \text{const.}$$

Choose $\xi = x - y$

Choose $\eta = y$ $\qquad J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \checkmark$ $\neq 0 \checkmark$

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$
$$= \frac{\partial w}{\partial \xi}\cdot 1 + \frac{\partial w}{\partial \eta}\cdot 0 = \frac{\partial w}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial \xi^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial w}{\partial \xi}\cdot(-1) + \frac{\partial w}{\partial \eta}(1)$$
$$= \left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) w$$

$$\frac{\partial^2 u}{\partial y} = \frac{\partial^2 w}{\partial \xi^2} - 2\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 \eta}{\partial w^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 w}{\partial \xi \partial \eta} - \frac{\partial^2 w}{\partial \xi^2}$$

Sub into equation: $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\cancel{\frac{\partial^2 w}{\partial \xi^2}} + 2\left(\cancel{\frac{\partial^2 w}{\partial \xi \partial \eta}} - \cancel{\frac{\partial^2 w}{\partial \xi^2}}\right) + \cancel{\frac{\partial^2 w}{\partial \xi^2}} - 2\cancel{\frac{\partial^2 w}{\partial \xi \partial \eta}} + \frac{\partial^2 \eta}{\partial w^2} = 0$$

$$\frac{\partial^2 w}{\partial \eta^2} = 0 \to \boxed{\frac{\partial^2 u}{\partial \eta^2} = 0}$$

Canonical form for parabolic case

hyperbolic case: $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} = 0$

$a = 1, \ b = 1, \ c = 0$

$b^2 - ac = 1 - 0 = 1 > 0$

algorithm to find canonical form for hyperbolic case:

$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a}$$

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{1 - 0}}{1} = \frac{1 \pm 1}{1} = 0 \text{ or } 2$$

$$\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = 2$$

$y(x) = \text{const}$ or $y = 2x + \text{Constant}$

const + const = const

$y - 2x = \text{const}$

$2x - y = \text{const.}$

$\xi = 2x - y$

$\eta = y$

$J \begin{vmatrix} 2 & -1 \\ 0 & +1 \end{vmatrix} = -2 \neq 0$ ✓

$$\frac{\partial u}{\partial x} = \frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \omega}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = 2\frac{\partial \omega}{\partial \xi} + 0 = 2\frac{\partial}{\partial \xi}\omega$$

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2}{\partial \xi^2}\omega$$

$$\frac{\partial u}{\partial y} = \frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \omega}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = -\frac{\partial \omega}{\partial \xi} + \frac{\partial \omega}{\partial \eta}$$

$$= \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\omega$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \omega}{\partial \xi^2} - 2\frac{\partial^2 \omega}{\partial \xi \partial \eta} + \frac{\partial^2 \omega}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)\left(2\frac{\partial \omega}{\partial \xi}\right) = 2\frac{\partial^2 \omega}{\partial \eta \partial \xi} - 2\frac{\partial^2 \omega}{\partial \xi^2}$$

sub into eqn

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} = \cancel{4\frac{\partial^2 \omega}{\partial \xi^2}} + 4\frac{\partial^2 \omega}{\partial \eta \partial \xi} - \cancel{4\frac{\partial^2 \omega}{\partial \xi^2}}$$

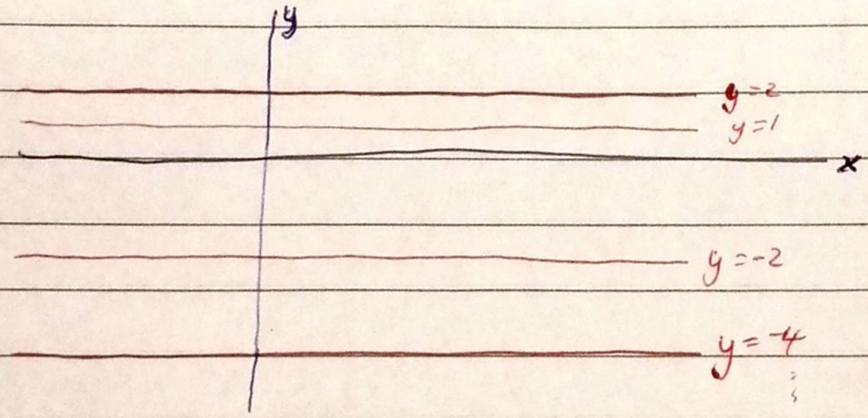
$$= 4\frac{\partial^2 \omega}{\partial \eta \partial \xi} \div 4 = \frac{\partial^2 \omega}{\partial \eta \partial \xi} = \boxed{\frac{\partial^2 u}{\partial \eta \partial \xi} = 0}$$

