6DOF Vehicle Handling Model with Nonlinear Tires

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1 Introduction

Vehicle handling models are key for many stages of vehicle development. They are used for mechanical design, control algorithm design and vehicle software. Many different models of varying fidelity exist. For example, the 2DOF bicycle model (also known as the 2DOF single-track model) [1] is used ubiquitously to analyze basic lateral handling characteristics (e.g. oversteer, understeer, etc). However, near the grip limit, the assumptions of this model begin to fall apart. Among other things, it fails to account for nonlinear tire behavior, combined slip and transient weight transfer due to body pitch, roll and heave.

In this project, a 6DOF vehicle handling model is developed. The entire vehicle is free to translate and yaw in the 2D plane. The vehicle chassis is modeled as four point masses at ground level representing each unsprung corner. The vehicle body is connected to the chassis at a pivot point (loosely referred to as a geometric "roll center"). This body is free to pitch, roll and heave relative to the chassis, and it is also coupled to each corner via a spring and damper in parallel. The nonlinear tire behavior is captured using the Fiala tire model.

2 Model Development

2.1 Kinematics

For this model we define 3 coordinate frames: the world frame ($\mathcal{W}$), the chassis frame ($\mathcal{C}$) and the body frame ($\mathcal{B}$). The world frame is an inertial frame fixed to the ground. The chassis frame is fixed to the vehicle at ground level, directly below the static center of mass of the body. The body frame is located above the chassis frame and is fixed to the body. We can define the rotations between frames:

$$ {}^{W}\mathbf{R}_C = \mathbf{R}_z(\psi) \qquad {}^{C}\mathbf{R}_B = \mathbf{R}_y(\theta)\mathbf{R}_x(\phi) \tag{1} $$

The location of the chassis (C) in the world frame is:

$$ {}^{W}[\mathbf{r}_C] = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{2} $$

The location of the roll center (P) in the chassis frame is:

$$ {}^{C}[\mathbf{r}_{CP}] = \begin{bmatrix} 0 \\ 0 \\ h_P + z \end{bmatrix} \tag{3} $$

The location of the body center of mass (G) in the body frame is:

$$ {}^{B}[\mathbf{r}_{PG}] = \begin{bmatrix} 0 \\ 0 \\ h_G - h_P \end{bmatrix} \tag{4} $$

Combining these, we can write the location of the body center of mass (G) in the world frame:

$$ {}^{W}[\mathbf{r}_G] = {}^{W}[\mathbf{r}_C] + {}^{W}\mathbf{R}_C({}^{C}[\mathbf{r}_{CP}] + {}^{C}\mathbf{R}_B {}^{B}[\mathbf{r}_{PG}]) \tag{5} $$

We can also write the angular velocity of the body in the body frame:

$$ {}^{B}[\boldsymbol{\omega}_B] = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}_x(-\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}_x(-\phi)\mathbf{R}_y(-\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \tag{6} $$

Additionally we define the positions of the tires (T_i):

$$ \begin{aligned} {}^{W}[\mathbf{r}_{T_{\mathrm{FrL}}}] &= {}^{W}[\mathbf{r}_C] + {}^{W}\mathbf{R}_C \begin{bmatrix} a \\ \frac{w}{2} \\ 0 \end{bmatrix} \\ {}^{W}[\mathbf{r}_{T_{\mathrm{FrR}}}] &= {}^{W}[\mathbf{r}_C] + {}^{W}\mathbf{R}_C \begin{bmatrix} a \\ -\frac{w}{2} \\ 0 \end{bmatrix} \\ {}^{W}[\mathbf{r}_{T_{\mathrm{ReL}}}] &= {}^{W}[\mathbf{r}_C] + {}^{W}\mathbf{R}_C \begin{bmatrix} -b \\ \frac{w}{2} \\ 0 \end{bmatrix} \\ {}^{W}[\mathbf{r}_{T_{\mathrm{ReR}}}] &= {}^{W}[\mathbf{r}_C] + {}^{W}\mathbf{R}_C \begin{bmatrix} -b \\ -\frac{w}{2} \\ 0 \end{bmatrix} \end{aligned} \tag{7} $$

