4. Bayesian Tracking

This is at the heart of many optimal, recursive estimation algorithms.

- Let $x(k) \in \mathcal{X}$, $\overbrace{k = 0, 1, \ldots}^{time}$, be the vector valued state we want to estimate – a discrete random variable which can only take on a finite number of values – $\mathcal{X}$ *is finite.*
- Let $z(k)$, $k = 1, 2, \ldots$, be a vector valued measurement, what we can observe. It can be a continuous or discrete random variable.

4.1 Model

$$x(k) = q_k(x(k-1), v(k)), \qquad k = 1, 2, \ldots$$

$$z(k) = h_k(x(k), w(k))$$

$x(0)$, $\{v(\cdot)\}$ and $\{w(\cdot)\}$ are independent. Note that even though the known input $u(k)$ is not explicitly included in the model, it can be implicitly modeled by absorbing it into q_k and h_k.

4.2 Recursive equations

Let $z(1:k)$ denote the set $\{z(1), \ldots, z(k)\}$. We want to calculate $f(x(k)|z(1:k))$ *efficiently*.

- Assume $f(x(k-1)|z(1:k-1))$ is known. For $k = 1$, we simply have $f(x(0))$, which is known. The recursion is then:

  Prior update:

  $$f(x(k)|z(1:k-1)) = \sum_{x(k-1)\in\mathcal{X}} f(x(k)|z(1:k-1), x(k-1))\, \underbrace{f(x(k-1)|z(1:k-1))}_{\text{Assumed to be known}}$$

  where the formula above follows from the total probability theorem. Note that $x(k)$ and $z(1:k-1)$ are conditionally independent, given $x(k-1)$:

  - $x(k) = q_k(x(k-1), v(k))$, a function of $v(k)$ only.
  - $z(k-1) = h_{k-1}(x(k-1), w(k-1))$
  - $z(k-2) = h_{k-2}(x(k-2), w(k-2))$, $x(k-2) = q_{k-2}(x(k-3), v(k-2))$, etc.
  - $\therefore z(1:k-1) = \text{FUNCTION}\big(x(k-1), \underbrace{v(1:k-2), w(1:k-1), x(0)}_{\text{independent of } v(k)}\big)$
  - Therefore $x(k)$ and $z(1:k-1)$ are conditionally independent, given $x(k-1)$. (**PSET 2: P10**)

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$$\boxed{f(x(k)|z(1:k-1)) = \sum_{x(k-1)\in\mathcal{X}} f(x(k)|x(k-1))\, f(x(k-1)|z(1:k-1))}$$

and $f(x(k)|x(k-1))$ can be calculated from $f(v(k))$ and $q_k(\cdot,\cdot)$ (although it may not be straight-forward to do so). This is an intuitive result: we use the process model to push our estimate forward in time. Note that conditional independence is crucial.

Measurement update:

$$\begin{aligned} f(x(k)|z(1:k)) &= f(x(k)|z(k), z(1:k-1)) \\ &= \frac{f(z(k)|x(k), z(1:k-1))\, f(x(k)|z(1:k-1))}{f(z(k)|z(1:k-1))} \qquad \text{(Bayes' rule)} \end{aligned}$$

Note that $z(k)$ and $z(1:k-1)$ are conditionally independent, given $x(k)$:

- $z(k) = h_k(x(k), w(k))$, a function of $w(k)$ only.
- $z(1:k-1) = \text{FUNCTION}\big(\underbrace{v(1:k-1), w(1:k-1), x(0)}_{\text{independent of } w(k)}\big)$.
- Therefore $z(k)$ and $z(1:k-1)$ are conditionally independent, given $x(k)$.

Therefore $f(z(k)|x(k), z(1:k-1)) = f(z(k)|x(k))$, and furthermore, it can be calculated from $f(w(k))$ and $h_k(\cdot,\cdot)$. Finally, $f(z(k)|z(1:k-1))$ is simply a normalization constant that can be calculated using the total probability theorem:

$$f(z(k)|z(1:k-1)) = \sum_{x(k)\in\mathcal{X}} f(z(k)|x(k))\, f(x(k)|z(1:k-1))$$

$$\boxed{f(x(k)|z(1:k)) = \frac{f(z(k)|x(k))\, f(x(k)|z(1:k-1))}{\displaystyle\sum_{x(k)\in\mathcal{X}} f(z(k)|x(k))\, f(x(k)|z(1:k-1))}}$$

Summary

$$f(x(k)|z(1:k-1)) = \sum_{x(k-1)\in\mathcal{X}} \overbrace{f(x(k)|x(k-1))}^{\text{process model}}\,\overbrace{f(x(k-1)|z(1:k-1))}^{\text{previous iteration}}, \qquad k = 1, 2, \ldots$$

$$f(x(k)|z(1:k)) = \frac{\overbrace{f(z(k)|x(k))}^{\text{measurement model}}\ \overbrace{f(x(k)|z(1:k-1))}^{\text{prior}}}{\displaystyle\sum_{x(k)\in\mathcal{X}} f(z(k)|x(k))\, f(x(k)|z(1:k-1))}$$