Canonical form for hyperbolic case

Q1 (c) Characteristics for hyperbolic case:

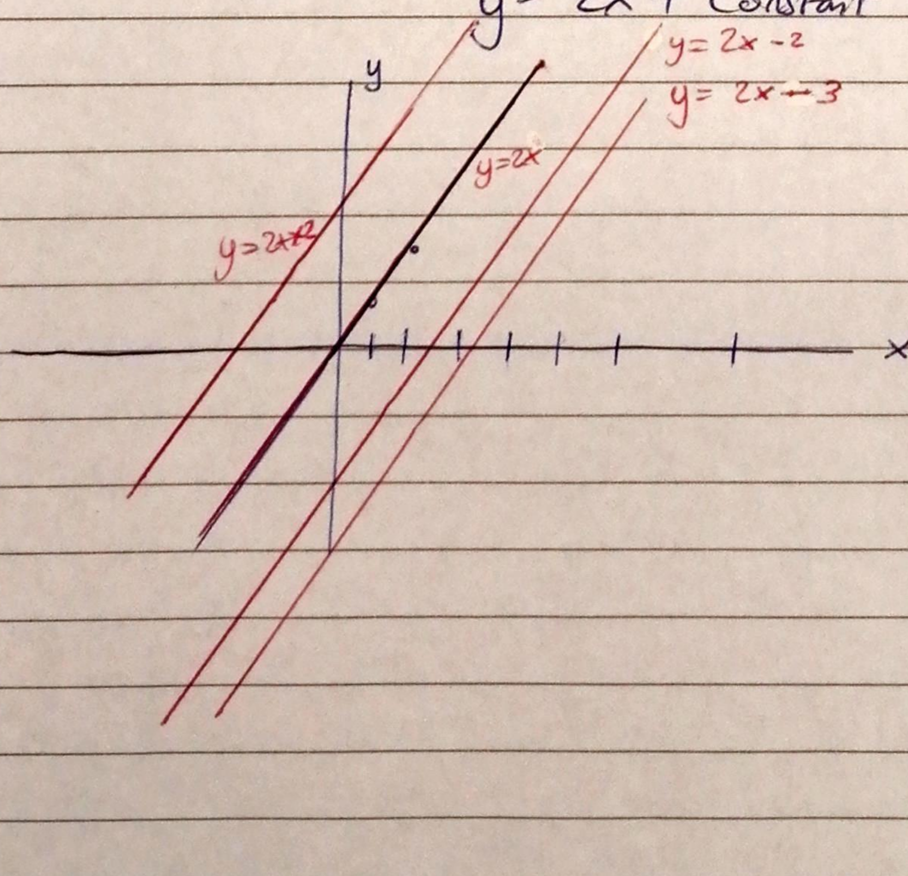
$$\xi = 2x - y = \text{constant}$$
$$\eta = y \quad = \text{constant}$$

Characteristics $\rightarrow$ $y = \text{constant}$



$2x - y = \text{constant}$

$y = 2x + \text{constant}$



Q2 $$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 4e^y = 0$$

(a) Find canonical form

$a = 1$, $b = \frac{-2}{2} = -1$, $c = 0$

$b^2 - ac = 1 - 0 = 1 > 0$

hyperbolic

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

$$\frac{dy}{dx} = \frac{-1 \pm \sqrt{1-0}}{1}$$

$$\frac{dy}{dx} = -1 \pm 1 = 0 \text{ or } -2$$

$y =$ const. or $y = -2x +$ const.

