Exercițiul 1

$$X \sim \begin{pmatrix} -1 & 5 \\ 0,42 & 0,58 \end{pmatrix}$$

$$Y \sim \begin{pmatrix} -5 & 6 \\ p_1 & p_2 \end{pmatrix} \quad , p_1, p_2 \in (0,1)$$

a) $p_1, p_2 = ?$

$P(X=-1, Y=6) = 0,084$

$E[X|Y=6] = 2$

X\Y	-5	6	Σ
-1	0,336	0,084	0,42
5	$p_1 - 0,336$	$p_2 - 0,084$	0,58
Σ	p_1	p_2	

$$P(X=-1, Y=-5) = P(X=-1) - P(X=-1, Y=-6)$$
$$= 0,42 - 0,084$$
$$= 0,336$$

$E[X|Y=6] = +2$

$$X|Y=6 \sim \begin{pmatrix} -1 & 5 \\ \frac{0,084}{p_2} & \frac{p_2 - 0,084}{p_2} \end{pmatrix}$$

$$(-1) \cdot \frac{0,084}{p_2} + 5 \cdot \frac{p_2 - 0,084}{p_2} = +2 \quad | \cdot p_2$$

$$(-1) \cdot 0,084 + 5(p_2 - 0,084) = +2p_2$$

$$-0,084 + 5p_2 - 0,42 = +2p_2$$

$$-0,504 = -3p_2 \Rightarrow p_2 = 0,168 \quad \Rightarrow p_1 = 0,832$$

b)

$X \backslash Y$	-5	6	Σ
-1	0,336	0,084	0,42
5	0,496	0,084	0,58
Σ	0,832	0,168	

$x \in \{-1, 5\}$, $y \in \{-5, 6\} \Rightarrow x+y \in \{-6, 5, 0, 11\}$

$$X+Y \sim \begin{pmatrix} -6 & 0 & 5 & 11 \\ 0{,}336 & 0{,}496 & 0{,}084 & 0{,}084 \end{pmatrix}$$

$P(X+Y=-6) = P(X=-1, Y=-5) = 0{,}336$

$P(X+Y=0) = 0{,}496$

$P(X+Y=5) = 0{,}084$

$P(X+Y=11) = 0{,}084$

$x-y \in \{+4, -7, 10, -1\}$

$$X-Y \sim \begin{pmatrix} -7 & -1 & 4 & 10 \\ 0{,}084 & 0{,}084 & 0{,}336 & 0{,}496 \end{pmatrix}$$

$P(X-Y=-7) = P(X=-1, Y=6) = 0{,}084$

$P(X-Y=-1) = 0{,}084$

$P(X-Y=4) = 0{,}336$

$P(X-Y=10) = 0{,}496$

$$2X^2+2Y^2 \sim \begin{pmatrix} 52 & 74 & 100 & 122 \\ 0{,}336 & 0{,}084 & 0{,}496 & 0{,}084 \end{pmatrix}$$

$2x^2+2y^2 \in \{2\cdot 1+2\cdot 25, 2\cdot 1+2\cdot 36, 2\cdot 25+2\cdot 25, 2\cdot 25+2\cdot 36\}$

$2x^2+2y^2 \in \{52, 74, 100; 122\}$

$E[X] = \sum_{x} x f(x)$

$E[X] = (-1) \cdot 0,42 + 5 \cdot 0,58 = 2,48$

$E[Y] = (-5) \cdot 0,832 + 6 \cdot 0,168 = -3,152$

$Var(X) = E[X^2] - (E[X])^2$

$X^2 \sim \begin{pmatrix} 1 & 25 \\ 0,42 & 0,58 \end{pmatrix}$

$Y^2 \sim \begin{pmatrix} 25 & 36 \\ 0,832 & 0,168 \end{pmatrix}$

$E[X^2] = 1 \cdot 0,42 + 25 \cdot 0,58 = 14,92$

$E[Y^2] = 25 \cdot 0,832 + 36 \cdot 0,168 = 26,846$

$Var(X) = 14,92 - (2,48)^2 = 8,7696$

$Var(Y) = 26,846 - (-3,152)^2 = 26,846 - 9,935 = 16,911$

$Var(6X - 6Y + 14) = Var(6(X - Y) + 14) = Var(6 \cdot (X - Y))$

$= 36 Var(X - Y) = 36(59,176 - 31,719) = 988,436$

$E[X-Y] = 5,632$

$E[(X-Y)^2] = 59,176$

$cov(X,Y) = E[X \cdot Y] - E[X] \cdot E[Y]$

$x \in \{-1, 5\}$, $y \in \{-5, 6\} \Rightarrow x \cdot y \in \{5, -6, -25, 30\}$