And the body corners (K_i):

$$ \begin{aligned} {}^{W}[\mathbf{r}_{K_{\mathrm{FrL}}}] &= {}^{W}[\mathbf{r}_G] + {}^{W}\mathbf{R}_B \begin{bmatrix} a \\ \frac{w}{2} \\ 0 \end{bmatrix} \\ {}^{W}[\mathbf{r}_{K_{\mathrm{FrR}}}] &= {}^{W}[\mathbf{r}_G] + {}^{W}\mathbf{R}_B \begin{bmatrix} a \\ -\frac{w}{2} \\ 0 \end{bmatrix} \\ {}^{W}[\mathbf{r}_{K_{\mathrm{ReL}}}] &= {}^{W}[\mathbf{r}_G] + {}^{W}\mathbf{R}_B \begin{bmatrix} -b \\ \frac{w}{2} \\ 0 \end{bmatrix} \\ {}^{W}[\mathbf{r}_{K_{\mathrm{ReR}}}] &= {}^{W}[\mathbf{r}_G] + {}^{W}\mathbf{R}_B \begin{bmatrix} -b \\ -\frac{w}{2} \\ 0 \end{bmatrix} \end{aligned} \tag{8} $$

In these equations there are 6 degrees of freedom, namely the x, y and z translations along with the yaw ψ, pitch θ and roll ϕ. h_P is the unloaded height of the roll center, h_G is the unloaded height of the body center of mass, a is the distance from the center of mass to the front axle, b is the distance to the rear axle and w is the track width.

2.2 Lagrangian

The generalized coordinates are the 6 degrees of freedom:

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ z \\ \psi \\ \theta \\ \phi \end{bmatrix} \qquad \dot{\mathbf{q}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{9}$$

We can write the kinetic energy in the following form using the mass matrix $\mathbf{M}(\mathbf{q})$:

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \tag{10}$$

To calculate the mass matrix, we need the Jacobians for body and tire motion:

$$\mathbf{J}_{Gv} = \frac{\partial}{\partial \mathbf{q}} \left({}^W[\mathbf{r}_G]\right) \tag{11}$$

$$\mathbf{J}_{G\omega} = \frac{\partial}{\partial \dot{\mathbf{q}}} \left({}^B[\boldsymbol{\omega}_B]\right) \tag{12}$$

$$\mathbf{J}_{T_i v} = \frac{\partial}{\partial \mathbf{q}} \left({}^W[\mathbf{r}_{T_i}]\right) \tag{13}$$

Then we can write the mass matrix using the body mass m_b, body inertia tensor $\mathbf{I}_b$ and corner mass m_c:

$$\mathbf{M}(\mathbf{q}) = m_b \mathbf{J}_{Gv}^T \mathbf{J}_{Gv} + \mathbf{J}_{G\omega}^T ({}^B[\mathbf{I}_b]_B) \mathbf{J}_{G\omega} + \sum_i m_c \mathbf{J}_{T_i v}^T \mathbf{J}_{T_i v} \tag{14}$$

The potential energy has contributions from gravity and the suspension springs:

$$V = m_b g({}^W[\mathbf{r}_G]_z) + \sum_i \frac{1}{2} k_s ({}^W[\mathbf{r}_{K_i}]_z)^2 \tag{15}$$

We can also define a dissipation function D representing the energy extracted from the system by the suspension dampers:

$$D = \sum_i \frac{1}{2} b_d ({}^W[\mathbf{v}_{K_i}]_z)^2 \tag{16}$$

In this case we are approximating the displacement of the suspension with the z travel of the body corners K_i, effective wheel rate k_s, and effective wheel damping b_d. Since the pitch and roll angles are small, this is a reasonable assumption.