$2x + y =$ const

$y =$ const

$\xi = 2x + y$

$\eta = y$

$(x, y) \longrightarrow (\xi, \eta)$

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial w}{\partial \xi} \cdot 2 + 0$$

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 w}{\partial \xi^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$= \frac{\partial w}{\partial \xi} \cdot 1 + \frac{\partial w}{\partial \eta} \cdot 1$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{2\partial w}{\partial \xi}\right)$$

$$= 2\frac{\partial^2 w}{\partial \xi^2} + 2\frac{\partial w}{\partial \xi \partial \eta}$$

eqn: $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 4e^y = 0$

$$\cancel{4\frac{\partial^2 w}{\partial \xi^2}} - \cancel{4\frac{\partial^2 w}{\partial \xi^2}} - 4\frac{\partial^2 w}{\partial \xi \partial \eta} + 4e^\eta = 0$$

$\div$ by 4 $\quad \boxed{-\frac{\partial^2 u}{\partial \xi \partial \eta} + e^\eta = 0}$ ①

(b) Find a solution which satisfies the conditions

$u(0,y) = f(y) \qquad \frac{\partial u}{\partial x}(0,y) = g(y)$

Solve ①: $-\frac{\partial^2 w}{\partial \xi \partial \eta} + e^\eta = 0$

$$\rightarrow \frac{\partial^2 w}{\partial \xi \partial \eta} = e^\eta$$

$$\int \frac{\partial^2 w}{\partial \xi \partial \eta} \partial \xi = \int e^\eta d\xi \rightarrow \frac{\partial w}{\partial \eta} = \xi e^\eta + G(\eta)$$

$$\int \frac{\partial w}{\partial \eta} d\eta = \int \xi e^\eta + G(\eta)$$

$$w(\xi, \eta) = \xi e^\eta + G(\eta) + F(\xi)$$

in original variables

$$u(x,y) = (2x+y)e^y + F(2x+y) + G(y)$$

Change from F,G. to avoid confusion with $f(y), g(y)$

Q2 (b) $u(x,y) = (2x+y)e^y + A(2x+y) + B(y)$

initial conditions:

$u(0,y) = f(y)$

$\frac{\partial u}{\partial x}(0,y) = g(y)$

① $u(0,y) = ye^y + A(y) + B(y) = f(y)$

② $\frac{\partial u}{\partial x}(0,y) = 2e^y + 2A'(y) + 0 = g(y)$

$A'(y) = \frac{1}{2}g(y) - e^y$

① $A(y) = f(y) - ye^y - B(y)$

$\int A'(y)\,dy = A(y)$

$\int A'(y)\,dy = \int \frac{1}{2}g(y) - e^y\,dy$

Solving for B: $\int \frac{1}{2}g(y)\,dy - e^y = f(y) - ye^y - B(y)$

$B(y) = -\int \frac{1}{2}g(y)\,dy + e^y + f(y) - ye^y$

$B(y) = -\int \frac{1}{2}g(y)\,dy + (1-y)e^y + f(y)$

sub into ①

$A(y) = \cancel{f(y)} - \cancel{ye^y} + \int \frac{1}{2}g(y)\,dy + \cancel{ye^y} - e^y - \cancel{f(y)}$

$A(y) = \int \frac{1}{2}g(y)\,dy - e^y$

now need to find $A(2x+y)$

$y \rightarrow 2x+y$

$A(2x+y) = \frac{1}{2}\int g(2x+y)\,d(2x+y) - e^{2x+y}$

$$u(x,y) = (2x+y)e^y + A(2x+y) + B(y)$$

$$u(x,y) = (2x+y)e^y + \tfrac{1}{2}\int g(2x+y)\,d2x+y - e^{2x+y}$$
$$+ -\tfrac{1}{2}\int g(y)\,dy + (1-y)e^y + f(y)$$

$$\tfrac{1}{2}\int g(2x+y)\,d(2x+y) - \tfrac{1}{2}\int g(y)\,dy$$
can be written as

$$\tfrac{1}{2}\int_y^{2x+y} g(s)\,ds$$ where s is used as a dummy variable.