$XY \sim \begin{pmatrix} -25 & -6 & 5 & 30 \\ 0,496 & 0,084 & 0,336 & 0,084 \end{pmatrix}$

$E[XY] = -12,4 - 0,504 + 1,68 + 2,52 = -8,704$

$cov(X,Y) = -8,704 - 2,48 \cdot (-3,152) = -8,704 + 7,816 = -0,88704$

$$\varphi(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\cdot\operatorname{Var}(Y)}} = \frac{-0{,}887}{\sqrt{148{,}29}} = \frac{-0{,}887}{12{,}177} = -0{,}072$$

2 X, Y. v.a indep. $X, Y > 0$, $c \geq 0$

1. $E[\log(X)] \leq \log(E[X])$

2. $E[X]$? $\sqrt{E[X]}$

nu putem decide pt. că

$E[X] \leq \sqrt{E[X]}$, $E[X] \in [0, 1]$

$E[X] \geq \sqrt{E[X]}$, $E[X] \in [1, \infty)$

3. $E[\sin^2(X)] + E[\cos^2(X)] = 1$

4. $P(X > c)$? $\dfrac{E[X^3]}{c^3}$

5. $P(X \leq Y) = P(X \geq Y)$

$$P(\cancel{Y} \geq \cancel{X}) = \sum_{i=1}^{m} P(Y \geq X \mid X = i)\, P(X = i)$$

$$= \sum_{i=1}^{m} P(Y \geq i)\, P(X = i) \quad (1)$$

$$P(X \geq Y) = \sum_{i=1}^{m} P(X \geq Y \mid Y = i)\, P(Y = i)$$

$$= \sum_{i=1}^{m} P(X \geq i)\, P(Y = i) \quad (2)$$

Dar (1), (2) $\Rightarrow P(Y \geq X) = P(X \geq Y)$

9. $E[X^2(X^2+1)] \;.\overset{?}{=}.\; E[X^2(Y^2+1)]$

$E[X^2(X^2+1)] = E[X^4+X^2]$

$E[X^2(Y^2+1)] = E[X^2 \cdot Y^2 + X^2]$ $\Rightarrow =$

X și Y au aceeași distribuție și i.i.d. $\Rightarrow X^2 = Y^2$

6. $P(X+Y>10) \leq P(X>5 \text{ sau } Y>5)$

$X+Y>10$

Ca să obținem o suma >10 trebuie să avem cel puțin o valoare care să fie >5

Ex

$1+\underline{10}$
$2+\underline{9}$
$3+\underline{8}$
$4+\underline{7}$
$5+\underline{6}$
$\underline{6}+5$
...

$\Rightarrow$ $X+Y>10 \leq X>5$ sau $Y>5$

$\Rightarrow \leq$

7. $E[\min(X,Y)] = \min(E[X], E[Y])$

8.

$$E\left[\frac{X}{Y}\right] = \frac{E[X]}{E[Y]}$$

10. $E\left[\frac{1}{X}\right] \; ? \; \frac{1}{E[X]}$

3.

9 telefoane

X - nr de teste pt. primul telefon

Y - nr de teste pt al doilea telefon

$P(\text{defect}) = \frac{2}{9}$

$2 \leq X+Y \leq 7$

$P(X=1, Y=1) = \frac{2}{9} \cdot \frac{1}{8} = \frac{2}{72}$

$P(X=1, Y=2) = \frac{2}{9} \cdot \frac{\cancel{7}}{8} \cdot \frac{1}{\cancel{7}} = \frac{2}{72}$

$P(X=1, Y=3) = \frac{2}{9} \cdot \frac{\cancel{7}}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{1}{\cancel{6}} = \frac{2}{72}$

$P(X=1, Y=4) = \frac{2}{9} \cdot \frac{\cancel{7}}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{1}{\cancel{5}} = \frac{2}{72}$

$P(X=1, Y=5) = \frac{2}{9} \cdot \frac{\cancel{7}}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{1}{\cancel{4}} = \frac{2}{72}$

$P(X=1, Y=6) = \frac{2}{9} \cdot \frac{\cancel{7}}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{\cancel{3}} = \frac{2}{72}$

$\left(P(X=1, Y=7) = \frac{2}{9} \cdot \frac{\cancel{7}}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} = \frac{2}{72}\right)$

$P(X=2, Y=1) = \frac{\cancel{7}}{9} \cdot \frac{2}{8} \cdot \frac{1}{\cancel{7}} = \frac{2}{72}$

$P(X=2, Y=2) = \frac{\cancel{7}}{9} \cdot \frac{2}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{1}{\cancel{6}} = \frac{2}{72}$

