Question 1 EM for Probabilistic PCA

(a) *E-step. Calculate the statistics of the posterior distribution $q(z) = p(z|\boldsymbol{x})$ which you'll need for the M-step.*

From the Appendix, we know how to get the distribution of z given $\mathbf{x}$, where z is drawn from Gaussian distribution and $\mathbf{x}$ is drawn from a spherical Gaussian distribution.
In our setting,

$$p(z) = \mathcal{N}(z|0,1)$$
$$p(\mathbf{x}|z) = \mathcal{N}(\mathbf{x}|z\mathbf{u}, \sigma^2\mathbf{I})$$

To apply the parameters in the formulae of the Appendix, we have

$$\begin{aligned}
&\boldsymbol{\mu} = 0, \boldsymbol{\Sigma} = 1, \\
&\boldsymbol{A} = \boldsymbol{u}, \boldsymbol{B} = 0, \boldsymbol{S} = \sigma^2\boldsymbol{I} \\
&\boldsymbol{C} = (1 + \boldsymbol{u}^T(\sigma^2)^{-1}\boldsymbol{u})^{-1} = \frac{\sigma^2}{\sigma^2 + \boldsymbol{u}^T\boldsymbol{u}}
\end{aligned}$$

Thus, we can obtain the folowing formulae:

$$\begin{aligned}
p(\boldsymbol{x}) &= \mathcal{N}(\boldsymbol{x}|0, \boldsymbol{u}^T\boldsymbol{u} + \sigma^2) \\
p(z|\boldsymbol{x}) &= \mathcal{N}(z|\boldsymbol{C}(\boldsymbol{u}^T(\sigma^2)^{-1}\boldsymbol{x}), \boldsymbol{C}) \\
&= \mathcal{N}(z|\frac{\boldsymbol{u}^T\boldsymbol{x}}{\sigma^2 + \boldsymbol{u}^T\boldsymbol{u}}, \frac{\sigma^2}{\sigma^2 + \boldsymbol{u}^T\boldsymbol{u}})
\end{aligned}$$

As a result,

$$\begin{aligned}
m = E[z|\boldsymbol{x}] &= \frac{\boldsymbol{u}^T\boldsymbol{x}}{\sigma^2 + \boldsymbol{u}^T\boldsymbol{u}} \\
Var[z|\boldsymbol{x}] &= \frac{\sigma^2}{\sigma^2 + \boldsymbol{u}^T\boldsymbol{u}} \\
s = E[z^2|\boldsymbol{x}] &= Var[z|\boldsymbol{x}] + E[z|\boldsymbol{x}]^2 \\
&= \frac{\sigma^4 + \sigma^2\boldsymbol{u}^T\boldsymbol{u} + (\boldsymbol{u}^T\boldsymbol{x})^2}{(\sigma^2 + \boldsymbol{u}^T\boldsymbol{u})^2}
\end{aligned}$$

(b) *M-step. Re-estimate the parameters, which consist of the vector $\boldsymbol{u}$. derive a formula for $\boldsymbol{u}_{new}$ that maximizes the expected log-likelihood, i.e.,*

$$\boldsymbol{u}_{new} = \arg\max_{\boldsymbol{u}} \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}_{q(z^{(i)})}[\log p(z^{(i)}, \boldsymbol{x}^{(i)})]$$

Denote the function to be maximized as

$$\begin{aligned}
\mathbb{F} &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}_{q(z^{(i)})}[\log p(z^{(i)}, \boldsymbol{x}^{(i)})] \\
&= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}_{q(z^{(i)})}[\log q(z^{(i)})p(\boldsymbol{x}^{(i)})]
\end{aligned}$$

Question 1 [(b)] continued on next page. . .