$$u(x,y) = (2x+y)e^y + \tfrac{1}{2}\int_y^{2x+y} g(s)\,ds - e^{2x+y}$$
$$+ (1-y)e^y + f(y)$$

Q3 $$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$$

classify eqn: $b^2 - ac < 0$ elliptic
$b^2 - ac = 0$ parabolic
$b^2 - ac > 0$ hyperbolic

$a = 1$, $b = 0$, $c = y$

$b^2 - ac = 0 - y = -y$
$b^2 - ac > 0$ for $-y \to y < 0$ hyperbolic
$b^2 - ac = 0$ for $y = 0$ parabolic
$b^2 - ac < 0$ for $+y \to$ elliptic
$y > 0$

find canonical form for equation for the domain where the equation is hyperbolic & the domain where it is elliptic

hyperbolic case: $y < 0$

$b^2 - ac = +y$

$a = 1$, $b = 0$, $c = -y$

algorithm for hyperbolic case:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} \qquad -(-y) = y$$

$$\frac{dy}{dx} = \pm\sqrt{+y} \quad \to y < 0 \to \frac{dy}{dx} = \pm\sqrt{-y}$$

define this way to get real soln.

$$\int \pm \frac{1}{\sqrt{-y}}\, dy = \int dx$$

$$\pm \int \frac{1}{\sqrt{y}}\, dy = \int y^{-1/2}\, dy = \int dx$$

$$\cancel{\text{term}} = \int \frac{1}{\sqrt{-y}}\, dy$$

let $u = \sqrt{-y} = y^{\frac{1}{2}}$

$$\frac{du}{dy} = \tfrac{1}{2} y^{-1/2}$$

$$2du = y^{-1/2}\, dy$$

$$\int y^{-1/2}\, dy = \int 2\, du = \int dx$$

$$2u = x + \text{constant}$$

$$\pm 2\sqrt{y} = x + \text{constant}$$

$$2\sqrt{-y} = x + \text{constant}$$

$$2\sqrt{-y} - x = \text{constant}$$

$$-2\sqrt{-y} - x = \text{constant}$$

$$\boxed{\begin{aligned}\xi &= 2\sqrt{-y} + x \\ \eta &= -2\sqrt{-y} + x\end{aligned}}$$

$$J = \begin{vmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{vmatrix} = \begin{vmatrix} 1 & (-y)^{-1/2} \\ +1 & +y^{-1/2} \end{vmatrix}$$

$$= -(-y)^{-1/2} - (-y)^{-1/2} \neq 0 \checkmark$$

$$(x, y) \to (\xi, \eta)$$

$$u(x,y) = u(x(\xi,\eta), y(\xi,\eta)) = w(\xi,\eta)$$

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial w}{\partial \xi} \cdot 1 + \frac{\partial w}{\partial \eta} \cdot +1 = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) w$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial \xi^2} + 2\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2}$$

$$\xi = 2\sqrt{-y} + x$$
$$\eta = -2\sqrt{-y} + x$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi} \cdot \frac{\partial}{\partial y}\left(2(-y)^{\frac{1}{2}} + x\right) + \frac{\partial w}{\partial \eta} \cdot \frac{\partial}{\partial y}\left(-2(-y)^{\frac{1}{2}} + x\right)$$

$$= \frac{\partial w}{\partial \xi}\left(\frac{2}{2}(-y)^{-\frac{1}{2}} \cdot -1\right) + \frac{\partial w}{\partial \eta} \cdot \left((-y)^{-\frac{1}{2}}\right)$$

$$= -\frac{\partial w}{\partial \xi}(-y)^{-\frac{1}{2}} + \frac{\partial w}{\partial \eta}(-y)^{-\frac{1}{2}}$$

$$= (-y)^{-\frac{1}{2}}\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) w$$

$$\xi = 2\sqrt{-y} + x \longrightarrow \xi - x = 2\sqrt{-y}$$
$$\eta = -2\sqrt{-y} + x \longrightarrow \rightarrow x - \eta = 2\sqrt{-y}$$