2.3 Equations of Motion

We can express the equations of motion in the following form:

$$\ddot{\mathbf{q}} = \mathbf{M}(\mathbf{q})^{-1} \left(-\mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{g}(\mathbf{q}) - \mathbf{d}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{Q}\right) \tag{17}$$

We already have the mass matrix from earlier. To calculate each element of the Coriolis vector, we use the following formula where Γ^i_{jk} are the Christoffel symbols:

$$
\begin{aligned}
c_i &= \sum_j \sum_k \Gamma^i_{jk} \dot{q}_j \dot{q}_k \\
\Gamma^i_{jk} &= \frac{1}{2} \left(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{jk}}{\partial q_i} \right)
\end{aligned} \tag{18}
$$

The gravity and damping vectors are simply:

$$
\mathbf{g}(\mathbf{q}) = \frac{\partial V}{\partial \mathbf{q}} \qquad \mathbf{d}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial D}{\partial \dot{\mathbf{q}}} \tag{19}
$$

In order to find the generalized forces $\mathbf{Q}$ we must first calculate the horizontal forces at each tire. These forces depend on the generalized coordinates $\mathbf{q}$ and generalized velocities $\dot{\mathbf{q}}$. They also depend on the control inputs which are wheel torques τ_i and steering angles δ_i. The vertical force at each tire is as follows:

$$
F_{zi} = -k_s({}^W[\mathbf{r}_{K_i}]_z) + m_c g \tag{20}
$$

We resolve the horizontal tire forces in the tire frame and assume that the longitudinal tire forces are specified as $F_{xi} = \frac{\tau_i}{R}$ where R is the radius of the wheel. This assumption is for two main reasons. First, the longitudinal tire dynamics often occur at a shorter time-scale than the lateral tire dynamics, so it makes the system easier to simulate while maintaining similar accuracy. Second, it is easier to model combined slip when one of the forces is specified. We calculate the effective lateral friction as:

$$
\mu_y = \mu \frac{\sqrt{F_{zi}^2 - F_{xi}^2}}{F_{zi}} \tag{21}
$$

By expressing the tire velocity in the chassis frame, we can calculate the slip angles relative to the tire heading. We calculate the slip angle at each wheel, taking into account the steering angle δ_i:

$$
\alpha_i = \text{arctan2}({}^C[\mathbf{v}_{T_i}]_y, {}^C[\mathbf{v}_{T_i}]_x) - \delta_i \tag{22}
$$

For a typical front steering car, $\delta_{\text{FrL}} = \delta_{\text{FrR}} = \delta$ and $\delta_{\text{ReL}} = \delta_{\text{ReR}} = 0$. We use the Fiala lateral tire model [2] where $C_{\alpha i}$ is the lateral cornering stiffness:

$$
F_{y_i} = \begin{cases} -C_{\alpha i} \tan \alpha_i + \frac{{C_{\alpha i}}^2}{3\mu_y F_{zi}} |\tan \alpha_i| \tan \alpha_i - \frac{{C_{\alpha i}}^3}{27(\mu_y F_{zi})^2} \tan^3 \alpha_i, & |\alpha_i| < \tan^{-1}\left(\frac{3\mu_y F_{zi}}{C_{\alpha i}}\right) \\ -\mu_y F_{zi} \operatorname{sgn} \alpha_i, & \text{otherwise} \end{cases} \tag{23}
$$

Finally we can express the tire forces in the world frame and compute the generalized force Q_j for each generalized coordinate q_j:

$$
Q_j = \sum_i \left({}^W\mathbf{R}_C \mathbf{R}_z(\delta_i) \begin{bmatrix} F_{xi} \\ F_{y_i} \\ 0 \end{bmatrix} \right) \cdot \frac{\partial}{\partial q_j} \left({}^W[\mathbf{r}_{T_i}] \right) \tag{24}
$$