$P(X=2, Y=3) = \frac{\cancel{7}}{9} \cdot \frac{2}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{1}{\cancel{5}} = \frac{2}{72}$

$P(X=2, Y=4) = \frac{\cancel{7}}{9} \cdot \frac{2}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{1}{\cancel{4}} = \frac{2}{72}$

$P(X=2, Y=5) = \frac{\cancel{7}}{9} \cdot \frac{2}{8} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{\cancel{3}} = \frac{2}{72}$

X\Y	0	1	2	3	4	5	6	7	Σ
1	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{8}{36}$
2	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	0	$\frac{7}{36}$
3	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	0	0	$\frac{6}{36}$
4	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	$\frac{5}{36}$
5	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0	$\frac{4}{36}$
6	0	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0	0	$\frac{3}{36}$
7	$\frac{1}{36}$	$\frac{2}{36}$	0	0	0	0	0	0	$\frac{3}{36}$
Σ	$\frac{1}{36}$	$\frac{8}{36}$	$\frac{7}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	

$$C_9^2 = \frac{9!}{2!\,7!} = \frac{9\cdot 8}{2} = \frac{36}{1}$$

$$\frac{1}{C_9^2} = \frac{1}{36}$$

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \frac{8}{36} & \frac{7}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{3}{36} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \frac{1}{36} & \frac{8}{36} & \frac{7}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} \end{pmatrix}$$

$$E[X] = \frac{1}{36}(8+14+18+20+20+18+21) = \frac{119}{36} = 3{,}305$$

$$E[Y] = \frac{1}{36}(8+14+18+20+20+18+14) = \frac{112}{36} = 3{,}111$$

$Var(X) = 3{,}65$

$Var(Y) = 3{,}54$

$$cov(X,Y) = E[XY] - E[X]\cdot E[Y]$$

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}}$$

$$X|Y=2 \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} & 0 \end{pmatrix}$$

$$Z = X|Y=2$$

$$E[Z] = \frac{1}{7}(1+2+3+4+5+12) = \frac{1}{7} \cdot 27$$

$$E[Z^2] = \frac{1}{7}(1+4+9+16+25+36 \cdot 2) = \frac{127}{7}$$

$$Var(Z) = \frac{127}{7} - \frac{729}{49} = \frac{889-729}{49} = \frac{160}{49} = 3{,}265$$

Exercițiul 4

X. v.a

$$f(x) = \frac{x}{64} \cdot e^{\frac{-x^2}{128}} \mathbb{1}_{\{x \geq 0\}}$$

$$f(x) = \begin{cases} \frac{x}{64} \cdot e^{-\frac{x^2}{128}} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

$$F(x) = \int_{-\infty}^{\infty} f(x)\, dx$$

$$F(x) = \int_{0}^{\infty} \frac{x}{64} \cdot e^{-\frac{x^2}{128}}\, dx$$

$$F(x) = \int_{0}^{\infty} \left(\frac{x^2}{128}\right)' \cdot e^{-\frac{x^2}{128}}\, dx$$

$$t = \frac{x^2}{128} \Rightarrow 128t = x^2$$

$$dt = \frac{2x}{128}\, dx = \frac{x}{64}\, dx$$

$$x = 0 \Rightarrow t = 0$$

$$x \to \infty \Rightarrow t \to \infty$$

$$F(x) = \int_{0}^{\frac{x^2}{128}} e^{-t}\, dt = -e^{-t}\Big|_{0}^{\frac{x^2}{128}} = -e^{-\frac{x^2}{128}} + 1$$

$$F(x) = \begin{cases} 1 - e^{-\frac{x^2}{128}} & , x \geq 0 \\ 0 & , \text{altfel} \end{cases}$$

$$Var(x) = E[x^2] - (E[x])^2$$

$$E[x] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x^2}{64} \cdot e^{-\frac{x^2}{128}}\,dx = 10,026$$

$$(E[x])^2 = 100,520$$

$$t = \frac{x^2}{128}$$

$$dt = \frac{2x}{128}\,dx = \frac{x}{64}\,dx$$

$$128t = x^2$$

$$\sqrt{128t} = 2x$$

$$\frac{\sqrt{128t}}{2} = x$$

$$E[x^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty \frac{x^3 \cdot e^{-\frac{x^2}{128}}}{64}\,dx =$$

$$= 128$$

$$Var(x) = 128 - 100,520 = 27,48$$

$$\sqrt{Var(x)} = 5,24$$

$$\frac{F^{-1}(0,75) - F^{-1}(0,25)}{\sqrt{Var(x)}} = \frac{13,32 - 6,068}{5,24} = \frac{7,252}{5,24} = 1,3839$$