$$\xi - x = x - \eta \rightarrow \quad x = \frac{\xi + \eta}{2}$$

$$(-y)^{\frac{1}{2}} = \frac{\xi - x}{2} \rightarrow \quad = \frac{\xi - \frac{\xi + \eta}{2}}{2}$$

$$(-y)^{-\frac{1}{2}} = \frac{2}{\xi - \left(\frac{\xi + \eta}{2}\right)} = \frac{2}{\left(\frac{\xi - \eta}{2}\right)}$$

$$(-y)^{-\frac{1}{2}} = \left(\frac{4}{\xi - \eta}\right) \quad ,$$

$$\frac{\partial u}{\partial y} = \left(\frac{4}{\xi - \eta}\right)\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) w$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{4}{\xi - \eta}\right)\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \cdot \left[\left(\frac{4}{\xi - \eta}\right) \cdot \frac{\partial w}{\partial \xi} + \left(\frac{4}{\eta - \xi}\right)\frac{\partial w}{\partial \eta}\right]$$

$$= \left(\frac{16}{\xi - \eta}\right)\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left[-(\xi - \eta)^{-1}\frac{\partial w}{\partial \xi} + (\xi - \eta)^{-1}\frac{\partial w}{\partial \eta}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{16}{\xi-\eta}\right)\left[\frac{\partial}{\partial\xi}(\xi-\eta)^{-1}\frac{\partial w}{\partial\xi} - \frac{\partial}{\partial\xi}(\xi-\eta)^{-1}\frac{\partial w}{\partial\eta}\right.$$
$$- \frac{\partial}{\partial\eta}(\xi-\eta)^{-1}\frac{\partial w}{\partial\xi} + \frac{\partial}{\partial\eta}(\xi-\eta)^{-1}\frac{\partial w}{\partial\eta}$$

$$\frac{\partial}{\partial\xi}(\xi-\eta)^{-1}\frac{\partial w}{\partial\xi} = -(\xi-\eta)^{-2}\frac{\partial w}{\partial\xi} + (\xi-\eta)^{-1}\frac{\partial^2 w}{\partial\xi^2}$$
$$-\frac{\partial}{\partial\xi}(\xi-\eta)^{-1}\frac{\partial w}{\partial\eta} = +(\xi-\eta)^{-2}\frac{\partial w}{\partial\eta} - (\xi-\eta)^{-1}\frac{\partial^2 w}{\partial\xi\partial\eta}$$
$$-\frac{\partial}{\partial\eta}(\xi-\eta)\frac{\partial w}{\partial\xi} = -(\xi-\eta)^{-2}\frac{\partial w}{\partial\xi} - (\xi-\eta)^{-1}\frac{\partial^2 w}{\partial\eta\partial\xi}$$
$$+\frac{\partial}{\partial\eta}(\xi-\eta)^{-1}\frac{\partial w}{\partial\eta} = +(\xi-\eta)^{-2}\frac{\partial w}{\partial\eta} + (\xi-\eta)^{-1}\frac{\partial^2 w}{\partial\eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{16}{(\xi-\eta)}\left[(\xi-\eta)^{-2}\left(2\frac{\partial w}{\partial\eta} - 2\frac{\partial w}{\partial\xi}\right)\right.$$
$$\left. + (\xi-\eta)^{-1}\left(\frac{\partial^2 w}{\partial\eta^2} - 2\frac{\partial^2 w}{\partial\xi\partial\eta} + \frac{\partial^2 w}{\partial\xi^2}\right)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{16}{(\xi-\eta)^3}\left[\left(2\frac{\partial w}{\partial\eta} - 2\frac{\partial w}{\partial\xi}\right)\right.$$
$$\left. + (\xi-\eta)\left(\frac{\partial^2 w}{\partial\eta^2} - 2\frac{\partial^2 w}{\partial\xi\partial\eta} + \frac{\partial^2 w}{\partial\xi^2}\right)\right] = 0$$

