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Tema 5

3. $K, K \geq 1$

$X \sim \mathcal{E}(\lambda)$ variabilă aleatoare repartizată exponențial

$f(x) = \lambda e^{-\lambda x}, \forall x \geq 0$

$F(x) = 1 - e^{-\lambda x}, \forall x \geq 0$

$P(X > x) = 1 - F(x) = e^{-\lambda x}$

$$E[X] = \int_0^\infty x f(x)\, dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x \cdot e^{-\lambda x}\, dx$$

Integrare prin părți:

$g(x) = x \Rightarrow g'(x) = 1$

$h'(x) = e^{-\lambda x} \Rightarrow h(x) = \frac{e^{-\lambda x}}{-\lambda}$

$$E[X] = \lambda \left(\frac{x e^{-\lambda x}}{-\lambda} \Big|_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right)$$

$$E[X] = \lambda \left(\frac{x}{(-\lambda) e^{\lambda x}} \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx \right)$$

$$E[X] = \lambda \left(0 - 0 + \frac{1}{\lambda} \cdot \frac{-e^{-\lambda x}}{\lambda} \Big|_0^\infty \right)$$

$$\frac{x}{e^{\lambda x}} \xrightarrow{x \to \infty} 0$$

$$\frac{x}{e^{\lambda x}} \xrightarrow{x = 0} \frac{0}{1} = 0$$

$\lambda > 0$

$$E[X] = \lambda \left(\frac{-1}{\lambda^2} \frac{1}{e^{\lambda x}} \Big|_0^\infty \right)$$

$$E[X] = \lambda \cdot \frac{-1}{\lambda^2} (0 - 1) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\Rightarrow \int_0^\infty x e^{-x\lambda} dx = \frac{\frac{1}{\lambda}}{\lambda} = \frac{1}{\lambda^2}$$

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$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} \, dx = \lambda \int_0^\infty x^2 e^{-\lambda x} \, dx =$$

$$= \lambda \left(x^2 \frac{-e^{-\lambda x}}{\lambda} \Big|_0^\infty - \int_0^\infty 2x \cdot \frac{e^{-\lambda x}}{-\lambda} \, dx \right)$$

$$= \lambda \left(\frac{x^2}{(-\lambda) e^{+\lambda x}} \Big|_0^\infty + \frac{2}{\lambda} \int_0^\infty x e^{-x\lambda} \, dx \right)$$

$$= \lambda \left(0 - 0 + \frac{2}{\lambda} \cdot \frac{1}{\lambda^2} \right)$$

$$= \lambda \cdot \frac{2}{\lambda^3}$$

$$= \frac{2}{\lambda^2}$$

$$= \frac{2!}{\lambda^2}$$

$$P(n): \; E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} \, dx = \frac{n!}{\lambda^n}$$

Demonstrat prin inducție.

1. $P(1) : E[X] = \int_0^\infty x \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$ (adevarat)

2. $P(k) \to P(k+1)$

$$P(k): \; E[X^k] = \int_0^\infty x^k \lambda \cdot e^{-\lambda x} \, dx = \frac{k!}{\lambda^k}$$

$$P(k+1): \; E[X^{k+1}] = \int_0^\infty x^{k+1} \cdot \lambda \cdot e^{-\lambda x} \, dx = \frac{(k+1)!}{\lambda^{k+1}}$$

Presupunem $P(k)$ adevarat. Dem. $P(k+1)$ adev.

$$E[X^{k+1}] = \int_0^{\infty} x^{k+1} \cdot \lambda \cdot e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x^{k+1} \cdot e^{-\lambda x}\, dx$$

$$= \lambda \left(x^{k+1} \cdot \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} (k+1) x^{k} \cdot e^{-\lambda x} dx \right)$$

$$= \lambda \cdot \left(0 - 0 + \frac{1}{\lambda} (k+1) \int_0^{\infty} x^{k} \cdot e^{-\lambda x}\, dx \right)$$

$$= \lambda \cdot \frac{1}{\lambda} (k+1) \cdot \frac{\frac{k!}{\lambda^k}}{\lambda}$$

$$= \lambda \cdot \frac{(k+1)\, k!}{\lambda^{k+2}}$$

$$= \frac{(k+1)!}{\lambda^{k+1}}$$

$\Rightarrow P(k+1)$ adevărat

$\Rightarrow$ Conform metodei inducției matematice, avem

$P(n)$ adevărat, $\forall n \geq 1$

$$\Rightarrow \quad E[X^k] = \frac{k!}{\lambda^k}, \quad \forall k \geq 1$$

4. X o variabilă aleatoare cu valori în $\mathbb{N}$

De dem că $E[X] = \sum_{m=1}^{\infty} P(X \geq m)$

X variabilă aleatoare cu valori în $\mathbb{N}$ deci:

$$X \sim \begin{pmatrix} 0 & 1 & \dots & K & \dots & N \\ p_0 & p_1 & \dots & p_K & \dots & p_N \end{pmatrix}, N \to \infty$$

$E[X] = \sum_{m=0}^{\infty} x_m \cdot p_m$ (conform definiției)

$= 0 \cdot p_0 + 1 \cdot p_1 + \dots + K \cdot p_K + \dots + N \cdot p_N$

$= 1 \cdot p_1 + 2 \cdot p_2 + \dots + K \cdot p_K + \dots + N \cdot p_N$

$= \sum_{m=1}^{\infty} m \cdot p_m$ (1)