(Am făcut integralele în R - se puteau face cu schimbare de variabila, dar pt rapiditate am optat pt. R)

$F^{-1}(x) = ?$

$$1 - e^{-\frac{x^2}{128}} = y$$

$$-e^{-\frac{x^2}{128}} = y - 1 \quad | \cdot (-1)$$

$$e^{-\frac{x^2}{128}} = 1 - y$$

$$-\frac{x^2}{128} = \ln(1-y) \quad | \cdot 128$$

$$-x^2 = 128 \ln(1-y)$$

$$x^2 = -128 \ln(1-y)$$

$$\left.\begin{array}{l} x = \pm\sqrt{-128 \ln(1-y)} \\ x \geq 0 \end{array}\right\} \Rightarrow F^{-1}(y) = \begin{cases} \sqrt{-128 \ln(1-y)}, & y \geq 0 \\ 0, & \text{altfel} \end{cases}$$

$F^{-1}(0{,}75) = 13{,}32$

$F^{-1}(0{,}25) = 6{,}068$

5.

$N \sim Pois(512)$

N - nr de votanți

A - votez pt orban

$P(A) = 0,41$

$P(A^c) = 0,59$

$E[N] = 512 = Var(N)$ (pt. rep. Poisson)

NC - nr. voturi câțu $= N \cdot P(A^c) \sim 0,59 \cdot Pois(512)$

NO - nr. voturi orban $= N \cdot P(A) \sim 0,41 \cdot Pois(512)$

$$V = NC - NO = N \cdot P(A^c) - N \cdot P(A)$$
$$= N(0,59 - 0,41)$$
$$= N \cdot 0,18$$

c) $E[V] = E[N \cdot 0,18] = 0,18 \cdot E[N] = 0,18 \cdot 512 = 92,16$

$Var(V) = Var(N \cdot 0,18) = (0,18)^2 \cdot Var(N) = (0,18)^2 \cdot 512 = 16,588$

b) $E[NC \cdot NO] = E[NC] \cdot E[NO]$ dc. sunt indep.

$E[NC] = 0,59 \cdot 512 = 302,08$

$E[NO] = 0,41 \cdot 512 = 209,92$

$E[NC \cdot NO] = [0,59 \cdot N \cdot 0,41 \cdot N] = E[0,2419 \cdot N^2] =$

$= 0,2419\, E[N^2]$

$= 0,2419 \cdot (E(N))^2$

$= 0,2419 \cdot 512^2$

$= 63412,6336$

$E[NC]\, E[NO] = 302,08 \cdot 209,92 = 63412,6336$

$\Rightarrow \quad NC \perp\!\!\!\perp NO$

Exercițiu 6

$p = 0{,}82$

Repartiția geometrica ne spune că avem k succese cu prob p

până la un eșec și avem formula

$$P(X=k) = p^k (1-p)$$

(pt. k succese și un eșec)

$$\underset{\substack{\downarrow\\ \text{succes}}}{SSS\ldots\ldots S}\ \underset{\substack{\downarrow\\ \text{eșec}}}{E}$$

Dăm cu banul până la obținerea de 4 eșecuri

SSS.... SE..... SE ...... SE....... SE.

Avem eșec la final - așa încheiem secvența

- pot să avem 3 eșecuri la început și apoi doar succese până la al 4-lea eșec sau 4 eșecuri consecutive etc.

$$P(X=k) = p^k \cdot (1-p)^4 \quad C_{k+3}^{3}$$

$$E[14X-15] = 14E[X] - 15 = 1515{,}62$$

$$Var(2X+16) = Var(2X) = 4\,Var(X)$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot p^k (1-p)^4 \cdot C_{k+3}^{3}$$

$$E[X] = (1-p)^4 \cdot \sum_{k=0}^{\infty} k \cdot p^k \cdot \frac{(k+3)!}{3!\cdot k!} = (1-p)^4 \sum_{k=0}^{\infty} p^k \cdot k(k+1)(k+2)(k+3$$

$$= 109{,}33$$

$$E[x^2] = \sum k^2 \; P(k)$$

$$= \sum k^2 \; p^k \; (1-p)^4 \cdot C_{k+3}^{3}$$

$$= (1-p)^4 \cdot \sum k^2 \; p^k \cdot C_{k+3}^{3}$$

$$= (1-p)^4 \cdot \sum p^k \cdot (k+3)(k+2)(k+1) \cdot k^2$$

$$= 2599,7 \cdot$$

$$Var(2X+16) = 4 \cdot Var(x) = -37413,3956$$

$$Var(x) = -9353,34$$