multiply across by $-\frac{(\xi-\eta)^2}{16}$ to get $+y\frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial^2 u}{\partial y^2} = (\xi-\eta)^{-1}\left[\left(2\frac{\partial w}{\partial\xi} - 2\frac{\partial w}{\partial\eta}\right)\right]$$
$$- \frac{\partial^2 w}{\partial\eta^2} + 2\frac{\partial^2 w}{\partial\xi\partial\eta} - \frac{\partial^2 w}{\partial\xi^2}$$

sub into equation $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2}$

$$(-y)^{-1/2} = \left(\frac{4}{\xi-\eta}\right), \quad (-y)^{1/2} = \frac{\xi-\eta}{4}$$
$$-y = \frac{(\xi-\eta)^2}{16}$$
$$y = -\frac{(\xi-\eta)^2}{16}$$

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{2\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial^2 w}{\partial \eta^2} + (\xi - \eta)^{-1} \cdot 2\left(\frac{\partial w}{\partial \xi} - \frac{\partial w}{\partial \eta}\right)$$

$$- \frac{\partial^2 w}{\partial \eta^2} + 2\frac{\partial^2 w}{\partial \xi \partial \eta} - \frac{\partial^2 w}{\partial \xi^2}$$

$$= \frac{4\partial^2 w}{\partial \xi \partial \eta} + \frac{2}{\xi - \eta}\left(\frac{\partial w}{\partial \xi} - \frac{\partial w}{\partial \eta}\right)$$

$$= \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{1}{2(\xi - \eta)}\left(\frac{\partial w}{\partial \xi} - \frac{\partial w}{\partial \eta}\right) = 0$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{2(\xi - \eta)}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = 0$$

Canonical form for hyperbolic domain.

elliptic case : $y > 0$

$$a = 1,\ b = 0,\ \cancel{y}\ c = y$$

$$b^2 - ac = -y$$

algorithm:

$$\frac{dy}{dx} = \frac{b + i\sqrt{ac - b^2}}{a}$$

$$\frac{dy}{dx} = i\sqrt{y}.$$

$$\int \frac{\cancel{-1}}{i\sqrt{y}}\, dy = \int dx$$

$$\int \frac{i}{\sqrt{y}}\, dy = \int d\cancel{x}$$

$$2i\sqrt{y} = x + \text{Constant}$$

$$\phi = x - 2i\sqrt{y} = \text{Constant}$$

$$Re(\phi) = \xi \qquad Im(\phi) = \eta$$

$$\xi = x \qquad \eta = -2\sqrt{y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial \xi}\cdot 1 + \frac{\partial w}{\partial \eta}\cdot(0) = \frac{\partial w}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial \xi^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial \xi}\cdot 0 + \frac{\partial w}{\partial \eta}\cdot -y^{-1/2}$$

$$\eta = -2\sqrt{y} \rightarrow y = \frac{\eta^2}{4} \quad y^{-1/2} = \frac{2}{\eta}$$

$$\frac{\partial u}{\partial y} = \frac{2}{\eta}\cdot\frac{\partial w}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2}{\eta}\cdot\frac{\partial}{\partial \eta}\left(\frac{2}{\eta}\cdot\frac{\partial w}{\partial \eta}\right)$$

$$= \frac{2}{\eta}\left(\frac{\partial^2 w}{\partial \eta^2}\cdot\frac{2}{\eta} + 2(-1)\eta^{-2}\frac{\partial w}{\partial \eta}\right)$$

$$= \frac{2}{\eta}\left(\frac{2}{\eta}\frac{\partial^2 w}{\partial \eta^2} - 2\eta^{-2}\frac{\partial w}{\partial \eta}\right)$$

$$= \frac{4}{\eta^2}\frac{\partial^2 w}{\partial \eta^2} - \frac{4}{\eta^3}\frac{\partial w}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{4}{\eta^2}\left(\frac{\partial^2 w}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial w}{\partial \eta}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0 \leftarrow \text{sub in}$$

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\eta^2}{4}\cdot\frac{4}{\eta^2}\left(\frac{\partial^2 w}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial w}{\partial \eta}\right)$$

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial w}{\partial \eta}$$

Canonical Form for elliptic case.

$$\boxed{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0}$$