Then,

$$\begin{aligned}
\log p(\boldsymbol{x}^{(i)})q(z^{(i)}) &= \log \frac{1}{\sqrt{2\pi(\boldsymbol{u}^T\boldsymbol{u}+\sigma^2)}} e^{-\frac{\boldsymbol{x}^{(i)2}}{2(\boldsymbol{u}^T\boldsymbol{u}+\sigma^2)}} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{\boldsymbol{u}^T\boldsymbol{u}+\sigma^2}}} e^{-\frac{(z^{(i)}-\frac{\boldsymbol{u}^T\boldsymbol{x}^{(i)}}{\sigma^2+\boldsymbol{u}^T\boldsymbol{u}})^2}{2\frac{\sigma^2}{\boldsymbol{u}^T\boldsymbol{u}+\sigma^2}}} \\
&\propto -\frac{\boldsymbol{x}^{(i)2}}{2(\boldsymbol{u}^T\boldsymbol{u}+\sigma^2)} - \frac{(z^{(i)}-\frac{\boldsymbol{u}^T\boldsymbol{x}^{(i)}}{\sigma^2+\boldsymbol{u}^T\boldsymbol{u}})^2}{2\frac{\sigma^2}{\boldsymbol{u}^T\boldsymbol{u}+\sigma^2}} \\
&\propto -\frac{\boldsymbol{x}^{(i)2}\sigma^2 + [\boldsymbol{u}^T\boldsymbol{x}^{(i)} - (\sigma^2+\boldsymbol{u}^T\boldsymbol{u})z^{(i)}]^2}{2\sigma^2(\boldsymbol{u}^T\boldsymbol{u}+\sigma^2)} \\
&\propto -\frac{z^{(i)2}(\sigma^2+\boldsymbol{u}^T\boldsymbol{u})}{2\sigma^2} + \frac{z^{(i)}\boldsymbol{u}^T\boldsymbol{x}^{(i)}}{\sigma^2} \\
&\propto -\frac{z^{(i)2}\boldsymbol{u}^T\boldsymbol{u}}{2\sigma^2} + \frac{z^{(i)}\boldsymbol{u}^T\boldsymbol{x}^{(i)}}{\sigma^2}
\end{aligned}$$

Apply the liearity of expectation,

$$\begin{aligned}
\mathbb{F} &= \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[\log p(\boldsymbol{x}^{(i)})q(z^{(i)})] \\
&= \frac{1}{N}\sum_{i=1}^{N}[-\frac{\mathbb{E}[z^{(i)2}|\boldsymbol{x}^{(i)}]\boldsymbol{u}^T\boldsymbol{u}}{2\sigma^2} + \frac{\mathbb{E}[z^{(i)}|\boldsymbol{x}^{(i)}]\boldsymbol{u}^T\boldsymbol{x}^{(i)}}{\sigma^2}] \\
&= \frac{1}{N}\sum_{i=1}^{N}[-\frac{m^{(i)}\boldsymbol{u}^T\boldsymbol{u}}{2\sigma^2} + \frac{s^{(i)}\boldsymbol{u}^T\boldsymbol{x}^{(i)}}{\sigma^2}]
\end{aligned}$$

To get the gradient with repect to $\boldsymbol{u}$,

$$\begin{aligned}
\frac{\partial\mathbb{F}}{\partial\boldsymbol{u}} &= -\frac{1}{N}\sum_{i=1}^{N}[\frac{m^{(i)}\boldsymbol{u}}{\sigma^2} + \frac{s^{(i)}\boldsymbol{x}^{(i)}\boldsymbol{I}}{\sigma^2}] = 0 \\
\boldsymbol{u} &\leftarrow \frac{\frac{1}{N}\sum_{i=1}^{N} s^{(i)}\boldsymbol{x}^{(i)}\boldsymbol{I}}{\frac{1}{N}\sum_{i=1}^{N} m^{(i)}} \\
\boldsymbol{u} &\leftarrow \frac{\sum_{i=1}^{N} s^{(i)}\boldsymbol{x}^{(i)}\boldsymbol{I}}{\sum_{i=1}^{N} m^{(i)}}
\end{aligned}$$

Question 2 Contraction Maps

(a)

Question 3 Q-Learning

(a)

(b)