3 Results

3.1 Implementation

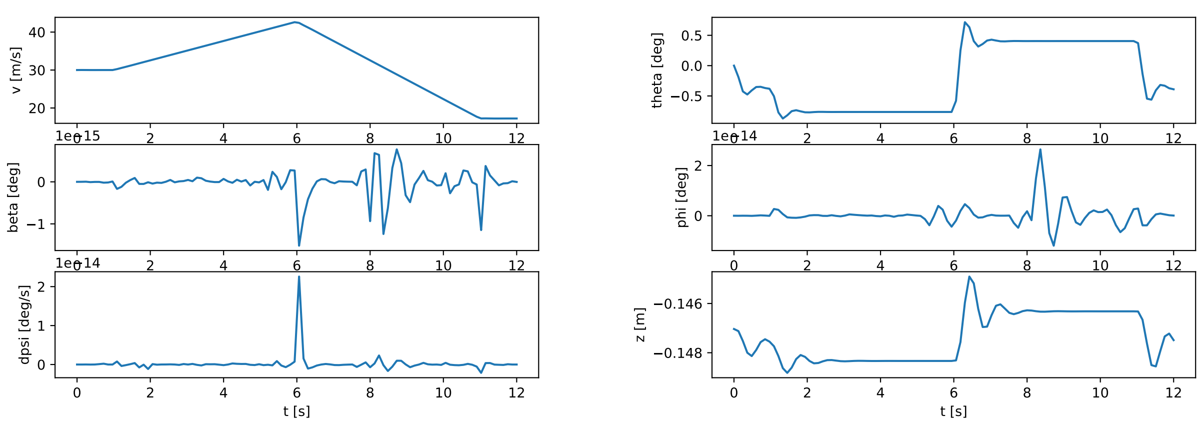
This model was simulated in Python using SciPy's `odeint` function. Derivatives and Jacobians were computed via autodiff with JAX. The vehicle parameters used are listed in Table 1.

Parameter	Symbol	Value
CG to front axle [m]	a	1.6
CG to rear axle [m]	b	1.4
CG height [m]	h_G	0.5
Roll center height [m]	h_P	0.1
Track width [m]	w	1.6
Body mass [kg]	m_b	1600
Roll inertia [kg*m^2]	$\mathbf{I}_{bxx}$	580
Pitch inertia [kg*m^2]	$\mathbf{I}_{byy}$	2300
Yaw inertia [kg*m^2]	$\mathbf{I}_{bzz}$	2300
Corner mass [kg]	m_c	70
Effective wheel rate [N/m]	k_s	26700
Effective wheel damping [N*s/m]	b_d	1960
Wheel radius [m]	R	0.25
Front cornering stiffness [N/rad]	$C_{\alpha \mathrm{Fr}}$	155000
Rear cornering stiffness [N/rad]	$C_{\alpha \mathrm{Re}}$	225000

Table 1: Parameters used for the simulation

3.2 Acceleration + Braking

For the first experiment, the vehicle was started at 30 m/s and accelerated forward with 300 Nm torque at each wheel. Then the vehicle was slowed with 600 Nm torque at each wheel. As shown in Figure 1, the vehicle correctly pitches up under acceleration and pitches down under braking. Note that the static pitch is nonzero since the CG is behind the geometric center.