4.3 Computer implementation

- Enumerate the state: $\mathcal{X} = \{1, 2, \ldots, N\}$
- Define $\omega^i_{k|k} := \Pr(x(k) = i|z(1:k)), \quad i = 1, \ldots, N$
- Define $\omega^i_{k|k-1} := \Pr(x(k) = i|z(1:k-1)), \quad i = 1, \ldots, N$

Then

Initialization, $k = 0:$

$$\omega^i_{0|0} = \Pr(x(0) = i), \quad i = 1, \dots, N$$

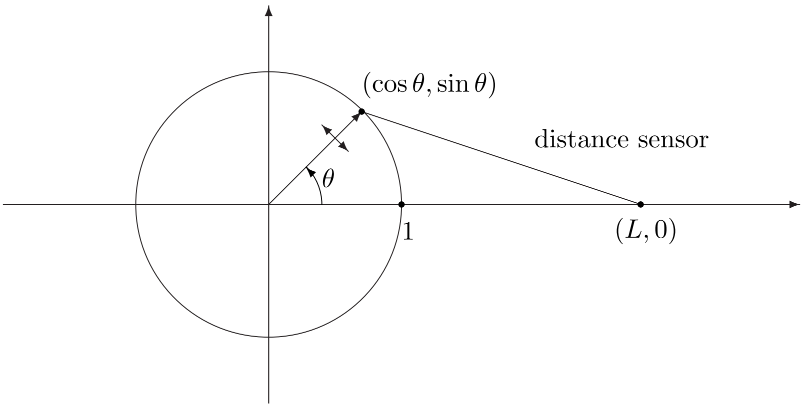
$k > 0:$

$$\left.\begin{aligned} \omega^i_{k|k-1} &= \sum_{j=1}^{N} \Pr(x(k) = i | x(k-1) = j)\, \omega^j_{k-1|k-1}, \quad i = 1, \dots, N \\ \omega^i_{k|k} &= \frac{f(z(k)|x(k) = i)\, \omega^i_{k|k-1}}{\sum_{j=1}^{N} f(z(k)|x(k) = j)\, \omega^j_{k|k-1}}, \quad i = 1, \dots, N \end{aligned}\right\} \quad \text{Iterate.}$$

Note that $\Pr(x(k) = i|x(k-1) = j)$ can be calculated from $x(k) = q_k(x(k-1), v(k))$ and $f(v(k))$. Similarly, $f(z(k)|x(k) = i)$ can be calculated from $z(k) = h_k(x(k), w(k))$ and $f(w(k))$.

4.4 Example

Consider an object moving randomly on a circle. We can measure the distance to the object and want to estimate its location.



- Define $x(k)$ as the object's location on the circle, $x(k) \in \{0, 1, \dots, N-1\}$, with

$$\theta(k) = 2\pi \frac{x(k)}{N}\,.$$

- We model the dynamics as $x(k) = x(k-1) + v(k)$, where

$$v(k) = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

$$\begin{aligned} (N-1) + 1 &:= 0 \\ 0 - 1 &:= N - 1\,. \end{aligned}$$

- Measurement:

$$z(k) = \sqrt{\left(L - \cos\theta(k)\right)^2 + \sin^2\theta(k)} + w(k),$$

where $w(k)$, uniformly distributed on $[-e, e]$, is the sensor noise.

- We can now construct the PDFs of the process and sensor models:

$$f(x(k)|x(k-1)) = \begin{cases} p & \text{if } x(k) = x(k-1)+1 \\ 1-p & \text{if } x(k) = x(k-1)-1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(z(k)|x(k)) = \begin{cases} \frac{1}{2e} & \text{if } \left| z(k) - \sqrt{(L-\cos\theta(k))^2 + \sin^2\theta(k)} \right| \le e \\ 0 & \text{otherwise.} \end{cases}$$

- Initialization: $f(x(0)) = \dfrac{1}{N}\ \forall\, x(0) \in \{0, 1, \ldots, N-1\}$. This captures the state of maximum ignorance (entropy).

Simulations

$N = 100 \Rightarrow 3.6°$. Initial location: $x(0) = N/4 = 25 \Rightarrow 90°$. $e = 0.50$.

1) $L = 2.0$, $p = 0.50$. Notice bimodal distribution. The mean is a horrible estimate!

2) $L = 2.0$, $p = 0.55$. Decay.

3) $L = 0.1$, $p = 0.55$. Works!

4) $L = 0.0$, $p = 0.55$. Uniform for all time.

How robust is this? Set the nominal parameters to $L = 2.0$, $e = 0.50$, $p = 0.55$, but use different values for the estimator.

5) $\hat{p} = 0.45$, $\hat{e} = e$. Of course we get it completely wrong.

6) $\hat{p} = 0.50$, $\hat{e} = e$. We can't differentiate.

7) $\hat{p} = 0.90$, $\hat{e} = e$. Makes incorrect assumption.

8) $\hat{p} = p$, $\hat{e} = 0.90$. Washes things out.

9) $\hat{p} = p$, $\hat{e} = 0.49$. Crashes!