Calculez $\sum_{m=1}^{\infty} P(X \geq m)$

$P(X \geq m) = \sum_{K=m}^{\infty} P(X = K) = \sum_{K=m}^{\infty} p_K = p_m + p_{m+1} + p_{m+2} + \dots + p_N$

$\sum_{m=1}^{\infty} P(X \geq m) = \sum_{m=1}^{\infty} (p_m + p_{m+1} + p_{m+2} + \dots + p_N) =$

$= (p_1 + p_2 + \dots + p_N) + (p_2 + p_3 + \dots + p_N) + \dots + (p_K + p_{K+1} + \dots + p_N) +$

$+ (p_{K+1} + \dots + p_N)$

$= p_1 + 2p_2 + 3p_3 + \dots + K \cdot p_K + \dots + N \cdot p_N$

$= \sum_{m=1}^{\infty} m \cdot p_m$ (2)

Din (1), (2) $\Rightarrow E[X] = \sum_{m=1}^{\infty} P(X \geq m)$

5.

X variabilă repartizată exponențial de parametru α

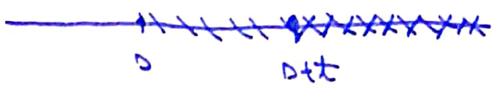
$$f(x) = \alpha \cdot e^{-\alpha x}$$

$$F(x) = 1 - e^{-\alpha x}$$

$$P(X > x) = 1 - F(x) = e^{-\alpha x}$$

„=>" X variabilă repartizată exponențial => prop. lipsei de memorie

$$P(X > t+s \mid X > s)$$



$$(X > s+t) \cap (X > s) = (X > s+t)$$

$$P(X > t+s \mid X > s) = \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\alpha(s+t)}}{e^{-\alpha s}} = \frac{e^{-\alpha s} \cdot e^{-\alpha t}}{e^{-\alpha s}} = e^{-\alpha t}$$

dar $P(X > t) = e^{-\alpha t}$ $\Big\}$ =>

$$\Rightarrow P(X > t+s \mid X > s) = P(X > t)$$

„<=" prop lipsei de memorie => X variabilă repartizată exponențial

$$P(X > t+s \mid X > s) = P(X > t)$$

$$\frac{P(X > t+s)}{P(X > s)} = P(X > t)$$

$P(X > t+s) = P(X > t)\, P(X > s)$

Consider $g(s) = P(X > s)$, $g: \mathbb{R} \to \mathbb{R}_+$ $\Big\}$ =>

$$g(s+t) = g(t)\, g(s)$$

$s = t \Rightarrow g(t+t) = g(2t) = g(t) \cdot g(t) = g^2(t)$

$s = 2t \Rightarrow g(2t+t) = g(2t) \cdot g(t) = g^2(t)\, g(t) = g^3(t)$

$\vdots$

$s = kt \Rightarrow g(kt+t) = g(kt)\, g(t) = g^k(t)\, g(t) = g^{k+1}(t)$

$P(n): \ g(nt) = g^n(t)$

$g(2t) = g^2(t)$, $t>0$

$P(k) \to P(k+1)$

$P(k): \quad g(kt) = g^k(t)$

$P(k+1): \quad g((k+1)t) = g^{k+1}(t)$

Presup. $P(k)$ adev. Dem $P(k+1)$ adev.

$g((k+1)t) = g(kt+t) = g(kt) \cdot g(t) = g^k(t)\, g(t) = g^{k+1}(t)$ adev.

$\Rightarrow P(n)$ adev (prin inducție)

$\Rightarrow \quad g(nt) = g^n(t)$, $t>0$

$t = \frac{1}{n} > 0 \Rightarrow g(n \cdot \frac{1}{n}) = g^n(\frac{1}{n}) \Rightarrow g(1) = g^n(\frac{1}{n})$

$t = 1 > 0 \Rightarrow g(n \cdot 1) = g^n(1) \Rightarrow g(n) = g^n(1)$

$\forall n>0, m>0$: $g(n \cdot \frac{1}{m}) = g^n(\frac{1}{m}) = \left(g(\frac{1}{m})\right)^n = \left(g^{\frac{1}{m}}(1)\right)^n = g^{\frac{n}{m}}(1) = (g(1))^{\frac{n}{m}}$

$g(\frac{n}{m}) = (g(1))^{\frac{n}{m}}$

$s = \frac{n}{m} > 0 \Rightarrow g(s) = (g(1))^s$, >0

$\log(g(s)) = \log(g(1))^s$

$\log(g(s)) = s \log(g(1))$

$\log(g(s)) = s \log\left(\left(\frac{1}{g(1)}\right)^{-1}\right)$

$\log(g(s)) = s\,(-1) \log\left(\frac{1}{g(1)}\right)$

$\Rightarrow e^{\log(g(s))} = e^{(-s)\log\left(\frac{1}{g(1)}\right)}$

$g(s) = e^{-s \log\left(\frac{1}{g(1)}\right)}$

Notez $\lambda = \log \frac{1}{g(1)}$ întrucât $\log \frac{1}{g(1)}$ e o constantă

$\Rightarrow g(s) = e^{-\lambda s}$

$P(X>s) = g(s) = e^{-\lambda s}$

$$P(X>s+t \mid X>s) = \frac{P(X>s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X>t)$$

Dar $P(X>k) = e^{-\lambda k}$ dacă X este o variabilă repartizată exponențial.

Din „$\Rightarrow$" și „$\Leftarrow$" $\Rightarrow$ X variabilă repartizată exponențial $\Leftrightarrow$ X satisface proprietatea lipsei de memorie.

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