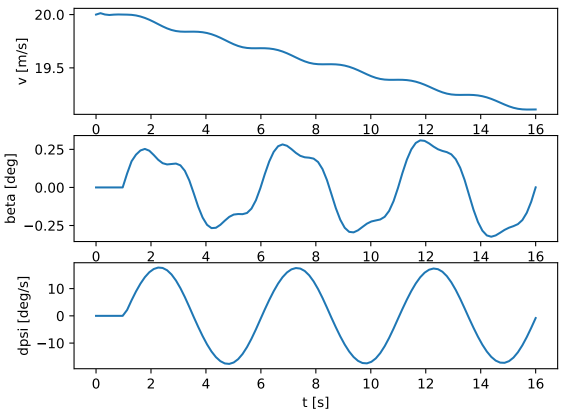
(a) Speed of the vehicle

(b) Vehicle body motion

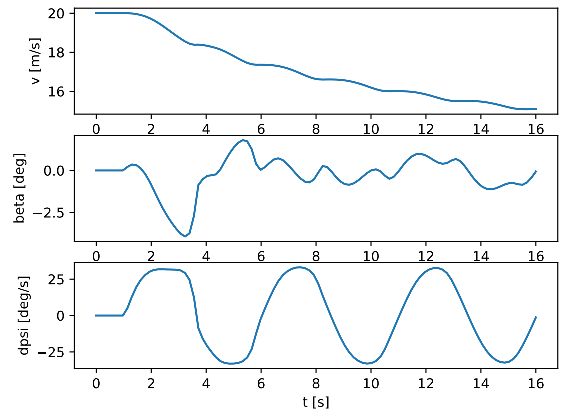
Figure 1: Motion of the vehicle under acceleration and braking

3.3 Sine Steering

For the second experiment, the vehicle was started at 20 m/s and a sine steering input was applied. The same experiment was performed for steering magnitudes of 3° (sub-limit) and 7° (past-limit). Figure 2 shows that the yaw rate closely resembles a sine wave for the sub-limit maneuver but clips at the peaks for the past-limit maneuver. This is also reflected in Figure 3 which shows the grip circle is saturated in the past-limit maneuver.

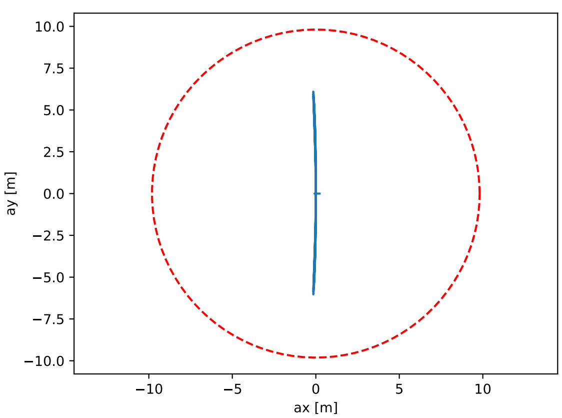


(a) Motion of the vehicle sub-limit

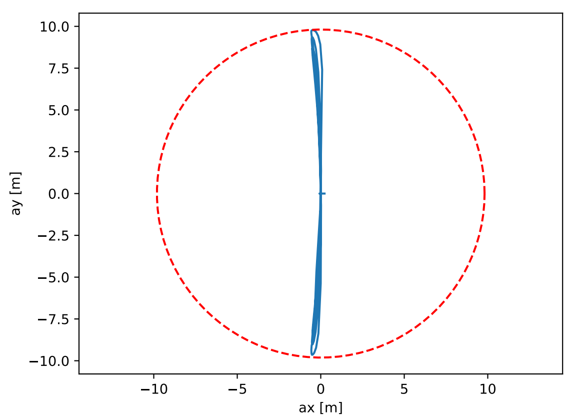


(b) Motion of the vehicle past-limit

Figure 2: Motion of the vehicle during sine steer maneuvers



(a) Grip circle sub-limit

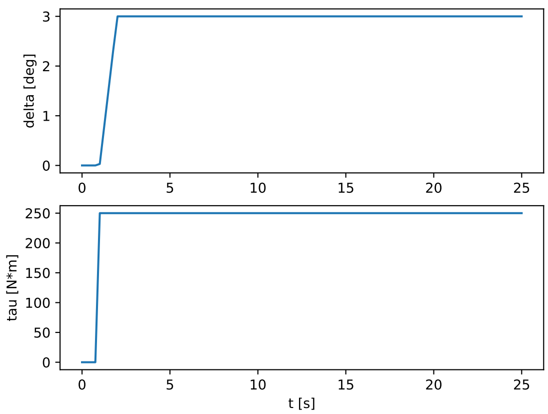


(b) Grip circle past-limit

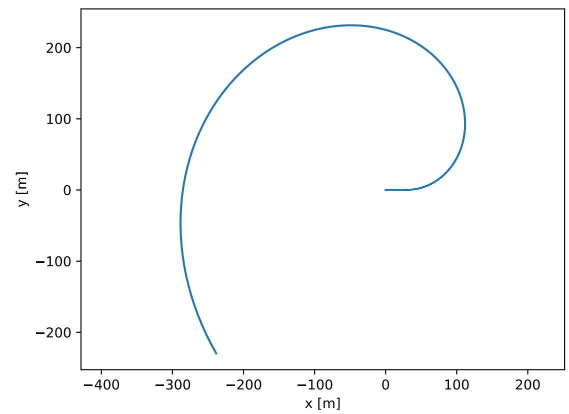
Figure 3: Grip circle for sine steer maneuvers

3.4 Step Steer

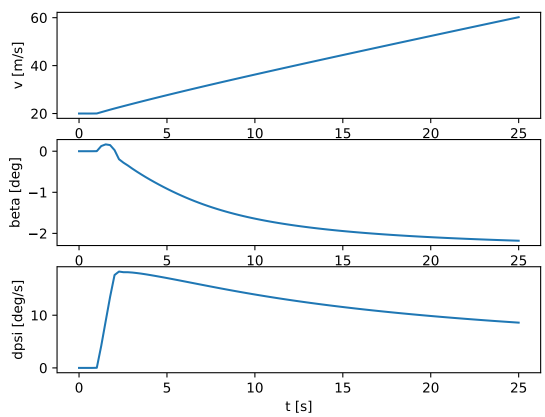
For the final experiment, the vehicle was started at 20 m/s and a constant steering input of 3° was applied along with a small torque of 250 Nm at each wheel. Figure 4 shows that the body slip angle magnitude gradually increases until the vehicle saturates the grip circle. With these parameters, the vehicle exhibits understeer behavior since it does not spin out but rather continues on a path of decreasing curvature.



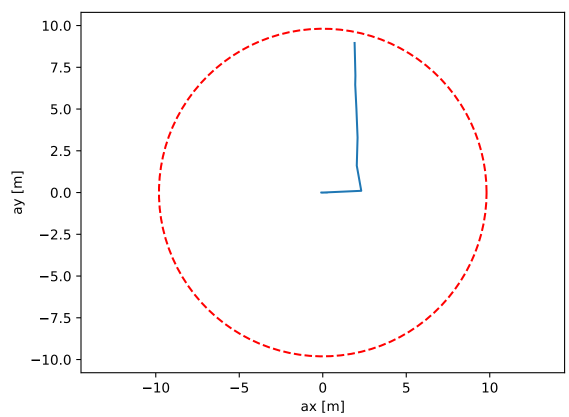
(a) Inputs during step steer



(b) Path of step steer



(c) Motion of the vehicle



(d) Grip circle

Figure 4: Motion of the vehicle during step steer maneuver

4 Conclusions and Future Work

In this project a 6DOF vehicle handling model was developed. The model was simulated for a variety of maneuvers and exhibits qualitatively correct behavior. Future work includes extending the model to consider longitudinal slip, as well as more accurately modeling the suspension geometry to capture effects like jacking force.

References

[1] Michael. Vehicle dynamics: The dynamic bicycle model. https://thef1clan.com/2020/12/23/vehicle-dynamics-the-dynamic-bicycle-model/.

[2] E. Fiala. Seitenkrafte am rollenden luftreifen. *Verein Deutscher Ingenieur*, 96:973–979, 1954.

A Appendix

A.1 Code

The code for this project can be found at https://github.com/dardeshna/me334-veh-dyn.